NASA CR-
147796

MCDONNELL DOUGLAS TECHNICAL SERVICES CO.
HOUSTON ASTRONAUTICS DIVISION

DESIGN NOTE NO. 1.4-7-38

PROPOSED POWERED EXPLICIT GUIDANCE THRUST INTEGRALS DERIVATION/IMPLEMENTATION

MISSION PLANNING, MISSION ANALYSIS AND SOFTWARE FORMULATION

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7 MAY 1976

This Design Note is Submitted to NASA Under Task Order No. D0509, in Partial Fulfillment of Contract NAS 9-14960.

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(NASA-CR-147796) PROPOSED POWERED EXPLICIT GUIDANCE THRUST INTEGRALS DERIVATION/IMPLEMENTATION. MISSION PLANNING, MISSION ANALYSIS AND SOFTWARE FORMULATION (McDonnell-Douglas Technical
N76-27292
HC $4.50
Unclas
G3/13 44222

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LIST OF SYMBOLS

Symbol	Definition
a	Thrust acceleration
$a(t)$	Thrust acceleration as a function of time, t.
a_L	Acceleration limit during SSME burn
F	Engine thrust
$\underline{G}$	Gravitational acceleration vector
$\underline{i}_f$	Unit thrust vector
K	Time about which the guidance solution is expanded
m	Mass of vehicle (function of time)
m_o	Mass of vehicle at current time
$\dot{m}$	Mass flow rate
$\underline{R}$	Current radius vector measured by navigation
R	Magnitude of $\underline{R}$
$\underline{R}_D$	Desired, target radius vector
$\underline{R}_{GN}$	Position to be gained by thrust (distance-to-go)
$\underline{R}_{grav}$	Change in position due to gravity
$\underline{R}_T$	Change in position imparted by thrust
t	Time
T_{Bi}	Burn time of the $i^{\underline{th}}$ guidance phase of a multi-phase burn
T_{GO}	Remaining burn time (time-to-go)
$\underline{V}$	Current velocity vector measured by navigation
$\underline{\dot{V}}$	Total vehicle acceleration vector
$\underline{\dot{V}}_T$	Vehicle acceleration due to thrust
$\underline{V}_D$	Desired, target velocity vector
$\underline{V}_{GN}$	Velocity to be gained by thrust (velocity-to-go) vector
V_{GO}	Magnitude of velocity-to-go vector
$\underline{V}_{grav}$	Change in velocity due to gravity
$\underline{V}_T$	Change in velocity imparted by thrust
V_{ex}	Engine exhaust velocity

LIST OF SYMBOLS (continued)

Symbol	Definition
$\underline{\lambda}$	Unit vector defining average thrust direction
$\underline{\dot{\lambda}}$	Rate vector defining desired turning rate
$\dot{\lambda}$	Magnitude of $\underline{\dot{\lambda}}$
μ	Gravitational constant
τ	Ratio of vehicle mass to mass flow rate
ω	A constant angular rate used in defining total thrust integrals

1.0 SUMMARY

A new exoatmospheric, powered explicit guidance (PEG) thrust integral formulation and a simple method of implementation is presented in this note. The new thrust integral formulation is significantly simpler than that currently used in PEG. Preliminary estimates indicate a computer storage savings of 220 words, which is approximately 10 percent of the current PEG ascent program. Alternate methods of implementation that could produce even more savings are noted.

The method of implementing the thrust integrals derived herein does not represent a departure from the current PEG implementation approach. Consequently, required simulation verification of the new equations is minimum. Essentially, one equation replaces the higher order thrust integrals of the current PEG equations, and this one equation can be verified analytically. The current PEG higher order integrals (higher than first order) formulation results from approximating the sine and cosine functions as series expansions. These approximations are eliminated in the formulation presented in this note. Therefore, the thrust integral formulation proposed herein is more accurate than the current formulation, when large steering angles are involved. A simple function or factor (f_1) is developed such that the higher order integrals are a product of the first order integrals and the quantity ($1-f_1$). The equation for the factor f_1 is verified analytically. The first order integrals of the current PEG equations are simplified and maintained, and therefore, require no verification.

The proposed equations and method of implementation offer the flexibility of applying a different guidance law by changing the value of one scalar parameter. This additional flexibility, producing more accuracy and optimality, is gained at significantly less cost, and is implemented in such a manner as to have no impact on other GN&C programs, e.g., the G&C steering interface routine

The new thrust integral formulation was tested and verified in simulations of Baseline Reference Missions 1 and 3A. Each simulation of the new integrals yielded virtually the same trajectory and performance as the current integrals, assuming the same guidance law (e.g., linear angle steering). However, the method of implementation proposed in this note allows high performance missions such as Mission 3A, to fly linear tangent steering. The new thrust integral formulation coupled with the proposed implementation method yielded slightly better performance than the current PEG equations for Mission 3A (i.e., +70 pounds MECO weight).

The new thrust integral formulation presented in this note reduces flight computer memory and processing loads for PEG guidance and is readily implemented within existing PEG. It is recommended that this thrust integral formulation be incorporated into the Shuttle powered flight guidance software requirements.

2.0 INTRODUCTION

The Mission Planning and Analysis Division, supported by MDTSCO, is engaged in the development and verification of guidance software requirements for implementation in the Shuttle onboard GN&C computers. This activity is being conducted as established in Track Task Agreements between Rockwell International and the Shuttle Program Office.

Development of the new powered explicit guidance thrust integrals, described herein, was initiated by request of the Shuttle Powered Flight Guidance Working Group, chaired by Aldo Bordano, FM7, in support of the guidance scrub activities started in April 1976.

3.0 DISCUSSION AND ANALYSIS

This section is divided into six subsections preceded by a discussion of the exoatmopsheric powered explicit guidance problem and the relationship of thrust integrals to the solution of this problem. The new thrust integral formulation is developed in the six subsections. Subsection 3.1 provides a slightly simplified development of the basic first order thrust integrals currently used to solve the powered explicit guidance problem. Since the total thrust integrals are integrals of the thrust acceleration vector, the equation defining the optimum unit thrust vector is derived in Subsection 3.2. The approach used in generating a new thrust integral formulation is explained in Subsection 3.3. Partial integrals, i.e., assuming that the time about which the guidance solution is expanded is one half of the time-to-go ($K = T_{GO}/2$), are developed in Subsection 3.4. Total thrust integrals, where $K = T_{GO}/2 + \Delta K$ (ΔK is a small perturbation term), are developed in Subsection 3.5. The proposed method of implementing the new thrust integrals, including flow charts, is presented in Subsection 3.6.

The basic rocket vehicle vacuum flight equation of motion is of the form

$$\dot{\underline{V}} = \frac{F}{m}\,\underline{i}_f + \underline{G}, \text{ where} \tag{1}$$

$\dot{\underline{V}}$ = Total acceleration vector,

F = Engine Thrust,

m = Vehicle mass,

F/m = Thrust acceleration,

$\underline{i}_f$ = Unit vector in desired thrust direction, and

$\underline{G}$ = Gravitational acceleration vector.

The typical powered flight guidance problem is to determine real time values of $\underline{i}_f$ (the unit thrust vector) and the rate of change of $\underline{i}_f$ that steer the vehicle to desired target conditions while expending minimum fuel, i.e., the fuel optimum path is determined between current state and target state.

Explicit guidance provides a closed-form predictor/corrector solution to the two point boundary-value problem discussed above. Closed-form integrals of the thrust acceleration vector, $(F/m)\underline{i}_f$, are involved. The thrust integrals are not exact, in general. However, thrust integrals are developed in the subsequent sections that produce more accuracy than is necessary in practice, since the integrals converge to exactness as the remaining burn time, T_{GO}, approaches the value of zero. The thrust integrals developed in this note are exact for a constant acceleration burn and virtually exact for a low thrust burn. The integrals are simple and provide a qualitative understanding of the exoatmospheric trajectory optimization and powered flight guidance problem. No rigorous mathematical proof is attempted in this note, i.e., a simple and practical engineering approach is taken for constructing a practical solution.

In Section 3.1 it is shown that by making small angle assumptions, the unit thrust vector can be approximated in the form $\underline{i}_f = \underline{a} + \underline{b}t$, where t is time and $\underline{a}$ and $\underline{b}$ are constant vectors, and the thrust acceleration vector ($\dot{\underline{V}}_T$) assumes the form $\dot{\underline{V}}_T = \frac{F}{m}(\underline{a} + \underline{b}t)$. Integration of the above equation involves integrals of the quantities $\frac{F}{m}$ and $\frac{F}{m}t$. These are conventional first order thrust integrals. Multi-stage equations for these basic thrust integrals are developed in Section 3.1.

In Section 3.2 it is shown that the fuel optimum acceleration vector is approximated in the form

$$\dot{\underline{V}}_T = \frac{F}{m}[\underline{a} \cos \omega t + (\underline{b}/\omega) \sin \omega t],$$

where $\underline{a}$ and $\underline{b}$ are constant vectors and ω is a constant angular rate. In practice, the independent variable time, t, in the above equations is redefined as $z = t - K$, where K is the time about which the solution is expanded and the value of t ranges from zero to time-to-go, T_{GO}, i.e., the integration limits are from zero to T_{GO}. The value of K is a function of the thrust integrals. Assuming constant acceleration, $K = T_{GO}/2$ and in any case $K \approx T_{GO}/2$. The explicit guidance problem reduces to determining values of K, $\underline{a}$, and $\underline{b}$ that satisfy the desired target conditions (in the following sections, $\underline{a}$ and $\underline{b}$ are redefined as $\underline{\lambda}$ and $\dot{\lambda}$). When dealing with the sine and cosine functions, it is convenient to define K as $K = T_{GO}/2 + \Delta K$ and solve for a value of ΔK, i.e., the following expansions are employed:

$$\cos(x - \delta) = \cos x \cos\delta + \sin x \sin\delta \text{ and}$$

$$\sin(x - \delta) = \sin x \cos\delta - \cos x \sin\delta, \text{ where}$$

$$x \equiv \omega(t - T_{GO}/2) \text{ and } \delta \equiv \omega\Delta K$$

Integration of the above equation for $\dot{\underline{V}}_T$ involves integrals of the quantities $\frac{F}{m}\cos\omega t$ and $\frac{F}{m}\frac{\sin\omega t}{\omega}$. Assuming constant thrust and mass flow rate, closed form integrals of these quantities do not exist. In the current PEG formulation, it is assumed that $\cos\omega t = 1 - \omega^2 t^2/2$ and that $(\sin\omega t)/\omega = t$. Therefore, the first order integrals, as developed in Section 3.1, as well as second order integrals of $\frac{F}{m}t^2$ are utilized in the current PEG equations. The approach taken in this note eliminates these second order integrals.

