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EIGENVALUE EXTRACTION IN NASTRAN BY THE TRIDIAGONAL REDUCTION (FEER) METHOD - REAL EIGENVALUE ANALYSIS

Malcolm Newman and Paul F. Flanagan

Prepared by
ANALYTICAL MECHANICS ASSOCIATES, INC.
Jericho, N.Y. 11753
for Langley Research Center

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50 Jericho Turnpike
Jericho, New York 11753

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16. Abstract

The Tridiagonal Reduction (FEER) Method is an automatic matrix reduction scheme whereby the eigensolutions in the neighborhood of a specified point in the eigenspectrum can be accurately extracted from a tridiagonal eigenvalve problem whose order is much lower than that of the full problem. The process is effected without orbitrary lumping of masses or other physical quantities at selected node points and thus avoids one of the basic weaknesses of other techniques. This report describes the development of the Tridiagonal Reduction Method and its implementation in NASTRAN for real eigenvalue analysis as typified by structural vibration and buckling problems.

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FOREWORD

The report describes the development of the Tridiagonal Reduction Method for real eigenvalue analysis and the implementation of this technique in NASTRAN. A follow-on report, Part II, will subsequently be issued for a complex eigenvalue version of the method.

All major steps in the analytical development are documented. In addition, a detailed summary of the computational procedures, explanatory flow diagrams and user data-preparation instructions are provided.

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SYMBOLS

Latin

a	general element of reduced eigenmatrix
c_i	participation factor for ith eigenvector, $\{X_i\}$
d	off-diagonal element of reduced tridiagonal matrix
$\bar{d}$	normalization factor for v-vectors and approximate off-diagonal element of reduced tridiagonal matrix
f	number of previously calculated modes
K_{ij}	element of $[K_{aa}]$ matrix
m	size of reduced eigenproblem
m_{ij}	element of [M] matrix
n	size of the full, unreduced eigenproblem
$\bar{n}$	number of non-null columns or rows of [M] matrix
q	total number of desired eigensolutions, including those previously computed
$\bar{q}$	desired number of new eigensolutions
r	rank of the matrix [M]
$\bar{r}$	maximum possible size of reduced eigenproblem
r_i	weighted root-mean square residual
t	number of decimal digits carried by computer

Greek

α^2	small negative shift parameter for vibration problems
ε	tolerance for rejecting small elements of [M]
Λ, $\bar{\Lambda}$	exact and approximate values of inverted and shifted eigenvalues
λ_a	physical eigenvalue

λ_o shift in physical eigenvalue

λ, $\bar{\lambda}$ exact and approximate buckling eigenvalues

λ' $\omega^2 - \omega_o^2$

ξ_i bound on the absolute relative error in ith physical eigenvalue

ω_o center of frequency range of interest

ω, $\bar{\omega}$ exact and approximate circular natural frequencies

Matrices and Vectors

$\{b\}$ vector of scalar coefficients

$[A]$ tridiagonal, reduced eigenmatrix

$[\bar{B}]$ eigenmatrix for the inverse problem

$[B]$ $[D][\bar{B}]$, symmetric matrix

$[C]$ lower triangular Cholesky factor of decomposed $[\bar{K}]$

$[d]$ diagonal factor of decomposed $[\bar{K}]$

$[D]$ symmetric orthogonality matrix for the inverse problem

$[G_{m+1}]$ defined by Equation (21), Section 2.2

$[H]$ upper Hessenberg matrix

$[I]$ identity matrix

$[K]$ stiffness matrix

$[\bar{K}]$ effective stiffness matrix after eigenvalue shift

$[K_{aa}]$ stiffness matrix for analysis set

$[K_{aa}^d]$ differential stiffness matrix for analysis set

$[L]$ lower triangular factor of decomposed $[\bar{K}]$

$[M]$ equal to $[M_{aa}]$ or $[K_{aa}^d]$

$[M_{aa}]$ mass matrix for analysis set

$\{R\}$ residual vector

$[V]$ matrix of v-vectors assembled column-wise

$\{v\}$ trial vector

$\{w\}$ pseudo-random vector

$\{X\}$ eigenvector of the symmetric inverse problem

$\{\tilde{X}\}$ previously generated eigenvector

$\{y\}$ eigenvector of the reduced problem

$[Z]$ exact modal matrix

$\{\phi\}$, $\{\bar{\phi}\}$ exact and approximate physical eigenvector

$[\tilde{\Lambda}_d\tilde{}]$ diagonal matrix of exact eigenvalues

1. INTRODUCTION

The Tridiagonal Reduction or FEER Method is an automatic matrix reduction scheme whereby the eigensolutions in the neighborhood of a specified point in the eigenspectrum can be accurately extracted from a tridiagonal eigenvalue problem whose order is much lower than that of the full problem. Specifically, the order, m, of the reduced problem is never greater than

$$m = 2\bar{q} + 10 \tag{1}$$

where $\bar{q}$ is the desired number of accurately computed eigenvalues. Thus, the intrinsic power of the method lies in the fact that the size of the reduced eigenvalue problem is of the same order of magnitude as the number of desired roots, even though the discretized system model may possess thousands of degrees of freedom. The process is effected without arbitrary lumping of masses or other physical quantities at selected node points and thus avoids one of the basic weaknesses of the Guyan Reduction Method (Reference [1]) and other techniques (References [2] and [3]) requiring a judicious selection of the degrees of freedom to be retained.

