NASA CR-145061 |51

(NASA-CR-145061) THE CRACK-CONTACT AND THE FREE END PROBLEM FOR A STRIP UNDER RESIDUAL STRESS (Lehigh Univ.) 22 p HC $3.50 N76-30606
CSCL 13M Unclas
G3/39 50401

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ABSTRACT. The plane problem for an infinite strip with two edge cracks under a given state of residual stress is considered. The residual stress is compressive near and at the surfaces and tensile in the interior of the strip. If the crack is deep enough to penetrate into the tensile zone, then the problem is one of crack-contact problem in which the depth of the contact area is an unknown which depends on the crack depth and the residual stress profile. The problem has applications to the static fatigue of glass plates and is solved for three typical residual stress profiles. In the limiting case of the crack crossing the entire plate thickness, the problem becomes a stress-free end problem for a semi-infinite strip under a given residual stress state away from the end. This is a typical stress diffusion problem in which decay behavior of the residual stress near and the nature of the normal displacement at the end of the semi-infinite strip are of special interest. For two typical residual stress states the solution is obtained and some numerical results are given.

## 1. Introduction

Introducing residual stresses into structural components which are compressive near and at the surfaces to improve their impact and fatigue resistance has been a design practice for many years. Some of the processes used for this purpose are tempering, cladding, ion exchange, and shot peening. In calculating the stress state in the part, these residual stresses must be superimposed on the stress state resulting from the applied loads. In some cases residual stresses may be the only stress state in the body. For example, in considering the problem of subcritical crack growth due to static fatigue in glass plates and other ceramics which normally do not carry any external loads, the crack driving force is mainly provided by the residual stresses. Such a subcritical crack propagation may take place if the surface crack accidentally introduced into the plate is deep enough for the initial crack front to be in the tensile zone and if the corresponding

(*)This work was supported by the National Science Foundation under Grant ENG 73-045053 A01 and by the Materials Division, NASA-Langley under Grant NGR 39-007-011.

stress intensity factor is greater than the threshold level which is the minimum stress intensity level required for crack growth under static loading (for a review of the subject see [1]). The correlation between the subcritical crack growth velocity and the stress intensity factor in ceramics, polymers, and certain metallic alloys under adverse environmental conditions appears to be quite well established [1,2]. Therefore, in principle it is possible to make a reliable prediction for the failure or crack arrest time in the material under residual stresses provided the proper fracture mechanics analysis is available.

In this paper a relatively simple problem of a plate under a known state of residual stress is considered. The plate is assumed to have two symmetric edge cracks. However, because of the compressive stresses, near and at the boundaries, the crack surfaces will be closed along a certain unknown distance from the boundary. Therefore, the problem is one of crack-contact problem rather than a conventional crack problem. After the cracks go through the entire plate thickness, the problem reduces to a stress-free end problem for a semi-infinite rectangular strip under residual stresses. This is a typical stress diffusion problem and is treated as the limiting case of the edge crack problem. To study the behavior of the displacement at the stress-free end and of the diffusion of residual stresses is one of the primary aims of this paper. The plane problem of an infinite strip with various crack geometries has been considered by many investigators who used a variety of techniques to solve the problem [e.g., 3-8].

## 2. On the Formulation and Solution of the Problem

Consider the elastostatic plane strain or generalized plane stress problem for an infinite strip under a symmetric state of residual stress satisfying

$$\sigma_{yy}(x,y) = \sigma_R(x) = \sigma_R(-x) \ , \quad \int_{-h}^{h} \sigma_R(x)\,dx = 0 \ , \tag{1}$$

where $\sigma_R(x)$ is a known function (Figure 1). Let the strip contain two symmetrically located edge cracks along $y=0$, $a<|x|\le h$. Since the stress component $\sigma_{yy}=\sigma_R$ perpendicular to the crack is compressive near and at the surfaces, the crack faces will be closed along $b<|x|\le h$, where the constant b is not known. The contact of the crack surfaces at $|x| = b$ is "smooth" and, consequently, the "ends" of the slits at $|x| = b$ have a cusp shape rather than the standard parabolic form. Note that the stress intensity factors at the ends of the slits shown in Figure 1b are defined by

$$k_a = \lim_{x \to a} [2(a-x)]^{\frac{1}{2}} \sigma_{yy}(x,0) = \frac{4\mu}{1+\kappa} \lim_{x \to a} [2(x-a)]^{\frac{1}{2}} \frac{\partial}{\partial x} v(x,0) \ ,$$

$$k_b = \lim_{x \to b} [2(x-b)]^{\frac{1}{2}} \sigma_{yy}(x,0) = - \frac{4\mu}{1+\kappa} \lim_{x \to b} [2(b-x)]^{\frac{1}{2}} \frac{\partial}{\partial x} v(x,0) \qquad (2a,b)$$

where v is the displacement component in y direction, and $\mu$ and $\kappa$ are the elastic constants, $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for generalized plane stress, $\nu$ being the Poisson's ratio. Thus, from (2b) it may be seen that the condition of "smooth closure" at the end of the slit x=b may be expressed as

$$k_b = 0 \ . \qquad (3)$$

Equation (3) provides the additional condition to determine the unknown constant b.

The solution of the edge crack problem may be expressed as the sum of two solutions: A) the homogeneous solution in the strip without any cracks which is essentially given by (1), and B) the perturbation solution for the strip with the edge cracks in which the crack surface traction $\sigma^B_{yy}(x,0) = -\sigma_R(x)$ is the only external load. It is clear that problem B contains all the important information for the stress-free end as well as the crack problem. The formulation of the problem is identical to that given in [5] where it was shown that the problem described in Figure 1b may be formulated in terms of the following integral equation which, considering the symmetry, is written for one quarter ($y>0$, $0 \le x < h$) of the medium only:

$$\frac{4\mu}{\pi(1+\kappa)} \int_a^b \left[\frac{1}{t-x} + \frac{1}{t+x} + k(x,t)\right] G(t)\,dt = \sigma^B_{yy}(x,0) = -\sigma_R(x) \ , \quad a<x<b \ , \qquad (4)$$

where

$$G(x) = \frac{\partial}{\partial x} v(x, +0) \ , \ a<x<b \ , \qquad (5)$$

$$k(x,t) = \int_0^\infty [K(x,t,s)e^{ts} - K(x,-t,s)e^{-ts}]ds \ , \qquad (6)$$

$$K(x,t,s) = \{[hs-2(2hs-3)(h-t)s-(2+ts)e^{-2hs}]\cosh(xs)$$

$$- [1-2s(h-t)+e^{-2hs}]xs\ \sinh(xs)\}[2hs+\sinh(2hs)]^{-1}. \qquad (7)$$

In the general case the index of the singular integral equation is +1 and hence its solution is determinate within an arbitrary constant [9]. On the other hand, in deriving (4) it was assumed that for $0\leq|x|<a$ and $b<|x|\leq h$ $G(x)$ rather than $v(x,0)$ vanishes. Thus from (5) it follows that

$$\int_a^b G(x)\,dx = 0 \qquad (8)$$

which is used to determine the constant arising from the solution of (4). The solution of the contact-crack problem is then obtained by evaluating $G(x)$ and $b$ from (4) and (3) subject to the verification that the resultant stress $\sigma_{yy}(x,0)$ obtained from the superposition of the solutions of problems A and B for $b<|x|\leq h$ be compressive.