Closed form integrals of $\frac{F}{m}\underline{i}_f$ exist if $\frac{F}{m}$ is expanded as a polynomial. The integrals involve integration of quantities of the form $x^n \cos x$ and $x^n \sin x$,* which have closed-form solutions. It is seen that the solution is exact assuming constant acceleration or linear acceleration, as is nearly the case for low thrust OMS burns. And in any case, the solution converges to exactness as T_{GO} approaches zero.

Using mean value considerations, it will be shown in Section 3.5 that the integrals of $\frac{F}{m}\cos\omega t$ can be adequately represented by assuming a constant, mean acceleration. However, this is not the case when dealing with the quantity $\frac{F}{m}\frac{\sin\omega t}{\omega}$. It is shown that the first integral of this quantity is adequately expressed by assuming linear acceleration. It

*Where n is the order of the polynomial.

is illustrated that the second integral of $\frac{F}{m}\frac{\sin \omega t}{\omega}$ requires an order greater than first*. However, this integral can be adequately expressed in terms of the first order integrals developed in Section 3.1. Consequently the integrals developed in the following sections assume a constant, mean acceleration or a linear acceleration profile. A second order acceleration profile is discussed for illustrative purposes.

*i.e., if $\frac{F}{m}$ is expanded as a polynomial.

3.1 Basic Thrust Integrals

The basic first order integrals used for solving the powered explicit guidance problem are developed below. The integrals are the same as in the current PEG equations except that they are somewhat simplified.

For the shuttle vehicle, either constant thrust (and mass flow rate) or constant acceleration is assumed. If constant thrust is assumed, the acceleration magnitude as a function of time is

$a\,(t) = F/(m_o - \dot{m}t)$, where

t = time,

F = constant thrust,

m_o = initial mass,

$\dot{m}$ = constant mass flow rate.

The above equation can be written in the form

$a(t) = (F/\dot{m})/[(m_o/\dot{m}) - t]$ or

$a(t) = V_{ex}/(\tau - t)$, where

$F/\dot{m} = V_{ex}$ (constant exhaust velocity), and

$m_o/\dot{m} \equiv \tau$.

It is convenient to define the parameter τ as

$\tau = m_o/\dot{m} = (F/\dot{m})/(F/m_o)$ or

$\tau = V_{ex}/a_o$, for

V_{ex} is one of the most constant parameters in the propulsion system (it is considered constant in the guidance problem) and the current acceleration, a_o, is measured by the navigation system, whereas m_o and $\dot{m}$ are not so easily determined.

The basic, constant thrust integrals are presented here for the i^{th} stage of an n stage vehicle.

$$L_i = \int_0^{T_{Bi}} a_i dt, \text{ and } S_i = \int_0^{T_{Bi}} \int_0^t a_i ds dt, \text{ where}$$

$$a_i = V_{exi}/(\tau_i - t) \text{ and}$$

$$T_{Bi} = \text{Burn time of } i^{th} \text{ stage.}$$

Employing integration by parts, the integrals are easily determined as

$$L_i = V_{exi} \ln[\tau_i/(\tau_i - T_{Bi})] \equiv -V_{exi} \ln (1 - T_{Bi}/\tau_i) \text{ and} \tag{2}$$

$$S_i = -L_i(\tau_i - T_{Bi}) + V_{exi} T_{Bi} \tag{3}$$

During the SSME constant acceleration burn

$$a_i = a_L \text{ (constant acceleration limit),}$$

and the thrust integrals are simply

$$L_i = a_L T_{Bi} \text{ and} \tag{2-a}$$

$$S_i = .5 L_i T_{Bi}. \tag{3-a}$$

In equation 1, it can be assumed that $\underline{i}_f \approx \underline{\lambda} + \underline{\dot{\lambda}}\,(t - K)$*, where

$K \approx T_{GO}/2$ (to be determined).

T_{GO} = Time-to-go, or remaining burn time.

$\underline{\lambda}$ = constant unit vector when t = K.

$\underline{\dot{\lambda}}$ = constant rate vector normal to $\underline{\lambda}$, i.e.,

$\underline{\lambda} \cdot \underline{\dot{\lambda}} = 0,$

$\underline{\lambda} \cdot \underline{\lambda} = 1.$

It is assumed that $\underline{\dot{\lambda}}K$ is small compared to unity (small angle approximation). Further assume that burn times (T_{Bi}) of all stages are known, and that K, $\underline{\lambda}$ and $\underline{\dot{\lambda}}$ have known values. Now, the thrust component of $\underline{\dot{V}}$, in equation 1, can be integrated to produce velocity and position changes due to thrust ($\underline{V}_T$ and $\underline{R}_T$), i.e.,

$$\underline{\dot{V}}_T = \frac{F}{m}\,[\underline{\lambda} + \underline{\dot{\lambda}}\,(t - K)],$$

$$\underline{V}_T = \int_0^{T_{GO}} \underline{\dot{V}}_T dt, \text{ and}$$

$$\underline{R}_T = \int_0^{T_{GO}} \int_0^{t} \underline{\dot{V}}_T ds dt.$$

The integrals of $\frac{F}{m}\,(t - K)$ are presented in this section, in addition to the integrals of $\frac{F}{m}$. These are conventional first order integrals used in the current PEG equations, and will be employed in the simplified thrust integrals presented in this note.

*This is known as linear tangent guidance (LTG).

Performing the above integration yields equations of the form

$$\underline{V}_T = L\underline{\lambda} + \dot{\underline{\lambda}}\ (J - LK), \text{ and} \tag{6}$$

$$\underline{R}_T = S\underline{\lambda} + \dot{\underline{\lambda}}\ (Q - SK), \text{ where} \tag{7}$$

L and S are as defined by equations 2, 3, 4, and 5 and J and Q are developed in the following paragraph.

Using the same nomenclature as for L_i and S_i,

$$J_i = \int_0^{T_{Bi}} a_i\ (t + t_{oi})\ dt \quad \text{and}$$

$$Q_i = \int_0^{T_{Bi}} \int_0^{t} a_i\ (s + t_{oi})\ dsdt,$$

where t_{oi} is the initial time of each stage (called t_{goa_i} in the current PEG equation) i.e.,

$$t_{oi} = \sum_{j=k}^{i-1} T_{Bj},$$

e.g.,

$$t_{o1} = 0,\ T_{o2} = T_{B1},\ T_{o3} = T_{B1} + T_{B2},\ \text{etc.}$$

Employing integration by parts, the constant thrust integrals are easily determined as

$$J_i = L_i\ \tau_i - V_{exi}\ T_{Bi} + L_i\ t_{oi}, \text{ and}$$

$$Q_i = S_i\ \tau_i - 1/2\ V_{exi}\ T^2_{Bi} + S_i\ t_{oi}.$$

From inspection of equation 3, it follows that

$$J_i = L_i T_{Bi} - S_i + L_i t_{oi} \qquad \text{or}$$

$$J_i = L_i t_{goi} - S_i, \text{ where } t_{goi} \equiv t_{oi} + T_{Bi}. \tag{8}$$

The equation for Q_i is written as

$$Q_i = S_i (\tau_i + t_{oi}) - 1/2\, V_{exi} T_{Bi}^2 . \tag{9}$$

For a constant acceleration phase, $a_i = a_L$ (constant acceleration limit), and the thrust integrals are

$$J_i = S_i + L_i t_{oi} \quad \text{and} \tag{10}$$

$$Q_i = 1/3\, S_i T_{Bi} + S_i t_{oi} \equiv S_i (TB_i/3 + t_{oi}), \text{ where} \tag{11}$$

$$L_i = a_L T_{Bi} \text{ and } S_i = .5\, L_i T_{Bi}.$$

Equation 10 can be written as

$$J_i = L_i T_{Bi} - .5\, L_i T_{Bi} + L_i t_{oi} \quad \text{or}$$

$$J_i = L_i t_{goi} - S_i , \tag{12}$$

where T_{goi} is defined by equation 8.

Inspection of equations 8 and 12 shows that the expression for J_i is the same for constant acceleration as for constant thrust. Summing over all stages (i = k, n), the multi-stage integrals are

$$J = \sum_{i=k}^{n} J_i \quad \text{and} \tag{13}$$

$$Q = \sum_{i=k}^{n} Q_i + J_{oi} T_{Bi}, \text{ where } J_{oi} = \sum_{j=k}^{i-1} J_j . \tag{14}$$

It is easily verified that

$$J = \sum_{i=k}^{n} (L_i t_{goi} - S_i) \equiv LT_{GO} - S, \text{ where}$$

$$T_{GO} = \sum_{i=k}^{n} T_{Bi} \text{ (Time-to-go), and L and S are multi-stage}$$

integrals as defined in equations 4 and 5, i.e.,

$$J = LT_{GO} - S. \tag{15}$$

In practice, K, $\underline{\lambda}$, and $\dot{\underline{\lambda}}$ are not known until after the thrust integrals are formed. It will be seen later that K is chosen such that the coefficient of $\dot{\underline{\lambda}}$ in equation 6 vanishes, i.e.,

$$J - LK = 0 \text{ or}$$

$$K = J/L.$$

Using this equation and equation 15

$$K = T_{GO} - S/L. \tag{16}$$

The first order integrals are summarized as follows:

CONSTANT THRUST

$$L_i = -V_{exi} \ln (1 - T_{Bi}/\tau_i) \quad (2)$$

$$S_i = -L_i (\tau_i - T_{Bi}) + V_{exi} T_{Bi} \quad (3)$$

$$Q_i = S_i (\tau_i + t_{oi}) - \frac{1}{2} V_{exi} T_{Bi}^2 \quad (9)$$

CONSTANT ACCELERATION

$$L_i = a_L T_{Bi} \quad (2\text{-a})$$

$$S_i = \frac{1}{2} L_i T_{Bi} \quad (3\text{-a})$$

$$Q_i = S_i (\frac{1}{3} T_{Bi} + t_{oi}) \quad (11)$$

CONSTANT THRUST OR ACCELERATION

$$J_i = L_i t_{goi} - S_i \quad (12)$$

The respective multi-stage equations for L, S, J, and Q are equations 4, 5, 13 and 14. The above equations are somewhat more simple than the current PEG thrust integral equations in that one equation for J_i applies to constant thrust or constant acceleration.