Tridiagonal reduction was first suggested by Crandall (Reference [4]) as a truncated version of the Lanczos Algorithm (Reference [5]). However, it was soon discovered that the original scheme possessed numerical instabilities (References [6] and [7]). The necessary improvements to correct these weaknesses were made by Ojalvo and Newman (Reference [8]) who were the first to develop a successful tridiagonal reduction program for large scale structural vibration problems. Further refinements were later introduced by Newman and Pipano in the FEER computer program (References [9] and [10]), including the following extended features:

1. Highly efficient numerical computation schemes which take advantage of matrix banding and sparsity.

2. Calculation of upper and lower error bounds on the extracted eigenvalue estimates.

3. Accommodation of singular mass matrices and stiffness matrix singularities associated with rigid body modes.

The last capability, which is also present in the Inverse Power Method with Shifts, overcomes a basic restriction of the Tridiagonal (Givens) Method, namely, the need to eliminate massless degrees of freedom.

From the standpoint of computational speed, the Tridiagonal Reduction Method is almost as fast as the Givens and Householder methods (References [11] and [12]) when all the eigensolutions are calculated, and becomes increasingly more efficient as the number of required eigensolutions is reduced. In addition, in order to avoid prohibitively long running times, both the Givens and Householder methods require the use of a relatively large computer central memory for even moderate problem sizes, while the Tridiagonal Reduction Method is extremely efficient with regard to core requirements.

As shown in Section 2.1, the Tridiagonal Reduction Method employs only a single initial shift of eigenvalues and hence usually requires only one matrix decomposition. It consequently tends to be much more efficient than the Inverse Power Method when more than one or two eigensolutions are required.

This report describes the development of the Tridiagonal Reduction Method and its implementation in NASTRAN for real eigenvalue analysis as typified by structural vibration and buckling problems.

The restrictions on the use of the method for real eigenvalue analysis in NASTRAN are as follows:

1. For structural vibration mode applications the method extracts a preselected number of eigenvalues which are closest to a specified shift value, λ_o, rather than computing the eigenvalues in a prescribed range.

2. In buckling problems, a preselected number of eigenvalues of smallest magnitude are obtained, i.e., no shifting is performed. Physically, this implies that the buckling load parameters, whether positive or negative, are computed in order of increasing magnitude.

The basic steps employed in the method are as follows. First, the initial eigenvalue problem

$$[K - \lambda_a M]\{\phi\} = 0 \tag{2}$$

is converted to a symmetric inverse form

$$[B]\{X\} = \Lambda[D]\{X\} \tag{3}$$

where

$$\Lambda = \frac{1}{\lambda_a - \lambda_o} \tag{4}$$

and λ_o is a shift value which is used only in structural vibration mode applications. Second, the tridiagonal reduction algorithm is employed to transform Equation (3) into a tridiagonal form of reduced order. Third, the eigenvalues of the reduced matrix are extracted using a Q-R algorithm similar to that described in Reference [13]. Fourth, the corresponding eigenvectors are computed and converted to physical form. Finally, upper and lower error bounds on the extracted eigenvalues are obtained.

The development of the method is set forth in Sections 2 to 6. A detailed summary of the computational procedures used in NASTRAN, explanatory flow diagrams and user data preparation instructions are provided in Sections 7 and 8. In addition, user information and error messages and optional diagnostic output relating to the Tridiagonal Reduction Method are described in Section 9.

2. THE TRIDIAGONAL REDUCTION METHOD

2.1 Preliminary Operations

The problem is to find a specified number of real eigenvalues and corresponding eigenvectors for

$$[K - \lambda_a M]\{\phi\} = 0 \qquad (1)$$

It is further required that these eigensolutions constitute the set lying closest to a specified point, λ_o, in the eigenspectrum.

The definitions of the eigenvalue, λ_a, the matrices [K] and [M], and their mathematical properties, depend on the type of problem being solved within the NASTRAN environment. For real analysis, which is the subject of the current report, only two separate problem types need be considered; structural vibration and buckling problems. The matrix definitions and mathematical distinctions for these two cases are summarized on the following page.

TABLE 1 - Problem Formulations

Problem Type	Quantity	Definition	NASTRAN Notation	Most General Properties
Structural Vibration Modes	[K]	Stiffness Matrix - analysis set	$[K_{aa}]$	Symmetric, non-negative, semidefinite matrix
	[M]	Mass Matrix - analysis set	$[M_{aa}]$	Same
	λ_a	Square of a circular natural frequency	ω^2	Positive
Buckling	[K]	Stiffness Matrix - analysis set	$[K_{aa}]$	Symmetric, positive-definite matrix
	[M]	Differential Stiffness Matrix - analysis set	$[K^d_{aa}]$	Symmetric, indefinite matrix
	λ_a	Buckling Load Parameter	$-\lambda$	Positive or negative

The essential mathematical differences between the two types of problems center around the properties of the [M] matrix, which is non-negative for vibration mode problems, but indefinite for buckling problems, thereby permitting the existence of both positive and negative eigenvalues in the latter case. In addition, the stiffness matrix may be singular for vibration problems while it is always positive definite in buckling applications, which implies that the buckling analysis is performed on a kinematically stable structure.

