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FRACTURE OF COMPOSITE ORTHOTROPIC

PLATES FOR MATERIALS TYPE II

by

Feridun Delale

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ABSTRACT

The fracture problem of laminated plates which consist of orthotropic layers is considered. The orthotropic material is assumed to be of type II. Symmetrical cracks are located normal to the bimaterial interfaces. The external loads are applied away from the crack region. Three cases are considered:

a) the case of internal cracks
b) the case of broken laminates
c) the case of a crack crossing the interface.

A general formulation of the problem is given for plane strain and generalized plane stress cases. The singular behavior of stresses at the crack tips and at the interfaces is studied. In each case the stress intensity factors are computed for various crack geometries.

1. INTRODUCTION

In recent years composites have attracted considerable attention largely because of their favorable crack propagation characteristics and of the flexibility they offer in designing a variety of structural components. In sheet structures experiments also show that the use of buffer strips with a relatively low stiffness may help to arrest a propagating crack. The composite materials are in general anisotropic. However, because of mathematical difficulties in most of the recent studies relating to composite laminates, the materials are assumed to be isotropic [1-5]. An orthotropic strip containing a crack and bonded to two orthotropic half planes were considered in [6,7]. The problem of periodically arranged orthotropic strips containing cracks has recently been investigated by Delale and Erdogan [8]. In this work it has been shown that regarding the roots of the characteristic equation orthotropic materials can be classified as of type I or of type II. In [8] the materials of both layers are assumed to be of type I. The study of the elastic properties of orthotropic materials shows that, materials of type II are as practical as of type I. In the present study it is assumed that the composite plate consists of periodically arranged orthotropic strips of type II. Symmetrical cracks are located normal to the bimaterial interfaces and the external loads are applied away from the crack region.

A general formulation of the problem is given for plane strain and generalized plane stress cases by the use of Fourier Integral Transform Technique. The resulting singular integral equations are solved numerically and the stress intensity factors are computed for various crack geometries.

2. DISPLACEMENT AND STRESS FIELDS FOR STRIPS

The composite plate shown in Figure 1 consists of two sets of periodically arranged strips of width $2h_1$ and $2h_2$. Symmetrical cracks of length 2a and 2b are located normal to the bimaterial interfaces. The external load is applied away from the crack region. Using the superposition technique the solution of the actual traction-free crack problem may be obtained by superposing the solution of an uncracked homogeneous strip and the solution of a cracked strip where the self-equilibrating crack surface tractions are the only external forces (see Figure 2). For an orthotropic solid the displacements satisfy the following field equations:

$$\begin{aligned} \beta_1 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \beta_3 \frac{\partial^2 v}{\partial y \partial x} &= 0 \\ \frac{\partial^2 v}{\partial x^2} + \beta_2 \frac{\partial^2 v}{\partial y^2} + \beta_3 \frac{\partial^2 u}{\partial x \partial y} &= 0 \end{aligned} \tag{2.1}$$

where β_i $(i=1,2,3)$ are elastic constants defined in Appendix A. The constants β_i have different expressions for plane strain and generalized plane stress cases. In our analysis, we will consider the generalized plane stress case. But the analysis is also valid for the plane strain case, simply by redefining the elastic constants.

In [8], it has been shown that solutions satisfying equations (2.1), yield the following characteristic equation:

$$s^4 + \beta_4 s^2 + \beta_5 = 0 \tag{2.2}$$

The roots of (2.2) are:

$$\begin{aligned} s_1 &= \omega_1 + i\omega_2 = \sqrt{(-\beta_4+\beta_6)/2} \\ s_2 &= \omega_3 + i\omega_4 = \sqrt{(-\beta_4-\beta_6)/2} \\ s_3 &= -s_1 \;, \quad s_4 = -s_2 \end{aligned} \tag{2.3}$$

s_1 and s_2 are both real or complex conjugates.

The material is defined as type I when both roots are real and as type II when they are complex conjugates. In [8] it has been assumed that the material of both strips is of type I. In this study, the orthotropic materials will be assumed to be of type II. For materials of type II the roots of equation (2.2) can be written as:

$$s_1 = \omega_o + i\omega_2$$

$$s_2 = \omega_o - i\omega_2$$

where ω_o is taken as positive. Furthermore noting that

$$u(x,y) = -u(-x,y) \quad ; \quad v(x,y) = -v(x,-y)$$

and u and v vanish when y goes to infinity, for y>0, the displacement expressions can be obtained by adding the solutions defined by equations (3.1) and (3.6) of [8].

Thus, for materials of type II the displacements for each strip or layer are:

$$u(x,y) = \frac{4}{\pi}\int_0^\infty [A(\alpha)\cos(\omega_2\alpha x)\sinh(\omega_o\alpha x)$$

$$+ C(\alpha)\sin(\omega_2\alpha x)\cosh(\omega_o\alpha x)]\cos\alpha y d\alpha$$

$$+ \frac{2}{\pi}\int_0^\infty [E(\alpha)\cos(\omega_2\alpha y/\sqrt{\beta_5})\, e^{-\omega_o\alpha y/\sqrt{\beta_5}}$$

$$+ G(\alpha)\sin(\omega_2\alpha y)/\sqrt{\beta_5})\, e^{-\omega_o\alpha y/\sqrt{\beta_5}}]\sin\alpha x d\alpha$$

$$v(x,y) = \frac{4}{\pi}\int_0^\infty \{A(\alpha)[\beta_7'\cos(\omega_2\alpha x)\cosh(\omega_o\alpha x)$$

$$- \beta_7''\sin(\omega_2\alpha x)\sinh(\omega_o\alpha x)] + C(\alpha)[\beta_7''\cos(\omega_2\alpha x)\cdot$$

$$\cdot\cosh(\omega_o\alpha x) + \beta_7'\sin(\omega_2\alpha x)\sinh(\omega_o\alpha x)]\}\sin\alpha y d\alpha$$

$$+ \frac{2}{\pi}\int_0^\infty \{E(\alpha)[-\beta_9'\cos(\omega_2\alpha y/\sqrt{\beta_5}) - \beta_9''\sin(\omega_2\alpha y/\sqrt{\beta_5})]$$

$$+ G(\alpha)[\beta_9''\cos(\omega_2\alpha y)/\sqrt{\beta_5}) - \beta_9'\sin(\omega_2\alpha y/\sqrt{\beta_5})]\}\cdot$$

$$\cdot\, e^{-\omega_o\alpha y/\sqrt{\beta_5}}\cos\alpha x d\alpha \qquad (2.4a,b)$$

Differentiating (2.4a,b) and using the stress-strain relations, the stress expressions for generalized plane stress case are obtained as:

$$\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_x}\sigma_x(x,y) = \int_0^\infty \{2A(\alpha)[-\Delta_2\sin(\omega_2\alpha x)\cdot$$

$$\cdot\sinh(\omega_o\alpha x) + \Delta_1\cos(\omega_2\alpha x)\cosh(\omega_o\alpha x)]$$

$$+ 2C(\alpha)[\Delta_1\sin(\omega_2\alpha x)\sinh(\omega_o\alpha x)$$

$$+ \Delta_2\cos(\omega_2\alpha x)\cosh(\omega_o\alpha x)]\}\ \alpha\cos\alpha y d\alpha$$

$$+ \int_0^\infty \{E(\alpha)[\Delta_3\cos(\omega_2\alpha y/\sqrt{\beta_5}) + \Delta_4\sin(\omega_2\alpha y/\sqrt{\beta_5})]$$

$$+ G(\alpha)[-\Delta_4\cos(\omega_2\alpha y/\sqrt{\beta_5}) + \Delta_3\sin(\omega_2\alpha y/\sqrt{\beta_5})]\} \cdot$$

$$\cdot\ e^{-\omega_o\alpha y/\sqrt{\beta_5}}\ \alpha\cos\alpha x d\alpha$$

$$\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y}\sigma_y(x,y) = \int^\infty \{2A(\alpha)[-\Delta_6\sin(\omega_2\alpha x) \cdot$$

$$\cdot \sinh(\omega_o\alpha x) + \Delta_5\cos(\omega_2\alpha x)\cosh(\omega_o\alpha x)]$$

$$+ 2C(\alpha)[\Delta_5\sin(\omega_2\alpha x)\sinh(\omega_o\alpha x)$$

$$+ \Delta_6\cos(\omega_2\alpha x)\cosh(\omega_o\alpha x)]\}\alpha\cos\alpha y d\alpha$$

$$+ \int_0^\infty \{E(\alpha)[\Delta_7\cos(\omega_2\alpha y/\sqrt{\beta_5}) + \Delta_8\sin(\omega_2\alpha y/\sqrt{\beta_5})]$$

$$+ G(\alpha)\ [-\Delta_8\cos(\omega_2\alpha y/\sqrt{\beta_5}) + \Delta_7\sin(\omega_2\alpha y/\sqrt{\beta_5})]\} \cdot$$

$$e^{-\omega_o\alpha y/\sqrt{\beta_5}}\ \alpha\cos\alpha x d\alpha$$

$$\frac{\pi}{2G_{xy}}\tau_{xy}(x,y) = \int_0^\infty \{2A(\alpha)[\Delta_9\cos(\omega_2\alpha x)\sinh(\omega_o\alpha x)$$

$$- \Delta_{10}\sin(\omega_2\alpha x)\cosh(\omega_o\alpha x)] + 2C(\alpha)[\Delta_9\sin(\omega_2\alpha x) \cdot$$

$$\cdot \cosh(\omega_o\alpha x) + \Delta_{10}\cos(\omega_2\alpha x)\sinh(\omega_o\alpha x)]\}\alpha\sin\alpha y d\alpha$$

$$+ \int_0^\infty \{E(\alpha)[-\Delta_{11}\sin(\omega_2\alpha y/\sqrt{\beta_5}) + \Delta_{12}\cos(\omega_2\alpha y/\sqrt{\beta_5})]$$

$$+ G(\alpha)[\Delta_{11}\cos(\omega_2\alpha y/\sqrt{\beta_5}) + \Delta_{12}\sin(\omega_2\alpha y/\sqrt{\beta_5})]\} \cdot$$

$$\cdot\ e^{-\omega_o\alpha y/\sqrt{\beta_5}}\ \alpha\sin\alpha x d\alpha\ .$$

β_i, β_j' and Δ_k are elastic constants given in Appendix A. A superscript * will be used for the elastic constants and the unknown functions when these expressions are used for the second strip.

3. FORMULATION OF THE PROBLEM

The unknown functions $A(\alpha)$, $C(\alpha)$ etc., which appear in the displacement and stress expressions can be determined by satisfying the following boundary and continuity conditions:

$$u_1(h_1,y) = u_2(-h_2,y) \quad ,$$

$$v_1(h_1,y) = v_2(-h_2,y) \quad , \quad (0\leq y<\infty) \quad . \tag{3.1a,b}$$

$$\sigma_{1x}(h_1 y) = \sigma_{2x}(-h_2,y) \quad ,$$

$$\tau_{1xy}(h_1,y) = \tau_{2xy}(-h_2,y) \quad , \quad (0\leq y<\infty) \quad . \tag{3.2a,b}$$

$$u_1(0,y) = 0 \quad , \quad \tau_{1xy}(0,y) = 0 \quad , \quad (0\leq y<\infty) \quad . \tag{3.3a,b}$$

$$u_2(0,y) = 0 \quad , \quad \tau_{2xy}(0,y) = 0 \quad , \quad (0\leq y<\infty) \quad . \tag{3.4a,b}$$

$$\tau_{1xy}(x_1,0) = 0 \quad , \quad |x_1|<h_1 \quad ,$$

$$\tau_{2xy}(x_2,0) = 0 \quad , \quad |x_2|<h_2 \quad . \tag{3.5a,b}$$

$$\sigma_{1y}(x_1,0) = -p_1(x_1) \quad , \quad |x_1|<a \quad ,$$

$$v_1(x_1,0) = 0 \quad , \quad a<|x_1|<h_1 \quad . \tag{3.6a,b}$$

$$\sigma_{2y}(x_2,0) = -p_2(x_2) \quad , \quad |x_2|<b \quad ,$$

$$v_2(x_2,0) = 0 \quad , \quad b<|x_2|<h_2 \quad . \tag{3.7a,b}$$

The conditions (3.3a,b) and (3.4a,b) are satisfied identically. Using (3.5a,b) we obtain

$$G(\alpha) = -\frac{\Delta_{12}}{\Delta_{11}} E(\alpha) \quad \text{and} \quad G^*(\alpha) = -\frac{\Delta_{12}^*}{\Delta_{11}^*} E^*(\alpha) \quad . \tag{3.8a,b}$$

Defining,

$$\frac{\partial v_1(x_1,0)}{\partial x_1} = \phi(x_1) \text{ such that } \phi(x_1) = 0 \text{ for } |x_1|>a$$

and

$$\frac{\partial v_2(x_2,0)}{\partial x_2} = \phi^*(x_2) \text{ such that } \phi^*(x_2) = 0 \text{ for } |x_2|>b \;,$$

the mixed boundary conditions (3.6a,b) and (3.7a,b) reduce to the following singular integral equations:

$$\Delta_{14} \int_{-a}^{a} \frac{\phi(t)}{t-x_1}\,dt + \int_{o}^{\infty} \{2A(\alpha)[-\Delta_6 \sin(\omega_2\alpha x_1)$$

$$\sinh(\omega_o\alpha x_1) + \Delta_5\cos(\omega_2\alpha x_1)\cosh(\omega_o\alpha x_1)]$$

$$+ 2C(\alpha)[\Delta_5\sin(\omega_2\alpha x_1)\sinh(\omega_o\alpha x_1)$$

$$+ \Delta_6\cos(\omega_2\alpha x_1)\cosh(\omega_o\alpha x_1)]\}\alpha d\alpha$$

$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y} p_1(x_1) \qquad -a<x_1<a$$

and

$$\Delta_{14}^* \int_{-b}^{b} \frac{\phi^*(t)}{t-x_2}\,dt + \int_{o}^{\infty} \{2A^*(\alpha)[-\Delta_6^* \sin(\omega_2^*\alpha x_2)$$

$$\sinh(\omega_o^*\alpha x_2) + \Delta_5^*\cos(\omega_2^*\alpha x_2)\cosh(\omega_o^*\alpha x_2)]$$

$$+ 2C^*(\alpha)[\Delta_5^*\sin(\omega_2^*\alpha x_2)\sinh(\omega_o^*\alpha x_2)$$

$$+ \Delta_6^*\cos(\omega_2^*\alpha x_2)\cosh(\omega_o^*\alpha x_2)]\}\alpha d\alpha$$

$$= -\frac{\pi(1-\nu_{xy}^*\nu_{yx}^*)}{2E_y^*} p_2(x_2) \qquad -b<x_2<b \qquad (3.9a,b)$$

Using the continuity conditions (3.1a,b) and (3.2a,b), the unknown functions $A(\alpha)$, $C(\alpha)$, $A^*(\alpha)$ and $C^*(\alpha)$ are found to be:

$$A(\alpha) = \frac{1}{\Delta_o} [R_1 f_1 + R_2 g_1 + R_3 h_1 + R_4 m_1]$$

$$C(\alpha) = \frac{1}{\Delta_o} [R_1 f_2 + R_2 g_2 + R_3 h_2 + R_4 m_2]$$

$$A^*(\alpha) = \frac{1}{\Delta_o} [R_1 f_3 + R_2 g_3 + R_3 h_3 + R_4 m_3]$$

$$C^*(\alpha) = \frac{1}{\Delta_o} [R_1 f_4 + R_2 g_4 + R_3 h_4 + R_4 m_4] \qquad (3.10a,b,c,d)$$

The expressions for $\Delta_o(\alpha)$, $R_i(\alpha)$, $f_i(\alpha)$, $g_i(\alpha)$, $h_i(\alpha)$, $m_i(\alpha)$, $(i=1,4)$ are given in Appendix B.

Substituting (3.10a,b,c,d) into (3.9a,b), we obtain:

$$\int_{-a}^{a} \frac{\phi(t)}{t-x_1} + \int_{-a}^{a} k_{11}(x_1,t)\phi(t)dt + \int_{-b}^{b} k_{12}(x_1,t)\phi^*(t)dt$$
$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y\Delta_{14}} p_1(x_1) \qquad -a<x_1<a$$

$$\int_{-b}^{b} \frac{\phi^*(t)}{t-x_2} dt + \int_{-a}^{a} k_{21}(x_2,t)\phi(t)dt + \int_{-b}^{b} k_{22}(x_2,t)\phi^*(t)dt \qquad (3.11a,b)$$
$$= -\frac{\pi(1-\nu^*_{xy}\nu^*_{yx})}{2E^*_y\Delta^*_{14}} p_2(x_2) \qquad -b<x_2<b$$

where

$$k_{11}(x_1,t) = \int_0^\infty k_1(x_1,t,\alpha)e^{-\omega_o\alpha(h_1-t)} d\alpha$$

$$k_{12}(x_1,t) = \int_0^\infty k_2(x_1,t,\alpha)e^{-\omega^*_o\alpha(h_2-t)} d\alpha$$

$$k_{21}(x_2,t) = \int_0^\infty k_3(x_2,t,\alpha)e^{-\omega_o\alpha(h_1-t)} d\alpha$$

$$k_{22}(x_2,t) = \int_0^\infty k_4(x_2,t,\alpha)e^{-\omega^*_o\alpha(h_2-t)} d\alpha$$

The expressions for k_i $(i=1,4)$ are given in Appendix B.