3.2 Fuel Optimum Unit Thrust Vector*

In this subsection, a well known method of the calculus of variations (the Euler-Lagrange method of multipliers) is employed to derive an equation defining the optimum unit thrust vector time history.

Assuming a spherical earth, it can be shown by employing the Euler-Lagrange method of multipliers that the fuel optimum thrust vector time history is defined by the differential equation

$$\ddot{\underline{\lambda}} = -\omega^2\underline{\lambda} + 3\,\omega^2(\underline{\lambda}\cdot\underline{u}_R)\,\underline{u}_R \text{ , where} \tag{17}$$

$\underline{i}_f$ = Unit ($\underline{\lambda}$) (unit thrust vector), and

$\omega^2 = \mu/R^3$ (ω is circular orbit rate at radius $R \equiv |\underline{R}|$)

μ = gravitational constant

$\underline{R}$ = Radius vector

$\underline{u}_R$ = unit ($\underline{R}$)

The following definitions are made:

$\underline{u} \equiv \underline{\lambda}\,(t)$ i.e., $\underline{\lambda}$ as a function of time.

$z \equiv t-K$, where K is the constant reference time.

$\underline{u}\,(K) \equiv \underline{\lambda}$, constant reference unit vector at $t = K$, or $z = 0$.

$\dot{\underline{u}}\,(K) \equiv \dot{\underline{\lambda}}$, constant reference value at $t = K$.

*This is sometimes called the primer vector.

The following simplifying assumptions and definitions are made:

$\underline{\lambda} \cdot \dot{\underline{\lambda}} = 0$ (i.e., $\dot{\underline{\lambda}}$ is normal to $\underline{\lambda}$).

ω^2 is constant.

$\underline{u}_R$ = unit ($\underline{R} + \underline{R}_D$) a constant mean value of $\underline{u}_R$.

$\underline{R}_D$ = Desired, terminal radius vector.

$\theta \equiv \sin^{-1} (\underline{\lambda} \cdot \underline{u}_R)$, constant mean value of the angle between the thrust vector and the horizontal. (It will be seen later that $\underline{\lambda}$ is approximately the mean value of $\underline{u} = \underline{\lambda}(t)$.)

With the above definition and assumptions, equation 17 can be written as

$$\ddot{\underline{u}} = -\omega^2 \underline{u} + 3\,\omega^2 \sin\theta \cos \omega z\, \underline{u}_R, \text{ where } 3\omega^2 \sin\theta\, \underline{u}_R \tag{18}$$

has a constant, mean value. The cos ωz term is added to compensate for the changing direction of that term when it is not assumed constant. Also, $\underline{\lambda}$ cos ωz is the dominant part of the homogeneous solution to equation 18.

The solution to equation 18 is

$$\underline{u} = \underline{\lambda} \cos \omega z + (\dot{\underline{\lambda}}/\omega) \sin \omega z + \frac{3}{2}(\sin\theta)\omega z \sin \omega z\, \underline{u}_R \tag{19}$$

and

$$\dot{\underline{u}} = -\omega\underline{\lambda} \sin \omega z + \dot{\underline{\lambda}} \cos \omega z \tag{20}$$

$$+\frac{3}{2} \sin \theta\, (\omega \sin \omega z + \omega^2 z \cos \omega z)\, \underline{u}_R$$

Differentiation of equation 20 verifies the solution. $\underline{u}$ is sometimes called the primer vector and $\dot{\underline{u}}$ the primer rate vector.

The term involving sin θ is usually small compared to the other terms, and is ignored in the following sections. For the shuttle, second stage boost to orbit, this term would increase the initial pitch angle by about 2 degrees, which would have very little effect on the burnout weight.

Considering the integral

$\Delta\underline{V} \equiv \int_0^{T_{GO}} \frac{F}{m} \text{ unit } (\underline{u})\, dt$, it is seen that

$\Delta\underline{V}$ (fuel expended) is minimum if $\sin\theta = 0$.

This implies that a maneuver involving no position constraints is optimum if the primer vector is normal to the radius vector at the midpoint of the maneuver, since $\underline{u}_R$ is the unit radius vector at the midpoint of the maneuver, and it will be seen later that the reference time, K, has the value of approximately $T_{GO}/2$, therefore, $\underline{\lambda}$ is approximately the value of the primer vector at the midpoint.

Inspection of equation 20 shows that the optimum primer rate can not have a constant value unless $\sin\theta = 0$. Assuming that the final optimum magnitude of $\dot{\underline{u}}$, $\dot{u}_f$, is known, equation 20 can be used to produce a value of $\dot{\lambda} \equiv |\dot{\underline{\lambda}}|$ to satisfy this final value, i.e.,

$\dot{\underline{u}}_f \cdot \dot{\underline{u}}_f = \dot{u}^2{}_f$ (Substituting $z = T_{GO} - K$ in equation 20).

This produces a quadratic equation from which $\dot{\lambda}$ is determined. The equation can be made very simple by replacing $\underline{u}_R$ with unit $(\dot{\underline{\lambda}})$ $[\underline{u}_R \cdot \text{unit } (\dot{\underline{\lambda}})]$, i.e., only considering the component of this term along the $\dot{\underline{\lambda}}$ vector, and the solution is of the form

$$\dot{\lambda} = \sqrt{f\ (\dot{u}_f,\ \omega)} - D.$$

This could be implemented in an approximate manner by assuming a constant, average primer rate of

$$\dot{\lambda} = \sqrt{f(\dot{u}_f, \omega)} - D/2,$$

and recomputing this parameter each guidance pass.

If $\dot{u}_f = \omega$* then $f(\dot{u}_f, \omega) = \omega^2$ and $\dot{\lambda} = \omega(1 - D/\omega)$, where the term D contains ω. The term D has the sign of $\underline{\dot{\lambda}} \cdot \underline{u}_R$. The term is time varying and vanishes when $T_{GO} = 0$. Also, if $\sin\theta = 0$, $D = 0$. The above equation can be implemented in an approximate manner as $\dot{\lambda} = \omega(1 - D/2\omega)$, where $\omega \equiv \mu/V_D R_D{}^2$.

If the term involving $\sin\theta$ is neglected, equations 19 and 20 become

$$\underline{u} = \underline{\lambda} \cos\omega z + (\underline{\dot{\lambda}}/\omega) \sin\omega z \text{ and} \tag{21}$$

$$\underline{\dot{u}} = -\omega\underline{\lambda} \sin\omega z + \underline{\dot{\lambda}} \cos\omega z\text{, where} \tag{22}$$

$z = t-K,$

$\omega^2 = \mu/R^3,$

$\underline{\lambda} \cdot \underline{\lambda} = 1$, and

$\underline{\lambda} \cdot \underline{\dot{\lambda}} = 0.$

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*i.e., $\omega = \sqrt{\mu/R^3} \sim \mu/V_D R_D{}^2$, where V_D and R_D are desired velocity and radius magnitudes.

Inspection of equation 21 yields interesting qualitative information:

a) If $\omega = \dot{\lambda}$, then $\underline{u} \cdot \underline{u} = 1$, i.e., $\underline{u}$ is a unit vector and the steering angle is linear with time.

b) For a flat earth assumption (i.e., ω approaches zero in equation 21), equation 21 becomes

$$\underline{u} = \underline{\lambda} + \underline{\dot{\lambda}}\,(t-K),$$

which is the well known linear tangent steering law. If $\dot{\lambda}$ is large compared to ω, the above conclusion is approximately true.

c) If there is a position constraint and $\dot{\lambda}$ is not related to ω, then the optimum steering is not necessarily linear angle or linear tangent.

In the following analysis, for simplicity in understanding the concept, it will be assumed that $\omega = \dot{\lambda}$. Later, it will be seen that for Shuttle second stage burn this is not necessary unless throttling is involved (i.e., Return to Launch Site abort (RTLS) and Baseline Reference Mission 3B). For example, experience has shown that linear tangent steering is somewhat more optimum than linear angle steering in cases where $\dot{\lambda} > \omega$ and throttling is not involved. This is especially true in Baseline Reference Mission 3A where the guidance assumes that an engine is going out at the abort mode boundary. Linear tangent steering does not loft the trajectory as much as linear angle, and this is more optimum (if an engine does not go out). However, in Mission 1 where there is no pseudo engine out the performance difference between linear steering and linear tangent steering is much less ($\approx$ 30 lbs. difference in weight at MECO). Saturn guidance used linear steering, which produced an orbital insertion weight loss of 100 pounds out of 250,000 pounds compared to the theoretically optimum solution.

If it is assumed that $\omega = \dot{\lambda}$ (where $\dot{\lambda} \equiv |\dot{\underline{\lambda}}|$), equation 21 becomes

$$\underline{i}_f = \underline{\lambda} \cos \dot{\lambda} (t - K) + (\dot{\underline{\lambda}}/\dot{\lambda}) \sin \dot{\lambda} (t - K), \tag{23}$$

and $\underline{i}_f \cdot \underline{i}_f = 1$, i.e., $\underline{i}_f$ is a unit vector.

Equation 23 is the assumed form of the optimum unit thrust vector that will be used in developing the total thrust integrals.

Since equation 23 produces a very nearly constant value of $\dot{\lambda}$ throughout the flight, it can be assumed in developing the total thrust integrals (in the following analysis), that the converged value of $\dot{\lambda}$ from the last guidance pass is used in defining the total integrals for the current pass. This is an important assumption since $\dot{\lambda}$ for the current pass is not known, if a position constraint is involved, until the value of the vector, $\dot{\underline{\lambda}}$, is determined, and the value of $\dot{\underline{\lambda}}$ is a function of the total integrals.