It should be noted that the left hand side of (4) gives the normal stress $\sigma_{yy}^B$ within as well as outside the crack on $y=0$. Therefore, after obtaining $G(t)$ and $b$, (4) may be used to evaluate $\sigma_{yy}^B(x,0)$ on $0\leq|x|<a$ and $b<|x|\leq h$. In the residual stress problem for $a>0$ $b$ is always less than $h$, and $b\to h$ for $a\to 0$. This limiting case of $a=0$, $b=h$ corresponds to the "end problem" for which (4) is still valid. However, in this case the kernel $k(x,t)$ contains additional singular parts which must be properly separated. The technique for doing so was described in [5] and will be omitted in this paper. The expression which will be needed to study the stress diffusion phenomenon is that of $\sigma_{yy}^B(x,y)$, $0\leq x\leq h$, $y>0$. From the general formulation of the problem it may be shown that [5,7]

$$\frac{1+\kappa}{4\mu}\pi\sigma_{yy}^B(x,y) = 2\int_a^b G(t)dt \int_0^\infty (1+ys)e^{-ys}\cos xs \sin ts\, ds$$
$$+ \int_a^b G(t)dt \int_0^\infty [K(x,t,s)e^{ts} - K(x,-t,s)e^{-ts}]\cos ys\, ds \qquad (9)$$

where $K(x,t,s)$ is given by (7). For $y=0$ and $a<x<b$ (9) reduces to (4). Evaluating the inner integrals, the kernel in the first term on the right-hand side of (9) may be expressed in closed form and the first term becomes

$$\int_a^b G(t)\left[\frac{t-x}{(t-x)^2+y^2} + \frac{t+x}{(t+x)^2+y^2} + \frac{2y^2(t-x)}{[(t-x)^2+y^2]^2} + \frac{2y^2(t+x)}{[(t+x)^2+y^2]^2}\right] dt \; .$$

For the limiting case b=h the kernel in the second term of (9) becomes unbounded as t and x approach h. For correct numerical (and singularity) analysis this unbounded part needs to be separated. After doing so, for b=h (9) may be expressed as follows:

$$\frac{1+\kappa}{4\mu}\,\pi\sigma^B_{yy}(x,y) = \int_a^h\Big[\frac{t-x}{(t-x)^2+y^2} + \frac{t+x}{(t+x)^2+y^2} + \frac{2y^2(t-x)}{[(t-x)^2+y]^2}$$

$$+ \frac{2y^2(t+x)}{[(t+x)^2+y^2]} - \frac{2(2h-x-t)}{(2h-x-t)^2+y^2} + \frac{(4h-x-3t)[(2h-x-t)^2-y^2]}{[(2h-x-t)^2+y^2]^2}$$

$$+ 4(h-t)(x-h)\,\frac{(2h-x-t)^3-3y^2(2h-x-t)}{[(2h-x-t)^2+y^2]^3}\Big]\; G(t)dt$$

$$+ \int_a^h G(t)\,dt \int_0^\infty\{[K(x,t,s) - K_\infty(x,t,s)]e^{ts}$$

$$- K(x,-t,s)e^{-ts}\}\cos ys\, ds \; , \quad (0\le x\le h,\; y>0) \qquad (10)$$

where $K_\infty(x,t,s)$ is the asymptotic behavior of $K(x,t,s)$ and may be obtained from (7) as

$$K_\infty(x,t,s) = [-2+s(4h-x-3t) + 2s^2(h-t)(x-h)]e^{-(2h-x)s} \quad . \qquad (11)$$

Another quantity of interest is the end displacement in the semi-infinite strip which referring to (5) may be evaluated from

$$v(x,0) - v(0,0) = \int_0^x G(t)\,dt \; . \qquad (12)$$

For $0<a<b<h$ the numerical solution of (4) is straightforward, although an interpolation scheme is needed to determine the unknown constant b from the condition (3). For $a\to 0$, $b\to h$ and the crack problem becomes a stress-free end problem. In this case G(t) may be evaluated by using a numerical technique similar to that described in [4].

## 4. Results for the Crack Problem

The solution of the problem is carried out for three different symmetric residual stress distributions given by(*)

$$\sigma_R^1(x) = \sigma_o(1-3x^2/h^2) \ , \tag{13}$$

$$\sigma_R^2(x) = \sigma_o(1-5x^4/h^4) \ , \tag{14}$$

$$\sigma_R^3(x) = \sigma_o(1-7x^6/h^6) \ , \tag{15}$$

where $\sigma_o$ is the magnitude of the tensile stress in the midplane x=0 (Figure 1). First the problem is solved routinely only for those values of a and b for which the stress intensity factors $k_a$ and $k_b$ obtained from (2) are both positive. The results corresponding to the residual stresses (13), (14), and (15) are given in Figures 2, 3, and 4, respectively. The stress intensity factors shown in these figures are normalized with respect to $\sigma_o\sqrt{\ell}$, where $\ell=(b-a)/2$. The figures also show the tensile portion of the numerical stress $\sigma_R^i/\sigma_o$, (i=1,2,3). Note that

$$\frac{k_a}{\sigma_o\sqrt{\ell}} \to \frac{\sigma_R^i}{\sigma_o} \ , \ \frac{k_b}{\sigma_o\sqrt{\ell}} \to \frac{\sigma_R^i}{\sigma_o} \quad \text{as} \quad a\to b \ , \ (i=1,2,3) \ . \tag{16}$$

This is due to the fact that for small values of $\ell=(b-a)/2$ the problem is equivalent to that of an infinite plane containing a crack of length $2\ell$ and subjected to crack surface traction $-\sigma_R^i$ for which $k_a=k_b=\sigma_R^i\sqrt{\ell}$, (i=1,2,3).

For a given value of a, the value of b for which $k_b=0$ (i.e., the location of crack closure and beginning of contact area) is shown in Figure 5. From the figure it may be observed that the point a=b on these curves corresponds to the point $x_o^i$ at which the residual stress is zero, $\sigma_R^i(x_o^i)=0$, (i=1,2,3), (Figure 1a). It may also be observed that the value of b for which $k_b=0$ is always greater than $x_o$. For a given value of the crack depth a the stress intensity factor $k_a$ corresponding to $k_b=0$ is shown in Figure 6.

(*) The parabolic distribution (13) seems to be typical for residual stresses in tempered glass and the 6th degree polynomial (15) more representative of the internal stress induced by ion exchange in glass plates.

Note that as expected $k_a \to \infty$ as $a \to 0$ and $k_a \to 0$ as $a \to x_o$.

Figure 7 shows a sample result giving the stress distribution $\sigma_{yy}(x,0)$ on y=0 plane for the crack-contact problem As pointed out before, the total stress is

$$\sigma_{yy}(x,0) = \sigma^A_{yy}(x,0) + \sigma^B_{yy}(x,0) \;, \tag{17}$$

where

$$\sigma^A_{yy}(x,0) = \sigma_R(x) \;, \tag{18}$$

The particular residual stress used in this example is given by (14) and is also shown in the figure. Since $\sigma_{yy}(b,0)=0$, the solution of the perturbation problem must give $\sigma^B_{yy}(b,0)=-\sigma_R(b)$. This was found to be the case in the numerical solution within four significant digits.

## 4. Results for the End Problem

In the end problem the main interest is in the diffusion of residual stress $\sigma_{yy}(x,y)$ in y direction, going from $\sigma_{yy}=0$ for y=0 to $\sigma_{yy}=\sigma_R(x)$ for $y \to \infty$. This is shown in Figures 8 and 9 for the residual stresses given by (13) and (15), respectively. These results as well as those given in Figures 10 and 11 must be considered in conjunction with (17) and (18). Figures 8 and 9 show $\sigma^B_{yy}(x,y)$ for y/h=0, 0.4, 0.8, 1.2. Perhaps a better description of the stress diffusion may be observed from Figures 10 and 11 where $\sigma^B_{yy}$ at x=0 and x=h is given as a function of y. Examining the figures one could make the following general remark: the stress diffusion rate on the surface is greater than that in the mid-plane, and a plate thickness away from the end, i.e., at y=2h the stress at the surface drops to approximately 1% of its maximum value(*) (which is at y=0).