The integrands of the kernels k_{ij} $(i,j=1,2)$ are bounded for all values of α, but are singular of order $1/\alpha$ when $\alpha=0$. In [8], it has been shown that using the single-valuedness conditions $\int_{-a}^{a}\phi(t)dt=0$ and $\int_{-b}^{b}\phi^*(t)dt=0$ the kernels k_{ij} can be integrated numerically.

4. CASE OF BROKEN LAMINATES

When one of the cracks touches the interface (i.e., $a=h_1$ or $b=h_2$) we obtain the case of broken laminates. In this case, for example, for $a=h_1$ the kernel k_{11} becomes unbounded as $(x_1,t)\to\pm h_1$ simultaneously. Therefore the kernel k_{11} must be considered in two parts:

$$k_{11}(x_1,t) = k_{11s}(x_1,t) + k_{11f}(x_1,t)$$

where k_{11s} is the singular part and k_{11f} is the bounded part.

After some lengthy algebra, the singular part of k_{11} is found to be:

$$k_{11s}(x_1,t) = F_s(x_1,t) + F_s(-x_1,t)$$

where

$$F_s(x_1,t) = \frac{1}{\lambda_{58}}\left\{\lambda_{63}\frac{\omega_2(t-x_1)}{[\omega_2(t-x_1)]^2+[\omega_o(2h_1-t-x_1)]^2}\right.$$

$$+\lambda_{64}\frac{\omega_2(2h_1-x_1-t)}{[\omega_2(2h_1-x_1-t)]^2+[\omega_o(2h_1-t-x_1)]^2}$$

$$+\lambda_{65}\frac{\omega_o(2h_1-t-x_1)}{[\omega_2(t-x_1)]^2+[\omega_o(2h_1-t-x_1)]^2}$$

$$\left.+\lambda_{66}\frac{\omega_o(2h_1-t-x_1)}{[\omega_2(2h_1-x_1-t)]^2+[\omega_o(2h_1-t-x_1)]^2}\right\} \tag{4.1}$$

The elastic constants λ_j are defined in Appendix A. The governing singular integral equations then become:

$$\int_{-h_1}^{h_1}\left[\frac{1}{t-x_1}+k_{11s}(x_1,t)\right]\phi(t)dt + \int_{-h_1}^{h_1}[k_{11}(x_1,t)-k_{11s}(x_1,t)]\cdot$$

$$\cdot\ \phi(t)dt + \int_{b}^{b} k_{12}(x_1,t)\phi^*(t)dt = -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y\Delta_{14}}\, p_1(x_1) \qquad -h_1<x_1<h_1$$

and

$$\int_{b}^{b}\frac{\phi^*(t)}{t-x_2}dt + \int_{h_1}^{h_1} k_{21}(x_2,t)\phi(t)dt + \int_{b}^{b} k_{22}(x_2,t)\phi^*(t)dt$$

$$= -\frac{\pi(1-\nu^*_{xy}\nu^*_{yx})}{2E^*_y\Delta^*_{14}}\, p_2(x_2) \qquad -b<x_2<b \tag{4.2a,b}$$

In equation (4.2b) since the only singular term is $\frac{1}{t-x_2}$, the singularity power is 1/2 at the tip of the crack in the second layer. But in equation (4.2a) we have further singular contribution from k_{11s}, which may result in a singularity power different than 0.5. To find this singularity power γ, let us first write equation (4.2a) in the form:

$$\frac{1}{\pi}\int_{h_1}^{h_1}\left[\frac{1}{t-x_1} + k_{11s}(x_1,t)\right]\phi(t)dt = P_1(x_1) \qquad -h_1<x_1<h_1 \tag{4.3}$$

where $P_1(x_1)$ is a bounded function for all values of x_1.

Assuming $\phi(t) = \frac{F(t)}{(h_1^2-t^2)^\gamma}$ and following the procedure described in [8], the following characteristic equation is found from which the singularity power γ can be determined:

$$H(\gamma) = \cos\pi\gamma + \frac{\lambda_{67}}{\lambda_{68}} + \frac{1}{(\omega_o^2+\omega_2^2)\lambda_{58}}\cos\left(2\gamma\tan^{-1}\frac{\omega_2}{\omega_o}\right)\cdot$$

$$\cdot\left[\omega_2\lambda_{63}-\omega_o\lambda_{65}\right] - \frac{1}{(\omega_o^2+\omega_2^2)\lambda_{58}}\cdot$$

$$\cdot\sin\left(2\gamma\tan^{-1}\frac{\omega_2}{\omega_o}\right)\left[\omega_o\lambda_{63}+\omega_2\lambda_{65}\right] = 0 \tag{4.4}$$

For practical orthotropic materials equation (4.4) has only one root between 0 and 1.

5. CASE OF A CRACK CROSSING THE INTERFACE

To formulate this problem we shall first consider the crack configuration shown in Figure 3. Noting that,

$$\int_{-b}^{b} k_{i2}(x_i,t)\phi^*(t)dt = \int_{o}^{b}[k_{i2}(x_i,t) - k_{i2}(x_i,-t)]\phi^*(t)dt \quad (i=1,2)$$

and

$$\int_{-b}^{b} \frac{\phi^*(t)}{t-x_2} dt = \int_{o}^{b} [\frac{1}{t-x_2} + \frac{1}{t+x_2}] \phi^*(t)dt$$

the governing singular integral equations for this case can be written as:

$$\int_{-a}^{a} \frac{\phi(t)}{t-x_1} + \int_{-a}^{a} k_{11}(x_1,t)\phi(t)dt + \int_{c}^{d}[k_{12}(x_1,t) - k_{12}(x_1,-t)]\phi^*(t)dt$$

$$= - \frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y\Delta_{14}} p_1(x_1) \qquad -a<x_1<a$$

and

$$\int_{c}^{d} [\frac{1}{t-x_2} + \frac{1}{t+x_2}]\phi^*(t)dt + \int_{-a}^{a} k_{21}(x_2,t)\phi(t)dt$$

$$+ \int_{-b}^{b}[k_{22}(x_2,t) - k_{22}(x_2,-t)]\phi^*(t)dt = - \frac{\pi(1-\nu^*_{xy}\nu^*_{yx})}{2E^*_y\Delta^*_{14}} \qquad c<x_2<d$$

(5.1a,b)

If $a=h_1$ and $d=h_2$ all the kernels k_{ij} $(i,j=1,2)$ in (5.1) become unbounded for $(x_1,t)=\pm h_1$ and $(x_2,t)=h_2$ simultaneously. Therefore, the singular part of kernels k_{ij} must be separated. The kernels can be written as:

$$k_{ij} = k_{ijs} + k_{ijf}$$

where k_{ijs} is the singular and k_{ijf} is the bounded part of k_{ij}. The expressions for the singular part of each kernel are:

$$k_{11s}(x_1,t) = F_s(x_1,t) + F_s(-x_1,t)$$

$$k_{12s}(x_1,t) = G_s(x_1,t) + G_s(-x_1,t)$$

$$k_{21s}(x_2,t) = H_s(x_2,t)$$

$$k_{22s}(x_2,t) = I_s(x_2,t)$$

where

$F_s(x_1,t)$ is defined by equation (4.1)

and

$$G_s(x_1,t) = \frac{1}{\lambda_{58}}\left\{\lambda_{81}\frac{\omega_2(h_1-x_1)-\omega_2^*(h_2-t)}{[\omega_2(h_1-x_1)-\omega_2^*(h_2-t)]^2+[\omega_o^*(h_2-t)+\omega_o(h_1-x_1)]^2}\right.$$

$$+\lambda_{82}\frac{\omega_2(h_1-x_1)+\omega_2^*(h_2-t)}{[\omega_2(h_1-x_1)+\omega_2^*(h_2-t)]^2+[\omega_o^*(h_2-t)+\omega_o(h_1-x_1)]^2}$$

$$+\lambda_{83}\frac{\omega_o^*(h_2-t)+\omega_o(h_1-x_1)}{[\omega_o^*(h_2-t)+\omega_o(h_1-x_1)]^2+[\omega_2(h_1-x_1)-\omega_2^*(h_2-t)]^2}$$

$$\left.+\lambda_{84}\frac{\omega_o^*(h_2-t)+\omega_o(h_1-x_1)}{[\omega_o^*(h_2-t)+\omega_o(h_1-x_1)]^2+[\omega_2(h_1-x_1)+\omega_2^*(h_2-t)]^2}\right\}$$

$$H_s(x_2,t) = \frac{1}{\lambda_{68}}\left\{\lambda_{73}\frac{\omega_2^*(h_2-x_2)-\omega_2(h_1-t)}{[\omega_2^*(h_2-x_2)-\omega_2(h_1-t)]^2+[\omega_o(h_1-t)+\omega_o^*(h_2-x_2)]^2}\right.$$

$$+\lambda_{74}\frac{\omega_2^*(h_2-x_2)+\omega_2(h_1-t)}{[\omega_2^*(h_2-x_2)+\omega_2(h_1-t)]^2+[\omega_o(h_1-t)+\omega_o^*(h_2-x_2)]^2}$$

$$+\lambda_{75}\frac{\omega_o(h_1-t)+\omega_o^*(h_2-x_2)}{[\omega_o(h_1-t)+\omega_o^*(h_2-x_2)]^2+[\omega_2^*(h_2-x_2)-\omega_2(h_1-t)]^2}$$

$$\left.+\lambda_{76}\frac{\omega_o(h_1-t)+\omega_o^*(h_2-x_2)}{[\omega_o(h_1-t)+\omega_o^*(h_2-x_2)]^2+[\omega_2^*(h_2-x_2)+\omega_2(h_1-t)]^2}\right\}$$

$$I_s(x_2,t) = \frac{1}{\lambda_{68}}\left\{\lambda_{89}\frac{\omega_2^*(t-x_2)}{[\omega_2^*(t-x_2)]^2+[\omega_o^*(2h_2-t-x_2)]^2}\right.$$

$$+\lambda_{90}\frac{\omega_2^*(2h_2-x_2-t)}{[\omega_2^*(2h_2-x_2-t)]^2+[\omega_o^*(2h_2-t-x_2)]^2}$$

$$+\lambda_{91}\frac{\omega_o^*(2h_2-t-x_2)}{[\omega_2^*(t-x_2)]^2+[\omega_o^*(2h_2-t-x_2)]^2}$$

$$+\lambda_{92}\frac{\omega_o^*(2h_2-t-x_2)}{[\omega_o^*(2h_2-t-x_2)]^2+[\omega_2^*(2h_2-x_2-t)]^2} \qquad (5.2a,b,c)$$

The singular integral equations then become:

$$\int_{-h_1}^{h_1}\left[\frac{1}{t-x_1}+k_{11s}(x_1,t)\right]\phi(t)dt+\int_{-h_1}^{h_1}k_{11f}(x_1,t)\phi(t)dt$$

$$+\int_c^{h_2}k_{12s}(x_1,t)\phi^*(t)dt+\int_c^{h_2}[k_{12f}(x_1,t)-k_{12}(x_1,-t)]\phi^*(t)dt$$

$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y\Delta_{14}}p_1(x_1) \qquad -h_1<x_1<h_1$$

and

$$\int_c^{h_2}\left[\frac{1}{t-x_2}+\frac{1}{t+x_2}\right]\phi^*(t)dt+\int_{-h_1}^{h_1}k_{21s}(x_2,t)\phi(t)dt$$

$$+\int_{-h_1}^{h_1}k_{21f}(x_2,t)\phi(t)dt+\int_c^{h_2}k_{22s}(x_2,t)\phi^*(t)dt$$

$$+\int_c^{h_2}[k_{22f}(x_2,t)-k_{22}(x_2,-t)]\phi^*(t)dt = -\frac{\pi(1-\nu_{xy}^*\nu_{yx}^*)}{2E_y^*\Delta_{14}^*}p_2(x_2)$$

$$c<x_2<h_2 \qquad (5.3a,b)$$

To find the singularity power at the interface, assume:

$$\phi(t) = \frac{F(t)}{(h_1^2-t^2)^\beta} \text{ and } \phi^*(t) = \frac{F^*(t)}{(t-c)^\delta (h_2-t)^\beta}$$

Considering the singular parts only, equations (5.3a,b) can be written as:

$$\frac{1}{\pi}\int_{-h_1}^{h_1} [\frac{1}{t-x_1} + k_{11s}(x_1,t)]\phi(t)dt + \frac{1}{\pi}\int_c^{h2} k_{12s}(x_1,t)\phi^*(t)dt$$

$$= Q_1(x_1) \qquad -h_1<x_1<h_1$$

and

$$\frac{1}{\pi}\int_{-h_1}^{h_1} k_{21s}(x_2,t)\phi(t)dt + \frac{1}{\pi}\int_c^{h2}[\frac{1}{t-x_2} + k_{22s}(x_2,t)]\phi^*(t)dt$$

$$= Q_2(x_2) \qquad c<x_2<h_2 \tag{5.4a,b}$$

where $Q_1(x_1)$ and $Q_2(x_2)$ are bounded for all x_1 and x_2.

Following the procedure described in [8], one can obtain the characteristic equations as:

$$\cot\pi\delta = 0$$

and

$$\Delta(\beta) = [\cos\pi\beta + \frac{\lambda_{67}}{\lambda_{58}} + \lambda_{94}\cos(\lambda_{96}\beta) + \lambda_{95}\sin(\lambda_{96}\beta)] \cdot$$

$$\cdot [\cos\pi\beta + \frac{\lambda_{93}}{\lambda_{68}} + \lambda_{108}\cos(\lambda_{107}\beta) + \lambda_{109}\sin(\lambda_{107}\beta)]$$

$$- \lambda_{110}[\lambda_{99}\cos(\lambda_{97}\beta) + \lambda_{100}\sin(\lambda_{97}\beta) + \lambda_{101}\cos(\lambda_{98}\beta)$$

$$+ \lambda_{102}\sin(\lambda_{98}\beta)][\lambda_{103}\cos(\lambda_{97}\beta) + \lambda_{104}\sin(\lambda_{97}\beta)$$

$$+ \lambda_{105}\cos(\lambda_{98}\beta) - \lambda_{106}\sin(\lambda_{98}\beta)] = 0 \qquad (5.5a,b)$$

Equation (5.5a) gives the expected 1/2 singularity power at the crack tip. But equation (5.5b) is rather complicated and its handling needs care. In [8] it has been shown that for some orthotropic materials of type I there is no singularity at the interface. This is also a possibility for materials of type II. For practical materials equation (5.5b) gives a root between 0 and 1. In deriving the characteristic equation (5.5b) one may note that there is a relation between $F(h_1)$ and $F^*(h_2)$.

$$a_1 F(h_1) + a_2 F^*(h_2) = 0 \qquad (5.6)$$

where

$$a_1 = -\frac{1}{(2h_1)^\beta}\left[\cos\pi\beta + \frac{\lambda_{67}}{\lambda_{68}} + \lambda_{94}\cos(\lambda_{96}\beta) + \lambda_{95}\sin(\lambda_{96}\beta)\right]$$

and

$$a_2 = -\frac{1}{\lambda_{58}\sqrt{h_2-c}\,(\omega_o^{*2}+\omega_2^{*2})}\left(\sqrt{\frac{\omega_o^{*2}+\omega_2^{*2}}{\omega_o^2+\omega_2^2}}\right)^\beta \cdot$$

$$\cdot[\lambda_{99}\cos(\lambda_{97}\beta) + \lambda_{100}\sin(\lambda_{97}\beta)$$

$$+ \lambda_{101}\cos(\lambda_{98}\beta) + \lambda_{102}\sin(\lambda_{98}\beta)]$$

This is a condition to be used in obtaining the solution.

6. THE SOLUTION AND THE RESULTS

6.1 Case of Internal Cracks

In this case the governing equations are (3.11a,b) with the single-valuedness conditions

$$\int_{-a}^{a}\phi(t)\,dt = 0 \quad , \quad \int_{-b}^{b}\phi^*(t)\,dt = 0 \quad .$$

The singular integral equations can be solved in a straight-forward manner by using the Gauss-Chebyshev integration technique described in [9]. In fracture problems one is generally interested in the stress intensity factors which can be expressed in terms of the density functions $F(t)$ and $F^*(t)$. The stress intensity factors are defined as follows:

$$\text{For } a<h_1: \quad k_a = \lim_{x_1 \to a} \sqrt{2(x_1-a)}\ \sigma_{1y}(x_1,0)$$
$$\text{and for } b<h_2: \quad k_b = \lim_{x_2 \to b} \sqrt{2(x_2-b)}\ \sigma_{2y}(x_2,0) \tag{6.1a,b}$$

By making use of (6.1a,b) and equations (3.11a,b) after some algebraic manipulations we obtain:

$$k_a = -2\,\frac{\Delta_{14} E_y \sqrt{a}}{(1-\nu_{xy}\nu_{yx})}\ F_o(1)$$

$$k_b = -2\,\frac{\Delta^*_{14} E^*_y \sqrt{b}}{(1-\nu^*_{xy}\nu^*_{yx})}\ F^*_o(1)$$

where the index "o" denotes the normalized quantities. The loads p_1, p_2 are not independent. Assuming that there is no constraint in x direction, p_1 and p_2 satisfy the following relation:

$$\frac{p_1}{p_2} = \frac{E_y}{E^*_y}$$

In the computation the following material combination is used:

Material of the first strip:

$E_x = 3.1 \times 10^6$ psi $(21.3745 \times 10^9\ N/m^2)$

$E_y = 9.7 \times 10^6$ psi $(66.8815 \times 10^9\ N/m^2)$

$G_{xy} = 2.6 \times 10^6$ psi $(17.927 \times 10^9\ N/m^2)$

$$\nu_{xy} = 0.200$$

Material of the second strip:

$$E_x = 2.5 \times 10^6 \text{ psi } (17.2375 \times 10^9 \text{ N/m}^2)$$

$$E_y = 2.5 \times 10^6 \text{ psi } (17.2375 \times 10^9 \text{ N/m}^2)$$

$$G_{xy} = 1 \times 10^6 \text{ psi } (6.895 \times 10^9 \text{ N/m}^2)$$

$$\nu_{xy} = 0.760$$

By letting a, b, h_1 or h_2 go to proper limits or choosing the materials close to isotropic, one can recover all the special cases done in [5], [6] and [10].