3.3 Approach to Development of Total Thrust Integrals

The approach used in generating a new thrust integral formulation is explained in this subsection.

Assuming that values of T_{GO} and $\dot{\lambda}$ are available, and that the total thrust integrals (here defined as L_T, J_T, S_T, and Q_T) are known, the thrust velocity and position assume the form,

$$\underline{V}_T = L_T\underline{\lambda} + J_T\,\dot{\underline{\lambda}} = \int_0^{T_{GO}} a(t)\,\underline{i}_f dt, \text{ and} \tag{24}$$

$$\underline{R}_T = S_T\underline{\lambda} + Q_T\dot{\underline{\lambda}} = \int_0^{T_{GO}}\int_0^t a(s)\underline{i}_f dsdt. \tag{25}$$

The total equation of motion is

$$\dot{\underline{V}} = \dot{\underline{V}}_T + \underline{G} \quad \text{(where } \dot{\underline{V}}_T = a(t)\underline{i}_f \text{ and } \underline{G} \text{ is the gravity vector).}$$

The following integrals are now considered:

$$\int_0^{T_{GO}} \dot{\underline{V}}\,dt = \int_0^{T_{GO}} (\dot{\underline{V}}_T + \underline{G})\,dt \quad \text{and}$$

$$\int_0^{T_{GO}}\int_0^t \dot{\underline{V}}dsdt = \int_0^{T_{GO}}\int_0^t (\dot{\underline{V}}_T + \underline{G})\,dsdt.$$

The final conditions are; when $t = T_{GO}$, $\underline{R} = \underline{R}_D$ (desired radius) and $\underline{V} = \underline{V}_D$ (desired velocity), and when $t = 0$, $\underline{R}_0 \equiv \underline{R}$ and $\underline{V}_0 \equiv \underline{V}$.

Performing the above integration results in

$$\underline{V}_D - \underline{V} = \underline{V}_T + \underline{V}_{grav}, \text{ and}$$

$$\underline{R}_D - (\underline{R} + \underline{V}T_{GO}) = \underline{R}_T + \underline{R}_{grav}, \text{ where}$$

$$\underline{V}_{grav} \equiv \int_0^{T_{GO}} \underline{G}dt \text{ and } \underline{R}_{grav} \equiv \int_0^{T_{GO}}\int_0^t \underline{G}dsdt.$$

For the development in this note, it is only necessary to assume that these gravity integrals exist. It is currently planned to make use of the numerical (average G) integration package (already in the navigation system) in order to extrapolate to the final state and obtain gravity effects over the trajectory, i.e., $\underline{V}_{grav}$ and $\underline{R}_{grav}$.

Rearranging the above equations and using equations 24 and 25 for $\underline{V}_T$ and $\underline{R}_T$ results in

$$L_T\underline{\lambda} + J_T\dot{\underline{\lambda}} = \underline{V}_{GN} \quad \text{and} \tag{26}$$

$$S_T\underline{\lambda} + Q_T\dot{\underline{\lambda}} = \underline{R}_{GN}, \quad \text{where} \tag{27}$$

$\underline{V}_{GN} \equiv \underline{V}_D - (\underline{V} + \underline{V}_{grav})$ (velocity-to-go) and

$\underline{R}_{GN} \equiv \underline{R}_D - (\underline{R} + \underline{V}\ T_{GO} + \underline{R}_{grav})$ (distance-to-go).

It can be shown that, from the standpoint of optimality, it is desirable to expand about the $\underline{V}_{GN}$ vector and have no turning rate control in that direction (i.e., $\dot{\underline{\lambda}} \cdot \underline{V}_{GN} = 0$). Therefore, it is desirable to determine a value of K such that $J_T = 0$.

Referring to equation 26, it is seen that if $J_T = 0$, then

$$\underline{\lambda} = \text{Unit}\ (\underline{V}_{GN}) \text{ and } \dot{\underline{\lambda}} \cdot \underline{V}_{GN} = 0, \text{ since } \underline{\lambda} \cdot \dot{\underline{\lambda}} = 0.$$

Performing the vector dot product of $\underline{\lambda}$ and equation 27 it follows that $S_T = \underline{\lambda} \cdot \underline{R}_{GN}$, and equation 27 is used to solve for $\underline{\dot{\lambda}}$ as $\underline{\dot{\lambda}} = \frac{S_T}{Q_T}\left(\frac{\underline{R}_{GN}}{\underline{\lambda}\cdot\underline{R}_{GN}} - \underline{\lambda}\right)$, which implies that $\underline{R}_{GN}$ in equation 27 is replaced with $S_T\left(\frac{\underline{R}_{GN}}{\underline{\lambda}\cdot\underline{R}_{GN}}\right)$. Assuming that all necessary values are known (including K), this completes the guidance solution for $\underline{\lambda}$ and $\underline{\dot{\lambda}}$.

The reference time K can be defined as $K = T_{GO}/2 + \Delta K$. The thrust acceleration vector is written as

$$\underline{\dot{V}}_T = a[\underline{\lambda} \cos \dot{\lambda} z + (\underline{\dot{\lambda}} / \dot{\lambda}) \sin \dot{\lambda} z] \text{ or}$$

$$\underline{\dot{V}}_T = (a \cos \dot{\lambda} z)\, \underline{\lambda} + [(a/\dot{\lambda}) \sin \dot{\lambda} z]\, \underline{\dot{\lambda}}, \text{ where } z = t - T_{GO}/2 - \Delta K,$$

from which it follows (referring to equations 24 and 25) that

$$L_T = \int_0^{T_{GO}} a \cos \dot{\lambda} z \, dt, \quad S_T = \int_0^{T_{GO}} \int_0^{t} a \cos \dot{\lambda} z \, ds\, dt$$

$$J_T = \int_0^{T_{GO}} (a/\dot{\lambda}) \sin \dot{\lambda} z \, dt, \quad Q_T = \int_0^{T_{GO}} \int_0^{t} (a/\dot{\lambda}) \sin \dot{\lambda} z \, ds\, dt.$$

If small angle approximations are made in the above equation for $\underline{\dot{V}}_T$, then

$$\underline{\dot{V}}_T = a(\underline{\lambda} + \underline{\dot{\lambda}} z)$$

and the thrust integrals are the first order integrals of Section 3.1, i.e., $L_T = L$, $S_T = S$, $J_T = J - LK$, and $Q_T = Q - SK$.

The purpose of the next two subsections (3.4 and 3.5) is to determine K (or ΔK) such that $J_T = 0$, and develop functions (F_1, F_2, and F_3) such that

$$L_T = \int_0^{T_{GO}} a \cos \dot{\lambda} z dt = F_1 \int_0^{T_{GO}} a\, dt = F_1 L,$$

$$S_T = \int_0^{T_{GO}} \int_0^{t} a \cos \dot{\lambda} z ds dt = F_3 \int_0^{T_{GO}} \int_0^{t} a ds dt = F_3 S, \text{ and}$$

$$Q_T = \int_0^{T_{GO}} \int_0^{t} (a/\dot{\lambda}) \sin \dot{\lambda} z ds dt = F_2 \int_0^{T_{GO}} \int_0^{t} a z ds dt = F_2 (Q-SK),$$

where L, S, and Q are the first order integrals of Section 3.1. F_1, F_2, and F_3 are simple function of $\dot{\lambda}$, T_{GO}, and ΔK. Therefore, the higher order thrust integrals of the current PEG equations are eliminated by introduction of the factors F_1, F_2, and F_3.

In Section 3.4, partial thrust integrals* are developed assuming $\Delta K = 0$, and in Section 3.5, total thrust integrals are readily derived from the partial integrals where

$$x \equiv \dot{\lambda} (t - T_{GO}/2), \ \delta \equiv \dot{\lambda} \Delta K, \text{ and}$$

$$\cos (x-\delta) = \cos x \cos \delta + \sin x \sin \delta, \text{ and}$$

$$\sin (x-\delta) = \sin x \cos \delta - \cos x \sin \delta.$$

It is important to remember when reading Section 3.4 that, for convenience, K is defined as $K = T_{GO}/2$. In Section 3.5, K is defined as $K = T_{GO}/2 + \Delta K$ and ΔK is chosen such that

$$J_T = \int_0^{T_{GO}} (a/\dot{\lambda})(\sin x \cos \delta - \cos x \sin \delta) dt = 0.$$

*Partial thrust integrals are here arbitrarily defined for convenience as integrals resulting from the assumption that $\Delta K = 0$, or $K = T_{GO}/2$. These integrals are partial in that, in general, $\Delta K \neq 0$, and additional terms are involved. Total integrals include the additional terms when it is assumed that $\Delta K \neq 0$. It will be seen that total integrals are simple functions of the partial integrals and ΔK.

3.4 Partial Thrust Integrals

Partial integrals are developed in this subsection, assuming that $K = T_{GO}/2$ (i.e., $\Delta K = 0$).

The thrust acceleration vector now assumes the form

$$\dot{\underline{V}}_T = a(t)\underline{i}_f \text{ , where} \tag{28}$$

$\underline{i}_f = \underline{\lambda} \cos \dot{\lambda}(t-K) + (\dot{\underline{\lambda}}/\dot{\lambda}) \sin \dot{\lambda}(t-K)$, and

$a(t) = V_{exk}/(\tau_k - t)$ or $a(t) = a_L$,

$K = T_{GO}/2$, and

k = Number of current thrusting phase.

For constant thrust phases, integration of equation 28 involves sine and cosine integrals, which are series expansions, i.e., exact closed form integrals do not exist. However, employing mean value considerations, thrust integrals are developed that are exact for constant acceleration and virtually exact for constant thrust.

Assume that acceleration can be represented as a linear function of time, i.e.,

$a(t) = A + B\,(t-K)$, where A and B are chosen such that

$$\int_0^{T_{GO}} a(t)\,dt = L \text{ and} \tag{29}$$

$$\int_0^{T_{GO}} \int_0^{t} a(s)\,ds\,dt = S\text{, where L and S are defined by} \tag{30}$$

equations 4 and 5.