In summary, the two problems under consideration are of the forms

$$[K_{aa} - \omega^2 M_{aa}]\{\phi\} = 0 \qquad \text{(structural vibrations)} \qquad (2a)$$

and

$$[K_{aa} + \lambda K^d_{aa}]\ \{\phi\} = 0 \qquad \text{(buckling)} \qquad (2b)$$

Further, if the user requests vibration modes in the neighborhood of a specified frequency, ω_o, Equation (2a) can be written as

$$[\bar{K}]\{\phi\} = \lambda'[M_{aa}]\{\phi\} \tag{3}$$

where

$$[\bar{K}] = [K_{aa} - \omega_o^2 M_{aa}] \tag{4a}$$

and

$$\lambda' = \omega^2 - \omega_o^2 \tag{4b}$$

The resulting effective stiffness matrix, $[\bar{K}]$, is indefinite in this case, since it possesses both positive and negative eigenvalues. This requires that a non-square root decomposition scheme be used in subsequent operations. However, $\omega_o = 0$ is taken as a default value, or it may be specified by the user. In this case, a specified number of natural frequencies starting with the lowest will be computed. In order to utilize a more efficient Cholesky decomposition of $[\bar{K}]$ under these conditions, a small negative shift $\lambda_o = -\alpha^2$ (see Section 5) is used, yielding

$$[\bar{K}] = [K_{aa} + \alpha^2 M_{aa}] \tag{5a}$$

and

$$\lambda' = \omega^2 + \alpha^2 \tag{5b}$$

It is easy to prove that the resulting effective stiffness matrix $[\bar{K}]$ is positive definite provided that the system masses generate positive kinetic energy due to any kinematically admissible rigid body motions of the structure. This requirement is always satisfied by the mass matrix in a physically well posed problem, thereby allowing a Cholesky square-root decomposition to be performed when the roots are computed in the neighborhood of zero. Since no shifting is performed in buckling problems, the effective stiffness matrix is $[\bar{K}] = [K_{aa}]$, which is always positive definite, again permitting the use of a Cholesky decomposition.

In any event, a decomposition or factoring of $[\bar{K}]$ is next performed:

$$[\bar{K}] = [L][\tilde{d}][L]^T \qquad \text{(shifted vibration mode problems)} \tag{6a}$$

or

$$[\bar{K}] = [C][C]^T \qquad \text{(buckling problems or vibration modes in the neighborhood of zero desired)} \tag{6b}$$

where [L] and [C] are lower triangular factors and [$\tilde{d}$] is a diagonal matrix.

To facilitate computation of eigenvalues closest to the point of interest within the eigenspectrum, inverse forms of the eigenvalue problems are employed, as in the Inverse Power Method with Shifts.

The general form of the inverse problem may be written as

$$[B]\{X\} = \Lambda[D]\{X\} \tag{7}$$

where the above terms are defined as follows:

TABLE 2 - Inverse Eigenproblem Definitions

Problem Type	[B]	[D]	{X}	Λ
1. Shifted Vibration Modes	$[M_{aa}][L^{-1}]^T[\tilde{d}]^{-1}[L^{-1}][M_{aa}]$	$[M_{aa}]$	$\{\phi\}$	$\frac{1}{\omega^2-\omega_o^2}$
2. Unshifted Vibration Modes (in the neighborhood of zero frequency	$[C^{-1}][M_{aa}][C^{-1}]^T$	[I] (Identity Matrix)	$[C]^T\{\phi\}$	$\frac{1}{\omega^2+\alpha^2}$
3. Buckling Modes	$[C^{-1}][K^d_{aa}][C^{-1}]^T$	[I]	$[C]^T\{\phi\}$	$-\frac{1}{\lambda}$

The above triangular matrix inverses are treated as purely operational symbols, since in actual numerical computations vectors defined, for example, by

$$\{a\} = [L^{-1}]\{b\} \tag{8a}$$

$$\{\bar{a}\} = [L^{-1}]^T\{\bar{b}\} \tag{8b}$$

are obtained from the solutions of

$$[L]\{a\} = \{b\} \tag{9a}$$

$$[L]^T\{\bar{a}\} = \{\bar{b}\} \tag{9b}$$

employing forward and backward passes.

2.2 The Reduction Algorithm

A reduction of the order of the eigenvalue problem, Equation (7), is effected through the transformation

$$\underset{nx1}{\{\bar{X}\}} = \underset{nxm}{[V]}\ \underset{mx1}{\{y\}} \tag{10}$$

where $\{\bar{X}\}$ is an approximation of $\{X\}$, n is the order of the unreduced problem, and $m \leq n$. The transformation matrix is taken to be orthonormal to $[D]$*, so that

$$[V]^T[D][V] = [I] \tag{11}$$

From Equations (7), (10), and (11) it is seen that

$$[A]\{y\} = \bar{\Lambda}\{y\} \tag{12}$$

where

$$\underset{mxm}{[A]} = [V]^T[B][V] \tag{13}$$

and $\bar{\Lambda}$ is an approximation of the eigenvalue, Λ.

