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A SPACECRAFT-BORNE GRADIOMETER MISSION ANALYSIS

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16. Abstract

Numerical simulations were performed to obtain the orbit- and attitude-determination requirements of a spacecraft-borne gradiometer mission. Results demonstrated that position determination of 300 meters in the along-track and cross-track directions and 50 meters in the radial direction are mission requirements. The optimal orientation of the gradiometer sensing plane is achieved when the spin vector elevation is 0 degrees. The attitude-determination requirements are 5 degree resolution for spin-vector azimuth and 0.2 degree resolution for spin-vector elevation. When these requirements are met, 3-degree gravity anomalies can be recovered globally with an accuracy of 0.025 mm/s^2 (2.5 mgals). The Appendix documents the mathematical procedures for estimating detailed gravity fields from gradiometer data.

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A SPACECRAFT-BORNE GRADIOMETER MISSION ANALYSIS

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INTRODUCTION

The spacecraft-borne gradiometer is an ideal instrument for globally mapping geopotential fine structure. Mean gravity anomalies exhibit considerable orthogonality in gradiometer data, and local blocks of gravity anomalies in local blocks of gradiometer data can therefore be successfully estimated (Reference 1). Thus, the data-reduction problem implicit in any attempt to obtain a global and detailed gravity-field mapping can be reduced to manageable proportions. Gradiometer data are the only known data type which have this property and which can be obtained on a global basis.

To ensure global coverage and sufficient sensitivity, the gradiometer should be mounted on a satellite in a polar, circular orbit with an altitude no greater than 300 km. Such orbits are difficult to determine. Consequently, to assess the ultimate cost of a gradiometer mission, it is necessary to obtain with some precision the orbit-determination requirements of the mission. Also, because the instrument output is sensitive to spacecraft orientation, the attitude-determination requirements must be specified. In this report, numerical simulations are used to determine the effect of orbit and attitude errors on the recovery of gravity anomalies from gradiometer data. The instrument simulated is a rotating gradiometer of the type under independent development by the Hughes Research Laboratory (Reference 2) and the Bell Aerospace Company (Reference 3). The two instruments are electromechanical analogs of each other, and their outputs therefore relate to the gravity field in a mathematically identical fashion. The Appendix provides a description of the mathematical procedures for extracting gravity-field estimates from gradiometer data.

ORBIT-DETERMINATION REQUIREMENTS FOR A SPACECRAFT-BORNE GRADIOMETER MISSION

Numerical simulation techniques were used to estimate the effect of orbit-determination errors on the extraction of gravity-field information from gradiometer data. A general description of the simulation procedures follows.

Attention was focused on an area of the globe between 0 and 60 degrees latitude and between 0 and 60 degrees longitude. If a reference field of degree and order 12 is used, mean gravity anomalies have a standard deviation of 20 milligals, and adjacent anomalies are virtually uncorrelated (Reference 4). Consequently, geopotential fine structure in this region was described by four hundred 3-degree gravity anomalies whose values in milligalileos (mgal)* were independently chosen from a normal population with a mean of zero and standard deviation of 20. A gradiometer was assumed to be mounted on a satellite in a circular, polar, 300-km orbit. The sensing plane of the instrument coincided with the orbital plane of the satellite. Observations were postulated every 15 seconds, implying a 1-degree latitude by 1.2-degree longitude grid of data. The output of the gradiometer is the sum of a component attributable to the reference field and a component attributable to the anomalous field as represented by the values of the gravity anomalies. The anomalous portion of the gradiometer output was computed by using Stokes' formula to obtain the contribution to the output from each gravity anomaly and summing the results. Instrument noise was assumed to be 0.1 etvos units (1 etvos unit = 10^{-9} gal/cm), and a random-number generator was used to simulate the effects of the noise on the data.

Computational considerations make it necessary to estimate local blocks of gravity anomalies in local blocks of gradiometer data. Reference 3 demonstrates that an efficient estimation strategy is achieved if the data block is the same size as, or somewhat smaller than, the block of estimated anomalies and if the estimates of anomalies in the outer three layers of the block are discarded because of aliasing. This strategy is implemented here. Attention is focused on obtaining good estimates of nine centrally located anomalies subtending a block 27- to 36-degrees longitude and 24- to 33-degrees latitude. These anomalies must be separated from unadjusted anomalies by three layers. Hence, the set of estimated anomalies must subtend a block 18- to 45-degrees longitude and 15- to 42-degrees latitude. Anomalies outside this block are fixed at zero and left unadjusted in the estimation process. Data inside a block 21- to 42-degrees longitude and 18- to 39-degrees latitude were processed by means of a standard least-squares estimator. Each gravity anomaly outside of the block of estimated anomalies was, in effect, fixed at a zero value and left unadjusted. Figure 1 displays the results of a simulation when the orbit determination is assumed to be perfect. The numbers shown represent the difference in mgal between estimated and true values of the gravity anomalies. The poor estimates of anomalies in the outer layers of the estimated block are, of course, attributable to the proximity of unadjusted anomalies. The shaded area represents the data block. The nine centrally located anomalies were recovered with an average accuracy of 0.025 mm/s^2 (2.5 mgal).