Performing the integration in equations 29 and 30 and solving for A and B results in

$A = L/T_{GO}$ and

$B = 12D/T_{GO}^3 = 3D/2K^3$, where

$D \equiv LK-S$.

Referring to the above equations for $\dot{\underline{V}}_T$, $\underline{i}_f$, and a(t), the resulting thrust acceleration is represented as

$$\dot{\underline{V}}_T = [A+(B/\dot{\lambda})\ \dot{\lambda}\ (t-K)][\underline{\lambda} \cos \dot{\lambda}\ (t-K) + (\dot{\underline{\lambda}}/\dot{\lambda}) \sin \dot{\lambda}\ (t-k)]^* .$$

Making a change in the independent variable:

$x \equiv \dot{\lambda}(t-K)$,

$\theta \equiv \dot{\lambda}K$, or $\dot{\lambda} = \theta/K$, and

$dt = dx/\dot{\lambda} = (K/\theta)dx$.

The integration limits are as follows:

when $t=0$, $x=-\theta$, and

when $t=T_{GO}$, $x = \theta$.

From the above, it follows that

$d\underline{V}_T = (K/\theta)[A+(BK/\theta)x][\underline{\lambda} \cos x + (\dot{\underline{\lambda}}K/\theta) \sin x]\ dx$, and

$d\underline{R}_T = (K/\theta)[\int_{-\theta}^{x} d\underline{V}\ ]dx$.

Integration of the above equations results in

$\underline{V}_T = \int_{-\theta}^{\theta} d\underline{V}$, and $\underline{R}_T = \int_{-\theta}^{\theta} d\underline{R}_T$, where

$\underline{V}_T$ is velocity change due to thrust and $\underline{R}_T$ is position change due to thrust.

This results in two linear equations of the form:

---

*The term in this equation involving the constant, B, is multiplied by $\dot{\lambda}$ and divided by $\dot{\lambda}$ such that this equation involves terms of the form x cos x and x sin x, where x is defined as $\dot{\lambda}(t - K)$.

$$\underline{V}_T = L_p\underline{\lambda} + J_p\dot{\underline{\lambda}} \quad \text{and} \tag{31}$$

$$\underline{R}_T = S_p\underline{\lambda} + Q_p\dot{\underline{\lambda}}, \quad \text{where} \tag{32}$$

L_p, J_p, S_p, and Q_p are thrust integrals developed below.

The following definitions are made:

$$A_1 \equiv AK/\theta = L/2\theta$$

$$A_2 \equiv A(K/\theta)^2 = LK/2\theta^2$$

$$A_3 \equiv A(K/\theta)^3 = LK^2/2\theta^3$$

$$B_1 \equiv B(K/\theta)^2 = 3D/2K\theta^2$$

$$B_2 \equiv B(K/\theta)^3 = 3D/2\theta^3$$

$$B_3 \equiv B(K/\theta)^4 = 3DK/2\theta^4$$

It follows that

$$d\underline{V}_T = [(A_1 \cos x + B_1 x \cos x)\underline{\lambda} + (A_2 \sin x + B_2 x \sin x)\dot{\underline{\lambda}}]dx,$$

$$d\underline{R}_T = \int_{-\theta}^{x} [(A_2 \cos s + B_2 s \cos s)\,\underline{\lambda} + (A_3 \sin s + B_3 s \sin s)\,\dot{\underline{\lambda}}]\,ds\,dx, \text{ and}$$

$$\underline{V}_T = \int_{-\theta}^{\theta} d\underline{V}_T, \quad \text{and} \quad \underline{R}_T = \int_{-\theta}^{\theta} d\underline{R}_T.$$

Comparing the above equations with equations 31 and 32, it follows that

$$L_p = A_1 \int_{-\theta}^{\theta} \cos x\, dx + B_1 \int_{-\theta}^{\theta} x \cos x\, dx$$

$$J_p = A_2 \int_{-\theta}^{\theta} \sin x\, dx + B_2 \int_{-\theta}^{\theta} x \sin x\, dx$$

$$S_p = A_2 \int_{-\theta}^{\theta}\int_{-\theta}^{x} \cos s\, ds dx + B_2 \int_{-\theta}^{\theta}\int_{-\theta}^{x} s \cos s ds\, dx$$

$$Q_p = A_3 \int_{-\theta}^{\theta}\int_{-\theta}^{x} \sin s\, ds dx + B_3 \int_{-\theta}^{\theta}\int_{-\theta}^{x} s \sin s\, ds dx.$$

Employing integration by parts,

$$\int x \cos x dx = \cos x + x \sin x \quad \text{and}$$

$$\int x \sin x dx = \sin x - x \cos x.$$

The following integrals are readily determined as

$$\int_{-\theta}^{\theta} \cos x dx = 2 \sin \theta, \quad \int_{-\theta}^{\theta} \sin x dx = 0$$

$$\int_{-\theta}^{\theta} x \cos x dx = 0, \quad \int_{-\theta}^{\theta} x \sin x dx = -2 (\theta \cos\theta - \sin\theta)$$

$$\int_{-\theta}^{\theta} \int_{-\theta}^{x} \cos s \, ds dx = 2\theta \sin \theta$$

$$\int_{-\theta}^{\theta} \int_{-\theta}^{x} s \cos s \, ds dx = -4 (\theta \cos\theta - \sin\theta) - 2\theta^2 \sin\theta$$

$$\int_{-\theta}^{\theta} \int_{-\theta}^{x} \sin s \, ds dx = 2 (\theta \cos \theta - \sin\theta)$$

$$\int_{-\theta}^{\theta} \int_{-\theta}^{x} s \sin s \, ds dx = -2\theta (\theta \cos \theta - \sin \theta)$$

From which it follows that

$$L_p = (L/2\theta) (2 \sin\theta), \text{ or}$$

$$L_p = L (1/\theta) \sin \theta.$$

Defining $f_1 = (1/\theta) \sin \theta$, then

$$L_p = f_1 L.$$

The expression for J_p is

$$J_p = (3D/2\theta^3)[-2(\theta \cos \theta - \sin \theta)] \text{ or}$$

$$J_p = D \left\{3[(1/\theta)\sin \theta - \cos \theta]/\theta^2]\right\} = D[3(f_1 - \cos \theta)/\theta^2].$$

Defining $f_2 = 3(f_1 - \cos\theta)/\theta^2$, then

$$J_P = f_2 D.$$

If it is assumed that $\sin\theta = \theta - \theta^3/6$ and $\cos\theta = 1 - \theta^2/2$, then $f_1 = 1 - \theta^2/6$ and $f_2 = 1$, therefore, it is seen that f_1 and f_2 approach the value of unity as θ approaches zero.

The expression for S_P becomes

$$S_P = f_1 LK + D(2f_2 - 3f_1) \quad \text{or}$$

$$S_P = [LK - D(3 - 2f_2/f_1)]f_1, \text{ and}$$

$$Q_P = (-LK^2/3 + DK)f_2.$$

If it is assumed that $f_2/f_1 = 1$ in the equation for S_P, then

$$S_P = f_1 S, \text{ since } D = LK - S.$$

This same result is obtained by assuming that $a = 2S/T_{GO}^2$ (constant, mean value of acceleration for distance) and

$$S_P = a \int_0^{T_{GO}} \int_0^{t} \cos\dot{\lambda}(s-K)\, ds\, dt = f_1 S.$$

This equation for S_P will be used in the following analysis.

In summary:

$\theta = \dot{\lambda}T_{GO}/2$	$J_P = f_2 D$
$f_1 = (1/\theta)\sin\theta$	$S_P = f_1 S$
$f_2 = 3(f_1 - \cos\theta)/\theta^2$	$Q_P = (-LT_{GO}^2/12 + DT_{GO}/2)f_2$,
$L_P = f_1 L$	since $K = T_{GO}/2$.
$D = LT_{GO}/2 - S$	

3.5 Total Thrust Integrals

Total integrals are developed below, where $K = T_{GO}/2 + \Delta K$ (i.e., $\Delta K \neq 0$). The total thrust integrals are now readily obtained, making the following definitions:

$$\delta = \dot{\lambda}\Delta K, \quad x = \dot{\lambda}(t - T_{GO}/2)$$

$$\cos (x-\delta) = \cos x \cos \delta + \sin x \sin \delta$$

$$\sin (x-\delta) = \sin x \cos \delta - \cos x \sin \delta$$

$$a(x) = A + (BK/\theta)x$$

$$J_T = (K/\theta)^2 \int_{-\theta}^{\theta} a(x) \sin (x-\delta)\, dx = 0$$

$$Q_T = (K/\theta)^3 \int_{-\theta}^{\theta} \int_{-\theta}^{x} a(s) \sin (s-\delta)\, ds dx$$

$$S_T = (K/\theta)^2 \int_{-\theta}^{\theta} \int_{-\theta}^{x} a(s) \cos (s-\delta)\, ds dx$$

$$L_T = (K/\theta) \int_{-\theta}^{\theta} a(x) \cos (x-\delta) dx$$

From inspection of the above equations, it follows that:

$$J_T = J_p \cos \delta - (L_p/\dot{\lambda}) \sin \delta = 0 \text{ (since } K/\theta = 1/\dot{\lambda}) \text{ or}$$

$$\tan \delta = \dot{\lambda} J_p/L_p = \tan (\dot{\lambda}\Delta K) = \dot{\lambda}(\dot{L}/L)(f_2/f_1).$$

If it is assumed that $\tan (\dot{\lambda}\Delta K) = \dot{\lambda}\Delta K$* and that $f_2/f_1 = 1$ in the above equation, then

$$\Delta K = D/L = (LT_{GO}/2 - S)/L, \text{ or}$$

---

*It can be shown that for a single-stage, constant thrust burn, ΔK is closely approximated as

$$\Delta K = (T_{GO}/6)(1-e^{-V_{GO}/V_{ex}})/(1+e^{-V_{GO}/V_{ex}}), \text{ i.e., the upper}$$

limit for ΔK is $T_{GO}/6$, and when V_{GO} is small compared to V_{ex}, $\Delta K \approx 0$. So, $\delta = \dot{\lambda}\Delta K$ has an upper limit of $\theta/3$. For a constant acceleration burn, $\Delta K = 0$. For a low thrust OMS burn $\Delta K \sim 0$, and ΔK converges to the value of zero in all cases. So, $\tan\delta = \delta$ is a valid approximation in all cases.

$$\Delta K = T_{GO}/2 - S/L, \text{ and} \tag{33}$$

$$K = T_{GO}/2 + \Delta K = T_{GO} - S/L,$$

which is identical to the current computation of equation 16.