Thus, Equation (12) is an mth order eigenvalue problem where $m \leq n$. The value of m is established according to the criteria given in Section 2.3.

The essence of the reduction scheme lies in the choice of the transformation matrix [V]. In the present case the Lanczos algorithm is used to build up the [V] matrix, vector by vector, i.e.,

*In problem types 2 and 3 (see Table 2), [D] is the identity matrix and, therefore, $[V]^T[V] = [I]$.

$$[V]_{nxm} = [\{v_1\}, \{v_2\}, \ldots, \{v_m\}], \tag{14}$$

such that the reduced mxm matrix [A] is tridiagonal and its eigenvalues accurately approximate the roots of Equation (7) having the largest magnitude (or, equivalently, the roots of the physical model closest to the specified point of interest in the eigenspectrum).

Define the matrix

$$[\bar{B}] = [D^{-1}][B]^*, \tag{15}$$

and let

$$\begin{aligned} d_{i+1}\{v_{i+1}\} = \{\bar{v}_{i+1}\} &= [\bar{B}]\{v_i\} - a_{i,i}\{v_i\} - a_{i,i-1}\{v_{i-1}\} \\ &\quad \ldots\ldots\ldots\ldots\ldots - a_{i,1}\{v_1\} \\ &= [\bar{B}]\{v_i\} - \sum_{j=1}^{i} a_{i,j}\{v_j\}; \quad i = 1, m-1 \end{aligned} \tag{16}$$

where $\{v_1\}$ is a starting vector (see Section 3) and all m $\{v\}$-vectors are orthonormal to [D]. This implies that

$$a_{i,j} = \{v_j\}^T[B]\{v_i\} \tag{17}$$

while d_{i+1} is a normalizing factor given by,

$$d_{i+1} = [\{\bar{v}_{i+1}\}^T[D]\{\bar{v}_{i+1}\}]^{1/2} \tag{18}$$

The recurrence relationship, Equation (16), when carried out to one additional step (i.e., over the range i = 1, m), can be expressed in the following matrix form:

*For shifted vibration mode problems, where $[D] = [M_{aa}]$, this operation is purely symbolic (since $[M_{aa}]$ may be singular) and Equation (15) merely implies that $[\bar{B}] = [L^{-1}]^T[\text{-d-}]^{-1}[L^{-1}][M_{aa}]$.

$$\underset{nxn}{[\bar{B}]}\ \underset{nxm}{[V]} = \underset{nxm}{[V]}\ \underset{mxm}{[H]} + d_{m+1}\underset{nxm}{[G_{m+1}]} \tag{19}$$

where [H] is an upper Hessenberg matrix given by,

$$[H] = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ d_2 & a_{22} & a_{32} & \cdots & a_{m2} \\ & d_3 & a_{33} & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & d_m & a_{mm} \end{bmatrix} \tag{20}$$

and

$$[G_{m+1}] = [\{0\} \,\vdots\, \{0\} \,\vdots\, \cdots\cdots \,\vdots\, \{0\} \,\vdots\, \{v_{m+1}\}]. \tag{21}$$

Premultiplying Equation (19) by $[V]^T[D]$, it can be seen that

$$[V]^T[B][V] = [V]^T[D][V][H] + [0] \tag{22}$$

or

$$[A] = [H]. \tag{23}$$

However, the matrix [A] is symmetric and hence [H] is symmetric, requiring that it be tridiagonal. It follows that

$$a_{i,j} = 0; \quad j < i-1 \tag{24}$$

and

$$d_i = a_{i,\, i-1}\,, \tag{25}$$

while Equation (19) assumes the form

$$[\bar{B}][V] = [V][A] + d_{m+1}[G_{m+1}]. \tag{26}$$

As a consequence of Equations (23) to (25), the reduced tridiagonal eigenvalue problem, Equation (12), is

$$[A]\{y\} = \begin{bmatrix} a_{11} & d_2 & & & & \\ d_2 & a_{22} & d_3 & & & \\ & d_3 & a_{33} & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & d_m \\ & & & & d_m & a_{mm} \end{bmatrix} \{y\} = \bar{\Lambda}\{y\} \qquad (27)$$

and the matrix coefficients are theoretically given by the simplified recurrence formulas

$$\left.\begin{aligned} a_{i,i} &= \{v_i\}^T[B]\{v_i\} \\ \{\bar{v}_{i+1}\} &= [\bar{B}]\{v_i\} - a_{i,i}\{v_i\} - d_i\{v_{i-1}\} \\ d_{i+1} &= [\{\bar{v}_{i+1}\}^T[D]\{\bar{v}_{i+1}\}]^{1/2} \end{aligned}\right\} \quad i = 1, m \qquad (28a)$$

$$\{v_{i+1}\} = \frac{1}{d_{i+1}}\{\bar{v}_{i+1}\}; \qquad i = 1, m-1 \qquad (28b)$$

where the sequence is initialized by choosing a random starting vector for $\{v_1\}$ and setting $d_1 = 0$, $\{v_o\} = \{0\}$.