Orbit-determination errors were simulated by associating with each data point a satellite longitude, latitude, and height which were different from the true values. The data which are actually processed are assumed to be the difference between the output of the instrument and the output predicted by the reference field. Therefore, an orbit-determination error leads to the subtraction of the wrong reference field. In the along-track and cross-track directions, this effect is negligible; in the radial direction, it is not negligible. At the nominal altitude of 300 km, a radial error will cause the reference value of the gradiometer

*Throughout the text of this document, the measurement unit of milligalileo (mgal) has been converted to a Standard International Unit of millimeters/sec^2 (mm/s^2). For convenience, the illustrations have not been converted. The simple conversion is 1 mgal = 0.01 mm/s^2.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	10	1	-1	1	-2	-2	0	8	-11	0	0	0	0	0
0	0	0	0	0	0	16	0	-3	4	-1	1	-2	2	2	0	0	0	0	0
0	0	0	0	0	0	0	-5	4	2	-1	4	-3	1	-5	0	0	0	0	0
0	0	0	0	0	0	18	2	-1	-3	3	-1	4	0	1	0	0	0	0	0
0	0	0	0	0	0	-19	2	-3	4	-1	-4	1	1	4	0	0	0	0	0
0	0	0	0	0	0	-5	6	-3	-2	0	2	-1	-5	14	0	0	0	0	0
0	0	0	0	0	0	24	-3	6	-1	5	10	4	-5	23	0	0	0	0	0
0	0	0	0	0	0	-82	15	-6	6	-9	-27	-1	15	26	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

SATELLITE ALTITUDE = 300 km, INSTRUMENT ACCURACY = 0.1 eV

Figure 1. Errors in the recovery of 3-degree gravity anomalies (in mgal) from gradiometer data.

output attributable to a 14^{th} degree and order reference field to be biased by 0.0018 etvos units for each meter of error in the radial direction (Reference 4). Therefore, a 50-meter radial error causes the reference field to bias the processed measurement by as much as the standard deviation of the instrument. The simulations took this effect into account. The results were that along-track and cross-track orbit-determination errors lead to very little degradation in the ability of a least-squares estimator to extract gravity information from gradiometer output. In fact, simulated along-track and cross-track errors of 300 meters had no noticeable effect on the recovery of gravity anomalies. Figure 2 shows the sensitivity of gravity/anomaly recovery caused by altitude error. The recoverability of gravity anomalies was computed by averaging the absolute values of the difference between estimated and actual values of the nine centrally located anomalies. Figure 2 shows that altitude errors of up to 50 meters cause no serious degradation in gravity anomaly recovery.

ATTITUDE-DETERMINATION REQUIREMENTS FOR A SPACECRAFT-BORNE GRADIOMETER MISSION

A rotating gradiometer senses gradients in a given plane, that is assumed to be perpendicular to the spin vector of the satellite. The instrument's sensitivity to a gravity anomaly will

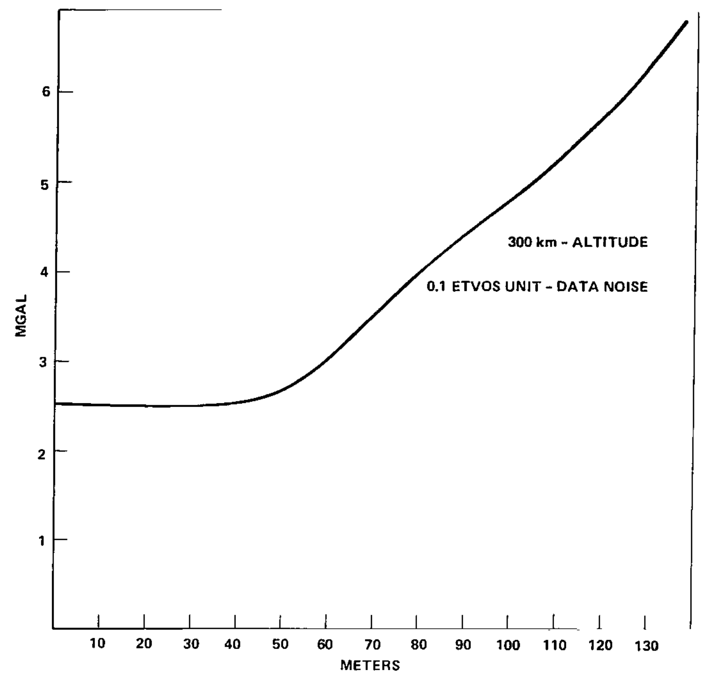

Figure 2. Three-degree gravity anomaly recovery versus altitude errror.

depend on the orientation of the sensing plane. Figure 3 shows the perturbation pattern of an 0.01-mm/s^2 (1-mgal) perturbation of a 3-degree anomaly in gradiometer data. Clearly, the strongest and most localized observability pattern is obtained when the spin vector has a zero elevation. At zero elevation, the perturbation pattern is virtually independent of azimuth angle.

The anomalous part of the gradiometer output is not sensitive to attitude-determination errors. Azimuth and elevation errors of as much as 10 degrees change the anomalous gradiometer output relative to an 0.01-mm/s^2 (1-mgal) perturbation of a 3-degree anomaly by less than 0.0006 etvos units. A more serious difficulty is that subtraction of the reference field leads to a data bias in the case of attitude-determination error. In Reference 4, the gravity-gradient tensor of a (14, 14) reference field at altitude 300 km, latitude 37 degrees, longitude 260.5 degrees is computed. Using this tensor as a reference, the data bias introduced can be computed by azimuth and elevation error at 300 km when a (14, 14) reference field is used. (See Appendix.) A 0.2-degree error in determination of elevation angle leads to a data bias of 0.05 etvos units. Results concerning the impact of orbit-determination errors indicate that such a bias does not lead to a serious degradation of results and that a 0.2-degree error level can be considered a reasonable requirement. The impact of azimuth errors is less severe. A 5-degree azimuth error yields a data bias of 0.03 etvos units. Hence, an error level of 5 degrees is the requirement for azimuth-angle resolution.