The expression for Q_T becomes

$$Q_T = Q_P \cos\delta - (1/\dot{\lambda})\, S_P \sin\delta.$$

From the above equation for J_T, it is seen that $\sin\delta = (\dot{\lambda} J_P/L_P)\cos\delta$, or $\sin\delta = (\dot{\lambda} D f_2/f_1 L)\cos\delta$, from which

$$Q_T = (-LT_{GO}^2/12 + DT_{GO}/2)\, f_2 \cos\delta - (Df_2 f_1 S/f_1 L)\cos\delta \text{ or}$$

$$Q_T = (-LT_{GO}^2/12 + DT_{GO}/2 - DS/L)\, f_2 \cos\delta. \tag{34}$$

The above equation can be written as

$$Q_T = L[-T_{GO}^2/12 + (\Delta K)^2] f_2 \cos\delta.$$

It follows that

$$S_T = S_P \cos\delta + \dot{\lambda} Q_P \sin\delta, \text{ where } \sin\delta = (\dot{\lambda} J_P/L_P)\cos\delta.$$

The second term in the above equation is small compared to the first term. If it is assumed that

$Q_P \approx -S_P T_{GO}/6$, and

$\sin\delta \approx \delta\cos\delta$, the above equation becomes

$$S_T = (S_P - \dot{\lambda}\delta S_P T_{GO}/6)\cos\delta \text{ or}$$

$$S_T = S(1-\theta\delta/3) f_1 \cos\delta \text{ (since } \dot{\lambda} T_{GO} = 2\theta). \tag{35}$$

The same result is obtained by assuming a constant, mean acceleration of

$$a(s) = 2S/T_{GO}^2$$

and the above approximation for Q_P is not necessary.

The integral L_T is simply

$L_T = L_P \cos\delta$ or

$L_T = Lf_1 \cos\delta$.

The same equation for L_T is obtained by assuming a constant, mean acceleration of $a(x) = L/T_{GO}$.

Equation 34 can be written as

$Q_T = Q_1 F_2$, where

$Q_1 \equiv -LT_{GO}^2/12 + DT_{GO}/2 - DS/L$, and $F_2 = f_2 \cos\delta$.

No approximations were made in deriving the above equation other than the original assumption of linear acceleration. Similarly, if a constant, mean acceleration of $2S/T_{GO}^2$ is assumed, it can be shown that $Q_T = Q_0 F_2$, where $Q_0 = -ST_{GO}/6 - DS/L$. The significance of the above equations for Q_T is that Q_0 and Q_1 are independent of the value of $\dot{\lambda}$, and the function F_2 factors out of Q_0 and Q_1. Simulations have shown that, in general, Q_0 is a fairly crude approximation for the first order integral, Q-SK. Q_1 is a signifant improvement over Q_0, however, an error of approximately 5% can exist, e.g., during the first PEG phase of Baseline Reference Mission 3A.* This reflects an error of approximately 5% in $\dot{\underline{\lambda}}$ and θ, since, from equation 27, it is seen that

$\dot{\underline{\lambda}} = (\underline{R}_{GN} - S_T\underline{\lambda})/Q_T$, i.e., $\dot{\underline{\lambda}}$ is inversely proportional to Q_T.

*In these simulations, Q_T was formed as a multi-stage integral, analogous to the multi-stage value of Q-SK.

Similar to the above considerations, a second order acceleration profile can be assumed, i.e.,

$$a(t) = A + (B/\dot{\lambda})\dot{\lambda}(t-K) + (C/\dot{\lambda}^2)\dot{\lambda}^2(t-K)^2$$

and the expression for Q_T can be expressed as $Q_T = Q_2F_2$, where Q_2 is independent of the value of $\dot{\lambda}$, and is a very good approximation for Q-SK. Closed form integrals exist for any order of a(t), however, the expression Q_n becomes complicated and the integral Q-SK is simpler to implement. By the process of induction, the integral Q_T can be expressed as

$$Q_T = (Q-SK)F_2\text{, where } F_2 = f_2 \cos \delta.$$

The above equations are summarized as:

$$\boxed{\begin{aligned}
&\theta = \dot{\lambda}\, T_{GO}/2 \\
&f_1 = (1/\theta) \sin \theta \\
&f_2 = 3(f_1 - \cos \theta)/\theta^2 \\
&K = T_{GO} - S/L \\
&\delta = \dot{\lambda}(K-T_{GO}/2) \\
&F_1 = f_1 \cos \delta \\
&F_2 = f_2 \cos \delta \\
&F_3 = F_1\ (1 - \theta\delta/3)\ \text{(From equation 35)} \\
&L_T = F_1 L \\
&S_T = F_3 S \\
&Q_T = F_2(Q-SK)
\end{aligned}}$$

where L, S, and Q are the first order, multi-stage integrals defined in Section 3.1.

The results of the above analysis can be summarized as follows:

a) Defining $\Delta K = K - T_{GO}/2$, where K is the constant, reference time about which the solution is expanded, it was shown that the resulting value of ΔK is the same whether a mean, linear acceleration profile is assumed or the actual acceleration profile is assumed. It can also be shown that (assuming linear acceleration or actual acceleration) the multi-stage expression for ΔK is the same as the single-stage expression. It can be concluded that the value of ΔK is directly proportional to the mean slope of the acceleration profile (multi-stage or single-stage), since $\Delta K = D/L$ and the expression for the mean slope is $B = 12D/T_{GO}^3$.

b) The total integrals L_T and S_T are obtained by assuming a constant, mean acceleration ($a = L/T_{GO}$ for deriving L_T, and $a \equiv 2S/T_{GO}^2$ for deriving S_T).

c) The integral Q_T is a higher order integral and approaches the value of $(Q-SK)F_2$ as the order of the assumed acceleration profile increases.

d) Since a constant acceleration can be assumed in deriving the integrals L_T and S_T, it can be concluded that the factors F_1 and F_3, as defined above, apply to an entire maneuver, i.e., do not have to be derived in a multi-stage manner. Since $F_2 \approx 1$, it will be assumed that the above conclusion also applies to this factor.

e) It was shown in Section 3.1 that the expression for the integral J_i, assuming constant acceleration, is the same as that when assuming constant thrust. This simplifies implementation of the thrust integrals, since two different expressions are unnecesssary.

f) The higher order thrust integrals in the current PEG program can be eliminated by introduction of the factors F_1, F_2, and F_3.

3.6 Implementation of Thrust Integrals

The proposed thrust integrals are presented in the flow chart of Figure 3.6-1. For purposes of comparison, the current thrust integrals are presented in Figure 3.6-2. The proposed integrals require about one half the code required by the current integrals and approximately 30 data words (assuming double precision) are saved by eliminating the variables S_i, J_i, Q_i, P_i, and H_i ($i = 1, 3$). Rationale for implementing the integrals as in Figure 3.6-1 was discussed in Section 3.1 and additional discussion is included in this section. The overall method of implementing the thrust integrals is presented in this section.

Currently the desired unit thrust vector is computed in the Guidance and Control (G&C) steering interface routine, not in the PEG routine. The parameters $\underline{\lambda}$, $\dot{\underline{\lambda}}$, and $T_\lambda = T_G + K$ are sent to the G&C steering interface from PEG (where T_G is the guidance time when K was computed). The unit thrust vector is then computed as

$$\underline{i}_f = \text{unit}\,[\underline{\lambda} + (t-T_\lambda)\dot{\underline{\lambda}}], \tag{36}$$

which is identical to

$$\underline{i}_f = \text{unit}\,[\underline{\lambda} - K\dot{\underline{\lambda}} + (t-T_G)\dot{\underline{\lambda}}], \text{ where the} \tag{37}$$

value of $t-T_G$ ranges from near zero to the value of the guidance cycle time (e.g., 2 seconds). Equation 21 of Section 3.2 can be written in an identical form to equation 37 by redefining the parameter T_λ, which is currently the last computation in the thrust integrals. Equation 21 can be used to define $\underline{i}_f$ as $\underline{i}_f = \text{unit}\,[\underline{\lambda} \cos \omega z + (\dot{\underline{\lambda}}/\omega) \sin \omega z]$* where $z \equiv t-T_G-K$. The above equation can be written as

* At this point, the parameter $\dot{\lambda}$ is replaced by ω, since in general $\omega \neq \dot{\lambda}$.

$$\underline{i}_f = \text{unit}\ [\underline{\lambda} + \dot{\underline{\lambda}}\ (\frac{1}{\omega}\tan \omega z)]\ \text{SIGN}\ (\cos \omega z).$$

If the angle ωz is limited to be less than $\pi/2$, SIGN ($\cos \omega z$) = +1, and since $\omega(t-T_G)$ is a small angle, the above equation can be closely approximated as

$$\underline{i}_f = \text{unit}\ [\underline{\lambda} - K(1/\omega k)(\tan \omega k)\dot{\underline{\lambda}} + (t-T_G)\dot{\underline{\lambda}}].$$

The above equation can be written as

$$\underline{i}_f = \text{unit}\ [\underline{\lambda} - K_P\dot{\underline{\lambda}} + (t-T_G)\dot{\underline{\lambda}}],\ \text{where}$$

$$K_P = F_4 K,\ \text{and}\ F_4 \equiv (1/\omega K)\ \tan \omega K).$$

Defining T_λ as $T_\lambda = T_G + K_P$, the above equation becomes

$$\underline{i}_f = \text{unit}\ [\underline{\lambda} + (t-T_\lambda)\dot{\underline{\lambda}}],$$

which is identical to equation 36. Implementation of the thrust integrals in the above manner has no impact on the current G&C steering interface equations, and only one scalar equation is modified in the PEG thrust integrals.