Although the final off-diagonal term, d_{m+1}, given by Equations (28a) is not needed to construct the tri-diagonal reduced matrix, it is calculated and saved for use in establishing error bounds (see Section 6). In addition, the above algorithm is modified in the computer program as follows:

1. Each vector $\{v_{i+1}\}$, calculated by Equations (28b) is reorthogonalized to all previously computed {v}-vectors and eigenvectors, as described in Section 4, before re-entering Equations (28a).

2. The actual size, m, of the reduced eigenproblem is established by the criteria given in Section 2.3, one of the restrictions being that it cannot exceed the rank of the matrix $[M_{aa}]$ for vibration mode problems or $[K^d_{aa}]$ for buckling problems.

The eigenvalues, $\bar{\Lambda}$, and eigenvectors, $\{y\}$ of Equation (27) are extracted using a Q-R algorithm and eigenvector computational procedure similar to that described in the NASTRAN Theoretical Manual (Reference [13]). They are then converted to physical form as follows:

$$\bar{\lambda}_i = \frac{-1}{\bar{\Lambda}_i} \qquad \text{(buckling problems)} \qquad (29a)$$

$$\bar{\omega}_i^2 = \frac{1}{\bar{\Lambda}_i} - \alpha^2 \qquad \text{(unshifted vibration mode problems)} \qquad (29b)$$

$$\bar{\omega}_i^2 = \frac{1}{\bar{\Lambda}_i} + \omega_o^2 \qquad \text{(shifted vibration mode problems)} \qquad (29c)$$

$$\{\bar{\phi}_i\} = [C^{-1}]^T[V]\{y_i\} \qquad \text{(buckling or unshifted vibration mode problems)} \qquad (29d)$$

$$\{\bar{\phi}_i\} = [V]\{y_i\} \qquad \text{(shifted vibration mode problems)} \qquad (29e)$$

In addition, the vectors $\{y_i\}$ are normalized ($\{y_i\}^T\{y_i\} = 1$) for convenience in establishing error bounds (see Section 6).

2.3 Criteria for the Size of the Reduced Eigenvalue Problem

The total number of eigensolutions, including any existing rigid body modes, is equal to the rank, r, of the matrix $[M] = [M_{aa}]$ or $[M] = [K^d_{aa}]$, depending on whether a buckling or vibration mode problem is being solved. Thus, the size of the reduced problem, m, cannot be greater than r. If, for example, [M] is diagonal, then the maximum permissible value of m is equal to the total number of non-zero diagonal entries. In addition, if f eigensolutions have previously been computed by NASTRAN, these must be swept out of the problem by making all

{v} vectors orthogonal to the previously computed eigenvectors. This implies that the maximum size of the reduced problem is further reduced to

$$\bar{r} = r - f \tag{30}$$

As a result of numerical experiments and application experiences (References [8, 9, 14]), it has been found that in cases where $m << \bar{r}$, a first grouping of more than $m/2$ eigenvalues closest to the shift point are in accurate agreement with the corresponding number of exact eigenvalues. The remaining reduced-system roots are spread across the remaining exact eigenspectrum.

Thus, if the user requests a total of q eigenvalues closest to a specified numerical value*, the order of the reduced problem solved by NASTRAN will be

$$m = \min[2\bar{q} + 10, \bar{r}] \tag{31a}$$

where

$$\bar{q} = q - f \tag{31b}$$

It should be noted that in all cases $m \leq \bar{r}$, and whenever m is set equal to $\bar{r}$, all the eigensolutions of the unreduced problem, which have not been previously computed, are generated.

From the above discussion it is apparent that, depending on the number of eigensolutions requested, the rank of [M] may have an influence on the order of the reduced problem which is generated by the program. Although the user should never request more than $\bar{r}$ eigensolutions, the value of $\bar{r}$ may not always be a simple matter to calculate, particularly in buckling problems. Therefore, the computational scheme has been designed to resolve the question of rank in the following manner:

*q includes the number of previously computed eigensolutions, f. These consist of modes generated prior to a restart plus rigid body modes generated by using a SUPØRT card in the bulk data deck.

1. The matrix [M] is first checked for inordinately small off-diagonal elements, i.e., those which lie in the round-off range of the computer. These terms can sometimes introduce artifically large physical eigenvalues and are therefore eliminated. Any off-diagonal element for which

$$|m_{ij}| \leq 10^{\frac{-2t}{3}} |m_{ii}|; \quad m_{ii} \neq 0, \tag{32}$$

is set equal to zero, where t is the number of decimal digits carried in the computation of the [M] matrix.

2. The number of non-null columns or rows of the above modified [M] matrix is counted and designated as $\bar{n}$. Since the rank, r, cannot be greater than $\bar{n}$, the program initially sets $\bar{r} = \bar{n} - f$ in Equations (31) to provide a tentative size, m, of the reduced eigenvalue problem. If the user has asked for more than $\bar{r}$ eigensolutions, he is given a message that too many eigensolutions have been requested and that the program will try to find all the existing solutions. (See Section 9 for a complete list of user messages and diagnostics).