Figures 4 and 5 show some of the computed results. In Figure 4 the stress intensity factor k_a is plotted versus h_2/h_1 for the case b=0 (i.e., there is no crack in the second strip). For h_2=0 one recovers the results for collinear cracks imbedded in a homogeneous medium. When $h_2 \to \infty$, k_a reaches an asymptotic value which can be found in [6]. Figure 5 shows the stress intensity factor k_b for the case when a=0 (i.e., there is no crack in the first layer).

6.2 Case of Broken Laminates

In this case the solution may be obtained by considering equations (4.2a,b) and the single-valuedness conditions

$$\int_{-h_1}^{h_1} \phi(t)\,dt = 0 \quad , \quad \int_{-b}^{b} \phi^*(t)\,dt = 0 \quad .$$

The singularity power γ at the bimaterial interface is obtained from the characteristic equation (4.4). The singular integral equations are solved by using the numerical method described in [9]. For this case the Gauss-Jacobi integration method is used and the fundamental functions are the weights of Jacobi Polynomials (for more details see [8]). The stress intensity factor is defined as:

$$\text{For } a = h_1: \quad k_a = \lim_{x_2 \to -h_2} 2^{\gamma}(x_2+h_2)^{\gamma}\sigma_{2y}(x_2,0) \tag{6.2}$$

Using definition (6.2) and equation (4.2b) k_a is found to be:

$$k_a = -\frac{2E_y^* \Delta_{14}^*}{(1-\nu_{xy}^* \nu_{yx}^*)} \frac{h_1 \gamma}{(\omega_o^2 + \omega_2^2)} \left(\frac{\sqrt{\omega_2^2+\omega_o^2}}{\sqrt{\omega_2^{*2}+\omega_o^{*2}}} \right)^{\gamma} \frac{F_o(1)}{\lambda_{68} \sin\pi\gamma} \cdot$$

$$\cdot \left\{ \cos[\gamma(\tan^{-1}\frac{\omega_2}{\omega_o} + \tan^{-1}\frac{\omega_2^*}{\omega_o^*})](\omega_2\lambda_{73}-\omega_o\lambda_{75}) \right.$$

$$+ \sin[\gamma(\tan^{-1}\frac{\omega_2}{\omega_o} + \tan^{-1}\frac{\omega_2^*}{\omega_o^*})]\ (-\omega_o\lambda_{73}-\omega_2\lambda_{75})$$

$$+ \cos[\gamma(-\tan\frac{\omega_2}{\omega_o} + \tan^{-1}\frac{\omega_2^*}{\omega_o^*})]\ (-\omega_2\lambda_{74}-\omega_o\lambda_{76})$$

$$\left. + \sin[\gamma(-\tan^{-1}\frac{\omega_2}{\omega_o} + \tan^{-1}\frac{\omega_2^*}{\omega_o^*})]\ (-\omega_o\lambda_{74}+\omega_2\lambda_{76}) \right\}$$

Figure 6 shows the variation of k_a when the first laminate is broken. The variation of k_b when the second laminate is broken is shown in Figure 7.

6.3 Case of a Crack Crossing the Interface

For this case the equations to be solved are equations (5.3a,b), the single-valuedness condition $\int_{-h1}^{h1}\phi(t)dt=0$ and relation (5.6). The singularity power β at the interface is obtained by solving equation (5.5b) numerically. Using the Newton-Raphson method for the material combination considered it is found that

$\beta = 0.0852.$

For the solution of the singular integral equations again the Gauss-Jacobi Integration Technique described in [9] is used. In this case the fundamental functions are the weights of Jacobi polynomials. The stress intensity factors can be defined as follows:

$$k_b = \lim_{x_2 \to c} \sqrt{2(c-x_2)}\ \sigma_{2y}(x_2,0) \tag{6.3a}$$

and at the bimaterial interfaces

$$\begin{aligned} k_{xx} &= \lim_{y \to 0^+} y^{\beta} \sigma_{1x}(h_1,y) \\ k_{xy} &= \lim_{y \to 0^+} y^{\beta} \tau_{1xy}(h_1,y) \end{aligned} \tag{6.3a,b}$$

By making use of definitions (6.3a,b,c) and after lengthy algebra we obtain:

$$k_b = \frac{2\Delta^*_{14} E^*_y}{2^{\beta}(1-\nu^*_{xy}\nu^*_{yx})} \sqrt{h_2-c}\ \ F^*_o(-1)$$

$$\begin{aligned} k_{xx} = \frac{2E_x}{(1-\nu_{xy}\nu_{yx})} \Bigg\{ & \frac{(\sqrt{\beta_5})^{\beta/2}\ h_1^{\ \beta} F_o(1)}{2^{\beta}\sin\pi\beta/2} \Big[\cos(\beta\tan^{-1}\frac{\omega_2}{\omega_o}) \cdot \\ & \cdot \Big(-\frac{\lambda_{111}}{2} + \frac{(\lambda_{115}\omega_2+\lambda_{114}\omega_o)}{\lambda_{113}(\omega_o^2+\omega_2^2)}\Big) \\ & + \sin(\beta\tan^{-1}\frac{\omega_2}{\omega_o})\Big(-\frac{\lambda_{112}}{2} + \frac{(-\omega_o\lambda_{115}+\omega_2\lambda_{114})}{\lambda_{113}(\omega_o^2+\omega_2^2)}\Big)\Big] \\ & + \frac{(\sqrt{\beta_5^*})^{\beta/2}(h_2-c)^{\beta} F^*_o(1)}{2^{\beta+\frac{1}{2}}\sin\pi\beta/2\,(\omega_o^{*2}+\omega_2^{*2})\lambda_{113}} \Big[(\omega^*_2\lambda_{117}+\omega^*_o\lambda_{116}) \cdot \\ & \cdot \cos(\beta\tan^{-1}\frac{\omega^*_2}{\omega^*_o}) + (-\omega^*_o\lambda_{117}+\omega^*_2\lambda_{116}) \cdot \\ & \cdot \sin(\beta\tan^{-1}\frac{\omega^*_2}{\omega^*_o})\Big] \Bigg\} \end{aligned}$$

and

$$\begin{aligned} k_{xy} = 2G_{xy} \Bigg\{ & \frac{(\sqrt{\beta_5})^{\beta/2}\ h_1^{\ \beta} F_o(1)}{2^{\beta}\cos\pi\beta/2} \Bigg[\frac{\Delta_{25}}{2}\sin(\beta\tan^{-1}\frac{\omega_2}{\omega_o}) \\ & + \frac{1}{\lambda_{113}(\omega_o^2+\omega_2^2)} \Big\{(-\lambda_{119}\omega_o+\lambda_{118}\omega_2)\sin(\beta\tan^{-1}\frac{\omega_2}{\omega_o}) \end{aligned}$$

$$+ (\lambda_{119}\omega_2+\lambda_{118}\omega_o)\cos(\beta\tan^{-1}\frac{\omega_2}{\omega_o})\Big]\Big]$$

$$+ \frac{(\sqrt{\beta_5^*})^{\beta/2}(h_2-c)^{\beta}F_o^*(1)}{2^{\beta+\frac{1}{2}}\cos\pi\beta/2(\omega_o^{*2}+\omega_2^{*2})\lambda_{113}} \; [(-\lambda_{121}\omega_o^*+\lambda_{120}\omega_2^*) \cdot$$

$$\cdot \sin(\beta\tan^{-1}\frac{\omega_2^*}{\omega_o^*}) + (\lambda_{121}\omega_2^*+\lambda_{120}\omega_o^*) \cdot$$

$$\cdot \cos(\beta\tan^{-1}\frac{\omega_2^*}{\omega_o^*})]\Big\}$$

Figure 8 shows the variation of k_b with c/h_2, for different values of (h_1/h_2) ratio. k_b increases as (h_1/h_2) increases. Figures 9 and 10 show the variation of k_{xx} and k_{xy} with respect to c/h_2.

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APPENDIX A

Definitions of the material constants: A superscript * will be used for the material in the second strip.

$$\beta_1 = \frac{E_x}{(1-\nu_{xy}\nu_{yx})G_{xy}}$$

$$\beta_2 = \frac{E_y}{(1-\nu_{xy}\nu_{yx})G_{xy}}$$

$$\beta_3 = 1 + \nu_{yx} \cdot \beta_1$$

$$\beta_4 = \frac{\beta_3^2-\beta_1\beta_2-1}{\beta_1}$$

$$\beta_5 = \frac{\beta_2}{\beta_1}$$

$$\beta_6 = \sqrt{\beta_4^2-4\beta_5}$$

$$s_1 = \omega_o + i\omega_2 = \sqrt{(-\beta_4+\beta_6)/2}$$

$$s_2 = \omega_o - i\omega_2 = \sqrt{(-\beta_4-\beta_6)/2}$$

$$\beta_7 = \frac{1-\beta_1 s_1^2}{\beta_3 s_1} = \beta_7' + i\beta_7''$$

$$\beta_8 = \frac{1-\beta_1 s_2^2}{\beta_3 s_2} = \beta_7' - i\beta_7''$$

$$\beta_9 = \frac{1}{\beta_3}\left[-\frac{\beta_1\sqrt{\beta_5}}{s_1} + \frac{s_1}{\sqrt{\beta_5}}\right] = \beta_9' + i\beta_9''$$

$$\beta_{10} = \frac{1}{\beta_3}\left[-\frac{\beta_1\sqrt{\beta_5}}{s_2} + \frac{s_2}{\sqrt{\beta_5}}\right] = \beta_9' - i\beta_9''$$

$$\Delta_1 = \omega_o + \nu_{yx}\beta_7'$$

$$\Delta_2 = \omega_2 + \nu_{yx}\beta_7''$$

$$\Delta_3 = 1 + \frac{\nu_{yx}}{\sqrt{\beta_5}} (-\beta_9''\omega_2+\beta_9'\omega_o)$$

$$\Delta_4 = \frac{\nu_{yx}}{\sqrt{\beta_5}} (\beta_9'\omega_2+\beta_9''\omega_o)$$

$$\Delta_5 = \omega_o\nu_{xy} + \beta_7'$$

$$\Delta_6 = \omega_2\nu_{xy} + \beta_7''$$

$$\Delta_7 = \nu_{xy} + \frac{1}{\sqrt{\beta_5}} (\omega_o\beta_9' - \beta_9''\omega_2)$$

$$\Delta_8 = \frac{1}{\sqrt{\beta_5}} (\beta_9'\omega_2 + \beta_9''\omega_o) = \frac{\Delta_4}{\nu_{yx}}$$

$$\Delta_9 = -1 + \omega_o\beta_7' - \omega_2\beta_7''$$

$$\Delta_{10} = \omega_2\beta_7' + \omega_o\beta_7''$$

$$\Delta_{11} = \frac{\omega_2}{\sqrt{\beta_5}} - \beta_9''$$

$$\Delta_{12} = -\frac{\omega_o}{\sqrt{\beta_5}} + \beta_9'$$

$$\Delta_{13} = -\beta_9' - \frac{\Delta_{12}}{\Delta_{11}} \beta_9''$$

$$\Delta_{14} = -\frac{1}{2\Delta_{13}} \left(\Delta_7 + \frac{\Delta_{12}}{\Delta_{11}} \Delta_8\right)$$

$$\Delta_{15} = \frac{\omega_o}{\sqrt{\beta_5}} - \frac{\Delta_{12}}{\Delta_{11}} \frac{\omega_2}{\sqrt{\beta_5}}$$

$$\Delta_{16} = \frac{1}{4\Delta_{13}} \frac{\Delta_{12}}{\Delta_{11}}$$

$$\Delta_{17} = \frac{\Delta_{11}}{\Delta_{12}}$$

$$\Delta_{18} = -\frac{\beta_9'}{\sqrt{\beta_5}} \left(\omega_2 + \frac{\Delta_{12}}{\Delta_{11}} \omega_o\right) + \beta_9''\Delta_{15}$$

$$\Delta_{19} = \frac{\sqrt{\beta_5}\,\Delta_{18}}{\Delta_{13}\omega_o} - \frac{\omega_2}{\omega_o}$$

$$\Delta_{20} = \Delta_3 + \Delta_4 \cdot \frac{\Delta_{12}}{\Delta_{11}}$$

$$\Delta_{21} = \Delta_4 - \Delta_3 \cdot \frac{\Delta_{12}}{\Delta_{11}}$$

$$\Delta_{22} = \frac{\Delta_{21}\sqrt{\beta_5}}{4\Delta_{13}\omega_o}$$

$$\Delta_{23} = -\Delta_{11} - \frac{\Delta_{12}^2}{\Delta_{11}}$$

$$\Delta_{24} = \frac{\Delta_{23}\omega_o}{4\Delta_{13}}$$

$$\Delta_{25} = \frac{1}{2\Delta_{13}} \left(\Delta_{11} + \frac{\Delta_{12}^2}{\Delta_{11}}\right)$$

$$\lambda_1 = \frac{E_x^*(1-\nu_{xy}\nu_{yx})}{E_x(1-\nu_{xy}^*\nu_{yx}^*)}$$

$$\lambda_2 = \frac{G_{xy}^*}{G_{xy}}$$

$$\lambda_3 = \Delta_9 - \lambda_2 \Delta_9^*$$

$$\lambda_4 = \Delta_{10} \beta_7' - \lambda_3 \beta_7''$$

$$\lambda_5 = -\Delta_{10} \beta_7'' - \lambda_3 \beta_7'$$

$$\lambda_6 = \lambda_2 \Delta_{10}^* \beta_7'$$

$$\lambda_7 = -\lambda_2 \Delta_{10}^* \beta_7''$$

$$\lambda_8 = \Delta_{10} \beta_7^{*'}$$

$$\lambda_9 = -\Delta_{10} \beta_7^{*''}$$

$$\lambda_{10} = \Delta_{10} \beta_7' - \Delta_9 \beta_7''$$

$$\lambda_{11} = -\Delta_{10} \beta_7'' - \Delta_9 \beta_7'$$

$$\lambda_{12} = \Delta_{10} \Delta_1 - \lambda_3 \Delta_2$$

$$\lambda_{13} = -\Delta_{10} \Delta_2 - \lambda_3 \Delta_1$$

$$\lambda_{14} = \lambda_2 \Delta_{10}^* \Delta_1$$

$$\lambda_{15} = -\lambda_2 \Delta_{10}^* \Delta_2$$

$$\lambda_{16} = \Delta_{10} \lambda_1 \Delta_1^*$$

$$\lambda_{17} = -\Delta_{10} \lambda_1 \Delta_2^*$$

$$\lambda_{18} = \Delta_{10} \Delta_1 - \Delta_9 \Delta_2$$

$$\lambda_{19} = -\Delta_{10} \Delta_2 - \Delta_9 \Delta_1$$

$$\lambda_{20} = \lambda_7 \lambda_{12} - \lambda_4 \lambda_{15}$$

$$\lambda_{21} = \lambda_5\lambda_{14} - \lambda_{16}\lambda_{13} = \lambda_{20}$$

$$\lambda_{22} = -\lambda_8\lambda_{14} + \lambda_{17}\lambda_5 + \lambda_6\lambda_{16} - \lambda_9\lambda_{13}$$

$$\lambda_{23} = -\lambda_{15}\lambda_8 - \lambda_{17}\lambda_4 + \lambda_7\lambda_{16} + \lambda_9\lambda_{12}$$

$$\lambda_{24} = \lambda_8\lambda_{12} - \lambda_{17}\lambda_7 - \lambda_{16}\lambda_4 + \lambda_9\lambda_{15}$$

$$\lambda_{25} = \lambda_8\lambda_{13} + \lambda_{17}\lambda_6 - \lambda_{16}\lambda_5 - \lambda_9\lambda_{14}$$

$$\lambda_{26} = -\lambda_8\lambda_{17} + \lambda_9\lambda_{16}$$

$$\lambda_{27} = \lambda_{14}\lambda_{11} - \lambda_{19}\lambda_6 = \lambda_{20}$$