To give an idea about the deformed shape of the stress-free end, Figure 12 shows some sample results. Here the relative displacement v(x,0)-v(0,0)

(*)Needless to say this depends on the undisturbed residual stress profile $\sigma_R(x)$. For example, for $\sigma_R$ given by (13) this figure is approximately 1.1% and for $\sigma_R$ given by (15) it is 0.83%.

is given for residual stresses (13) and (15). The normalization factor which appears in the figure is

$$v_o = \frac{(1+\kappa)h\sigma_o}{4\mu} . \tag{19}$$

The examination of the results shown in Figures 8-12 appears to indicate that in relative terms the difference between two end displacements is much greater than that between two corresponding residual stresses.

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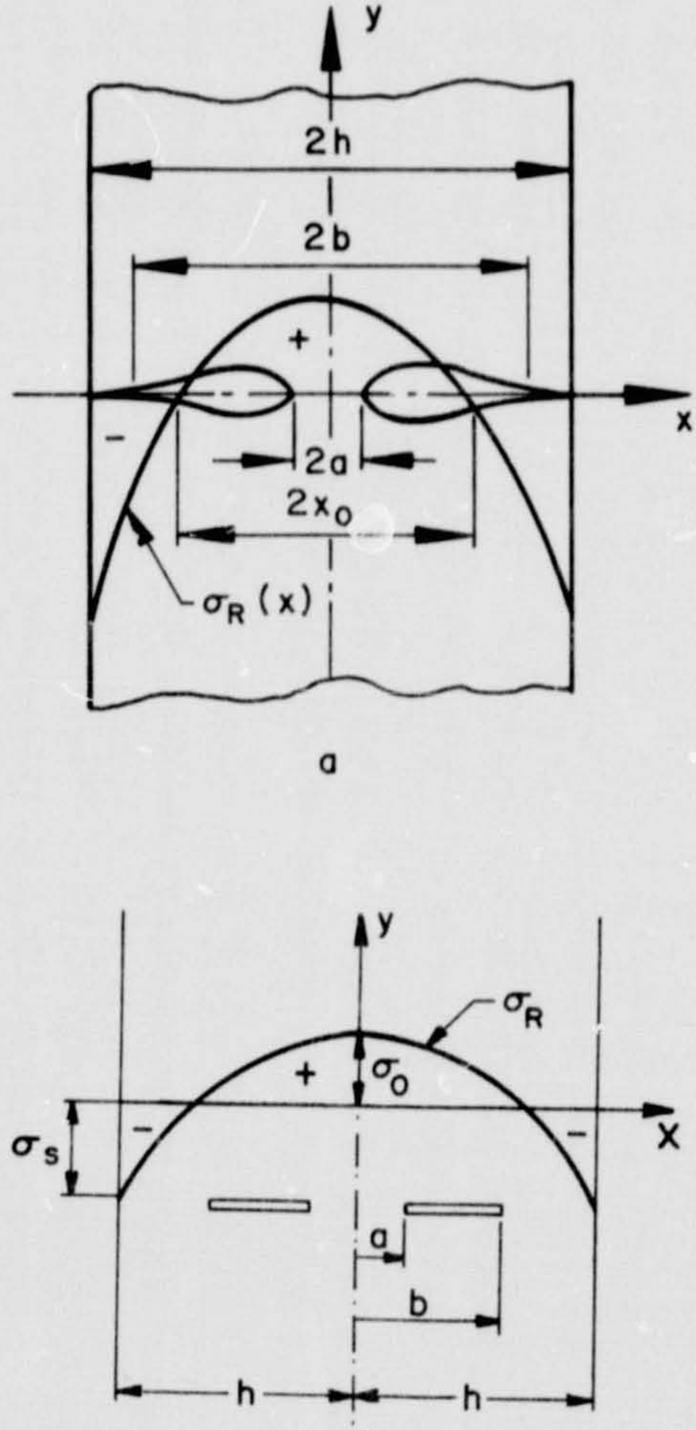


Figure 1. Geometry of cracks in the infinite strip.

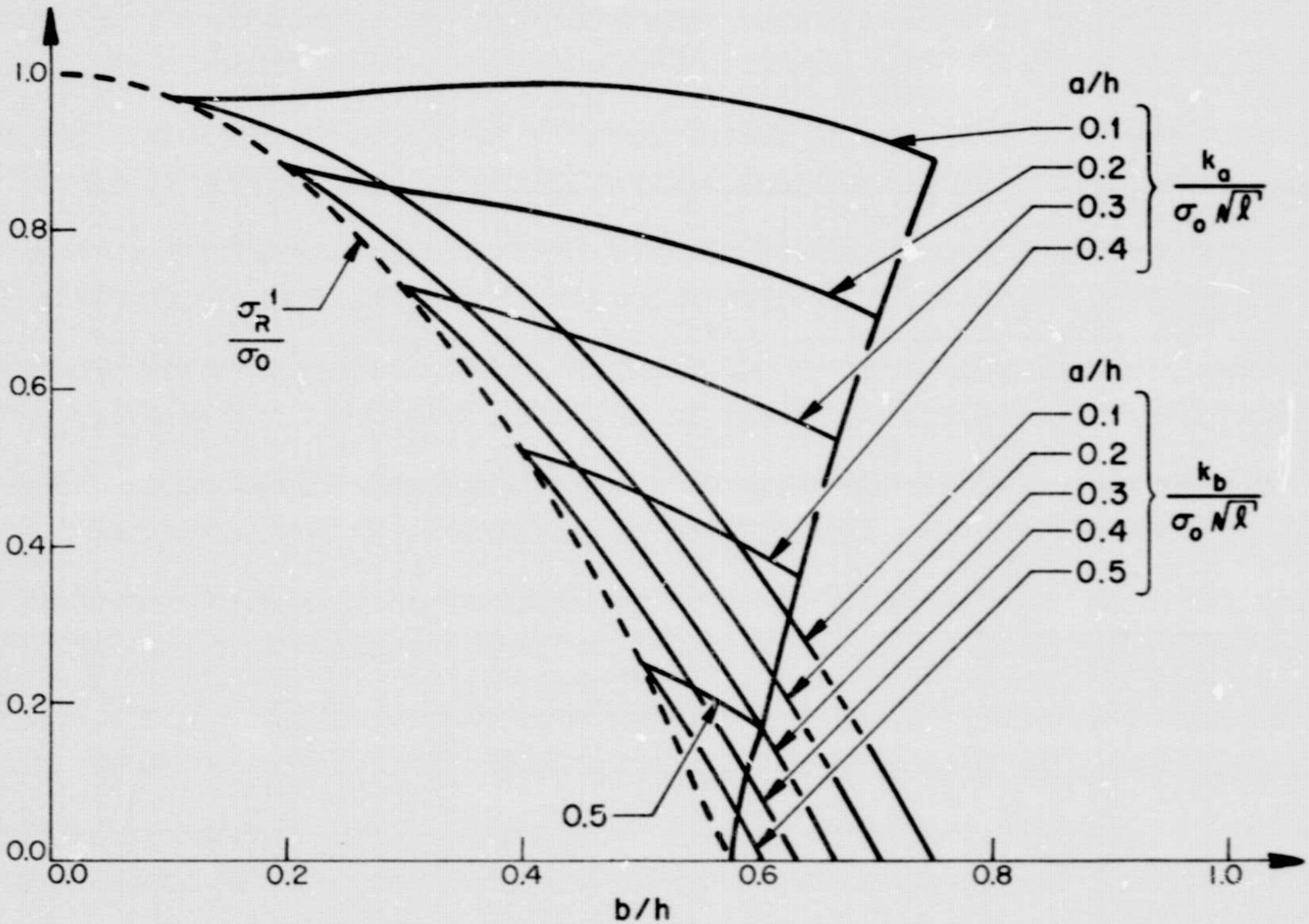


Figure 2. Stress intensity factors at the crack tips a and b due to the residual stress field $\sigma_R^1(x) = \sigma_o(1-3x^2/h^2)$.