As discussed in Section 3.2, using linear angle steering (i.e., $\omega = \dot{\lambda}$) produces a MECO weight loss for Baseline Reference Mission 3A. Experience has shown that linear tangent steering is more optimum for Mission 3A. The steering is linear tangent if $\omega \approx 0$, e.g., $\omega = .00001$. This capability is implemented by initializing $\omega = .00001$ (in the PEG initialization block) and making the following test in the PEG turning rate block.

If $\omega <$ .00001 or n=3, $\omega = .00001$, where n is the number of phases, and Mission 3A is the only Shuttle maneuver with as many as 3 phases. The above test without "or n = 3" is equivalent to: If $\omega = 0$, $\omega > 0$.

This test is recommended because the parameter f_1 is defined as $f_1 = \frac{1}{\theta}\sin\theta$, where $\theta = \omega T_{GO}/2$, i.e., division by zero is avoided. This results in linear tangent steering and linear tangent prediction. Simulations have shown that this is more optimum for Mission 3A (+70 pounds MECO weight) than the current linear tangent steering and linear sine prediction.

Experience has shown that when the thrust integrals are computed in single precision, they become unstable during the last 40 seconds of an OMS burn. This is probably due to instability in the J and Q integrals. These integrals can be written as

$J = \tau(L - a_o T_{GO})$ and

$Q = \tau(S - \frac{1}{2} a_o T_{GO}^2)$, where a_o is the value of current acceleration. The coefficients of τ, in the above equations, approach the value of zero as T_{GO} approaches zero. Single precision probably produces randomness in these coefficients. If acceleration is expanded as a quadratic, the above equations are approximated as

$$J = \frac{1}{2} a_o T_{GO}^2 \left(1 + \frac{2}{3a_o} \frac{T_{GO}}{\tau}\right) \text{ and}$$

$$Q = \frac{1}{6} a_o T_{GO}^3 \left(1 + \frac{1}{2a_o} \frac{T_{GO}}{\tau}\right).$$

The value of T_{GO}/τ approaches zero as T_{GO} approaches zero, and the expressions for J and Q approach $J = \frac{1}{2} a_o T_{GO}^2$ and $Q = \frac{1}{6} a_o T_{GO}^3$. If an average acceleration of V_{GO}/T_{GO} is assumed the values of J and Q can be approximated in a virtually exact manner during the last 40 seconds of an OMS burn, i.e.,

$$J = \frac{1}{2} V_{GO} T_{GO} \text{ and}$$

$$Q = \frac{1}{6} V_{GO} T_{GO}^2 \quad \text{(at this time } \Delta K \approx 0\text{).}$$

Single precision is adequate if the equations for J and Q are expressed as above, and this result is accomplished if the constant acceleration thrust integrals are used when T_{GO} is less than 40 seconds (for an OMS burn). This capability is implemented in the thrust integrals of Figure 3.6-1.

The equations for the total thrust integrals, as summarized in Section 3.5, produce more accuracy than is necessary in practice. For example, assumptions used for Saturn V/Apollo guidance were $F_1 = F_2 = F_3 = 1$. However, in cases where accurate prediction is desirable (e.g., throttling is involved or deorbit targeting) the F_1 term is important. The angle δ is at most about 4 degrees during shuttle second stage burn, therefore it can be assumed that $\cos \delta = 1$. The quantity $\theta\delta/3$ is at most about .01, and can be neglected*. As observed in Section 3.4, $f_2 = 1$ for third order assumptions on sine and cosine expansions. Also, this term has no direct effect on velocity prediction, therefore, it can be assumed that $f_2 = 1$.

Experience has shown that the most stable manner of implementing the f_1 term is to feed back information from the previous guidance cycle. Currently the quantity $\underline{R}_{BIAS}$ is fed back. The $\underline{R}_{BIAS}$ term can be eliminated by initializing $f_1 = 1$ and defining $S = f_1 S$.

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*However, if throttle control is used, it may be desirable not to neglect this term.

$\underline{R}_{GO}$ is formed without the $\underline{R}_{BIAS}$ term as $\underline{R}_{GO} = \underline{R}_{GN}$, where $\underline{R}_{GN}$ is defined as in Section 3.3. The above form is somewhat more accurate than current implementation of the $\underline{R}_{BIAS}$ term. S is computed in the range-to-go block and f_1, in the predictor block.

The $\underline{V}_{THRUST}$ computation and variable name is eliminated in the predictor block by replacing this quantity with $f_1\underline{V}_{GO}$, and $\underline{R}_{THRUST}$ is replaced with R_{GO}, where f_1 has the value of the current guidance pass.

Currently, the thrust integral Q-SK is used at six places in the PEG equations, and recomputed each time. This is avoided by replacing Q-SK with Q $\equiv$ Q-SK.

The test, if Q-SK $\neq$ 0, can be eliminated in the turning rate block since Q-SK can not be zero unless $T_{GO} = V_{GO} = 0$, and if this is the case the guidance routine has already been aborted.

The ϕ_{MAX} equations are simplified by elimination of the equations for recomputing $\underline{\dot{\lambda}}$ and $\underline{R}_{GO}$, since the main purpose for this is for the computation of $\underline{R}_{BIAS}$, and this computation is eliminated. However, it is desirable to limit the value of f_1, and this is accomplished by maintaining the ϕ_{MAX} test.

The above discussion is summarized below and detailed implementation is discussed. Modifications required in each PEG block are presented. An estimation of code saved in each block is included.

Initialization Block

Eliminate: $\underline{R}_{BIAS} = 0$

Add: $f_1 = 1$ and $\omega = .00001$

(code saved: ≈0 words)

Integrals Block

Replace the equations of Figure 3.6-2 with the equations of Figure 3.6-1.

(code saved: 100 words)

Turning Rate and Range-To-Go-Blocks

Replace Q-SK with Q.

(code saved: 30 words)

Turning Rate Block

Eliminate the following equations:

If Q-SK $\neq$ 0, Then $\dot{\underline{\lambda}} = (\underline{R}_{GO} - S\underline{\lambda})/(Q\text{-}SK)$

, Else $\dot{\underline{\lambda}} = 0$

$\dot{\lambda} = |\dot{\underline{\lambda}}|$

If $K\dot{\lambda} > \phi_{MAX}$, $\dot{\lambda} = \phi_{MAX}/K$

, $\dot{\underline{\lambda}} = \dot{\lambda}$ unit $(\dot{\underline{\lambda}})$

, $\underline{R}_{GO} = S\underline{\lambda} + (Q\text{-}SK)\,\dot{\underline{\lambda}}$

Replace the above equations with:

$\dot{\underline{\lambda}} = (\underline{R}_{GO} - S\underline{\lambda})/Q$

$\omega = |\dot{\underline{\lambda}}|$

If $\omega K > \phi_{MAX}$, $\omega = \phi_{MAX}/K$

If $\omega < .00001$ or $n = 3$, $\omega = .00001$

(code saved: 50 words)

Range-To-Go Block

Define: $\underline{R}_{GN} = \underline{R}_D - (\underline{R} + \underline{V}\, T_{GO} + \underline{R}_{grav})$.

Replace $\underline{R}_{GO} = \underline{R}_{GN} + \underline{R}_{BIAS}$ with

$\underline{R}_{GO} = \underline{R}_{GN}$ and $S = f_1 S$.

(code saved: -10 words)

Predictor Block

Eliminate:

$$\underline{V}_{THRUST} = [L - \frac{1}{2}\dot{\lambda}^2(H-JK)]\underline{\lambda}$$

$$\underline{R}_{THRUST} = [S - \frac{1}{2}\dot{\lambda}^2(P-2QK + SK^2)]\underline{\lambda} + (Q-SK)\dot{\underline{\lambda}}$$

$$\underline{R}_{BIAS} = \underline{R}_{GO} - \underline{R}_{THRUST}$$

Replace with:

$$f_1 = (2/\omega T_{GO}) \sin(\omega T_{GO}/2)$$

Replace $\underline{R}_{THRUST}$ with $\underline{R}_{GO}$ and $\underline{V}_{THRUST}$ with $f_1\, \underline{V}_{GO}$.

(code saved: 50 words)

The total estimated code saved from the above PEG modifications is 220 words. This amounts to 10 percent of the current PEG program.

For OMS maneuvers, involving no position constraints, the thrust integrals could be eliminated completely by sending the total acceleration (thrust and gravity) to the average G routine, returning with predicted velocity and position. An OMS (one stage) version of PEG would be extremely simple implemented in this manner. In general, analytical thrust integrals could be eliminated by use of numerical integration, however, implementation is more difficult if position constraints are involved.

The most accurate and optimum and possibly the simplest manner to implement PEG is to update $\underline{R}_{GO}$ in the PEG update block as $\underline{R}_{GO} = \underline{R}_{GO} - \Delta\underline{V}_S[T_{GO} - \Delta t_G/2]$ and recompute the parameter in the PEG correction block as $\underline{R}_{GO} = \underline{R}_{GO} + \underline{R}_D - \underline{R}_P$. Then, the thrust acceleration vector, $(F/m)\underline{i}_f$, where $\underline{i}_f$ is a unit vector, is sent to the average G routine,* along with the gravity vector, and values of predicted velocity and position are obtained. This eliminates the requirement for gravity integrals and higher order thrust integrals, since these quantities are implicit in the resulting values of $\underline{V}_{GO}$ and $\underline{R}_{GO}$. The entire PEG predictor block would consist of defining a value of the thrust acceleration vector and the integration time step and calling the average G routine.* This would entail multi-stage logic and logic for defining the integration time step. If PEG were implemented in this manner, the term involving sin θ defined in Subsection 3.2, could be incorporated into the thrust acceleration vector, and the PEG program would be equivalent to an accurate onboard calculus of variations trajectory optimization program.

*or some other numerical integration routine.