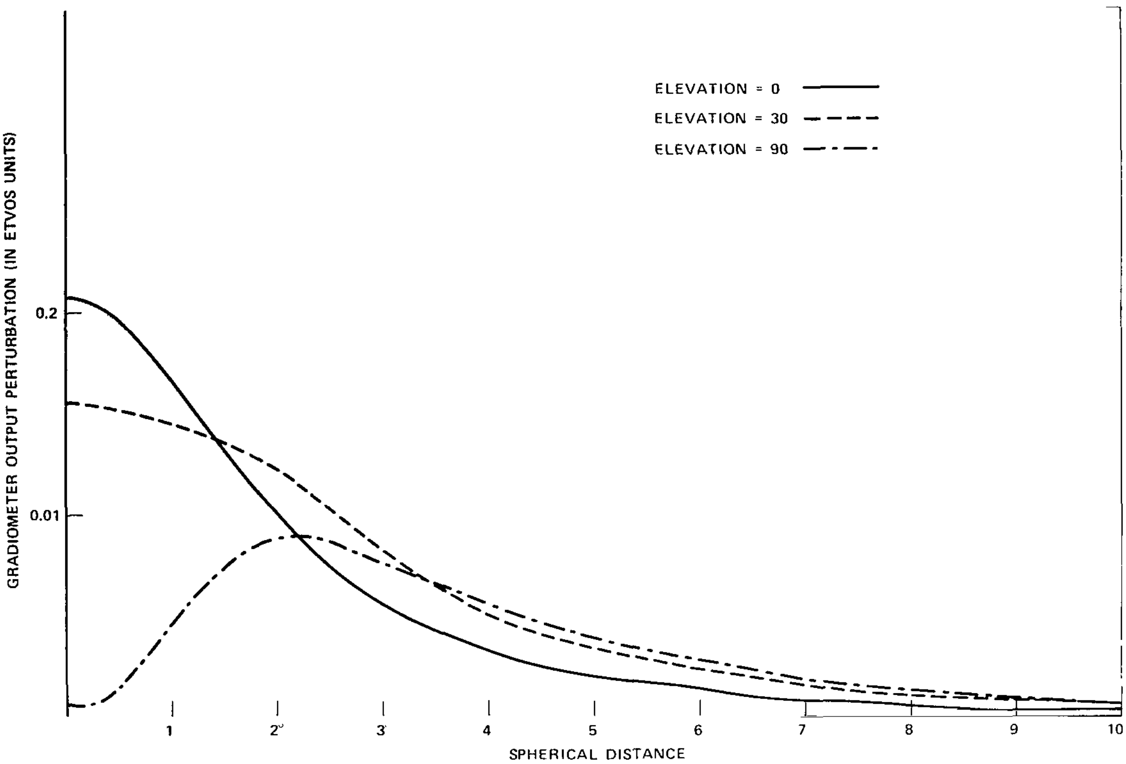

Figure 3. Gradiometer output perturbation caused by 0.01 mm/s^2 (1-mgal) perturbation of 3-degree gravity anomaly (satellite altitude = 300 km)

CONCLUSIONS

Numerical simulations were performed to obtain the orbit- and attitude-determination requirements of a spacecraft-borne gradiometer mission. The simulations were made with the assumption that the spacecraft was in a circular, 300-kilometer, polar orbit. Observations of a 0.1-etvos unit accuracy were assumed every 15 seconds. Aliasing effects of distant, unadjusted gravity anomalies were included. Results demonstrate that position determination of 300 meters in the along-track and cross-track directions and 50 meters in the radial direction are required for a successful spacecraft-borne gradiometer mission. When these requirements are met, the degree gravity anomalies can be recovered with an accuracy of 0.025 mm/s^2 (2.5 mgal).

The optimal orientation of the gradiometer sensing plane is achieved when the spin-vector elevation is 0 degrees. The attitude-determination requirements are 5-degree spin-vector azimuth and 0.2-degree spin vector-elevation.

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Greenbelt, Maryland June 7, 1976

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APPENDIX

MATHEMATICAL PROCEDURES FOR OBTAINING DETAILED GRAVITY FIELDS FROM GRADIOMETER DATA

The following is an outline of the mathematical procedures for using conventional parameter-estimation techniques to recover gravity anomalies from the output of a spacecraft-borne rotating gradiometer. The next section details the precise relationship between the gradiometer output and the ambient gravity field. In succeeding sections, a convenient parameterization of the gravity field is displayed, and the sensitivity matrix of this parameterization relative to gradiometer data is derived. Finally, an algorithm for combining gradiometer data with surface-gravity data and satellite-perturbation data to yield an optimal estimate of the gravity field is displayed.

DERIVATION OF THE OUTPUT SIGNAL OF A ROTATING GRADIOMETER

Figure A-1 represents a rotating gradiometer of the type described in Reference 1. Assume that accelerometers are at positions P_1, P_2, P_3, and P_4 and that the system is rotating in the plane of the figure with angular velocity ω. The output of accelerometer A_i at time T is $\alpha_i(T)$. The four signals are electronically combined to yield an output of the system at time T of

$$\text{output} = [\alpha_1(T) + \alpha_3(T)] - [\alpha_2(T) + \alpha_4(T)] \tag{1}$$

Represent the force attributable to gravity in the sensing plane of the instrument as

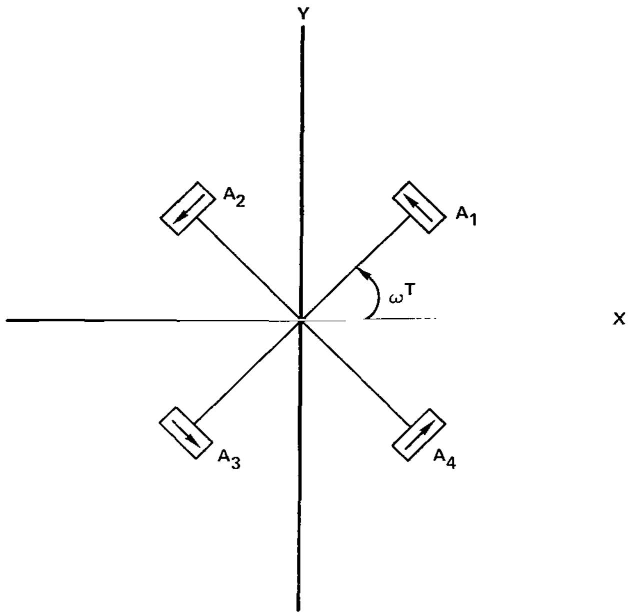
$$\bar{F}(x, y) = F_x \bar{I}_x + F_y \bar{I}_y \tag{2}$$