$$\lambda_{28} = \lambda_{10}\lambda_{12} - \lambda_4\lambda_{18}$$

$$\lambda_{29} = \lambda_9\lambda_{18} - \lambda_{10}\lambda_{17}$$

$$\lambda_{30} = \lambda_9\lambda_{19} - \lambda_{11}\lambda_{17}$$

$$\lambda_{31} = \lambda_{16}\lambda_{10} - \lambda_8\lambda_{18}$$

$$\lambda_{32} = \lambda_{16}\lambda_{11} - \lambda_8\lambda_{19}$$

$$\lambda_{33} = \beta_7''\lambda_{12} - \lambda_4\Delta_2$$

$$\lambda_{34} = \lambda_9\Delta_2 - \beta_7''\lambda_{17}$$

$$\lambda_{35} = \lambda_9\Delta_1 - \beta_7'\lambda_{17}$$

$$\lambda_{36} = \lambda_{16}\beta_7'' - \lambda_8\Delta_2$$

$$\lambda_{37} = \lambda_{16}\beta_7' - \lambda_8\Delta_1$$

$$\lambda_{38} = \lambda_{18}\lambda_4 - \lambda_{10}\lambda_{12} = -\lambda_{28}$$

$$\lambda_{39} = \lambda_{18}\lambda_7 - \lambda_{10}\lambda_{15} = \lambda_{20}$$

$$\lambda_{40} = \lambda_{23} - \lambda_{29}$$

$$\lambda_{41} = \lambda_{30} + \lambda_{22}$$

$$\lambda_{42} = \lambda_{32} + \lambda_{25}$$

$$\lambda_{43} = -\lambda_{31} - \lambda_{24}$$

$$\lambda_{44} = -\frac{\Delta_9}{\Delta_{10}}\lambda_{26}$$

$$\lambda_{45} = -\lambda_{14}\cdot\lambda_{10}$$

$$\lambda_{46} = -\lambda_{15}\cdot\lambda_{10}$$

$$\lambda_{47} = \lambda_{17}\lambda_3 - \lambda_2\Delta_{10}^{*}\lambda_{16}$$

$$\lambda_{48} = -\lambda_{17}\Delta_{10}$$

$$\lambda_{49} = -\lambda_3\lambda_{16} - \lambda_2\Delta_{10}^{*}\lambda_{17}$$

$$\lambda_{50} = \Delta_{10}\cdot\lambda_{16}$$

$$\lambda_{51} = \Delta_{10}\cdot\lambda_6$$

$$\lambda_{52} = \lambda_7\cdot\Delta_{10}$$

$$\lambda_{53} = -\lambda_9\lambda_3 + \lambda_2\Delta_{10}^{*}\lambda_8$$

$$\lambda_{54} = \lambda_9\Delta_{10}$$

$$\lambda_{55} = \lambda_3\lambda_8 + \lambda_2\Delta_{10}^{*}\lambda_9$$

$$\lambda_{56} = -\Delta_{10}\cdot\lambda_8$$

$$\lambda_{57} = \frac{\lambda_{26}}{\Delta_{10}}$$

$$\lambda_{58} = \Delta_{14}(\lambda_{20} + \lambda_{23} + \lambda_{26})$$

$$\lambda_{59} = -\Delta_6 \ (\lambda_{26}+\lambda_{40})\Delta_{16}\Delta_{17} - \frac{1}{4}\,\Delta_6(\lambda_{46}+\lambda_{48})$$

$$+ \Delta_6\lambda_{34}\Delta_{24}\,\frac{\omega_2}{\omega_o} + \Delta_5\Delta_{16}(\lambda_{44}+\lambda_{41})\Delta_{17} + \frac{1}{4}\,\Delta_5(\lambda_{45}+\lambda_{47})$$

$$+ \Delta_5(\lambda_{57}+\lambda_{35})\ \Delta_{24}\,\frac{\omega_2}{\omega_o}$$

$$\lambda_{60} = -\Delta_6(\lambda_{26}+\lambda_{40})\Delta_{16} - \frac{1}{4}\,\Delta_6(\lambda_{46}+\lambda_{48})\Delta_{19}$$

$$- \Delta_6\Delta_{22}(\lambda_{52}+\lambda_{54}) - \Delta_6\lambda_{34}\Delta_{24}$$

$$+ \Delta_5\Delta_{16}(\lambda_{44}+\lambda_{41}) + \frac{1}{4}\,\Delta_5(\lambda_{45}+\lambda_{47})\Delta_{19}$$

$$+ \Delta_5(\lambda_{51}+\lambda_{53})\Delta_{22} - \Delta_5(\lambda_{57}+\lambda_{35})\Delta_{24}$$

$$\lambda_{61} = \Delta_6(\lambda_{44}+\lambda_{41})\Delta_{16}\Delta_{17} + \frac{1}{4}\,\Delta_6(\lambda_{45}+\lambda_{47})$$

$$+ \Delta_6\Delta_{24}(\lambda_{57}+\lambda_{35})\,\frac{\omega_2}{\omega_o} + \Delta_5\Delta_{16}(\lambda_{26}+\lambda_{40})\Delta_{17}$$

$$+ \frac{1}{4}\,\Delta_5(\lambda_{46}+\lambda_{48}) - \Delta_5\lambda_{34}\Delta_{24}\,\frac{\omega_2}{\omega_o}$$

$$\lambda_{62} = \Delta_6(\lambda_{44}+\lambda_{41})\Delta_{16} + \frac{1}{4}\,\Delta_6(\lambda_{45}+\lambda_{47})\Delta_{19}$$

$$+ \Delta_6\Delta_{22}(\lambda_{51}+\lambda_{53}) - \Delta_6(\lambda_{57}+\lambda_{35})\Delta_{24}$$

$$+ \Delta_5(\lambda_{26}+\lambda_{40})\Delta_5 + \frac{1}{4}\,\Delta_5(\lambda_{46}+\lambda_{48})\Delta_{19}$$

$$+ \Delta_5(\lambda_{52}+\lambda_{54})\Delta_{22} + \Delta_5\lambda_{34}\Delta_{24}$$

$$\lambda_{63} = -\lambda_{59} - \lambda_{62}$$

$$\lambda_{64} = -\lambda_{59} + \lambda_{62}$$

$$\lambda_{65} = -\lambda_{60} + \lambda_{61}$$

$$\lambda_{66} = \lambda_{60} + \lambda_{61}$$

$$\lambda_{67} = -\lambda_{64} \frac{\omega_2}{\omega_2^2+\omega_o^2} - \lambda_{66} \frac{\omega_o}{\omega_2^2+\omega_o^2}$$

$$\lambda_{68} = \Delta_{14}^*(\lambda_{20}+\lambda_{23}+\lambda_{26})$$

$$\lambda_{69} = -\Delta_6^*(\lambda_{27}+\lambda_{29})\Delta_{16}\Delta_{17} + \frac{1}{4}\Delta_6^*(\lambda_{46}+\lambda_{48})$$

$$- \Delta_6^* \lambda_{34}\Delta_{24} \frac{\omega_2}{\omega_o} - \Delta_5^*\Delta_{16}(\lambda_{28}+\lambda_{31})\Delta_{17}$$

$$+ \frac{1}{4}\Delta_5^* (\Delta_{10}\lambda_{12}+\lambda_{50}) - \Delta_5^*(\lambda_{33}+\lambda_{36})\Delta_{24} \frac{\omega_2}{\omega_o}$$

$$\lambda_{70} = -\Delta_6^*(\lambda_{27}+\lambda_{29})\Delta_{16} + \frac{1}{4}\Delta_6^*(\lambda_{46}+\lambda_{48})\Delta_{19}$$

$$+ \Delta_6^*\Delta_{22}(\lambda_{52}+\lambda_{54}) + \Delta_6^* \lambda_{34}\Delta_{24} - \Delta_5^*\Delta_{16}(\lambda_{28}+\lambda_{31})$$

$$+ \frac{1}{4}\Delta_5^*(\Delta_{10}\lambda_{12}+\lambda_{50})\Delta_{19} + \Delta_5^*(-\Delta_{10}\lambda_4+\lambda_{56})\Delta_{22} + \Delta_5^*(\lambda_{33}+\lambda_{36})\Delta_{24}$$

$$\lambda_{71} = -\Delta_6^*(\lambda_{28}+\lambda_{31})\Delta_{16}\Delta_{17} + \frac{1}{4}\Delta_6^*(\Delta_{10}\lambda_{12}+\lambda_{50})$$

$$- \Delta_6^*\Delta_{24}(\lambda_{33}+\lambda_{36}) \frac{\omega_2}{\omega_o} + \Delta_5^*\Delta_{16}(\lambda_{27}+\lambda_{29})\Delta_{17}$$

$$- \frac{1}{4}\Delta_5^*(\lambda_{46}+\lambda_{48}) + \Delta_5^*\lambda_{34}\Delta_{24} \frac{\omega_2}{\omega_o}$$

$$\lambda_{72} = -\Delta_6^*(\lambda_{28}+\lambda_{31})\Delta_{16} + \frac{1}{4}\Delta_6^*(\Delta_{10}\lambda_{12}+\lambda_{50})\Delta_{19}$$

$$+ \Delta_6^*\Delta_{22}(-\Delta_{10}\lambda_4+\lambda_{56}) + \Delta_6^*(\lambda_{33}+\lambda_{36})\Delta_{24}$$

$$+ \Delta_5^*(\lambda_{27}+\lambda_{29})\Delta_{16} - \frac{1}{4}\Delta_5^*(\lambda_{46}+\lambda_{48})\Delta_{19} - \Delta_5^*(\lambda_{52}+\lambda_{54})\Delta_{22} - \Delta_5^*\lambda_{34}\Delta_{24}$$

$$\lambda_{73} = -\lambda_{69} - \lambda_{72}$$

$$\lambda_{74} = -\lambda_{69} + \lambda_{72}$$

$$\lambda_{75} = -\lambda_{70} + \lambda_{71}$$

$$\lambda_{76} = \lambda_{70} + \lambda_{71}$$

$$\lambda_{77} = -\Delta_6(\lambda_{26}+\lambda_{40})\Delta_{16}^*\Delta_{17}^* + \frac{1}{4}\,\Delta_6(\lambda_{46}+\lambda_{48})$$

$$+ \Delta_6\lambda_{34}\lambda_2\Delta_{24}^*\,\frac{\omega_2^*}{\omega_o^*} + \Delta_5\Delta_{16}^*\Delta_{17}^*(\lambda_{44}+\lambda_{41})$$

$$- \frac{1}{4}\,\Delta_5(\lambda_{45}+\lambda_{47}) + \Delta_5(\lambda_{57}+\lambda_{35})\lambda_2\Delta_{24}^*\,\frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{78} = -\Delta_6(\lambda_{26}+\lambda_{40})\Delta_{16}^* + \frac{1}{4}\,\Delta_6(\lambda_{46}+\lambda_{48})\Delta_{19}^*$$

$$+ \Delta_6\lambda_1\Delta_{22}^*(\lambda_{52}+\lambda_{54}) - \Delta_6\lambda_{34}\lambda_2\Delta_{24}^*$$

$$+ \Delta_5\Delta_{16}^*(\lambda_{44}+\lambda_{41}) - \frac{1}{4}\,\Delta_5(\lambda_{45}+\lambda_{47})\Delta_{19}^*$$

$$- \Delta_5(\lambda_{51}+\lambda_{53})\lambda_1\Delta_{22}^* - \Delta_5(\lambda_{57}+\lambda_{35})\lambda_2\Delta_{24}^*$$

$$\lambda_{79} = \Delta_6(\lambda_{44}+\lambda_{41})\Delta_{16}^*\Delta_{17}^* - \frac{1}{4}\,\Delta_6(\lambda_{45}+\lambda_{47})$$

$$+ \Delta_6\lambda_2\Delta_{24}^*(\lambda_{57}+\lambda_{35})\,\frac{\omega_2^*}{\omega_o^*}$$

$$+ \Delta_5\Delta_{16}^*(\lambda_{26}+\lambda_{40})\Delta_{17}^* - \frac{1}{4}\,\Delta_5(\lambda_{46}+\lambda_{48}) - \Delta_5\lambda_{34}\lambda_2\Delta_{24}^*\,\frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{80} = \Delta_6(\lambda_{44}+\lambda_{41})\Delta_{16}^* - \frac{1}{4}\,\Delta_6(\lambda_{45}+\lambda_{47})\Delta_{19}^*$$

$$- \Delta_6\lambda_1\Delta_{22}^*(\lambda_{51}+\lambda_{53}) - \Delta_6(\lambda_{57}+\lambda_{35})\lambda_2\Delta_{24}^*$$

$$+ \Delta_5(\lambda_{26}+\lambda_{40})\Delta_{16}^* - \frac{1}{4}\,\Delta_5(\lambda_{46}+\lambda_{48})\Delta_{19}^*$$

$$- \Delta_5(\lambda_{52}+\lambda_{54})\lambda_1\Delta_{22}^* + \Delta_5\lambda_{34}\lambda_2\Delta_{24}^*$$

$$\lambda_{81} = -\lambda_{77} - \lambda_{80}$$

$$\lambda_{82} = -\lambda_{77} + \lambda_{80}$$

$$\lambda_{83} = -\lambda_{78} + \lambda_{79}$$

$$\lambda_{84} = \lambda_{78} + \lambda_{79}$$

$$\lambda_{85} = -\Delta_6^*(\lambda_{27}+\lambda_{29})\Delta_{16}^*\Delta_{17}^*$$

$$- \frac{1}{4}\,\Delta_6^*(\lambda_{46} + \lambda_{48}) - \Delta_6^*\lambda_{34}\lambda_2\Delta_{24}^*\,\frac{\omega_2^*}{\omega_o^*}$$

$$- \Delta_5^*\Delta_{16}^*(\lambda_{28}+\lambda_{31})\Delta_{17}^* - \frac{1}{4}\,\Delta_5^*(\Delta_{10}\lambda_{12}+\lambda_{50}) - \Delta_5^*(\lambda_{33}+\lambda_{36})\lambda_2\Delta_{24}^*\,\frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{86} = -\Delta_6^*(\lambda_{27}+\lambda_{29})\Delta_{16}^* - \frac{1}{4}\,\Delta_6^*(\lambda_{46}+\lambda_{48})\Delta_{19}^*$$

$$- \Delta_6^*\lambda_1\Delta_{22}^*(\lambda_{52}+\lambda_{54}) + \Delta_6^*\lambda_{34}\lambda_2\Delta_{24}^* - \Delta_5^*\Delta_{16}^*(\lambda_{28}+\lambda_{31})$$

$$- \frac{1}{4}\,\Delta_5^*(\Delta_{10}\lambda_{12}+\lambda_{50})\Delta_{19}^* - \Delta_5^*(-\Delta_{10}\lambda_4+\lambda_{56})\lambda_1\Delta_{22}^* + \Delta_5^*(\lambda_{33}+\lambda_{36})\lambda_2\Delta_{24}^*$$

$$\lambda_{87} = -\Delta_6^*(\lambda_{28}+\lambda_{31})\Delta_{16}^*\Delta_{17}^* - \frac{1}{4}\,\Delta_6^*(\Delta_{10}\lambda_{12}+\lambda_{50})$$

$$- \Delta_6^*\lambda_2\Delta_{24}^*(\lambda_{33}+\lambda_{36})\,\frac{\omega_2^*}{\omega_o^*} + \Delta_5^*\Delta_{16}^*(\lambda_{27}+\lambda_{29})\Delta_{17}^*$$

$$+ \frac{1}{4}\,\Delta_5^*(\lambda_{46}+\lambda_{48}) + \Delta_5^*\lambda_{34}\lambda_2\Delta_{24}^*\,\frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{88} = -\Delta_6^*(\lambda_{28}+\lambda_{31})\,\Delta_{16}^* - \frac{1}{4}\,\Delta_6^*(\Delta_{10}\lambda_{12}+\lambda_{50})\Delta_{19}^* - \Delta_6^*\lambda_1\Delta_{22}^*(-\Delta_{10}\lambda_4+\lambda_{56})$$

$$+ \Delta_6^*(\lambda_{33}+\lambda_{36})\lambda_2\Delta_{24}^* + \Delta_5^*(\lambda_{27}+\lambda_{29})\Delta_{16}^*$$

$$+ \frac{1}{4}\,\Delta_5^*(\lambda_{46}+\lambda_{48})\Delta_{19}^* + \Delta_5^*(\lambda_{52}+\lambda_{54})\lambda_1\Delta_{22}^* - \Delta_5^*\lambda_{34}\lambda_2\Delta_{24}^*$$

$$\lambda_{89} = -\lambda_{85} - \lambda_{88}$$

$$\lambda_{90} = -\lambda_{85} + \lambda_{88}$$

$$\lambda_{91} = -\lambda_{86} + \lambda_{87}$$

$$\lambda_{92} = \lambda_{86} + \lambda_{87}$$

$$\lambda_{93} = -\frac{1}{(\omega_o^{*2}+\omega_2^{*2})}(\lambda_{90}\omega_2^{*}+\lambda_{92}\omega_o^{*})$$

$$\lambda_{94} = \frac{1}{\lambda_{58}(\omega_o^2+\omega_2^2)}(\omega_2\lambda_{63}-\omega_o\lambda_{65})$$

$$\lambda_{95} = \frac{1}{\lambda_{58}(\omega_o^2+\omega_2^2)}(-\omega_o\lambda_{63}-\omega_2\lambda_{65})$$

$$\lambda_{96} = 2\tan^{-1}\frac{\omega_2}{\omega_o}$$

$$\lambda_{97} = \tan^{-1}\frac{\omega_2}{\omega_o} + \tan^{-1}\frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{98} = \tan^{-1}\frac{\omega_2}{\omega_o} - \tan^{-1}\frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{99} = \lambda_{81}\omega_2^* - \lambda_{83}\omega_o^*$$