3. If the reorthogonalization tests (see Section 4) fail for some vector $\{v_{i+1}\}$, this is an indication that a null vector has been generated because the maximum number of r-f linearly independent $\{v\}$ vectors have already been obtained. The recurrence sequence is then terminated and the order of the reduced eigenproblem is further reduced to $m = i$.

3. CHOICE OF THE INITIAL TRIAL VECTOR AND RESTART VECTORS

Prior to tridiagonal reduction, the original eigenvalue problem

$$[K - \lambda_a M]\{\phi\} = 0 \tag{1}$$

is cast in the inverse form

$$[B]\{X\} = \Lambda[D]\{X\} \tag{2}$$

or

$$[\bar{B}]\{X\} = \Lambda\{X\} \tag{3}$$

where

$$\Lambda = \frac{1}{\lambda_a - \lambda_o} . \tag{4}$$

From Equation (1) it can be seen that if the problem size is designated as n and r is the rank of the [M] matrix, then there are (n-r) spurious eigensolutions corresponding to $\Lambda = 0$ or, equivalently, $\lambda_a \to \infty$. If, for example, a relatively small number of nodal masses or differential stiffness elements are employed, then $[\bar{B}]$ contains a large multiplicity of zero eigenvalues which are of no interest and cause numerical difficulties. These are eliminated from the reduced tridiagonal problem by employing a constrained sub-set of v-vectors having zero projection on the set of eigenvectors associated with $\Lambda = 0$.

To accomplish this, use is made of the fact that any non-null vector $\{\bar{v}_1\}$, generated from any other non-null vector $\{w\}$, through

$$\{\bar{v}_1\} = [\bar{B}]\{w\} \tag{5}$$

will contain no components of the eigenvectors corresponding to $\Lambda = 0$. This can be seen as follows.

First, express $\{w\}$ as a linear combination of all the eigenvectors of $[\bar{B}]$:

$$\{w\} = \sum_{i=1}^{r} c_i\{X_i\} + \sum_{i=r+1}^{n} c_i\{X_i\} \tag{6}$$

where $\{X_i\}$, (i = 1, r), are the eigenvectors for $\Lambda \neq 0$ and $\{X_i\}$, (i = r + 1, n), are the eigenvectors for $\Lambda = 0$.

Next, substitute Equation (6) into Equation (5), giving

$$\{\bar{v}_1\} = \sum_{i=1}^{r} c_i[\bar{B}]\{X_i\} + \sum_{i=r+1}^{n} c_i[\bar{B}]\{X_i\} \; . \tag{7}$$

Since

$$[\bar{B}]\{X_i\} = \Lambda_i\{X_i\}; \quad i = 1, r \tag{8}$$

and

$$[\bar{B}]\{X_i\} = 0 \quad ; \quad i = r+1, n \tag{9}$$

it follows that

$$\{\bar{v}_1\} = \sum_{i=1}^{r} c_i\Lambda_i\{X_i\} \tag{10}$$

Thus, $\{\bar{v}_1\}$ contains no eigenvectors corresponding to $\Lambda = 0$, and is a null vector only if $\{w\}$ happens to be a $\Lambda = 0$ eigenvector. Further, it is easy to see that after $\{\bar{v}_1\}$ is normalized

$$\{v_1\} = \left[\frac{1}{\{\bar{v}_1\}^T[D]\{\bar{v}_1\}}\right]^{1/2}\{\bar{v}_1\}, \tag{11}$$

the next vector generated,

$$\{\bar{v}_2\} = [\bar{B}]\{v_1\} - a_{1,1}\{v_1\}, \tag{12}$$

as well as all subsequent trial vectors will be free of $\Lambda = 0$ eigenvectors.

Employing a somewhat similar argument, it can be shown that the most desirable initial trial vector, $\{\bar{v}_1\}$, and hence the vector $\{w\}$ from which it is

generated, should contain all components of the eigenvectors for $\Lambda \neq 0$. However, since there is generally no a priori knowledge of the modal matrix, {w} should be selected in such a way so as to make it as "irregular" as possible with respect to the system of eigenvectors so that it is most likely to contain a mixture of all the mode shapes. It has been found that this is best achieved by using a random or pseudo-random number generator to obtain the elements of {w}.

If the vector {w} selected in the above manner should, by some chance, be deficient in eigenvector components, then a null v-vector may be generated at some point by the recurrence algorithm (Equations (28), Section 2.2). In the context of a finite digit computer, this is indicated by the appearance of an off-diagonal term, d_{i+1}, which is exceedingly small compared to the corresponding diagonal term, $a_{i,i}$, in the reduced tridiagonal matrix. The test used is that

$$|d_{i+1}| \leq 10^{2-t} |a_{i,i}| \tag{13}$$

implies that the newly generated vector $\{\bar{v}_{i+1}\}$, is null, where t is the number of decimal digits carried by the computer. In this event, d_{i+1} is set exactly equal to zero and a new restart vector is employed for $\{\bar{v}_{i+1}\}$. This vector is generated exactly as in the case of the initial trial vector $\{v_1\}$, but using a different pseudo-random number seed. The recurrence algorithm for generation of the v-vectors is subsequently continued in the usual manner until the required number of vectors has been generated.