The output of accelerometer A_i can be expressed in vector cross-produce notation as

$$\alpha_i(T) = \| \bar{F}(x(T), y(T)) \otimes \bar{P}_i(x(T), y(T)) \| \tag{3}$$

For simplicity, assume that $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$, and $\bar{P}_4$ are unit vectors. Then

$$\begin{aligned}
\alpha_1(T) &= \| \bar{F}(\cos \omega T, \sin \omega T) \otimes (\cos \omega T\, \bar{I}_x + \sin \omega T\, \bar{I}_y) \| \\
\alpha_2(T) &= \| \bar{F}(-\sin \omega T, \cos \omega T) \otimes (-\sin \omega T\, \bar{I}_x + \cos \omega T\, \bar{I}_y) \| \\
\alpha_3(T) &= \| \bar{F}(-\cos \omega T, -\sin \omega T) \otimes (-\cos \omega T\, \bar{I}_x + -\sin \omega T\, \bar{I}_y) \| \\
\alpha_4(T) &= \| \bar{F}(\sin \omega T, -\cos \omega T) \otimes (\sin \omega T\, \bar{I}_x + -\cos \omega T\, \bar{I}_y) \|
\end{aligned} \tag{4}$$



$$\text{SIGNAL} = (A_1 + A_3) - (A_2 + A_4)$$
$$= 2\,(\nabla_{xx} - \nabla_{yy})\sin 2\omega T - 4\,\nabla_{xy}\cos 2\omega T$$
$$\text{AMP} = 2\,[(\nabla_{xx} - \nabla_{yy})^2 + 4\nabla_{xy}^2]^{1/2}$$

Figure A-1. Output signal of rotating gradiometer.

From equation 4,

$$\begin{aligned}
\alpha_1(T) &= F_x(\cos\omega T, \sin\omega T)\sin\omega T - F_y(\cos\omega T, \sin\omega T)\cos\omega T \\
\alpha_2(T) &= F_x(-\sin\omega T, \cos\omega T)\cos\omega T + F_y(-\sin\omega T, \cos\omega T)\sin\omega T \\
\alpha_3(T) &= -F_x(-\cos\omega T, -\sin\omega T)\sin\omega T + F_y(-\cos\omega T, -\sin\omega T)\cos\omega T \\
\alpha_4(T) &= -F_x(\sin\omega T, -\cos\omega T)\cos\omega T - F_y(\sin\omega T, -\cos\omega T)\sin\omega T
\end{aligned} \tag{5}$$

Expand $\overline{F}$ (x, y) in a first-order Taylor series about the origin as

$$\begin{aligned}
F_x(x, y) &= F_x(0, 0) + \nabla_x^x x + \nabla_y^x y \\
F_y(x, y) &= F_y(0, 0) + \nabla_x^y x + \nabla_y^y y
\end{aligned} \tag{6}$$

where

$$\nabla_x^x = \frac{\partial F_x}{\partial x}, \quad \nabla_y^x = \frac{\partial F_x}{\partial y}, \quad \nabla_x^y = \frac{\partial F_y}{\partial x}, \quad \nabla_y^y = \frac{\partial F_y}{\partial y}$$

In a conservative force field, $\nabla\frac{x}{y} = \nabla\frac{y}{x}$. Hence, from equations 5 and 6,

$$\begin{aligned}
\alpha_1(T) &= [F_x(0,0) + \nabla_x^x \cos \omega T + \nabla_y^x \sin \omega T] \sin \omega T \\
&\quad - [F_y(0,0) + \nabla_y^x \cos \omega T + \nabla_y^y \sin \omega T] \cos \omega T \\
\alpha_2(T) &= [F_x(0,0) - \nabla_x^x \sin \omega T + \nabla_y^x \cos \omega T] \cos \omega T \\
&\quad + [F_y(0,0) - \nabla_y^x \sin \omega T + \nabla_y^y \cos \omega T] \sin \omega T \\
\alpha_3(T) &= -[F_x(0,0) - \nabla_x^x \cos \omega T - \nabla_y^x \sin \omega T] \sin \omega T \\
&\quad + [F_y(0,0) - \nabla_y^x \cos \omega T - \nabla_y^y \sin \omega T] \cos \omega T \\
\alpha_4(T) &= -[F_x(0,0) + \nabla_x^x \sin \omega T - \nabla_y^x \cos \omega T] \cos \omega T \\
&\quad - [F_y(0,0) + \nabla_y^x \sin \omega T - \nabla_y^y \cos \omega T] \sin \omega T
\end{aligned} \tag{7}$$

Combining equation 7 with equation 1 yields

$$\text{output} = 2(\nabla_x^x - \nabla_y^y) \sin \alpha\, \omega T - 4\nabla_y^x \cos \alpha\, \omega T \tag{8}$$

The maximum output, which is taken to be the essential measurement of the instrument, is

$$\text{AMP} = 2[(\nabla_x^x - \nabla_y^y)^2 + 4\nabla_y^x]^{1/2} \tag{9}$$

Finally, the differential form on the right side of equation 9 is invariant under rotations in the sensing plane of the instrument. This implies, as is necessary, that the gradiometer output is independent of a particular choice of coordinate set.