$$\lambda_{100} = -\lambda_{81}\omega_o^* - \lambda_{83}\omega_2^*$$

$$\lambda_{101} = -\lambda_{82}\omega_2^* = \lambda_{84}\omega_o^*$$

$$\lambda_{102} = -\lambda_{82}\omega_o^* + \lambda_{84}\omega_2^*$$

$$\lambda_{103} = \omega_2\lambda_{73} - \omega_o\lambda_{75}$$

$$\lambda_{104} = -\omega_o\lambda_{73} - \omega_2\lambda_{75}$$

$$\lambda_{105} = -\omega_2\lambda_{74} - \omega_o\lambda_{76}$$

$$\lambda_{106} = -\omega_o\lambda_{74} + \omega_2\lambda_{76}$$

$$\lambda_{107} = 2\tan^{-1}\frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{108} = \frac{1}{\lambda_{68}(\omega_o^{*2}+\omega_2^{*2})}(\omega_2^*\lambda_{89}-\omega_o^*\lambda_{91})$$

$$\lambda_{109} = \frac{1}{\lambda_{68}(\omega_o^{*2}+\omega_2^{*2})}(-\omega_o^*\lambda_{89}-\omega_2^*\lambda_{91})$$

$$\lambda_{110} = \frac{1}{\lambda_{58}\lambda_{68}(\omega_o^2+\omega_2^2)(\omega_o^{*2}+\omega_2^{*2})}$$

$$\lambda_{111} = -\frac{1}{2\Delta_{13}}\left(\Delta_3+\Delta_4\ \frac{\Delta_{12}}{\Delta_{11}}\right)$$

$$\lambda_{112} = -\frac{1}{2\Delta_{13}}\left(\Delta_4-\Delta_3\ \frac{\Delta_{12}}{\Delta_{11}}\right)$$

$$\lambda_{113} = \lambda_{20} + \lambda_{23} + \lambda_{26}$$

$$\lambda_{114} = \Delta_2(\lambda_{44}+\lambda_{41})\Delta_{16}\Delta_{17} + \frac{1}{4}\Delta_2(\lambda_{45}+\lambda_{47})$$

$$+ \Delta_2\Delta_{24}(\lambda_{57}+\lambda_{35})\frac{\omega_2}{\omega_o} + \Delta_1\Delta_{16}(\lambda_{26}+\lambda_{40})\Delta_{17}$$

$$+ \frac{1}{4}\Delta_1(\lambda_{46}+\lambda_{48}) - \Delta_1\lambda_{34}\Delta_{24}\frac{\omega_2}{\omega_o}$$

$$\lambda_{115} = \Delta_2(\lambda_{44}+\lambda_{41})\Delta_{16} + \frac{1}{4}\Delta_2(\lambda_{45}+\lambda_{47})\Delta_{19}$$

$$+\Delta_2\Delta_{22}(\lambda_{51}+\lambda_{53}) - \Delta_2(\lambda_{57}+\lambda_{35})\Delta_{24}$$

$$+ \Delta_1(\lambda_{26}+\lambda_{40})\Delta_{16} + \frac{1}{4}\Delta_1(\lambda_{46}+\lambda_{48})\Delta_{19}$$

$$+ \Delta_1(\lambda_{52}+\lambda_{54})\Delta_{22} + \Delta_1\lambda_{34}\Delta_{24}$$

$$\lambda_{116} = \Delta_2(\lambda_{44}+\lambda_{41})\Delta_{16}^*\Delta_{17}^* - \frac{1}{4}\Delta_2(\lambda_{45}+\lambda_{47}) + \Delta_2\lambda_2\Delta_{24}^*(\lambda_{57}+\lambda_{35})\ \frac{\omega_2^*}{\omega_o^*}$$

$$+ \Delta_1\Delta_{16}^*(\lambda_{26}+\lambda_{40})\Delta_{17}^* - \frac{1}{4}\Delta_1(\lambda_{46}+\lambda_{48}) - \Delta_1\lambda_{34}\lambda_2\Delta_{24}^*\ \frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{117} = \Delta_2(\lambda_{44}+\lambda_{41})\Delta_{16}^* - \frac{1}{4}\Delta_2(\lambda_{45}+\lambda_{47})\Delta_{19}^*$$

$$- \Delta_2\lambda_1\Delta_{22}^*(\lambda_{51}+\lambda_{53}) - \Delta_2(\lambda_{57}+\lambda_{35})\lambda_2\Delta_{24}^*$$

$$+ \Delta_1(\lambda_{26}+\lambda_{40})\Delta_{16}^* - \frac{1}{4}\Delta_1(\lambda_{46}+\lambda_{48})\Delta_{19}^*$$

$$- \Delta_1(\lambda_{52}+\lambda_{54})\lambda_1\Delta_{22}^* + \Delta_1\lambda_{34}\lambda_2\Delta_{24}^*$$

$$\lambda_{118} = \Delta_{10}(\lambda_{44}+\lambda_{41})\Delta_{16}\Delta_{17} + \frac{1}{4}\Delta_{10}(\lambda_{45}+\lambda_{47}) + \Delta_{10}\Delta_{24}(\lambda_{57}+\lambda_{35})\frac{\omega_2}{\omega_o}$$

$$+ \Delta_9\Delta_{16}(\lambda_{26}+\lambda_{40})\Delta_{17} + \frac{1}{4}\Delta_9(\lambda_{46}+\lambda_{48}) - \Delta_9\lambda_{34}\Delta_{24}\frac{\omega_2}{\omega_o}$$

$$\lambda_{119} = \Delta_{10}(\lambda_{44}+\lambda_{41})\Delta_{16} + \frac{1}{4}\Delta_{10}(\lambda_{45}+\lambda_{47})\Delta_{19}$$

$$+ \Delta_{10}\Delta_{22}(\lambda_{51}+\lambda_{53}) - \Delta_{10}(\lambda_{57}+\lambda_{35})\Delta_{24}$$

$$+ \Delta_9(\lambda_{26}+\lambda_{40})\Delta_{16} + \frac{1}{4}\Delta_9(\lambda_{46}+\lambda_{48})\Delta_{19}$$

$$+ \Delta_9(\lambda_{52}+\lambda_{54})\Delta_{22} + \Delta_9\lambda_{34}\Delta_{24}$$

$$\lambda_{120} = \Delta_{10}(\lambda_{44}+\lambda_{41})\Delta_{16}^*\Delta_{17}^* - \frac{1}{4}\Delta_{10}(\lambda_{45}+\lambda_{47}) + \Delta_{10}\lambda_2\Delta_{24}^*(\lambda_{57}+\lambda_{35})\frac{\omega_2^*}{\omega_o^*}$$

$$+ \Delta_9\Delta_{16}^*(\lambda_{26}+\lambda_{40})\Delta_{17}^* - \frac{1}{4}\Delta_9(\lambda_{46}+\lambda_{48}) - \Delta_9\lambda_{34}\lambda_2\Delta_{24}^*\frac{\omega_2^*}{\omega_o^*}$$

$$\lambda_{121} = \Delta_{10}(\lambda_{44}+\lambda_{41})\Delta_{16}^* - \frac{1}{4}\Delta_{10}(\lambda_{45}+\lambda_{47})\Delta_{19}^*$$

$$- \Delta_{10}\lambda_1\Delta_{22}^*(\lambda_{51}+\lambda_{53}) - \Delta_{10}(\lambda_{57}+\lambda_{35})\lambda_2\Delta_{24}^*$$

$$+ \Delta_9(\lambda_{26}+\lambda_{40})\Delta_{16}^* - \frac{1}{4}\Delta_9(\lambda_{46}+\lambda_{48})\Delta_{19}^*$$

$$- \Delta_9(\lambda_{52}+\lambda_{54})\lambda_1\Delta_{22}^* + \Delta_9\lambda_{34}\lambda_2\Delta_{24}^*$$

APPENDIX B

Expressions of the functions used in equations (3.10a,b,c,d):

$$\Delta_o(\alpha) = \lambda_{20}(h^2(\alpha)+m^2(\alpha))(f^{*2}(\alpha)+g^{*2}(\alpha))$$

$$+ (\lambda_{24}f(\alpha)h(\alpha) + \lambda_{25}f(\alpha)m(\alpha) - \lambda_{25}g(\alpha)h(\alpha)$$

$$+ \lambda_{24}g(\alpha)m(\alpha))(g^*(\alpha)h^*(\alpha) - m^*(\alpha)f^*(\alpha))$$

$$+ (\lambda_{22}g(\alpha)h(\alpha) - \lambda_{22}f(\alpha)m(\alpha) + \lambda_{23}g(\alpha)m(\alpha)$$

$$+ \lambda_{23}f(\alpha)h(\alpha))(m^*(\alpha)g^*(\alpha) + f^*(\alpha)h^*(\alpha))$$

$$+ \lambda_{26}(g^2(\alpha)+f^2(\alpha))(h^{*2}(\alpha)+m^{*2}(\alpha))$$

$$R_1(\alpha) = \frac{\Delta_{16}}{\alpha}\int_{-a}^{a}\{\Delta_{17}\cos[\omega_2\alpha(h_1-t)]$$

$$+ \sin[\omega_2\alpha(h_1-t)]\}e^{-\omega_o\alpha(h_1-t)}\phi(t)dt$$

$$+ \frac{\Delta_{16}^*}{\alpha}\int_{-b}^{b}\{\Delta_{17}^*\cos[\omega_2^*\alpha(h_2-t)]$$

$$+ \sin[\omega_2^*\alpha(h_2-t)]\}e^{-\omega_o^*\alpha(h_2-t)}\phi^*(t)dt$$

$$R_2(\alpha) = \frac{1}{4\alpha}\int_{-a}^{a}\{\cos[\omega_2\alpha(h_1-t)] + \Delta_{19}\cdot$$

$$\sin[\omega_2\alpha(h_1-t)]\}e^{-\omega_o(h_1-t)\alpha}\phi(t)dt$$

$$- \frac{1}{4\alpha}\int_{-b}^{b}\{\cos[\omega_2^*\alpha(h_2-t)] + \Delta_{19}^*\sin[\omega_2^*\alpha(h_2-t)]\}\cdot$$

$$\cdot\ e^{-\omega_o^*(h_2-t)\alpha}\phi^*(t)dt$$

$$R_3(\alpha) = \frac{\Delta_{22}}{\alpha}\int_{-a}^{a}\sin[\omega_2\alpha(h_1-t)]e^{-\omega_o(h_1-t)\alpha}\phi(t)dt$$

$$- \frac{\Delta_{22}^{*}\lambda_1}{\alpha} \int_{-b}^{b} \sin[\omega_2^{*}\alpha(h_2-t)] e^{-\omega_o^{*}\alpha(h_2-t)} \phi^{*}(t)\,dt$$

$$R_4(\alpha) = \frac{\Delta_{24}}{\alpha} \int_{-a}^{a} \{\frac{\omega_2}{\omega_o} \cos[\omega_2\alpha(h_1-t)] - \sin[\omega_2\alpha(h_1-t)]\} \cdot$$
$$e^{-\omega_o\alpha(h_1-t)} \phi(t)\,dt$$

$$+ \frac{\lambda_2\Delta_{24}^{*}}{\alpha} \int_{-b}^{b} \{ \frac{\omega_2^{*}}{\omega_o^{*}} \cos[\omega_2\alpha(h_2-t)] - \sin[\omega_2^{*}\alpha(h_2-t)]\} \cdot$$
$$e^{-\omega_o^{*}\alpha(h_2-t)} \phi^{*}(t)\,dt \quad .$$

$$f_1(\alpha) = (h^{*2}(\alpha)+m^{*2}(\alpha))(\lambda_{26}f(\alpha)-\lambda_{44}g(\alpha))$$
$$+ (\lambda_{40}h(\alpha)-\lambda_{41}m(\alpha))(f^{*}(\alpha)h^{*}(\alpha)+g^{*}(\alpha)m^{*}(\alpha))$$
$$+(\lambda_{43}h(\alpha)-\lambda_{42}m(\alpha))(f^{*}(\alpha)m^{*}(\alpha)-g^{*}(\alpha)h^{*}(\alpha))$$

$$g_1(\alpha) = (f^{*2}(\alpha)+g^{*2}(\alpha))(\lambda_{46}h(\alpha)-\lambda_{45}m(\alpha))$$
$$+ (f^{*}(\alpha)h^{*}(\alpha)+g^{*}(\alpha)m^{*}(\alpha))(\lambda_{48}f(\alpha)-\lambda_{47}g(\alpha))$$
$$+ (f^{*}(\alpha)m^{*}(\alpha)-g^{*}(\alpha)h^{*}(\alpha))(\lambda_{50}f(\alpha)-\lambda_{49}g(\alpha))$$

$$h_1(\alpha) = (f^{*2}(\alpha)+g^{*2}(\alpha))(\lambda_{52}h(\alpha)-\lambda_{51}m(\alpha))$$
$$+ (f^{*}(\alpha)h^{*}(\alpha)+g^{*}(\alpha)m^{*}(\alpha))(\lambda_{54}f(\alpha)-\lambda_{53}g(\alpha))$$
$$+ (f^{*}(\alpha)m^{*}(\alpha)-g^{*}(\alpha)h^{*}(\alpha))(\lambda_{56}f(\alpha)-\lambda_{55}g(\alpha))$$

$$m_1(\alpha) = -\lambda_{57}g(\alpha)(h^{*2}(\alpha)+m^{*2}(\alpha))$$
$$- (f^{*}(\alpha)h^{*}(\alpha)+g^{*}(\alpha)m^{*}(\alpha))(\lambda_{34}h(\alpha)+\lambda_{35}m(\alpha))$$
$$- (f^{*}(\alpha)m^{*}(\alpha)-g^{*}(\alpha)h^{*}(\alpha))(\lambda_{36}h(\alpha)+\lambda_{37}m(\alpha))$$

$$f_2(\alpha) = (h^{*2}(\alpha)+m^{*2}(\alpha))(\lambda_{44}f(\alpha)+\lambda_{26}g(\alpha))$$

$$(\lambda_{40}m(\alpha)+\lambda_{41}h(\alpha))(f^*(\alpha)h^*(\alpha)+g^*(\alpha)m^*(\alpha))$$

$$+ (\lambda_{42}h(\alpha)+\lambda_{43}m(\alpha))(f^*(\alpha)m^*(\alpha)-g^*(\alpha)h^*(\alpha))$$

$$g_2(\alpha) = (\lambda_{45}h(\alpha)+\lambda_{46}m(\alpha))(f^{*2}(\alpha)+g^{*2}(\alpha))$$

$$+ (\lambda_{47}f(\alpha)+\lambda_{48}g(\alpha))(f^*(\alpha)h^*(\alpha)+g^*(\alpha)m^*(\alpha))$$

$$+ (\lambda_{49}f(\alpha)+\lambda_{50}g(\alpha))(f^*(\alpha)m^*(\alpha)-g^*(\alpha)h^*(\alpha))$$

$$h_2(\alpha) = (\lambda_{51}h(\alpha)+\lambda_{52}m(\alpha))(f^{*2}(\alpha)+g^{*2}(\alpha))$$

$$+ (\lambda_{53}f(\alpha)+\lambda_{54}g(\alpha))(f^*(\alpha)h^*(\alpha)+g^*(\alpha)m^*(\alpha))$$

$$+ (\lambda_{55}f(\alpha)+\lambda_{56}g(\alpha))(f^*(\alpha)m^*(\alpha)-g^*(\alpha)h^*(\alpha))$$

$$m_2(\alpha)=(\lambda_{35}h(\alpha)-\lambda_{34}m(\alpha))(f^*(\alpha)h^*(\alpha)+g^*(\alpha)m^*(\alpha))$$

$$+ (\lambda_{37}h(\alpha)-\lambda_{36}m(\alpha))(f^*(\alpha)m^*(\alpha)-g^*(\alpha)h^*(\alpha))$$

$$+ \lambda_{57}f(\alpha)(h^{*2}(\alpha)+m^{*2}(\alpha))$$

$$f_3(\alpha) + (\lambda_{27}f^*(\alpha)+\lambda_{28}g^*(\alpha))(h^2(\alpha)+m^2(\alpha))$$

$$+ h^*(\alpha)(\lambda_{29}f(\alpha)h(\alpha)+\lambda_{30}f(\alpha)m(\alpha) - \lambda_{30}g(\alpha)h(\alpha) + \lambda_{29}g(\alpha)m(\alpha))$$

$$+ m^*(\alpha)(\lambda_{31}f(\alpha)h(\alpha) + \lambda_{32}f(\alpha)m(\alpha) - \lambda_{32}g(\alpha)h(\alpha) + \lambda_{31}g(\alpha)m(\alpha))$$

$$g_3(\alpha) = -\Delta_{10}[f^*(\alpha)(-\lambda_{14}g(\alpha)h(\alpha) + \lambda_{14}f(\alpha)m(\alpha)$$

$$- \lambda_{15}g(\alpha)m(\alpha) - \lambda_{15}f(\alpha)h(\alpha))$$

$$+ g^*(\alpha)(\lambda_{12}f(\alpha)h(\alpha) + \lambda_{13}f(\alpha)m(\alpha) - \lambda_{13}g(\alpha)h(\alpha) + \lambda_{12}g(\alpha)m(\alpha))$$

$$+ (-\lambda_{17}h^*(\alpha) + \lambda_{16}m^*(\alpha))(g^2(\alpha) + f^2(\alpha))]$$