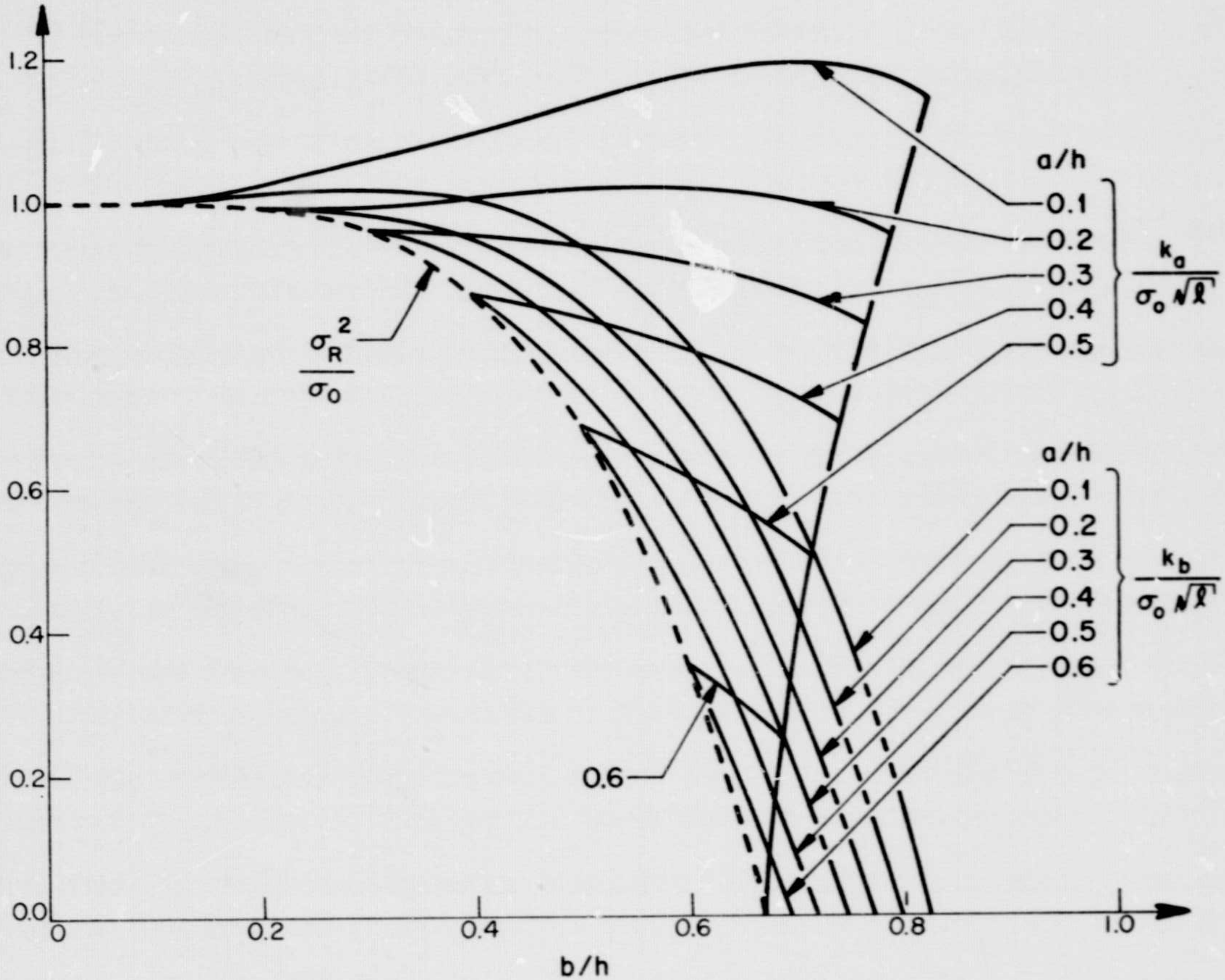


Figure 3. Stress intensity factors at the crack tips a and b due to the residual stress field $\sigma_R^2(x) = \sigma_o(1-5x^4/h^4)$.

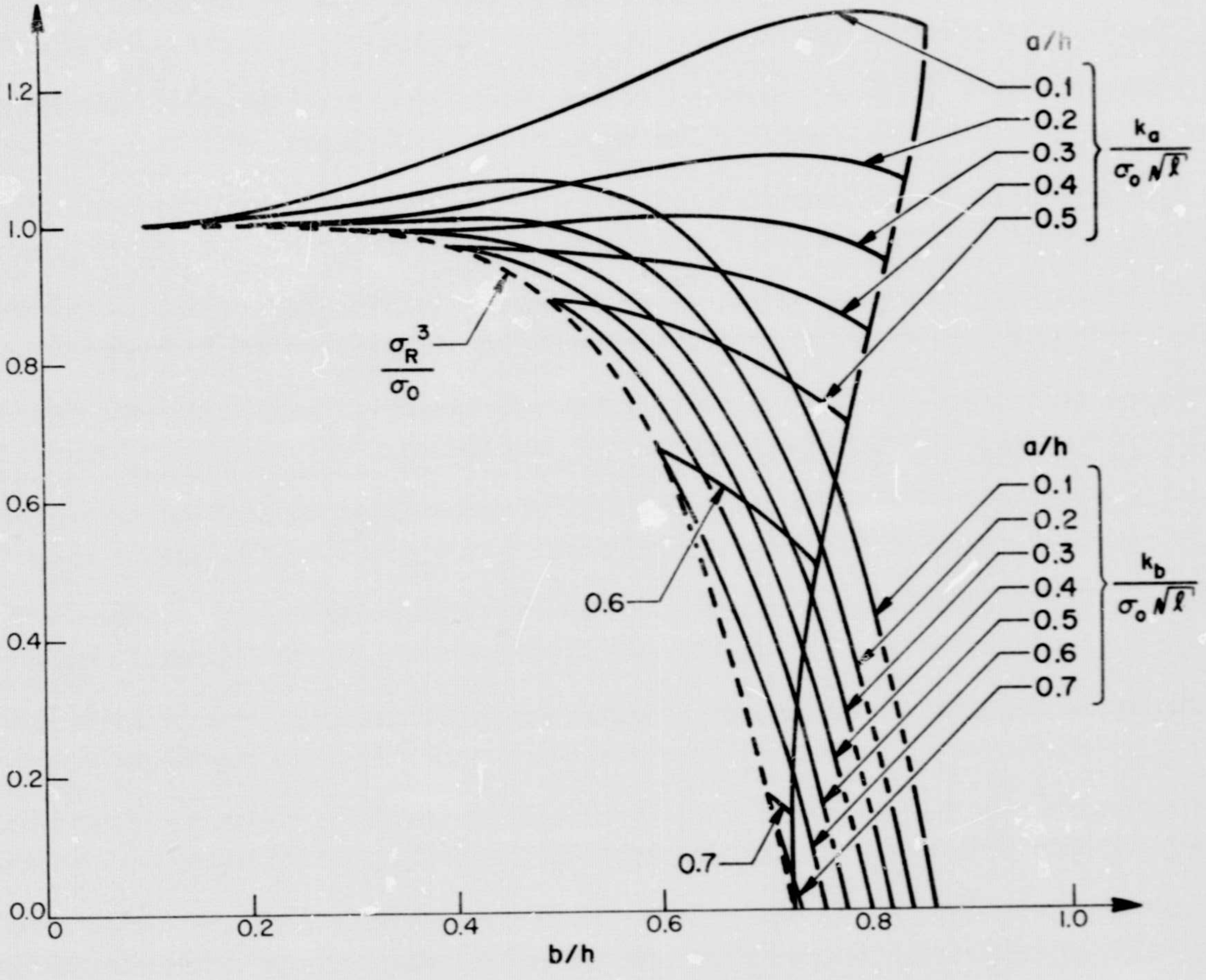


Figure 4. Stress intensity factors at the crack tip a and b due to the residual stress field $\sigma_R^3(x) = \sigma_o(1-7x^6/h^6)$.