A PARAMETERIZATION OF THE GEOPOTENTIAL FIELD

To apply standard parameter-estimation techniques to recover the gravity field from gradiometer data, it is necessary to parameterize the gravity field. However, in this context, the parameterization must be carefully chosen. The global recovery of gravity with any degree of detail requires the estimation of many parameters. Unless the parameter set can be decomposed into smaller subsets which are separately estimated, the data-reduction problem is insurmountable. What is required is a representation of the gravity field, the representation of which is perturbed only in a given localized area if a given parameter of the representation is perturbed. Because the output of the gradiometer is an *in-situ* observation, with this type of representation it is possible to estimate local blocks of parameters in local blocks of gradiometer data. There are several such parameterizations. The one presented here is considered to be the most convenient in terms of optimally combining gradiometer data with surface-gravity data and satellite-perturbation data to yield the best estimate of the gravity field.

Represent the geopotential field as

$$W = U + T \tag{10}$$

where U is a reference geopotential generally defined by a low degree and order spherical harmonic expansion and T is the so-called anomalous potential. The anomalous potential T at any point above the surface of the Earth can be expressed by means of the discrete form of Stokes' formula as

$$T(r, \phi, \lambda) = \frac{R}{4\pi} \sum_i S(r, \phi, \lambda, \phi_i', \lambda_i') \cos(\phi_i') \Delta\phi_i' \Delta\lambda_i' \delta g_i(\phi_i', \lambda_i') \tag{11}$$

where $\delta g_i(\phi_i', \lambda_i')$ is a mean gravity anomaly over a block, centered at latitude ϕ_i' and longitude λ_i', and referenced to an equipotential surface defined by the nominal field U. The expression $\cos(\phi_i') \Delta\phi_i' \Delta\lambda_i'$ represents the area of the block on which the i^{th} gravity anomaly is defined, and $S(r, \phi, \lambda, \phi_i', \lambda_i')$ is Stokes' function given by

$$S(r, \phi, \lambda, \phi', \lambda') = t\left[\frac{2}{D} + 1 - 3D - t\cos(\psi)\left(5 + 3\ell n\left[\frac{1 + t\cos(\psi) + D}{2}\right]\right)\right] \tag{12}$$

where

$$\psi = \cos^{-1}[\sin(\phi)\sin(\phi') + \cos(\phi)\cos(\phi')\cos(\lambda' - \lambda)]$$

$$t = \frac{R}{r}$$

$$D = (1 - 2t\cos(\psi) + t^2)^{1/2}$$

and R is the mean radius of the Earth.

It will be useful to obtain the gravity-gradient tensor for the anomalous field in a local topocentric coordinate system. Define an orthogonal coordinate set $\{\bar{1}_r, \bar{1}_\phi, \bar{1}_\lambda\}$ where 1_r is directed outward from the Earth, 1_ϕ is directed northward, and 1_λ is directed eastward. Because of the solenoidal and irrotational nature of a conservative force field, the tensor matrix is symmetric, and the diagonal elements must add to zero. The tensor assumes the form

$$\begin{bmatrix} \dfrac{\partial^2 T}{\partial r^2} & \dfrac{1}{r}\dfrac{\partial^2 T}{\partial r\,\partial\phi} & \dfrac{1}{r}\dfrac{\partial^2 T}{\partial r\,\partial\lambda} \\ & \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial\phi^2} & \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial\phi\,\partial\lambda} \\ & & -\dfrac{\partial^2 T}{\partial r^2} - \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial\lambda^2} \end{bmatrix} \tag{13}$$

From equation 11, we obtain

$$\begin{aligned}
\frac{\partial^2 T}{\partial r^2} &= \frac{R}{4\pi}\sum_i \frac{\partial^2 S}{\partial r^2}(r, \phi, \lambda, \phi_i', \lambda_i') \cos(\phi_i')\, \Delta\phi_i'\, \Delta\lambda_i'\, \delta g_i(\phi_i', \lambda_i') \\
\frac{\partial^2 T}{\partial\phi^2} &= \frac{R}{4\pi}\sum_i \frac{\partial^2 S}{\partial\phi^2}(r, \phi, \lambda, \phi_i', \lambda_i') \cos(\phi_i')\, \Delta\phi_i'\, \Delta\lambda_i'\, \delta g_i(\phi_i', \lambda_i') \\
\frac{\partial^2 T}{\partial\lambda^2} &= \frac{R}{4\pi}\sum_i \frac{\partial^2 S}{\partial\lambda^2}(r, \phi, \lambda, \phi_i', \lambda_i') \cos(\phi_i')\, \Delta\phi_i'\, \Delta\lambda_i'\, \delta g_i(\phi_i', \lambda_i') \\
\frac{\partial^2 T}{\partial\phi\,\partial\lambda} &= \frac{R}{4\pi}\sum_i \frac{\partial^2 S}{\partial\phi\,\partial\lambda}(r, \phi, \lambda, \phi_i', \lambda_i') \cos(\phi_i')\, \Delta\phi_i'\, \Delta\lambda_i'\, \delta g_i(\phi_i', \lambda_i') \\
\frac{\partial^2 T}{\partial r\,\partial\phi} &= \frac{R}{4\pi}\sum_i \frac{\partial^2 S}{\partial r\,\partial\phi}(r, \phi, \lambda, \phi_i', \lambda_i') \cos(\phi_i')\, \Delta\phi_i'\, \Delta\lambda_i'\, \delta g_i(\phi_i', \lambda_i') \\
\frac{\partial^2 T}{\partial r\,\partial\lambda} &= \frac{R}{4\pi}\sum_i \frac{\partial^2 S}{\partial r\,\partial\lambda}(r, \phi, \lambda, \phi_i', \lambda_i') \cos(\phi_i')\, \Delta\phi_i'\, \Delta\lambda_i'\, \delta g_i(\phi_i', \lambda_i')
\end{aligned} \tag{14}$$