$$h_3(\alpha) = \Delta_{10}[f^*(\alpha)(-\lambda_6 g(\alpha)h(\alpha) + \lambda_6 f(\alpha)m(\alpha)$$

$$- \lambda_7 g(\alpha)m(\alpha) - \lambda_7 f(\alpha)h(\alpha)) + g^*(\alpha)(\lambda_4 f(\alpha)h(\alpha)$$

$$+ \lambda_5 f(\alpha)m(\alpha) - \lambda_5 g(\alpha)h(\alpha) + \lambda_4 g(\alpha)m(\alpha))$$

$$+ (f^2(\alpha) + g^2(\alpha))(\lambda_8 m^*(\alpha) - \lambda_9 h^*(\alpha))]$$

$$m_3(\alpha) = \lambda_{33} g^*(\alpha)(h^2(\alpha) + m^2(\alpha))$$

$$+ h^*(\alpha)(\lambda_{34} f(\alpha)h(\alpha) + \lambda_{35} f(\alpha)m(\alpha) - \lambda_{35} g(\alpha)h(\alpha)$$

$$+ \lambda_{34} g(\alpha)m(\alpha)) + m^*(\alpha)(\lambda_{36} f(\alpha)h(\alpha) + \lambda_{37} f(\alpha)m(\alpha)$$

$$- \lambda_{37} g(\alpha)h(\alpha) + \lambda_{36} g(\alpha)m(\alpha))$$

$$f_4(\alpha) = (h^2(\alpha) + m^2(\alpha))(\lambda_{38} f^*(\alpha) + \lambda_{39} g^*(\alpha))$$

$$+ h^*(\alpha)(-\lambda_{31} f(\alpha)h(\alpha) - \lambda_{32} f(\alpha)m(\alpha)$$

$$+ \lambda_{32} g(\alpha)h(\alpha) - \lambda_{31} g(\alpha)m(\alpha)) + m^*(\alpha)(\lambda_{29} f(\alpha)h(\alpha)$$

$$+ \lambda_{30} f(\alpha)m(\alpha) - \lambda_{30} g(\alpha)h(\alpha) + \lambda_{29} g(\alpha)m(\alpha))$$

$$g_4(\alpha) = \Delta_{10}[f^*(\alpha)(\lambda_{12} f(\alpha)h(\alpha) + \lambda_{13} f(\alpha)m(\alpha)$$

$$- \lambda_{13} g(\alpha)h(\alpha) + \lambda_{12} g(\alpha)m(\alpha))$$

$$+ g^*(\alpha)(\lambda_{14} g(\alpha)h(\alpha) - \lambda_{14} f(\alpha)m(\alpha)$$

$$+ \lambda_{15} g(\alpha)m(\alpha) + \lambda_{15} f(\alpha)h(\alpha))$$

$$+(f^2(\alpha) + g^2(\alpha))(\lambda_{16}h^*(\alpha) + \lambda_{17}m^*(\alpha))$$

$$h_4(\alpha) = -\Delta_{10}[f^*(\alpha)(\lambda_4 f(\alpha)h(\alpha) + \lambda_5 f(\alpha)m(\alpha)$$

$$-\lambda_5 g(\alpha)h(\alpha) + \lambda_4 g(\alpha)m(\alpha)) + g^*(\alpha)(\lambda_6 g(\alpha)h(\alpha)$$

$$-\lambda_6 f(\alpha)m(\alpha) + \lambda_7 g(\alpha)m(\alpha) + \lambda_7 f(\alpha)h(\alpha))$$

$$+ (f^2(\alpha) + g^2(\alpha))(\lambda_8 h^*(\alpha) + \lambda_9 m^*(\alpha))]$$

$$m_4(\alpha) = -\lambda_{33}f^*(\alpha)(h^2(\alpha) + m^2(\alpha)) + h^*(\alpha)($$

$$\lambda_{37}g(\alpha)h(\alpha) - \lambda_{37}f(\alpha)m(\alpha) - \lambda_{36}g(\alpha)m(\alpha)$$

$$- \lambda_{36}f(\alpha)h(\alpha)) + m^*(\alpha)(-\lambda_{35}g(\alpha)h(\alpha)$$

$$+ \lambda_{35}f(\alpha)m(\alpha) + \lambda_{34}g(\alpha)m(\alpha) + \lambda_{34}f(\alpha)h(\alpha))$$

$$f(\alpha) = \cos(\omega_2\alpha h_1)\sinh(\omega_o\alpha h_1)$$

$$g(\alpha) = \sin(\omega_2\alpha h_1)\cosh(\omega_o\alpha h_1)$$

$$h(\alpha) = \cos(\omega_2\alpha h_1)\cosh(\omega_o\alpha h_1)$$

$$m(\alpha) = \sin(\omega_2\alpha h_1)\sinh(\omega_o\alpha h_1)$$

$$f^*(\alpha) = \cos(\omega_2^*\alpha h_2)\sinh(\omega_o^*\alpha h_2)$$

$$g^*(\alpha) = \sin(\omega_2^*\alpha h_2)\cosh(\omega_o^*\alpha h_2)$$

$$h^*(\alpha) = \cos(\omega_2^*\alpha h_2)\cosh(\omega_o^*\alpha h_2)$$

$$m^*(\alpha) = \sin(\omega_2^*\alpha h_2)\sinh(\omega_o^*\alpha h_2)$$

The expressions of $k_i (i=1,4)$ used in equations (3.11a,b):

$$k_1(x_1,t,\alpha) = \frac{1}{\Delta_{14}\Delta_o(\alpha)} \{2[-\Delta_6 \sin(\omega_2 \alpha x_1)\sinh(\omega_o \alpha x_1)$$

$$+ \Delta_5 \cos(\omega_2 \alpha x_1)\cosh(\omega_o \alpha x_1)] \cdot [f_1(\alpha)\Delta_{16} \cdot$$

$$(\Delta_{17}\cos[\omega_2\alpha(h_1-t)] + \sin[\omega_2\alpha(h_1-t)])$$

$$+ g_1(\alpha)\ \frac{1}{4}\ (\cos[\omega_2\alpha(h_1-t)] \ + \Delta_{19}\sin[\omega_2\alpha(h_1-t)])$$

$$+ h_1(\alpha)\Delta_{22}(\sin[\omega_2\alpha(h_1-t)])$$

$$+ m_1(\alpha)\ \Delta_{24}\ (\frac{\omega_2}{\omega_o}\cos[\omega_2\alpha(h_1-t)] - \sin[\omega_2\alpha(h_1-t)])]$$

$$+ 2[\Delta_5\sin(\omega_2\alpha x_1)\sinh(\omega_o\alpha x_1) + \Delta_6\cos(\omega_2\alpha x_1) \cdot$$

$$\cosh(\omega_o\alpha x_1)] \cdot$$

$$\cdot\ [f_2(\alpha)\Delta_{16}(\Delta_{17}\cos[\omega_2\alpha(h_1-t)] \ + \sin[\omega_2\alpha(h_1-t)])$$

$$+ g_2(\alpha)\ \frac{1}{4}\ (\cos[\omega_2\alpha(h_1-t)] \ + \Delta_{19}\sin[\omega_2\alpha(h_1-t)])$$

$$+ h_2(\alpha)\Delta_{22}\sin[\omega_2\alpha(h_1-t)] \ + m_2(\alpha)\Delta_{24}\ ($$

$$\frac{\omega_2}{\omega_o}\cos[\omega_2\alpha(h_1-t)] \ - \sin[\omega_2\alpha(h_1-t)])]\}$$

$$k_2(x_1,t,\alpha) = \frac{1}{\Delta_{14}\Delta_o(\alpha)} \{ 2[-\Delta_6\sin(\omega_2\alpha x_1) \ \cdot$$

$$\sinh(\omega_o\alpha x_1) + \Delta_5\cos(\omega_2\alpha x_1)\cosh(\omega_o\alpha x_1)] \cdot$$

$$[f_1(\alpha)\Delta_{16}^*(\Delta_{17}^*\cos[\omega_2^*\ \alpha(h_2-t)] + \sin[\omega_2^*\alpha(h_2-t)])$$

$$- g_1(\alpha)\ \frac{1}{4}\ (\cos[\omega_2^*\alpha(h_2-t)] + \Delta_{19}^*\sin[\omega_2^*\alpha\ (h_2-t)])$$

$$-h_1(\alpha)\Delta_{22}^{*}\lambda_1\sin[\omega_2^{*}\alpha(h_2-t)]$$

$$+ m_1(\alpha)\lambda_2\Delta_{24}^{*}(\frac{\omega_2^{*}}{\omega_o^{*}}\cos[\omega_2^{*}\alpha(h_2-t)] - \sin[\omega_2^{*}\alpha(h_2-t)])]$$

$$+ 2[\Delta_5\sin(\omega_2\alpha x_1)\sinh(\omega_o\alpha x_1)$$

$$+ \Delta_6\cos(\omega_2\alpha x_1)\cosh(\omega_o\alpha x_1)]\cdot[f_2(\alpha)\Delta_{16}^{*}(\Delta_{17}^{*}\cdot$$

$$\cos[\omega_2^{*}\alpha(h_2-t)] + \sin[\omega_2^{*}\alpha(h_2-t)])$$

$$- g_2(\alpha)\ \frac{1}{4}\ (\cos[\omega_2^{*}\alpha(h_2-t)] + \Delta_{19}^{*}\ \sin[\omega_2^{*}\alpha(h_2-t)])$$

$$- h_2(\alpha)\Delta_{22}^{*}\lambda_1\sin[\omega_2^{*}\alpha(h_2-t)]$$

$$+ m_2(\alpha)\lambda_2\Delta_{24}^{*}(\frac{\omega_2^{*}}{\omega_o^{*}}\cos[\omega_2^{*}\alpha(h_2-t)]$$

$$- \sin[\omega_2^{*}\alpha(h_2-t)])]\}$$

$$k_3(x_2,t,\alpha) = \frac{1}{\Delta_{14}^{*}\Delta_o(\alpha)}\ \{2[-\Delta_6^{*}\ \sin(\omega_2^{*}\alpha x_2)\ \cdot$$

$$\sinh(\omega_o^{*}\alpha x_2) + \Delta_5^{*}\ \cos(\omega_2^{*}\alpha x_2)\cosh(\omega_o^{*}\alpha x_2)]\ \cdot$$

$$[f_3(\alpha)\Delta_{16}(\Delta_{17}\cos[\omega_2\alpha(h_1-t)] + \sin[\omega_2\alpha(h_1-t)])$$

$$+ g_3(\alpha)\ \frac{1}{4}(\cos[\omega_2\alpha(h_1-t)] + \Delta_{19}\sin[\omega_2\alpha(h_1-t)]\ )$$

$$+ h_3(\alpha)\Delta_{22}\sin[\omega_2\alpha(h_1-t)]$$

$$+ m_3(\alpha)\Delta_{24}(\frac{\omega_2}{\omega_o}\cos[\omega_2\alpha(h_1-t)] - \sin[\omega_2\alpha(h_1-t)])]$$

$$+ 2[\Delta_5^{*}\ \sin(\omega_2^{*}\alpha x_2)\sinh(\omega_o^{*}\alpha x_2)$$

$$+ \Delta_6^{*}\cos(\omega_2^{*}\alpha x_2)\cosh(\omega_o^{*}\alpha x_2)]\ \cdot$$

$$\cdot \; [f_4(\alpha)\Delta_{16}(\Delta_{17}\cos[\omega_2\alpha(h_1-t)] + \sin[\omega_2\alpha(h_1-t)])$$

$$+ g_4(\alpha)\; \frac{1}{4}\; (\cos[\omega_2\alpha(h_1-t)] + \Delta_{19}\sin[\omega_2\alpha(h_1-t)])$$

$$+ h_4(\alpha)\Delta_{22}\sin[\omega_2\alpha(h_1-t)]$$

$$+ m_4(\alpha)\Delta_{24}(\frac{\omega_2}{\omega_o}\cos[\omega_2\alpha(h_1-t)]$$

$$- \sin[\omega_2\alpha(h_1-t)])]\}$$

$$k_4(x_2,t,\alpha) = \frac{1}{\Delta_{14}^*\Delta_o(\alpha)}\{2\; [-\Delta_6^*\sin(\omega_2^*\alpha x_2)\cdot$$

$$\sinh(\omega_o^*\alpha x_2) + \Delta_5^*\cos(\omega_2^*\alpha x_2)\cosh(\omega_o^*\alpha x_2)]\; \cdot$$

$$\cdot\; [f_3(\alpha)\Delta_{16}^*(\Delta_{17}^*\cos[\omega_2^*\alpha(h_2-t)] + \sin[\omega_2^*\alpha(h_2-t)])$$

$$- g_3(\alpha)\; \frac{1}{4}\; (\cos[\omega_2^*\alpha(h_2-t)] + \Delta_{19}^*\sin[\omega_2^*\alpha(h_2-t)])$$

$$- h_3(\alpha)\Delta_{22}^*\lambda_1\sin[\omega_2^*\alpha(h_2-t)]$$

$$+ m_3(\alpha)\lambda_2\Delta_{24}^*(\frac{\omega_2^*}{\omega_o^*}\cos[\omega_2^*\alpha(h_2-t)]$$

$$- \sin[\omega_2^*\alpha(h_2-t)])]$$

$$+ 2[\Delta_5^*\sin(\omega_2^*\alpha x_2)\sinh(\omega_o^*\alpha x_2)$$

$$+ \Delta_6^*\cos(\omega_2^*\alpha x_2)\cosh(\omega_o^*\alpha x_2)]\; \cdot$$

$$[f_4(\alpha)\Delta_{16}^*(\Delta_{17}^*\cos[\omega_2^*\alpha(h_2-t)] + \sin[\omega_2^*\alpha(h_2-t)])$$

$$- g_4(\alpha)\; \frac{1}{4}\; (\cos[\omega_2^*\alpha(h_2-t)] + \Delta_{19}^*\sin[\omega_2^*\alpha(h_2-t)])$$

$$- h_4(\alpha)\Delta_{22}^*\lambda_1\sin[\omega_2^*\alpha(h_2-t)]$$

$$+ m_4(\alpha)\lambda_2\Delta_{24}^*\; (\frac{\omega_2^*}{\omega_o^*}\cos[\omega_2^*\alpha(h_2-t)]$$

$$- \sin[\omega_2^*\alpha(h_2-t)])]\}$$

REFERENCES

1. Hilton, P. D. and Sih, G. C., "A Laminate Composite with a Crack Normal to the Interfaces", *Int. J. Solids Structures*, Vol. 7 (1971) pp. 913-930.

2. Bogy, D. B., "The Plane Elastostatic Solution for a Symmetrically Loaded Crack in a Strip Composite", *Int. J. Eng. Sci.*, Vol. 11 (1973) pp. 985-996.

3. Ashbaugh, N. E., "Stresses in Laminated Composites Containing a Broken Layer", ASME Paper No. 72-WA/APM-14.

4. Gupta, G. D., "A Layerd Composite with a Broken Laminate", *Int. J. Solids Structures*, Vol. 9 (1973) pp. 1141-1154.

5. Erdogan, F. and Bakioglu, M., "Fracture of Composite Plates Containing Periodic Buffer Strips", The National Aeronautics and Space Administration, Technical Report, Grant NGR 39-007-011, October 1974.

6. Arin, K., "An Orthotropic Laminate Composite Containing a Layer with a Crack", Technical Report, NASA TR74-1, March 1974.

7. Arin, K., "A Note on a Broken Layer in an Orthotropic Laminate Composite", Technical Report, NASA TR-74-4, March 1974.

8. Delale, F. and Erdogan, F., "Fracture of Orthotropic Composite Plates Containing Periodic Buffer Strips", The National Aeronautics and Space Administration, Technical Report, Grant NGR 39-007-011, January 1976.

9. Erdogan, F., Gupta, G. D. and Cook, T. S., "The Numerical Solutions of Singular Integral Equations", Methods of Analysis and Solutions to Crack Problems, ed., G. C. Sih, Wolters-Noordhoff Publishing, 1972.

10. Isida, M., Methods of Solutuion of Crack Problems, ed., G. C. Sih, Noordhoff International Publishing (1973) pp. 56-130.

11. Erdogan, F. and Biricikoglu, V., "Two Bonded Half Planes with a Crack Going Through the Interface", *Int. J. Engrg. Sci.*, Vol. 11 (1973) pp. 745-766.

12. Erdogan, F. and Bakioglu, M., "Fracture of Composite Panels", The National Aeronautics and Space Administration, Technical Report, Grant NGR 39-007-011, February 1976.

13. Lekhnitskii, S. G., Theory of Elasticity of an Anisotropic Elastic Body, Holden-Day, Inc., 1963.

14. Muskhelishvili, N. I., Singular Integral Equations, Wolters-Noordhoff Publishing, Groningen, 1958.

15. Krenk, S., "A Note on the Use of the Interpolation Polynomial for Solutions of Singular Integral Equations", Lehigh University, Report IFSM-73-48, August 1973.