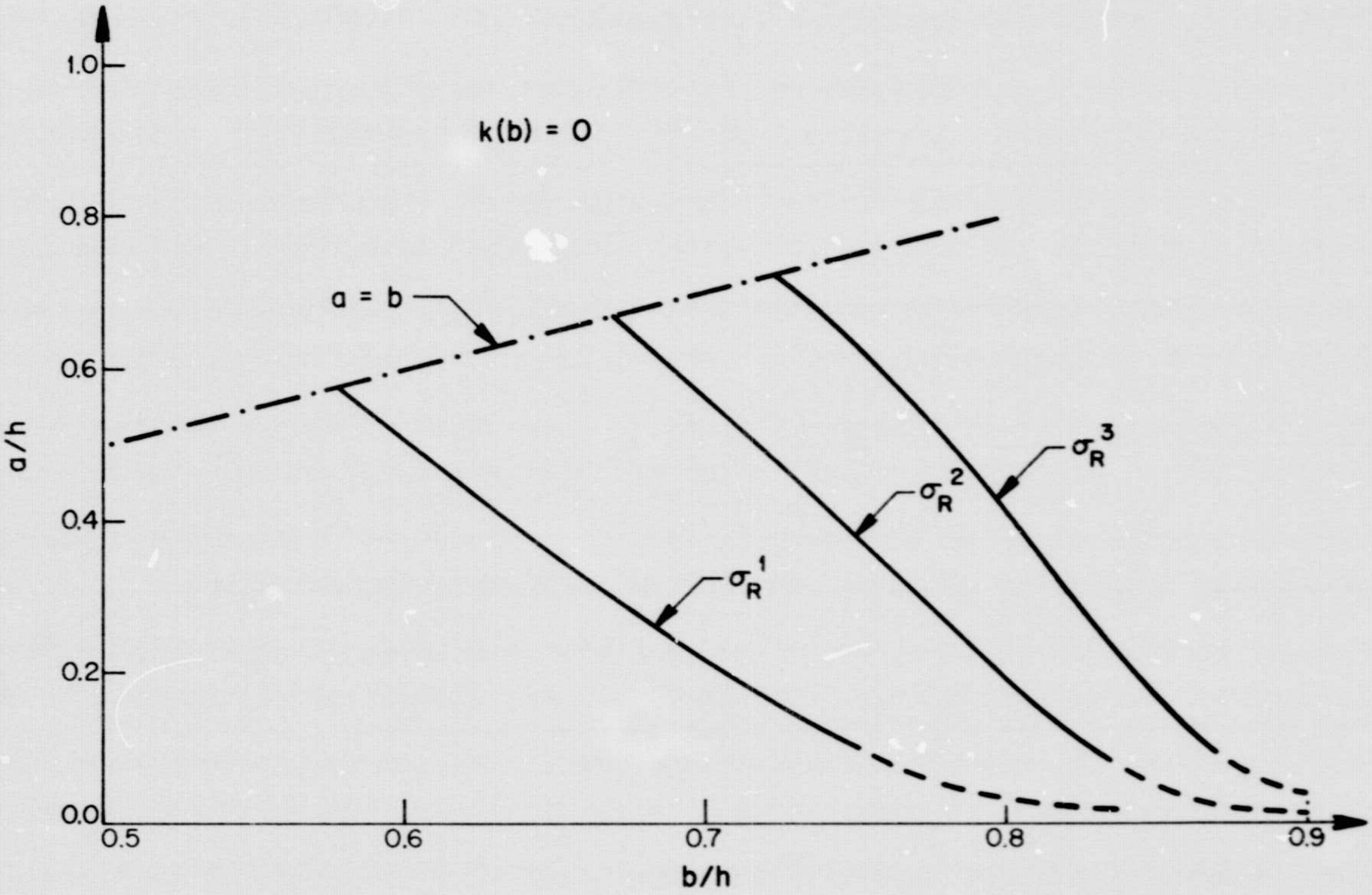


Figure 5. The value of the crack closure distance b for a given crack depth a in the crack-contact problem.

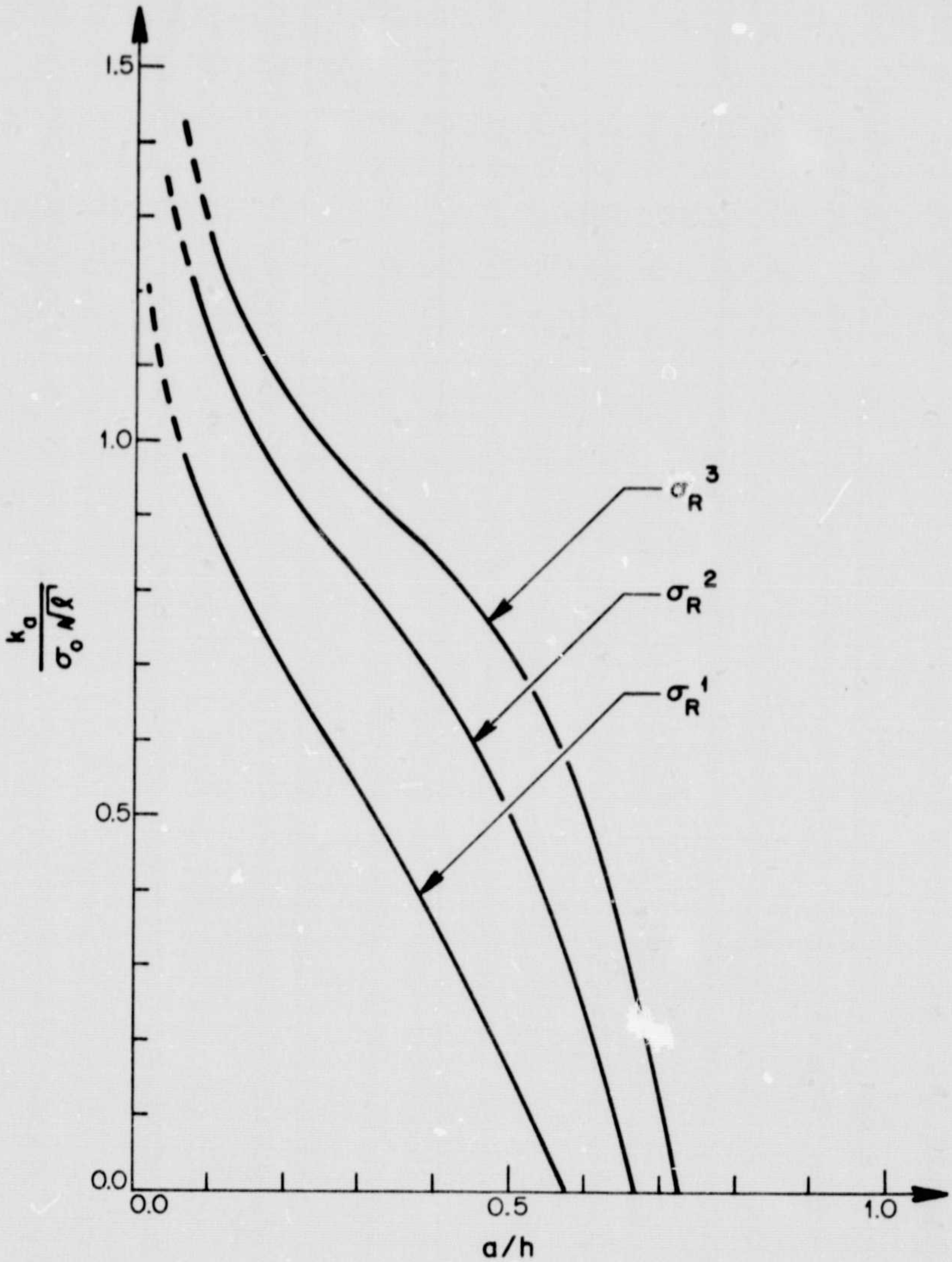


Figure 6. The stress intensity factor $k_a$ at the tip of the edge cracks for the crack-contact problem under residual stresses $\sigma_R^1$, $\sigma_R^2$, and $\sigma_R^3$.