An application of the usual chain rule for derivatives yields the following recursion relations for the necessary partial derivatives of Stokes' function

$$
\begin{aligned}
\alpha &= \sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\lambda' - \lambda) \\
\psi &= \cos\alpha \\
\frac{d\alpha}{d\phi} &= \cos\phi \sin\phi' - \sin\phi \cos\phi' \cos(\lambda' - \lambda) \\
\frac{d^2\alpha}{d\phi^2} &= -\alpha \\
\frac{d^2\alpha}{d\lambda d\phi} &= \sin\phi \cos\phi' \sin(\lambda' - \lambda) \\
\frac{d\alpha}{d\lambda} &= \cos\phi \cos\phi' \sin(\lambda' - \lambda) \\
\frac{d^2\alpha}{d\lambda^2} &= -\cos\phi \cos\phi' \cos(\lambda' - \lambda) \\
\frac{d\psi}{d\phi} &= (1 - \alpha^2)^{-1/2} \frac{d\alpha}{d\phi} \\
\frac{d^2\psi}{d\phi^2} &= \left(\frac{d\alpha}{d\phi}\right)^2 (1 - \alpha^2)^{-3/2} \alpha + (1 - \alpha^2)^{-1/2} \frac{d^2\alpha}{d\phi^2} \\
\frac{d^2\psi}{d\lambda d\phi} &= -\alpha (1 - \alpha^2)^{-3/2} \frac{d\alpha}{d\phi} \frac{d\alpha}{d\lambda} + (1 - \alpha^2)^{-1/2} \frac{d^2\alpha}{d\lambda d\phi} \\
\frac{d\psi}{d\lambda} &= (1 - \alpha^2)^{-1/2} \frac{d\alpha}{d\lambda} \\
\frac{d^2\psi}{d\lambda^2} &= \left(\frac{d\alpha}{d\lambda}\right)^2 (1 - \alpha^2)^{-3/2} \alpha + (1 - \alpha^2)^{-1/2} \frac{d^2\alpha}{d\lambda^2}
\end{aligned}
\qquad (15)
$$

$$\frac{dS}{d\psi} = -t^2 \sin\psi \left[\frac{2}{D^3} + \frac{6}{D} - 8 - 3\,\frac{1 - t\cos\psi - D}{D\sin^2\psi} - 3\,\ell n\,\frac{1 - t\cos\psi + D}{2} \right]$$

$$\frac{\partial^2 S}{\partial\psi^2} = -t^2 \cos\psi \left[\frac{2}{D^3} + \frac{6}{D} - 8 - 3\,\frac{1 - t\cos\psi - D}{D\sin^2\psi} - 3\,\ell n\,\frac{1 - t\cos\psi + D}{2} \right]$$

$$+ t^3 \sin^3\psi \left[\frac{6}{D^5} + \frac{6}{D^3} + 3\,\frac{(D-1)}{D^2\sin^2\psi} + 3\,\frac{D+1}{D(1 - t\cos\psi + D)} \right.$$

$$\left. - 3\,\frac{1 - t\cos\psi - D}{D\sin^2\psi}\left(\frac{2\cos\psi}{t\sin^2\psi} + \frac{1}{D^2}\right) \right]$$

$$\frac{\partial^2 S}{\partial r\,\partial\psi} = \frac{t^3\sin\psi}{R}\left[\frac{3(1-t^2)}{D^5} + \frac{4}{D^3} + \frac{6}{D} - 13 - 6\,\ell n\left(\frac{1 - t\cos\psi + D}{2}\right) \right.$$

$$\left. + \frac{t\cos t\,(D+1)}{D(1 - t\cos\psi + D)} \right]$$

$$\frac{\partial^2 S}{\partial r^2} = \frac{t^3}{R^2}\left[(1 - t\cos\psi)\left(\frac{3(1-t^2)}{D^5} - \frac{4}{D^3}\right) - \frac{(1+t^2)}{D^3} - \frac{10}{D} - 18D + 2 \right.$$

$$\left. - 3t\cos\psi\,(15 + 6\,\ell n\left(\frac{1 - t\cos\psi + D}{2}\right) \right]$$

(15 continued)

$$\frac{\partial^2 S}{\partial\phi^2} = \frac{\partial^2 S}{\partial\psi^2}\left(\frac{\partial\psi}{\partial\phi}\right)^2 + \frac{\partial S}{\partial\psi}\,\frac{\partial^2\psi}{\partial\phi^2}$$

$$\frac{\partial^2 S}{\partial\lambda^2} = \frac{\partial^2 S}{\partial\psi^2}\left(\frac{\partial\psi}{\partial\lambda}\right)^2 + \frac{\partial S}{\partial\psi}\,\frac{\partial^2\psi}{\partial\lambda^2}$$

$$\frac{\partial^2 S}{\partial\phi\partial\lambda} = \frac{\partial^2 S}{\partial\psi^2}\,\frac{\partial\psi}{\partial\phi}\,\frac{\partial\psi}{\partial\lambda} + \frac{\partial S}{\partial\psi}\,\frac{\partial^2\psi}{\partial\phi\partial\lambda}$$

$$\frac{\partial^2 S}{\partial r\,\partial\phi} = \frac{\partial^2 S}{\partial r\,\partial\psi}\,\frac{\partial\psi}{\partial\phi}$$

$$\frac{\partial^2 S}{\partial r\,\partial\lambda} = \frac{\partial^2 S}{\partial r\,\partial\lambda}\,\frac{\partial\psi}{\partial\lambda}$$

The parameterization of the gravity field in terms of gravity anomalies has some advantages. The anomalies are, in fact, measurables. Hence, if this parameterization is used to reduce gradiometer data, it becomes easy to optimally combine the output of the gradiometer with surface-gravity data to yield a best-estimate gravity field. Significant data-reduction advantages also exist. Figure A-2 displays the perturbation pattern of a gravity anomaly in gradiometer data. The localized nature of this pattern suggests that if two gravity anomalies are sufficiently separated, their perturbation patterns would not overlap, and the anomalies could be estimated separately without serious aliasing. Local blocks of gravity anomalies can therefore be estimated in local blocks of gradiometer data.