16. Erdelyi, A., ed., Tables of Integral Transforms, Vol. 1, McGraw-Hill, New York (1953).

17. Abraniowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, Dover Publications, Inc., New York, 1965.

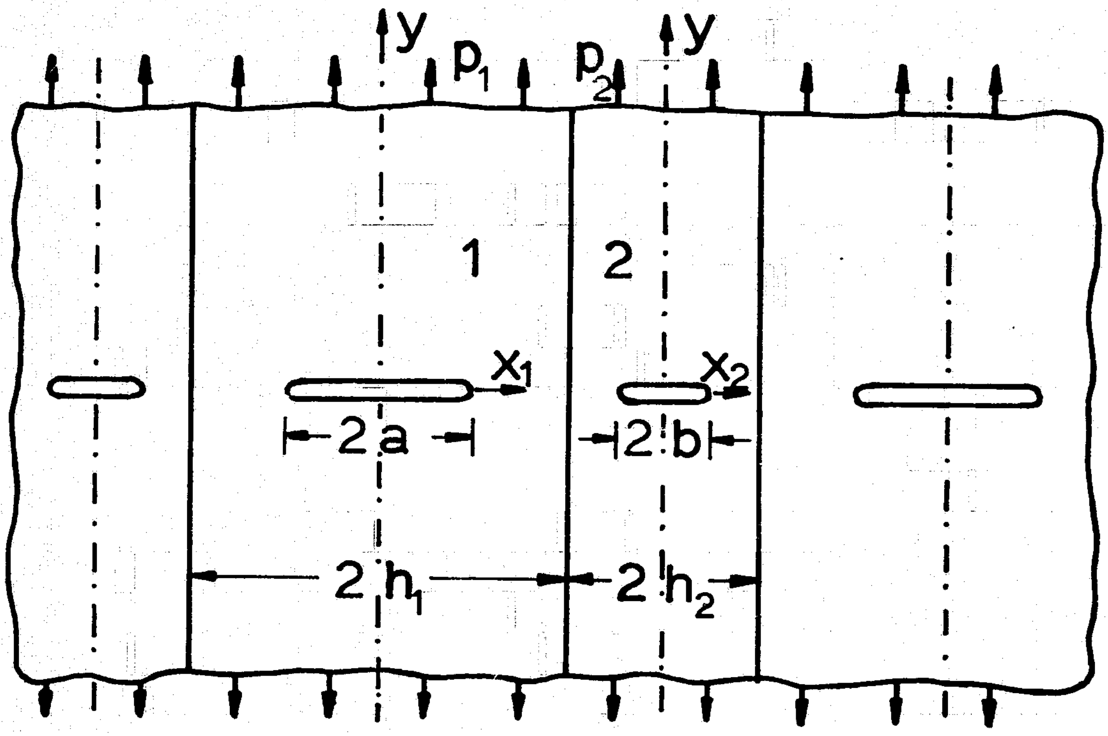

Figure 1. Geometry of the composite plate.

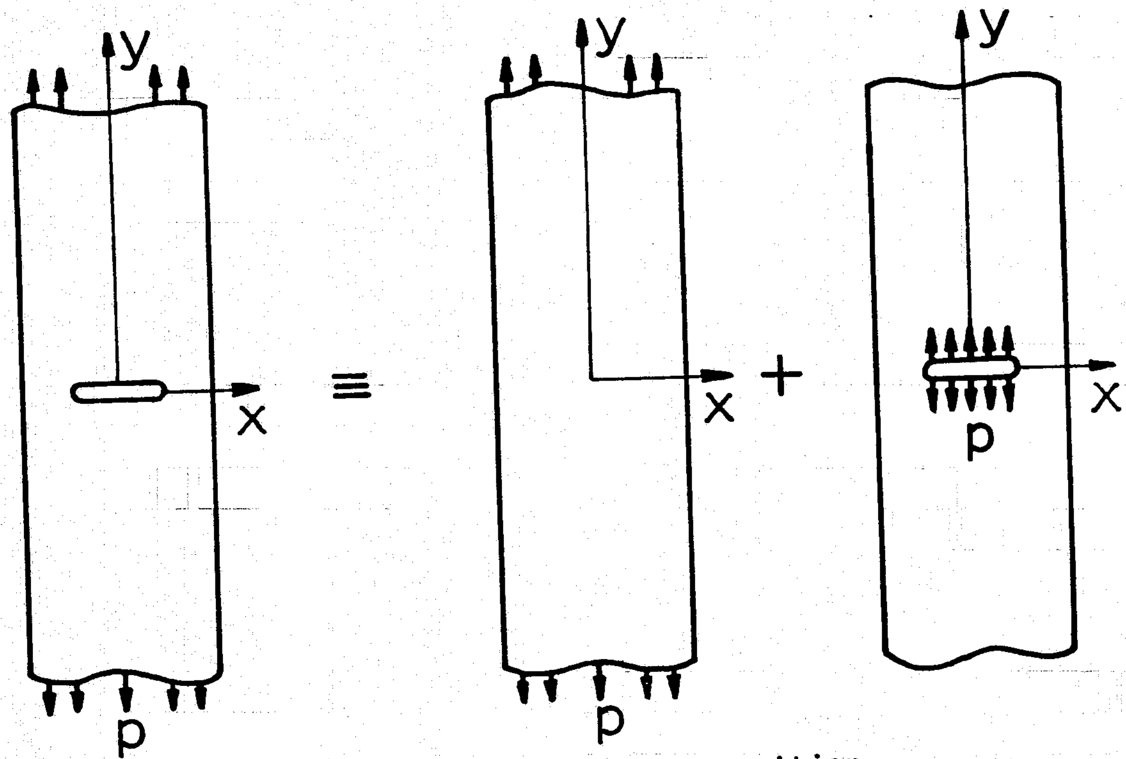

Figure 2. The superposition.

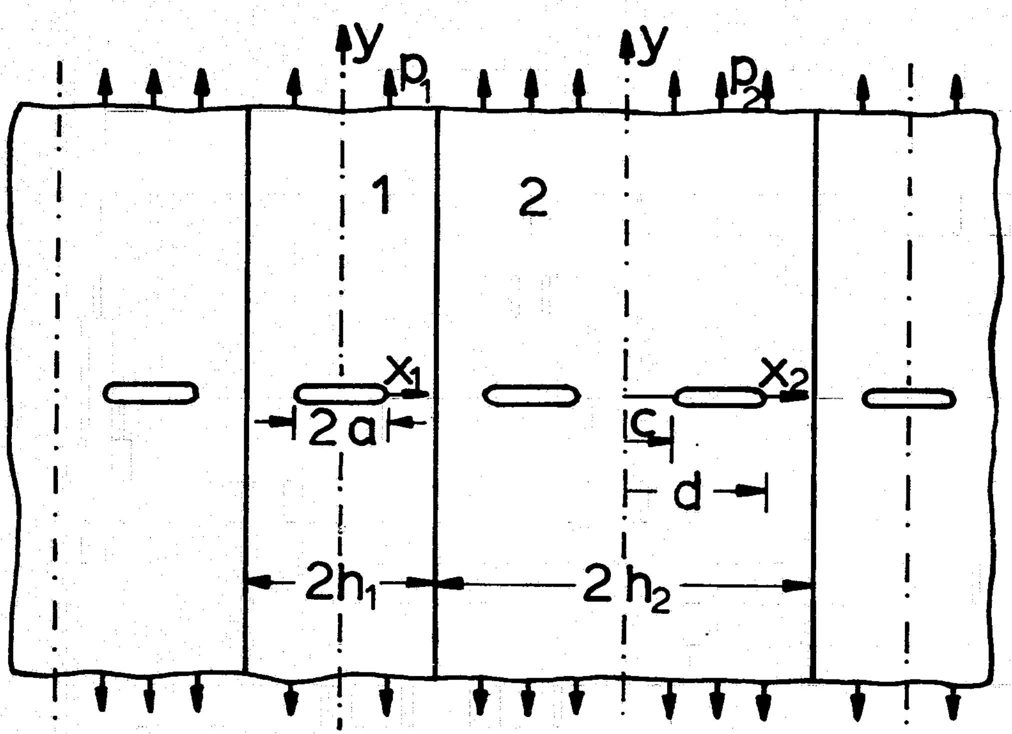

Figure 3. Initial geometry for the case of a crack crossing the interface.

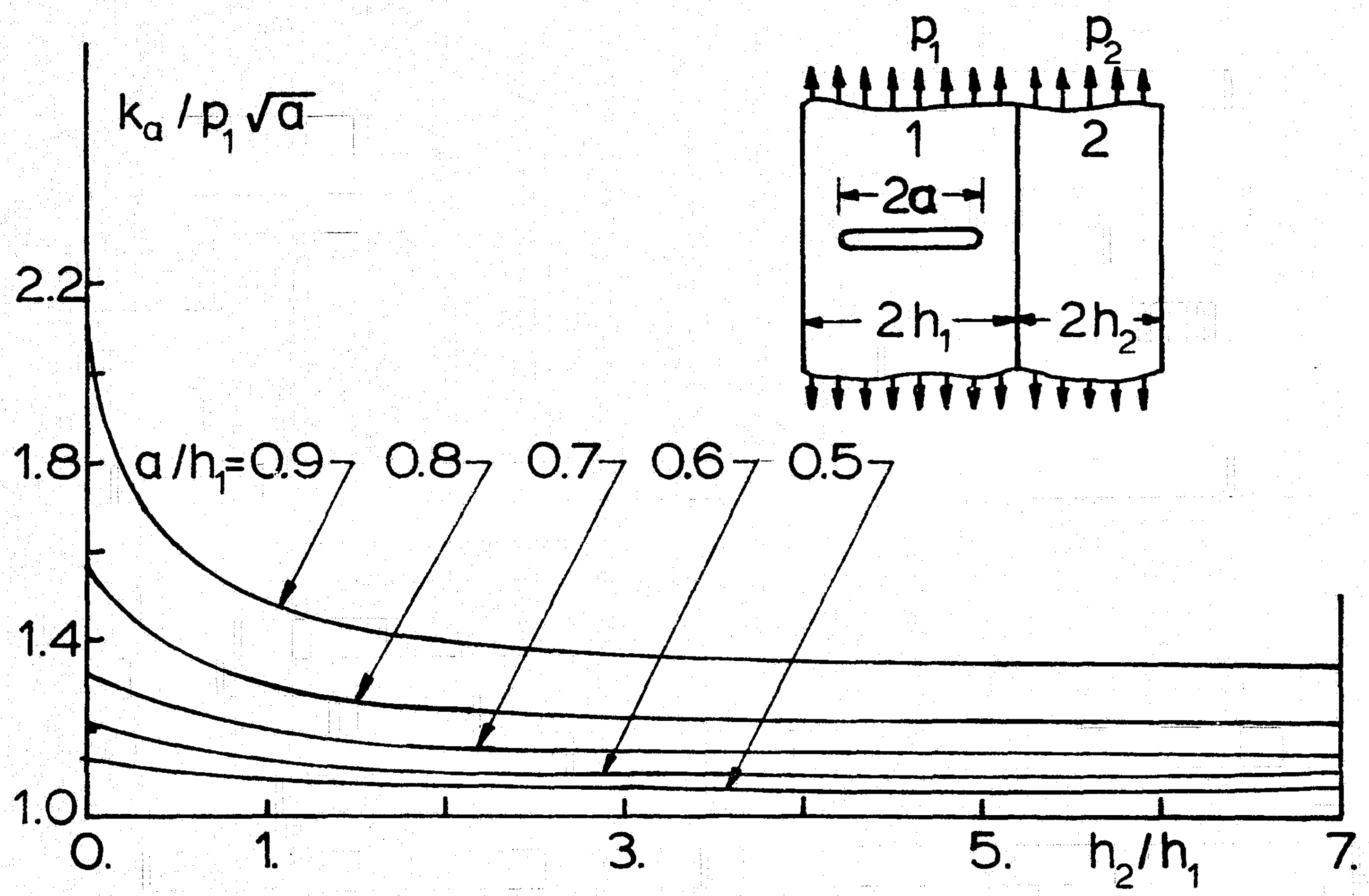

Figure 4. The stress intensity factor k_a for the crack in the first strip.

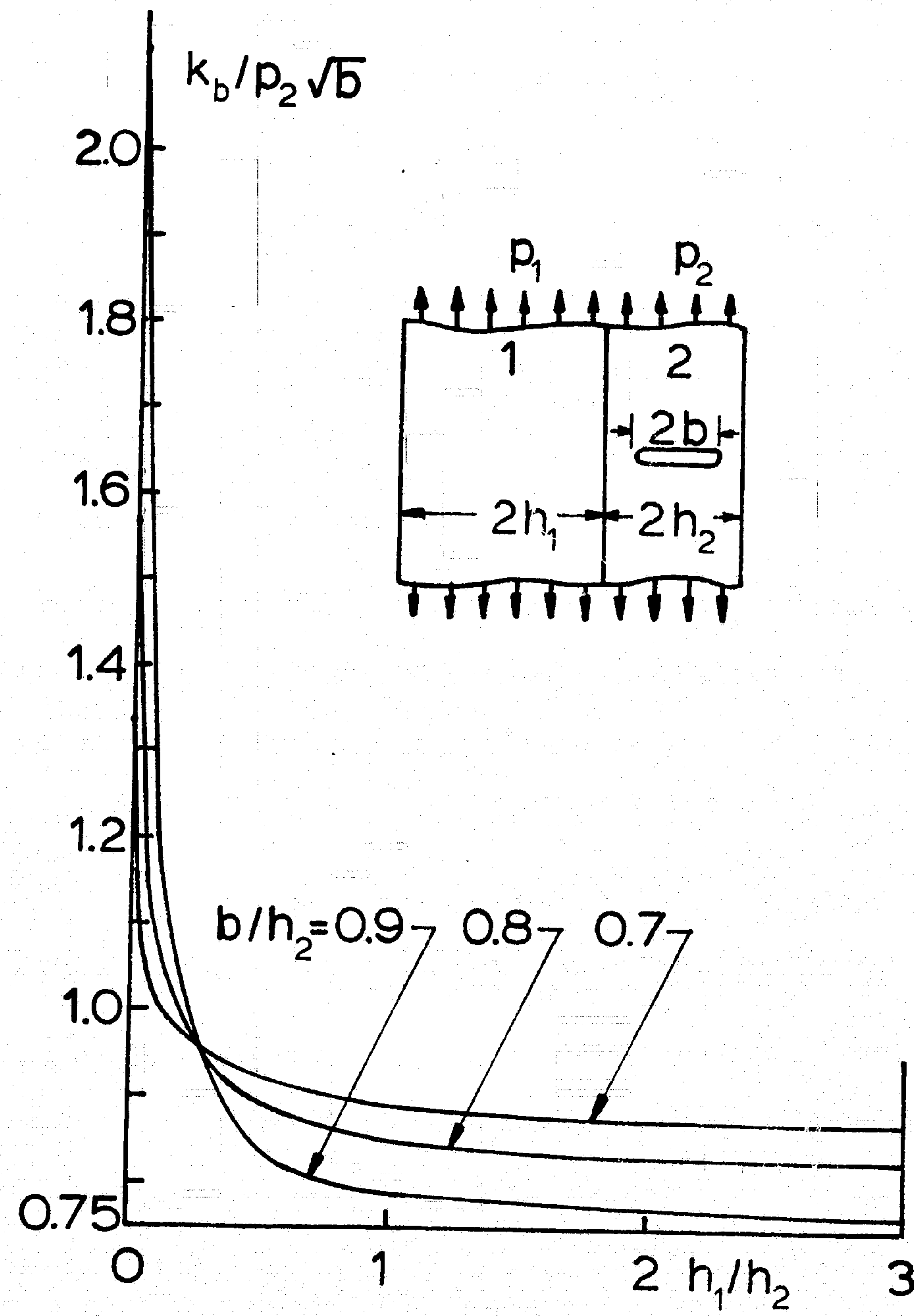

Figure 5. The stress intensity factor k_b for the crack in buffer strip.

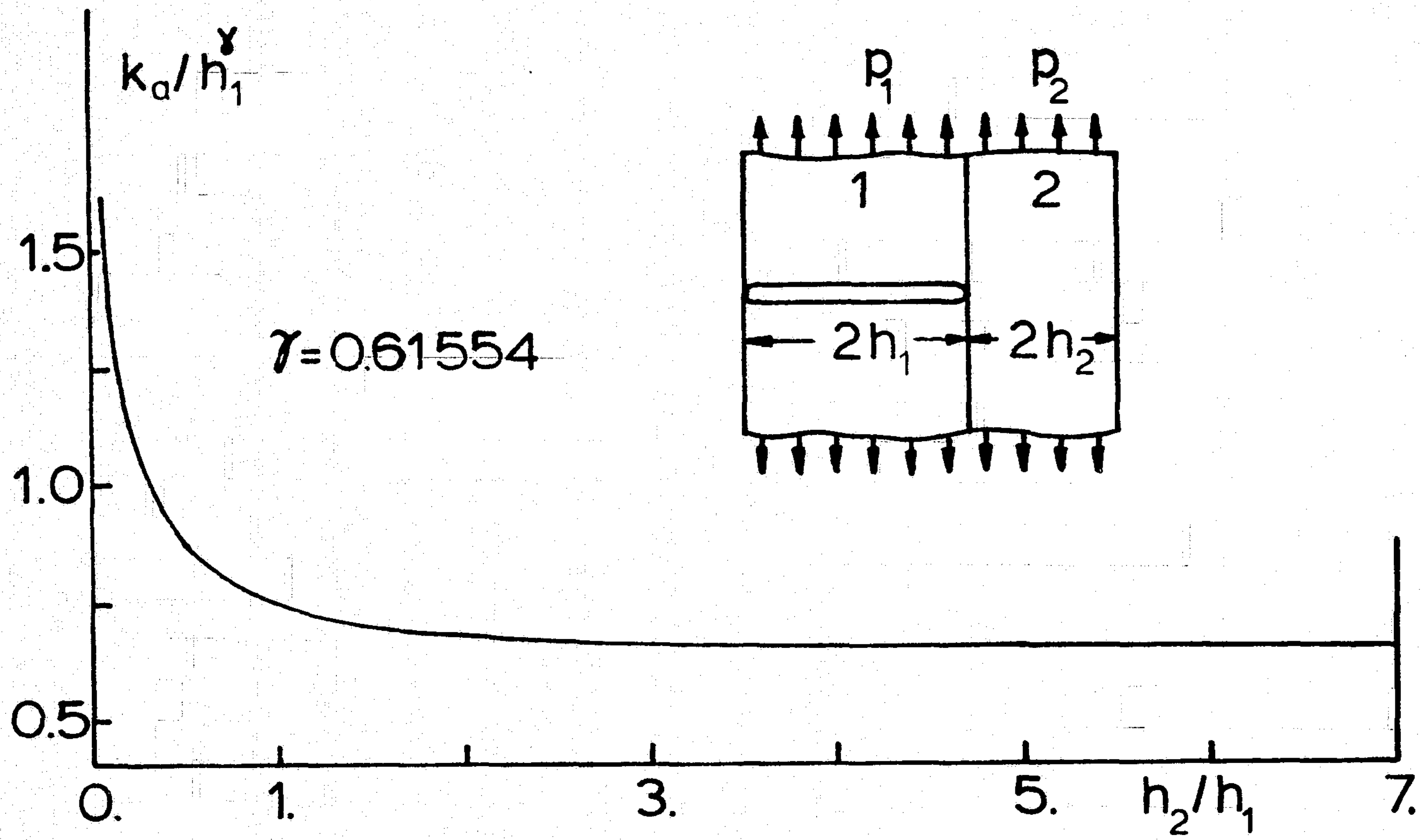

Figure 6. The stress intensity factor k_a when the first laminate is broken.