A further constraint on each v-vector, including the initial one, is that it be orthogonal to all previously generated v-vectors and previously calculated eigenvectors*. The imposition of these additional constraints is discussed in the next section.

*Previously calculated eigenvectors may be available from the following sources:
(1) The specification of fictitious free body supports on a SUPØRT card, which causes an equal number of rigid body modes to be automatically generated by NASTRAN prior to entering an eigenvalue extraction routine.
(2) Checkpointing of previously obtained eigensolutions followed by a restart to obtain additional eigensolutions.

4. SWEEPING-OUT OF PREVIOUSLY OBTAINED EIGENVECTORS AND REORTHOGONALIZATION OF THE TRIAL VECTORS

Assume that a combination of f rigid body and non-rigid body eigenvectors have already been extracted prior to the current application of the tridiagonal reduction method. Let these vectors be designated by $\{\tilde{X}_1\}$, $\{\tilde{X}_2\}$, ...,$\{\tilde{X}_f\}$. In order to avoid regenerating these previous eigensolutions, which would be inefficient, the initial trial vector, $\{v_1\}$, obtained in the manner just described, should be made orthogonal or "swept clean" of these eigenvector components. This is theoretically accomplished by setting

$$\{v_1^*\} = \{v_1\} - \sum_{j=1}^{f} [\{\tilde{X}_j\}^T[D]\{v_1\}]\{\tilde{X}_j\} \qquad (1)$$

where $\{v_1^*\}$ is the swept initial vector to be used in place of $\{v_1\}$ following its normalization. It can then be shown that all succeeding v-vectors generated by the recurrence algorithm (Equations 28, Section 2.2) form a theoretically orthogonal set which does not contain components of $\{\tilde{X}_j\}$; j = 1, f. However, it has been shown (Reference [15]) that the v-vectors degrade rapidly as the computations proceed, such that the later vectors are far removed from orthogonality to the earlier ones. This is caused by unavoidable computational round-off which, because of repeated multiplications by the unreduced eigenmatrix, $[\bar{B}]$, tends to amplify the contributions of the eigenvector components nearest the shift point in the calculated trial vectors. Thus, unless sufficiently accurate orthogonality of the trial vectors is maintained, they will be excessively rich in the modes near the shift point, and the solution of the reduced tridiagonal eigenvalue problem will yield a false bunching of eigenvalues around this point in the eigenspectrum.

To correct this problem, Gregory (Reference [6]) experimented with the use of higher-precision computer operations, but found only marginal improvement in the final results. Later, Lanczos suggested a reorthogonalization of the type

$$\{v_{i+1}^*\} = \{v_{i+1}\} - \sum_{j=1}^{i} [\{v_j\}^T[D]\{v_{i+1}\}]\{v_j\} - \sum_{j=1}^{f} [\{\tilde{X}_j\}^T[D]\{v_{i+1}\}]\{\tilde{X}_j\} \qquad (2)$$

where $\{v_{i+1}\}$ is calculated by the unmodified recurrence algorithm and $\{v^*_{i+1}\}$ is an improved vector. While this improves matters substantially, it still does not eliminate the precision problem adequately. However, Ojalvo and Newman (Reference [8]) found that the introduction of an iterative reorthogonalization loop can make the trial vectors as orthogonal as necessary for extremely large systems. The procedure is as follows:

The vector $\{v_{i+1}\}$, obtained from either the recurrence algorithm or the pseudo-random number generator (see Section 3) is denoted as $\{v^{(0)}_{i+1}\}$ and reorthogonalized with respect to all the previously obtained vectors. This is accomplished by iterating,

$$\{v^{(1)}_{i+1}\} = \{v^{(0)}_{i+1}\} - \sum_{j=1}^{i} [\{v_j\}^T[D]\{v^{(0)}_{i+1}\}]\{v_j\} - \sum_{j=1}^{f} [\{\tilde{X}_j\}^T[D]\{v^{(0)}_{i+1}\}]\{\tilde{X}_j\}$$

$$\{v^{(2)}_{i+1}\} = \{v^{(1)}_{i+1}\} - \sum_{j=1}^{i} [\{v_j\}^T[D]\{v^{(1)}_{i+1}\}]\{v_j\} - \sum_{j=1}^{f} [\{\tilde{X}_j\}^T[D]\{v^{(1)}_{i+1}\}]\{\tilde{X}_j\} \qquad (3)$$

$$\vdots$$

until an acceptable vector

$$\{v^{(s+1)}_{i+1}\} = \{v^{(s)}_{i+1}\} - \sum_{j=1}^{i} [\{v_j\}^T[D]\{v^{(s)}_{i+1}\}]\{v_j\} - \sum_{j=1}^{f} [\{\tilde{X}_j\}^T[D]\{v^{(s)}_{i+1}\}]\{\tilde{X}_j\} \qquad (4)$$

is found which satisfies the orthogonality criteria

$$\max_{1 \le j \le i} |\{v_j\}^T[D]\{v^{(s)}_{i+1}\}| \le 10^{2-t} \qquad (5a)$$

$$\max_{1 \le j \le f} |\{\tilde{X}_j\}^T[D]\{v^{(s)}_{i+1}\}| \le 10^{2-t} \qquad (5b)$$

where t is the number of decimal digits carried by the computer. If, for some vector, the above criteria are not satisfied after a set number of iterations, then the program assumes that a new trial vector cannot be generated, and a reduced eigen-

value problem of order $m = i$ is solved, as discussed in Section 2.3.