SENSITIVITY MATRIX OF GRAVITY ANOMALIES IN GRADIOMETER DATA

To apply standard parameter-estimation techniques for recovering gravity anomalies from gradiometer data, it is necessary to obtain the so-called sensitivity matrix. The elements of this matrix are the partial derivatives of each gradiometer observation with respect to each estimated gravity anomaly.

To derive the sensitivity matrix, choose a coordinate set $\{\bar{I}_x, \bar{I}_y, \bar{I}_z\}$ so that $\bar{I}_z$ is coincident to the satellite spin vector. From equations 9 and 10, the amplitude of the output signal of the gradiometer can be expressed as

$$\mathrm{AMP} = 2[(W_{xx} + T_{xx} - W_{yy} - T_{yy})^2 + 4(W_{xy} + T_{xy})^2]^{1/2} \tag{16}$$

where W_{xx} and T_{xx} are the second partial derivatives of the reference and anomolous fields in the x direction, W_{yy} and T_{yy} are the second partial derivatives of the reference and anomalous fields, and W_{xy} and T_{xy} are the cross-derivatives of the reference and anomalous fields in the x and y directions. The derivative of the signal amplitude of a particular point (r, ϕ, λ) in space relative to a gravity anomaly centered at (ϕ', λ') is

$$\frac{\partial \mathrm{AMP}}{\partial g} = \frac{\partial \mathrm{AMP}}{\partial T_{xx}} \frac{\partial T_{xx}}{\partial g} + \frac{\partial \mathrm{AMP}}{\partial T_{yy}} \frac{\partial T_{yy}}{\partial g} + \frac{\partial \mathrm{AMP}}{\partial T_{xy}} \frac{\partial T_{xy}}{\partial g} \tag{17}$$

where

$$\begin{aligned}
\frac{\partial \mathrm{AMP}}{\partial T_{xx}} &= \frac{(W_{xx} - W_{yy})}{[(W_{xx} - W_{yy})^2 + 4\,W_{xy}^2]^{1/2}} \\
\frac{\partial \mathrm{AMP}}{\partial T_{yy}} &= \frac{-(W_{xx} - W_{yy})}{[(W_{xx} - W_{yy})^2 + 4\,W_{xy}^2]^{1/2}} \\
\frac{\partial \mathrm{AMP}}{\partial T_{xy}} &= \frac{4\,W_{xy}}{[(W_{xx} - W_{yy})^2 + 4\,W_{xy}^2]^{1/2}}
\end{aligned} \tag{18}$$

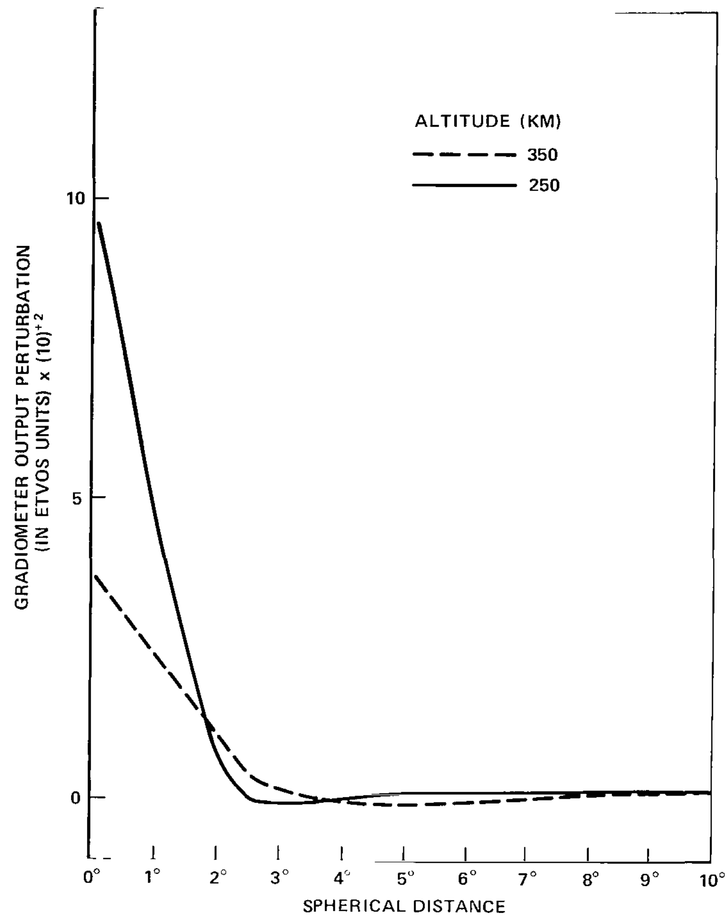


Figure A-2. Gradiometer output perturbation attributable to 0.01 mm/s^2 (1 mgal).