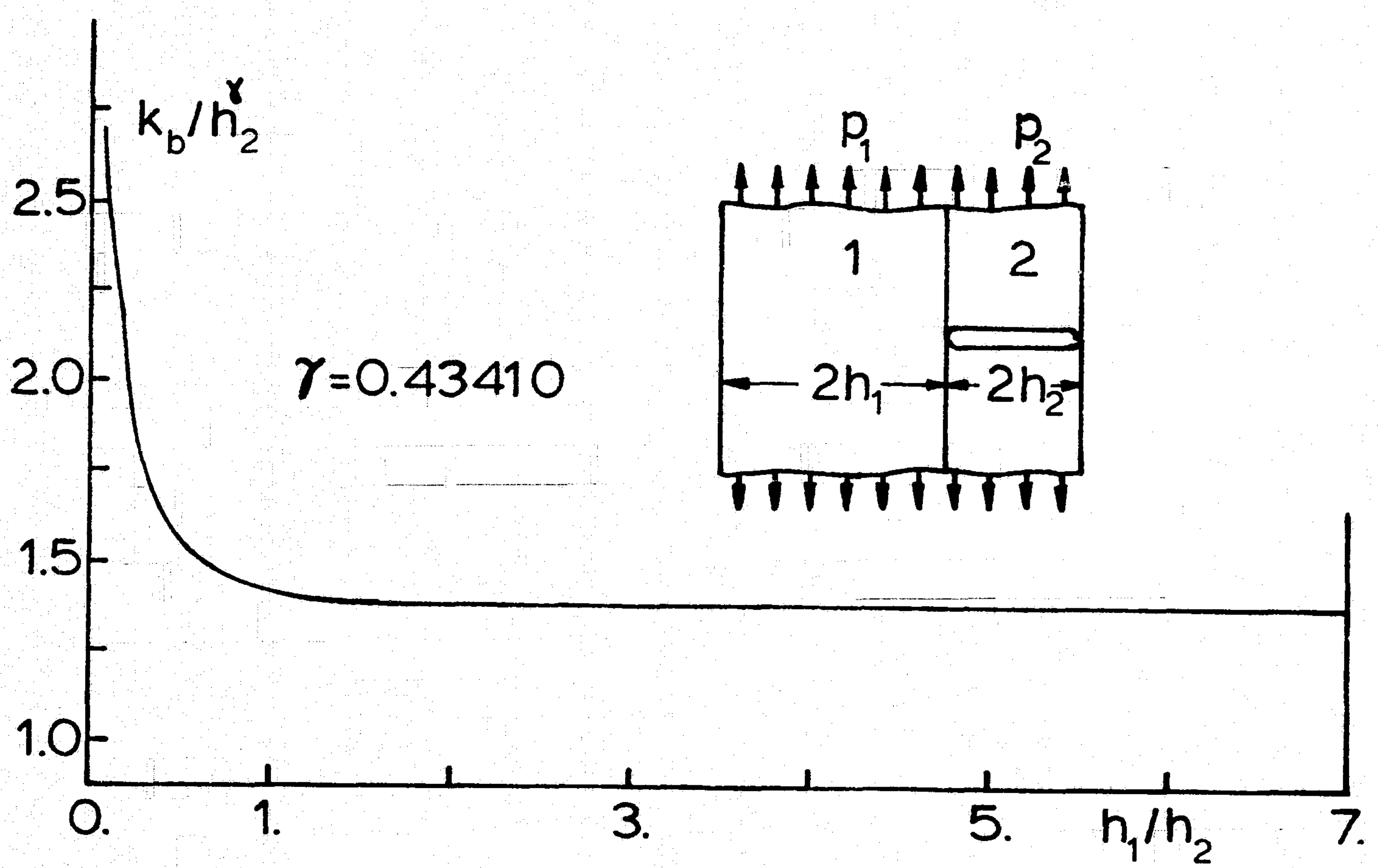

Figure 7. The stress intensity factor k_b when the second laminate is broken.

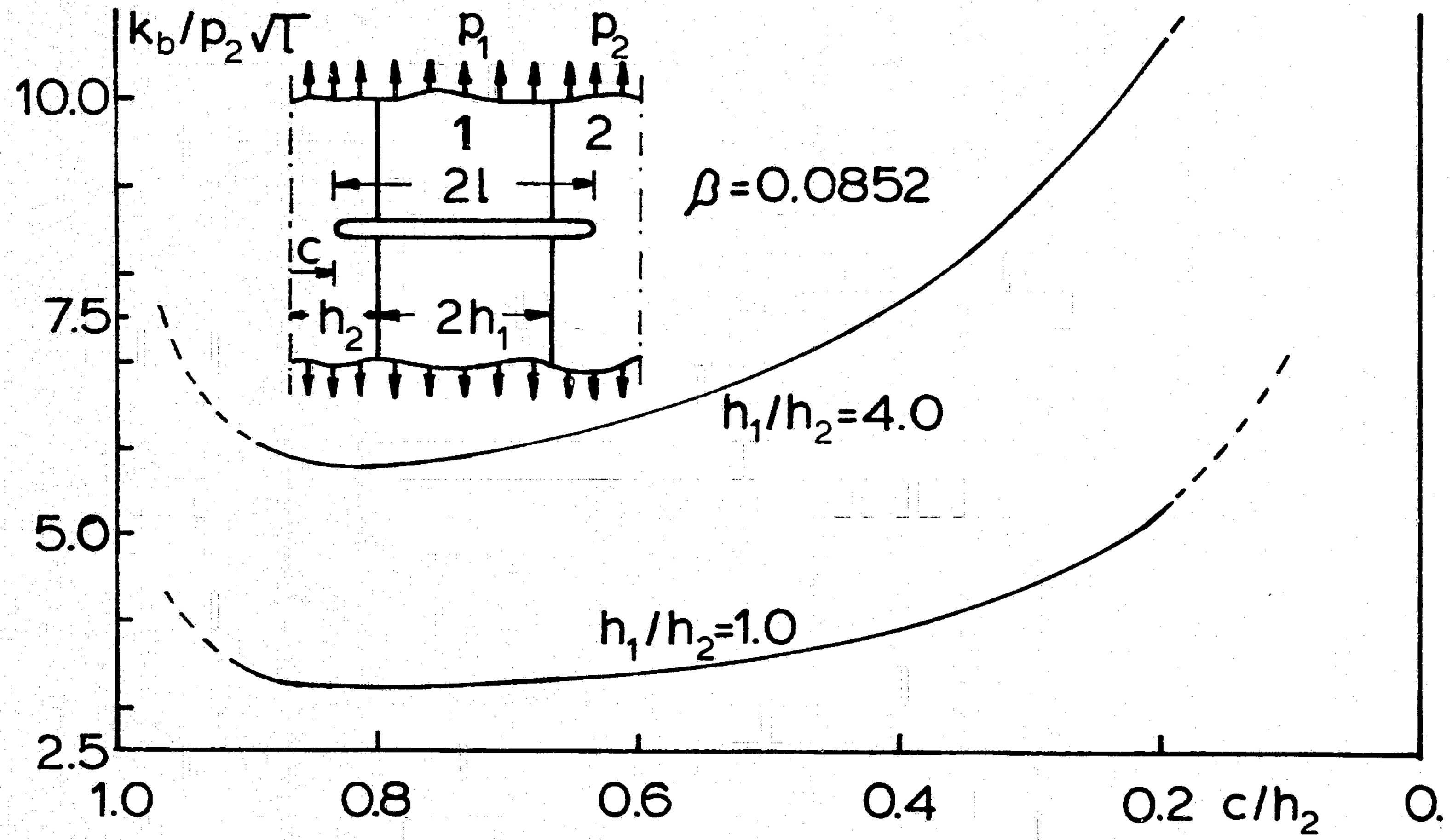

Figure 8. The stress intensity factor k_b for a crack crossing the interface.

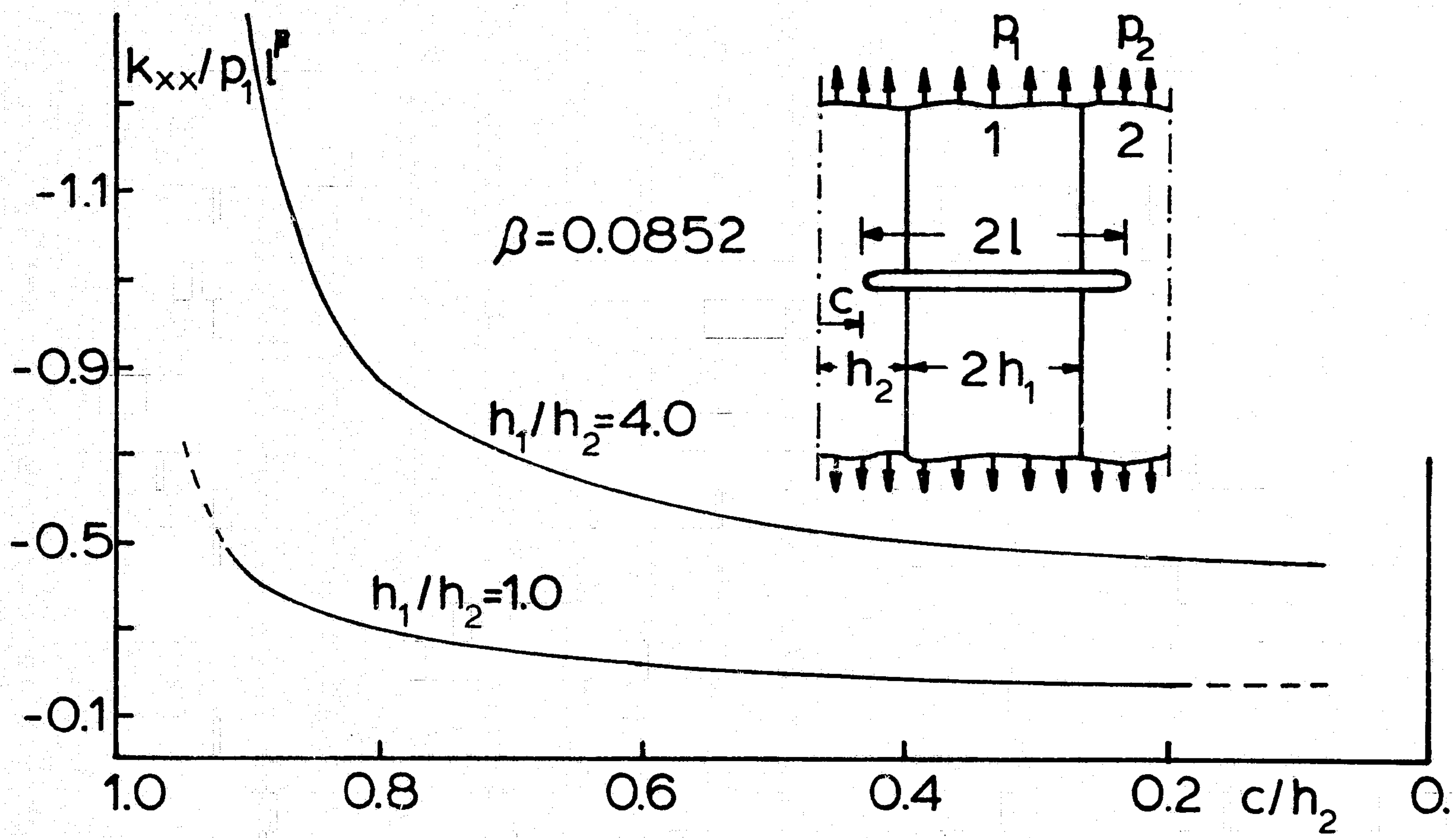

Figure 9. The stress intensity factor k_{xx} for a crack crossing the interface.

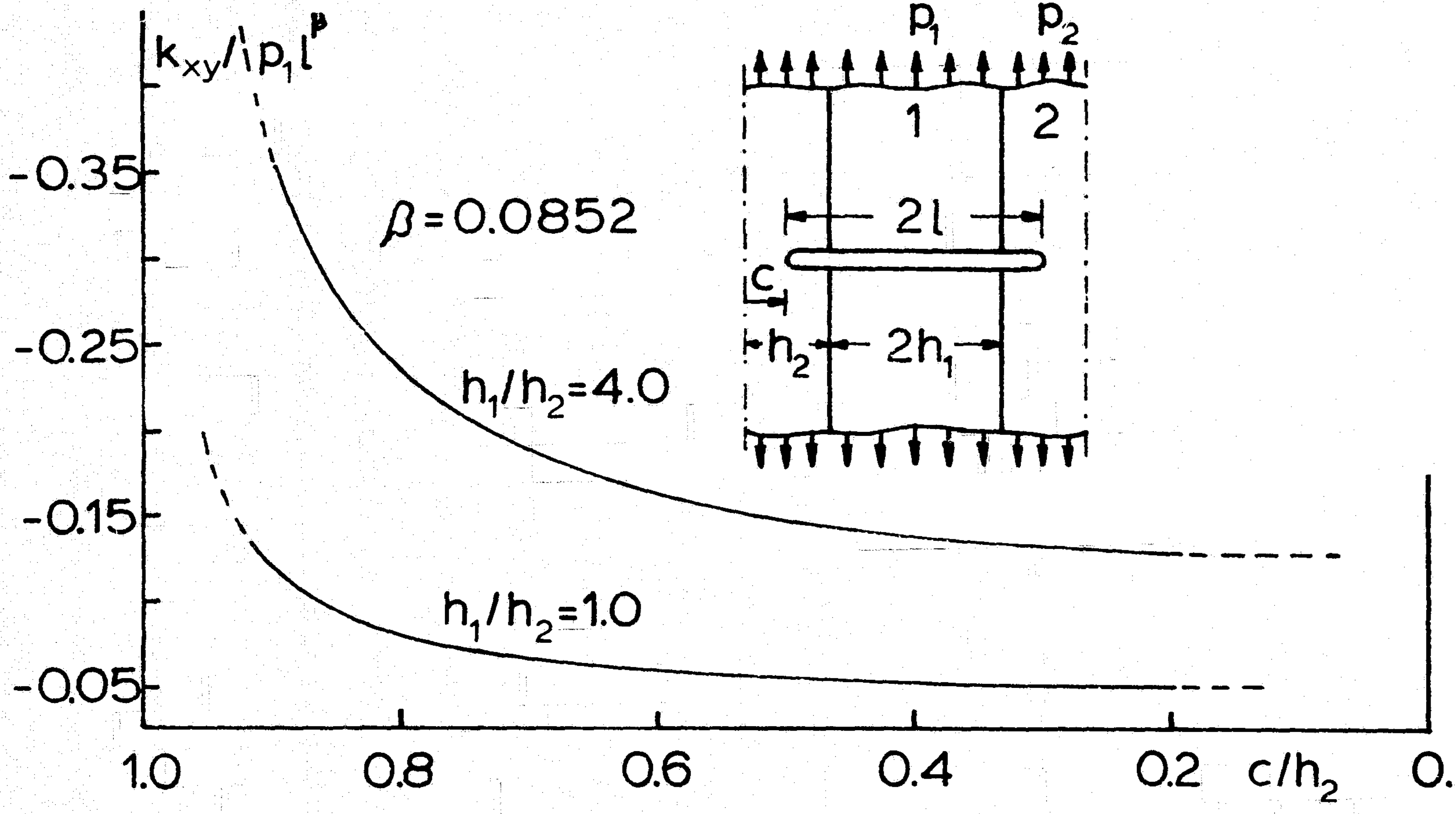

Figure 10. The stress intensity factor k_{xy} for a crack crossing the interface.