An alternate consideration is to compute $\underline{R}_{GO}$ as defined above and maintain the current PEG predictor method. However, the variable names and equations for $\underline{V}_{grav}$ and $\underline{R}_{grav}$ can be eliminated and the equations for predicted velocity ($\underline{V}_p$) and position ($\underline{R}_p$) can be written as

$$\underline{V}_p = \underline{V}_{c2} + f_1 \underline{V}_{GO} + \underline{V} - \underline{V}_{c1} \text{ and}$$

$$\underline{R}_p = \underline{R}_{c2} + \underline{R}_{c1} - \underline{R} + f_1 \underline{V}_{GO} T_{GO}/6.$$

In addition, the variable names $\underline{V}_{c2}$ and $\underline{R}_{c2}$ can be eliminated by replacement with $\underline{V}_p$ and $\underline{R}_p$. This method of implementation would result in a savings of approximately 50 words, in addition to the 220 words discussed earlier in this section.

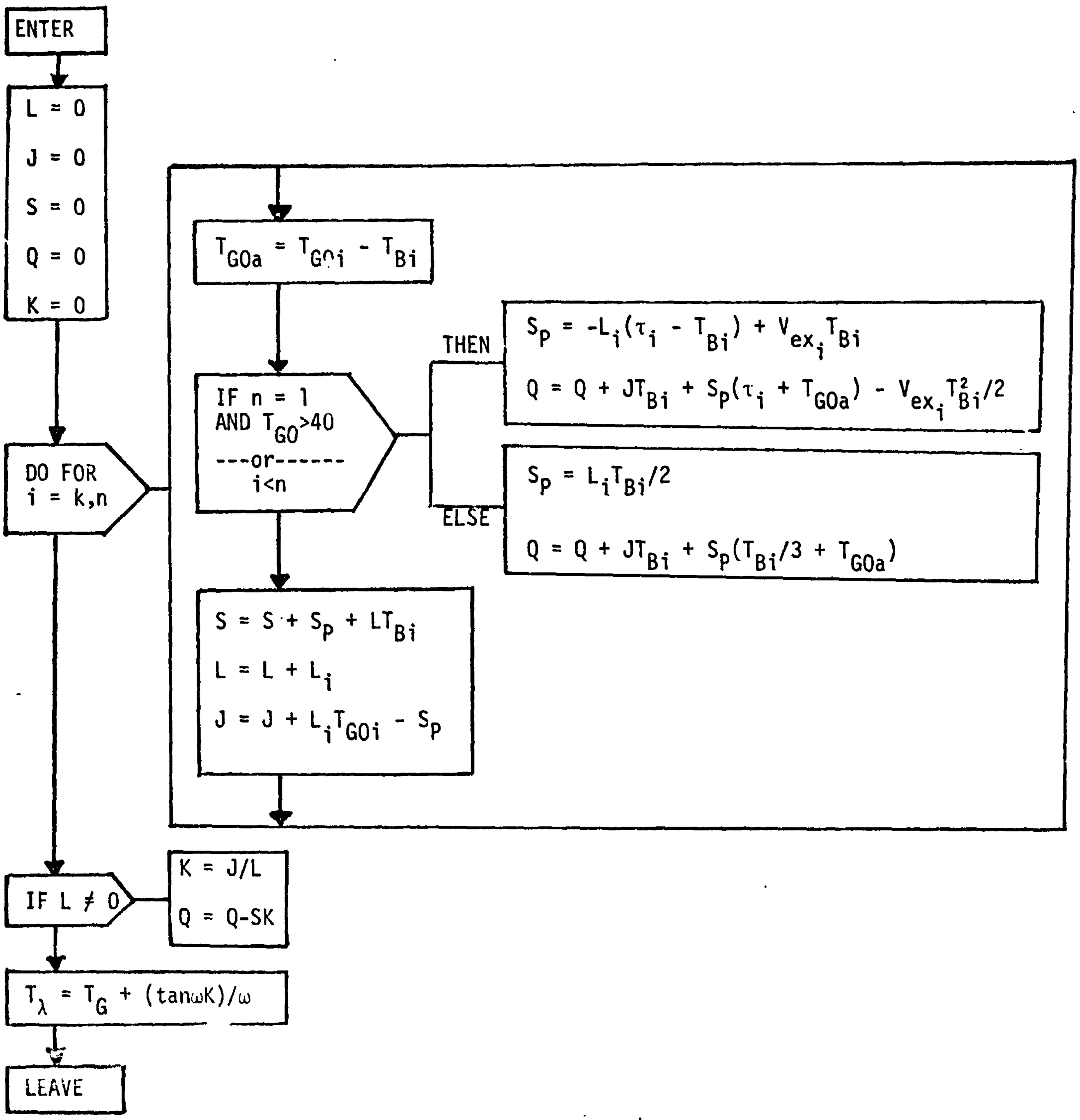

FIGURE 3.6-1 PROPOSED PEG THRUST INTEGRALS

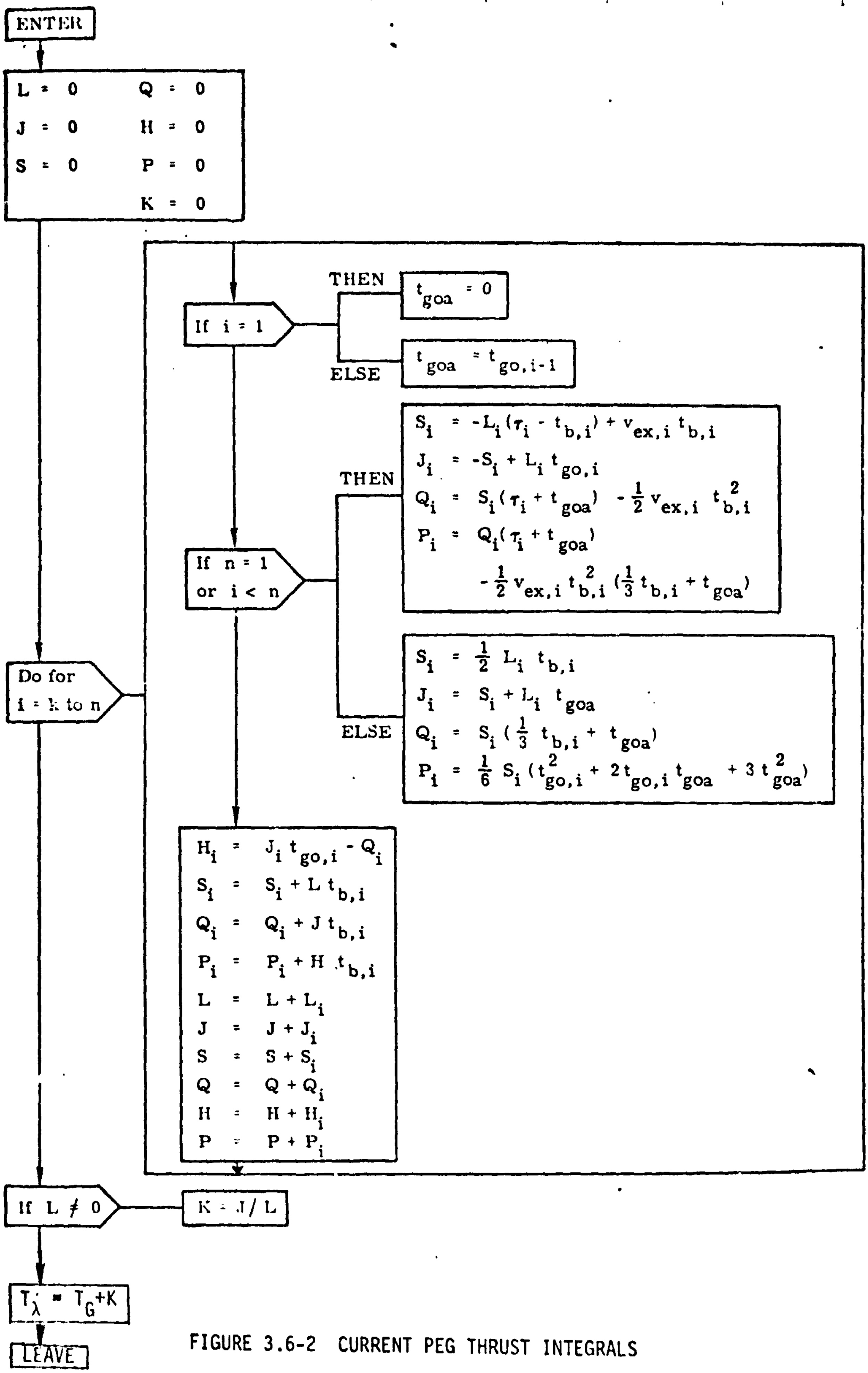

FIGURE 3.6-2 CURRENT PEG THRUST INTEGRALS

4.0 CONCLUSIONS

The new PEG thrust integral formulation and method of implementation derived and presented in this note have the following advantages:

a) Economy - Preliminary estimates indicate a computer storage savings of 220 words, approximately 10 percent of the current PEG program.

b) Ease of implementation - The implementation method proposed in this note does not represent a departure from the current PEG implementation approach, rather a simplification of current equations. The new equations are implemented with very minimum effort.

c) Ease of verification - The implementation approach (logic, sequence of computations, etc.) requires no verification, since it is analogous to current implementation, i.e., stability and convergence are not affected. The first order integrals of the current PEG equations are maintained, therefore, require no verification. The simplified higher order integrals are verified analytically. The primary motive for verification is to demonstrate the additional flexibility.

d) Flexibility - The proposed equations represent a generalized guidance law, i.e., different guidance laws are applied by changing the value of one scalar parameter. This offers optimality and accuracy of prediction for various types of maneuvers.

e) Accuracy - The new equations produce more accuracy than current equations if large central turning angles are involved (e.g., greater than 90 degrees), since sine and cosine approximations are eliminated.

f) There are no known disadvantages to the proposed equations.

Alternate implementation methods, which could result in even more computer storage savings, are noted for possible consideration. However, the alternate methods represent a larger deviation from the current PEG equations than the primary method presented.

The proposed equations were verified in simulations of Baseline Reference Missions 1 and 3A. Slightly improved performance was demonstrated for high performance missions such as Mission 3A (i.e., +70 pounds weight at MECO).

It is recommended that these equations be incorporated into the Shuttle powered explicit guidance software.