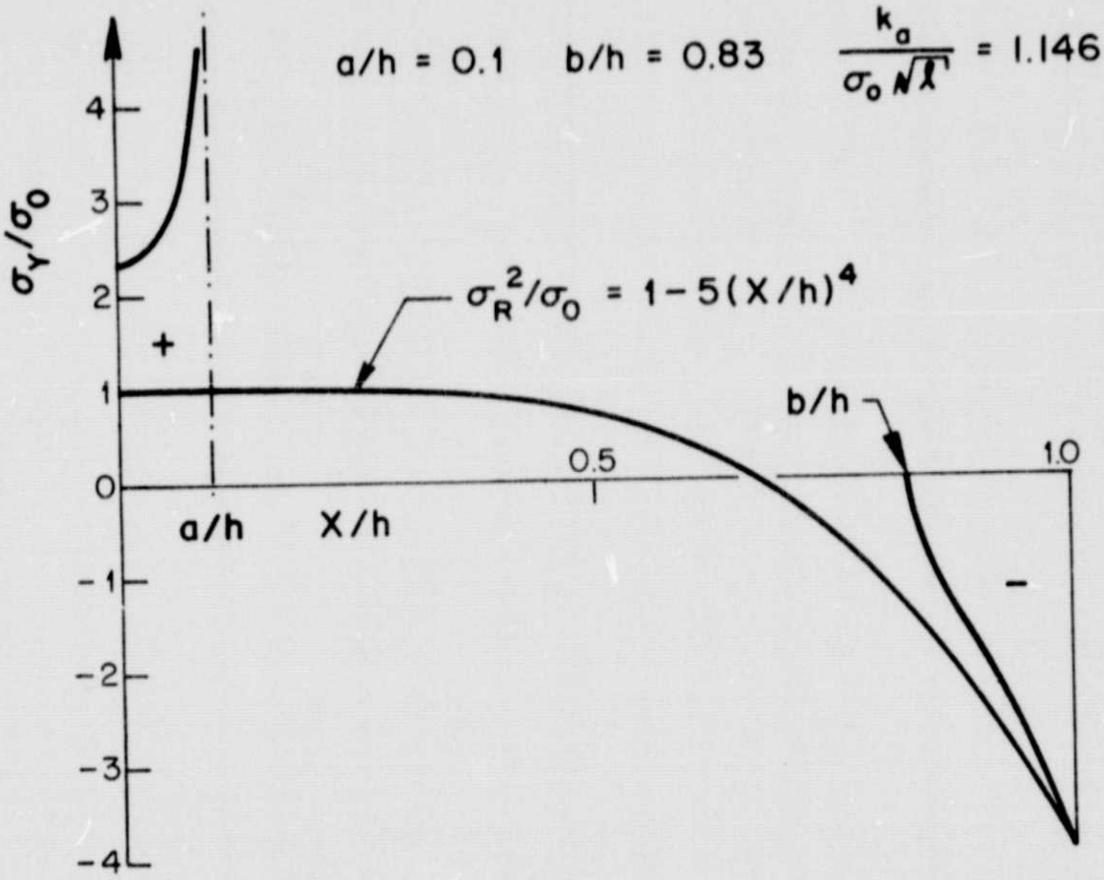


Figure 7. The normal stress on the y=0 plane for the crack-contact problem crack: a<x<b, contact: b<x<h.

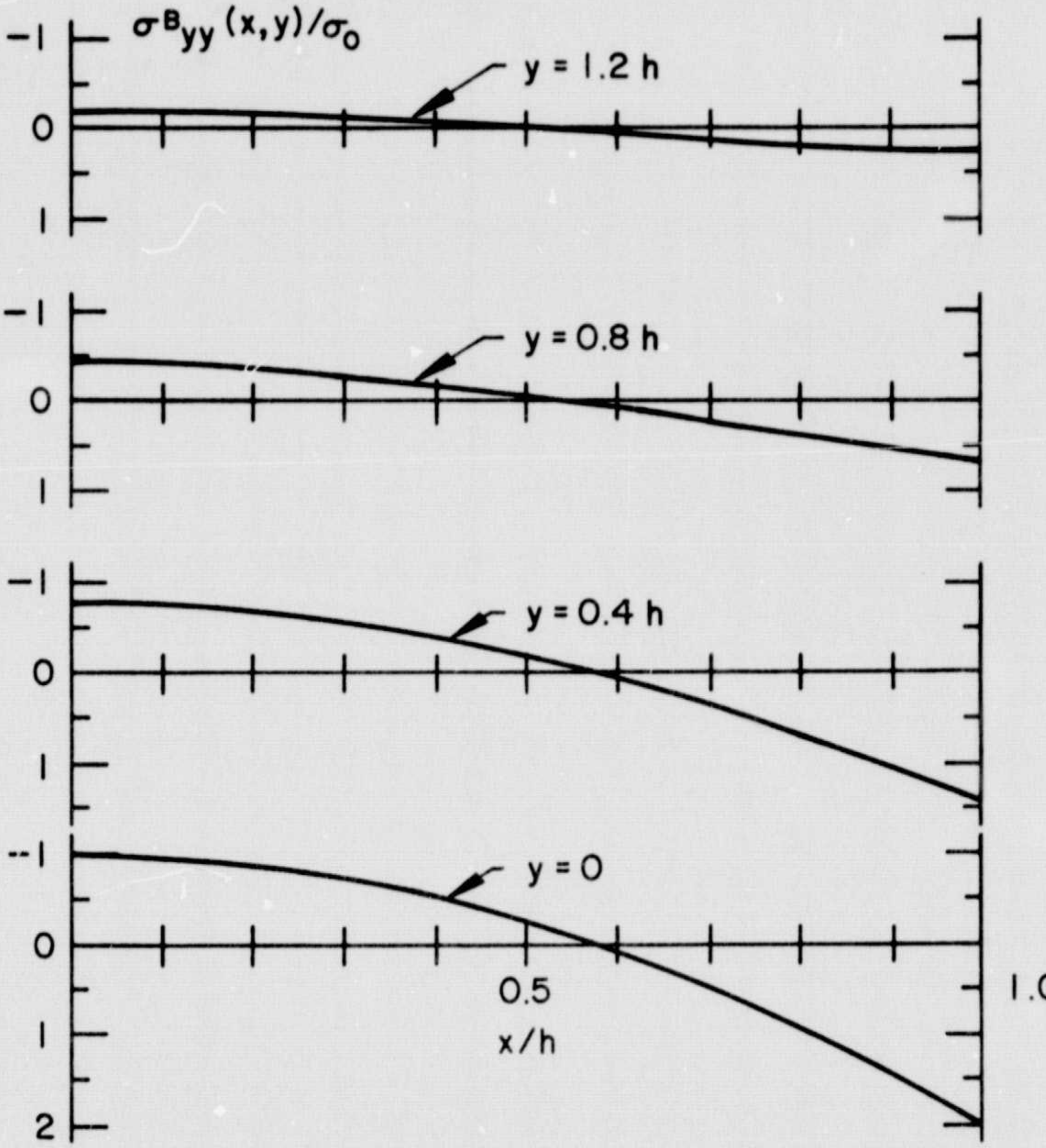


Figure 8. Diffusion of the normal stress $\sigma^B_{yy}$ for the perturbation component of the end problem in a semi-infinite strip under the residual stress $\sigma^1_R(x) = \sigma_o(1-3x^2/h^2)$.