Assuming that the sensing plane of the instrument is near its optimal orientation, the above derivatives can be computed from the gradient tensor of a (14, 14) reference field provided in Reference 2. To five-place accuracy, these derivatives are given as

$$\frac{\partial AMP}{\partial T_{xx}} = 1, \quad \frac{\partial AMP}{\partial T_{yy}} = -1, \quad \frac{\partial AMP}{\partial T_{xy}} = 0.01 \tag{19}$$

Hence, to sufficient accuracy, the necessary derivative can be written as

$$\frac{\partial AMP}{\partial g} = \frac{\partial T_{xx}}{\partial g} - \frac{\partial T_{yy}}{\partial g} \tag{20}$$

The derivatives T_{xx} and T_{yy} are elements of the gradient tensor of the anomalous field in the spacecraft fixed-coordinate set $\left\{ \overline{I}_x, \overline{I}_y, \overline{I}_z \right\}$. Equations 13, 14, and 15 provide the gradient tensor of the anomalous field in a local topocentric-coordinate set. Define

C = gravity-gradient tensor of the anomalous field in a local topocentric-coordinate system

C = gravity-gradient tensor of the anomalous field in the satellite fixed-coordinate set

A = matrix of the linear transformation which maps vectors from the local topocentric system to the spacecraft fixed system

The rules for the tensor calculus assert that

$$C' = A\,C\,A^T \tag{21}$$

If α and β represent the azimuth and elevation respectively of the satellite spin vector, then A is

$$A = \begin{bmatrix} \cos\beta & -\sin\beta\sin\alpha & \cos\alpha \\ 0 & \cos\alpha & \sin\alpha \\ -\sin\beta & -\cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix} \tag{22}$$

Equations 13, 14, 21, and 22 yield

$$\begin{aligned} \frac{\partial T_{xx}}{\partial g} = \frac{R\cos\phi'\,\Delta\phi'\,\Delta\lambda'}{4\pi} \Bigg[& \left(\cos\beta\,\frac{\partial^2 S}{\partial r^2} - \frac{1}{r}\sin\beta\sin\alpha\,\frac{\partial^2 S}{\partial r\,\partial\phi} \right. \\ & \left. + \frac{1}{r}\cos\alpha\,\frac{\partial^2 S}{\partial r\,\partial\lambda} \right) \cos\beta - \left(\frac{1}{r}\cos\beta\,\frac{\partial^2 S}{\partial r\,\partial\phi} - \frac{1}{r^2}\sin\beta\sin\alpha\,\frac{\partial^2 S}{\partial\phi^2} \right. \\ & \left. + \frac{1}{r^2}\cos\alpha\,\frac{\partial^2 S}{\partial\phi\partial\lambda} \right) \sin\beta\sin\alpha + \left(\frac{1}{r}\cos\beta\,\frac{\partial^2 S}{\partial r\,\partial\lambda} - \frac{1}{r^2}\sin\beta\sin\alpha\,\frac{\partial^2 S}{\partial\phi\partial\lambda} \right. \\ & \left. - \cos\alpha\,\frac{\partial^2 S}{\partial r^2} - \frac{1}{r^2}\cos\alpha\,\frac{\partial^2 S}{\partial\lambda^2} \right) \cos\alpha \Bigg] \end{aligned} \tag{23}$$

$$\frac{\partial T_{yy}}{\partial g} = \frac{R\cos\phi'\,\Delta\phi'\,\Delta\lambda'}{4\pi} \Bigg[\left(\frac{1}{r^2}\cos\alpha\,\frac{\partial^2 S}{\partial\phi^2} + \frac{1}{r^2}\sin\alpha\,\frac{\partial^2 S}{\partial\phi\partial\lambda} \right) \cos\alpha$$

$$+\left(\frac{1}{r^2}\cos\alpha\frac{\partial^2 S}{\partial\phi\partial\lambda} - \sin\alpha\frac{\partial^2 S}{\partial r^2} - \frac{1}{r^2}\sin\alpha\frac{\partial^2 S}{\partial\lambda^2}\right)\sin\alpha\Bigg] \quad \text{(23 continued)}$$

Equations 20 and 23, together with recursion relations 15, provide a computational algorithm for obtaining the needed sensitivity matrix.

MINIMUM VARIANCE ESTIMATOR FOR OBTAINING GRAVITY ANOMALIES FROM GRADIOMETER DATA

Let $\widetilde{\delta g}$ be the difference between the true values of the gradiometer observations and nominal values as computed from a reference field. Assume that the i^{th} component of $\widetilde{\delta y}$ is the anomalous gradiometer reading obtained when the position of the satellite is given in spherical coordinates as (r_i, ϕ_i, λ_i). Let g be a fector of numerical values of a global set of gravity anomalies. The ν^{th} component of $\tilde{g}$ is the numerical value of the gravity anomaly centered at latitude ϕ_j and longitude λ_i'. The functional relationship between $\widetilde{\delta y}$ and $\tilde{g}$ can be approximated as

$$\widetilde{\delta y} = A\tilde{g} \tag{24}$$

where A is the sensitivity matrix, the number of whose rows is the number of observations and the number of whose columns is the number of gravity anomalies. The element in the i^{th} row and j^{th} column is the partial derivative of the i^{th} gradiometer measurement with respect to the j^{th} gravity anomaly.

The actual output of the gradiometer, minus the values computed from the reference field, provides direct observations δy of $\widetilde{\delta y}$ with statistics

$$\delta y = \widetilde{\partial y} + \nu,\ E(\nu) = 0,\ E(\nu\nu^T) = Q \tag{25}$$

Estimates of mean free-air gravity anomalies obtained from satellite tracking and gravimetry measurements are available. Unless this information is correctly factored into the gravity-anomaly estimates obtained from gradiometer data, the resultant estimates are not optimal. Consequently, assume the existence of an *a priori* estimate g' of $\tilde{g}$ with statistics

$$g' = \tilde{g} + \alpha,\ E(\alpha) = 0,\ E(\alpha\alpha^T) = P_1 \tag{26}$$

Under these assumptions, the minimum variance estimator of g is known to be

$$\hat{g} = (A^T Q^{-1} A + P_1^{-1})^{-1} (A^T Q^{-1} \delta y + P_1^{-1} g') \tag{27}$$

And the covariance matrix of this estimator is given by

$$E[(\hat{g} - \tilde{g})(\hat{g} - \tilde{g})^T] = (A^T Q^{-1} A + P_1^{-1})^{-1} \tag{28}$$

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