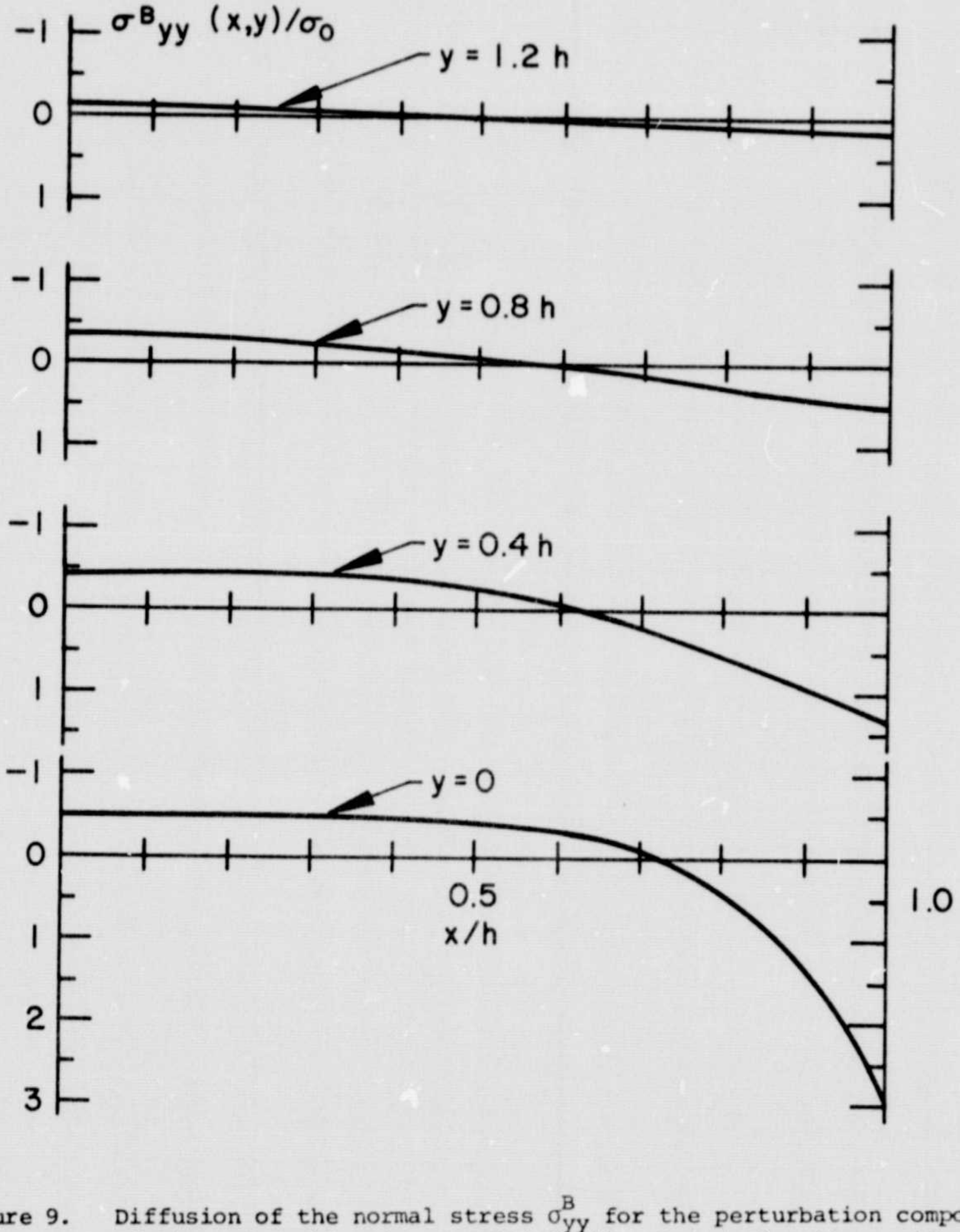


Figure 9. Diffusion of the normal stress $\sigma_{yy}^{B}$ for the perturbation component of the end problem in a semi-infinite strip under the residual stress $\sigma_R^3(x) = \sigma_o(1-7x^6/h^6)$.

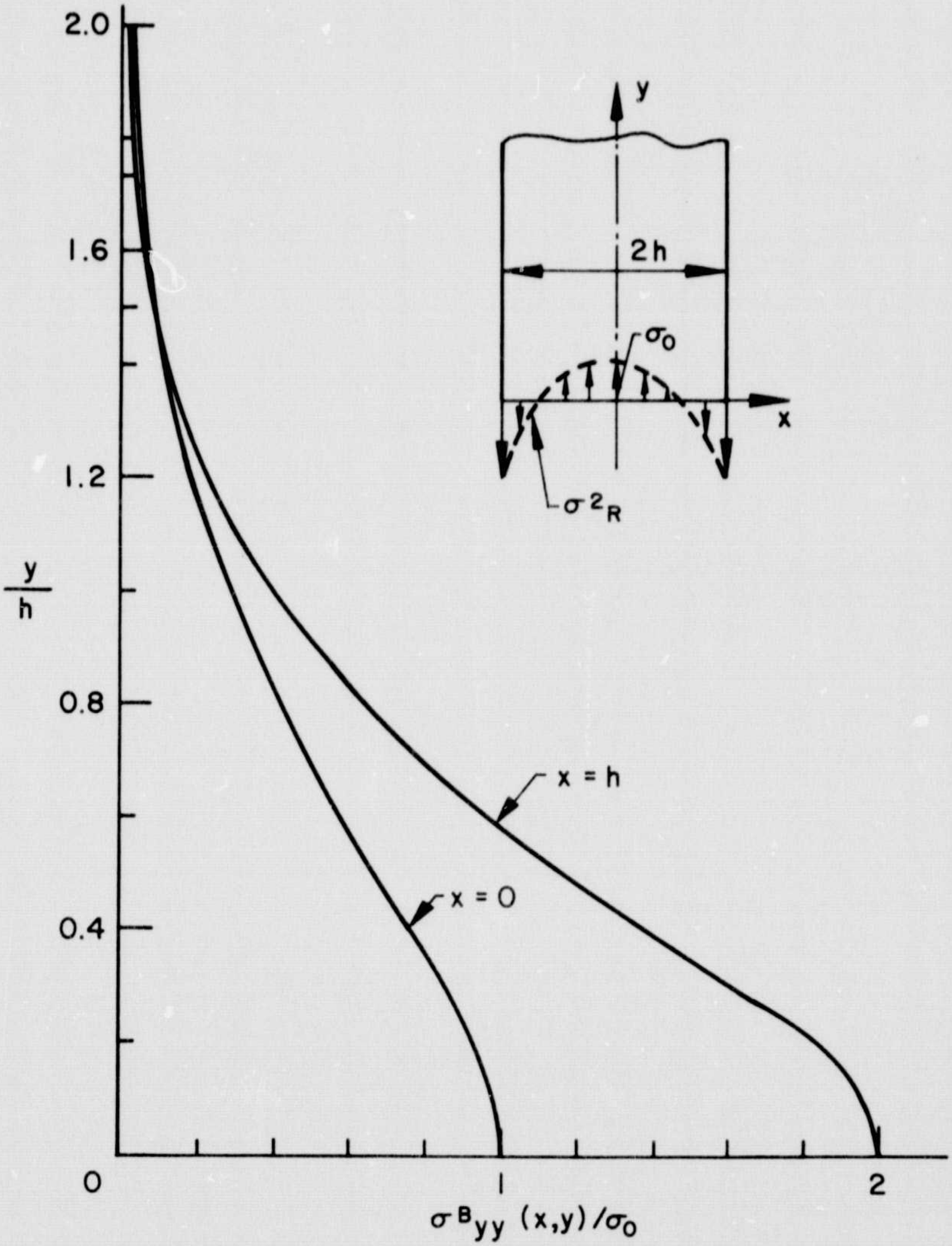


Figure 10. Variation of the normal stress $\sigma^B_{yy}$ in the midplane and on the surface of a semi-infinite strip under the residual stress $\sigma^1_R$.

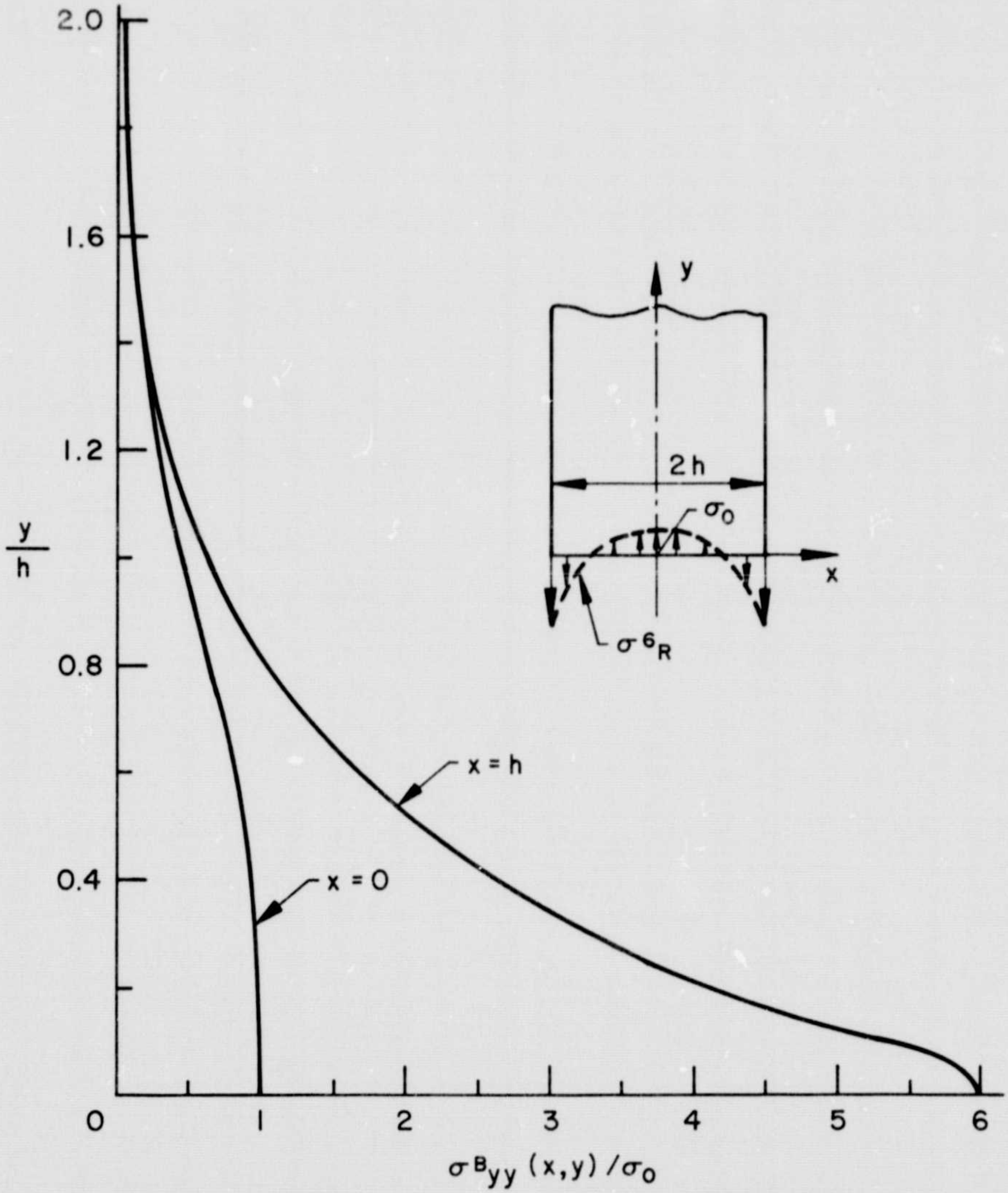


Figure 11. Variation of $\sigma_{yy}^B$ on x=0 and x=h in a semi-infinite strip under the residual stress $\sigma_R^3$.

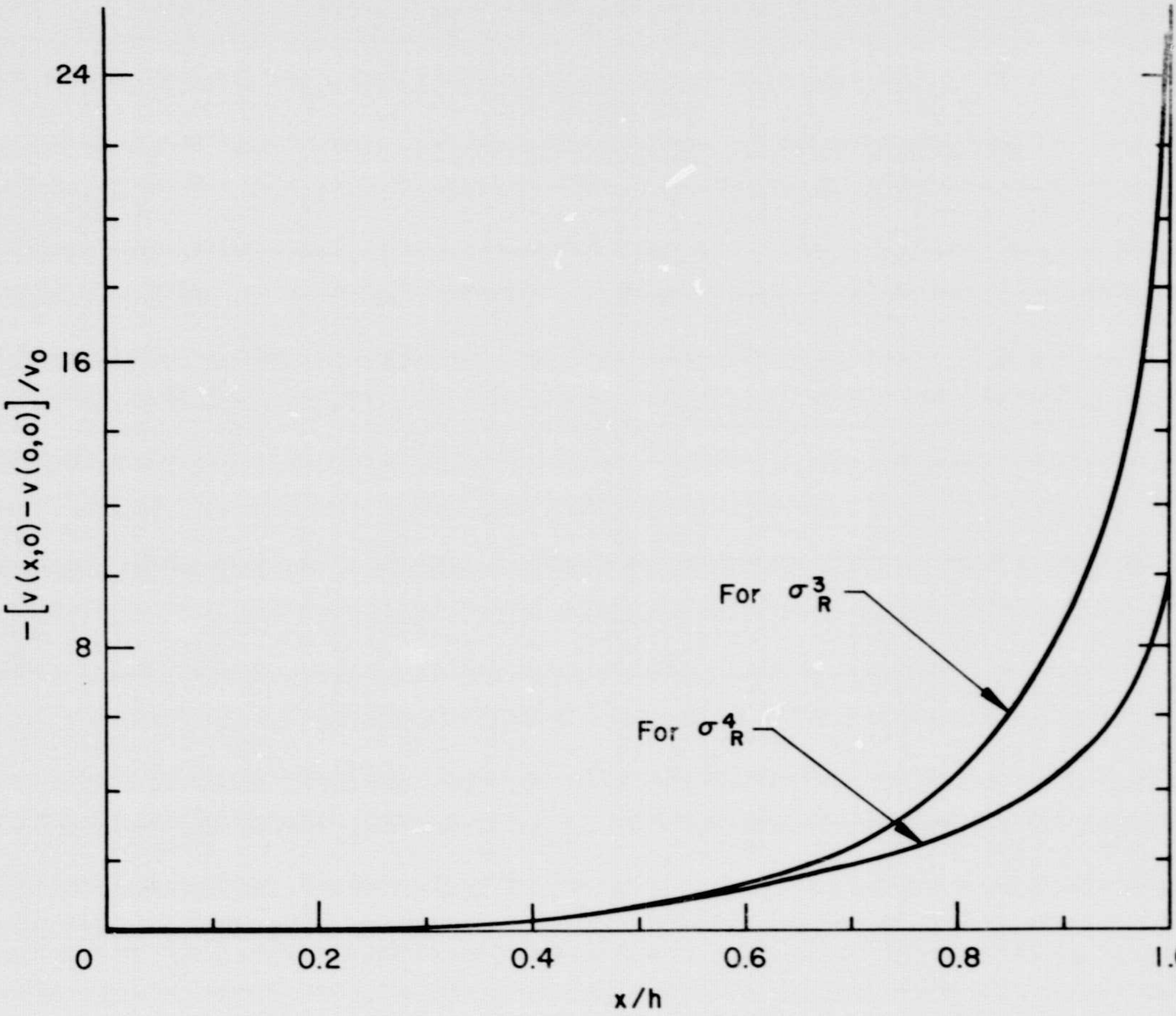


Figure 12. Relative normal displacement of the end surface in a semi-infinite strip under the residual stresses $\sigma_R^1$ and $\sigma_R^3$ ($v_o=(1+\kappa)h\sigma_o/(4\mu)$).