If the above criteria are met, then the resulting vector is normalized and set equal to the new normalized trial vector, i.e.,

$$\{v_{i+1}\} = \frac{\{v_{i+1}^{(s+1)}\}}{[\{v_{i+1}^{(s+1)}\}^T[D]\{v_{i+1}^{(s+1)}\}]^{1/2}} . \tag{6}$$

This new vector is used to compute the next off-diagonal term in the reduced tridiagonal matrix from the formula

$$d_{i+1} = \{v_{i+1}\}^T[B]\{v_i\} . \tag{7}$$

However, if

$$|d_{i+1}| \leq 10^{2-t} |a_{i,i}|, \tag{8}$$

it is probable that $\{v_{i+1}\}$ is a null vector, possibly because the maximum number of linearly independent vectors, corresponding to the rank of the problem, has been exceeded. In this event a reduced eigenvalue problem of order $m = i$ is solved, as above.

If the criteria given by Equations (5) and (8) are all met, then the new normalized vector $\{v_{i+1}\}$ is used to continue the reduction algorithm.

5. CRITERION FOR THE NEGATIVE SHIFT PARAMETER, α^2

Before tridiagonal reduction, the eigenvalue problem for natural frequencies in the neighborhood of zero is

$$[B]\{X\} = \Lambda\{X\} \tag{1}$$

where

$$[B] = [C^{-1}][M_{aa}][C^{-1}]^T \tag{2}$$

$$\{X\} = [C]^T\{\phi\} \tag{3}$$

$$[K_{aa} + \alpha^2 M_{aa}] = [C][C]^T \tag{4}$$

and

$$\Lambda = \frac{1}{\omega^2 + \alpha^2} \tag{5}$$

The criterion involved in the choice of the shift parameter, α^2, is that it be large enough to render a possibly singular stiffness matrix non-singular (to the extent that Cholesky symmetric decomposiion can be performed accurately in a finite digit computer), and small enough to prevent troublesome clustering of the eigenvalues, Λ_i. As an approach toward solving this problem, it is helpful to note that, given a symmetric positive-definite matrix [J], we really obtain

$$[C][C]^T = [J] + [\delta J] \tag{6}$$

when it is factored, because of computer rounding errors. According to Wilkinson (Reference [16]), the following inequality is almost always satisfied,

$$||\delta J||_1 \leq n(10^{1-t})\ ||J||_1 \tag{7}$$

where the above Holder-one matrix norms are equal to the maximum row sums of absolute values, n is the order of the matrix, and t is the number of decimal digits carried by the computer.

Thus, in order to render the matrix $[K_{aa} + \alpha^2 M_{aa}]$ non-singular, even when $[K_{aa}]$ itself is singular, some matrix norm of the modification $\alpha^2[M_{aa}]$ should be appreciably larger than the corresponding norm of $[\delta K_{aa}]$, where the latter matrix is in the "noise level" of the computer. On the basis of Equation (7), the minimum α^2 should therefore satisfy the requirement

$$\alpha^2_{min} \, ||M_{aa}||_1 \geq n(10^{1-t}) \, ||K_{aa}||_1 \tag{8}$$

or

$$\alpha^2_{min} \geq n(10^{1-t}) \frac{\left[\max\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} |K_{ij}|\right]}{\left[\max\limits_{1 \leq i \leq n} \sum\limits_{j=1}^{n} |m_{ij}|\right]} \tag{9}$$

where K_{ij} and m_{ij} are the elements of $[K_{aa}]$ and $[M_{aa}]$.

To further enhance the removal of possible singularities in the stiffness matrix, a factor of ten is applied to the right side and the maximum ratio of diagonal matrix elements is used. The resulting value of α^2_{min} is

$$\alpha^2_{min} = n(10^{2-t}) \left|\frac{K_{ii}}{m_{ii}}\right|_{max}; \quad m_{ii} \neq 0 \tag{10}$$

A final requirement imposed on the shift parameter is that it be large enough to effect alterations in the last two-thirds of the significant digits in each diagonal term of the unmodified stiffness matrix or, equivalently,

$$\alpha^2_o = 10^{-t/3} \left|\frac{K_{ii}}{m_{ii}}\right|_{min}; \quad m_{ii} \neq 0 \tag{11}$$

Consequently, the actual value of the shift used by the program is

$$\alpha^2 = \max\,(\alpha^2_{min}, \alpha^2_o) \tag{12}$$

If the resulting modified stiffness matrix is still singular, as indicated by a failure of the Cholesky decomposition process, the above value of α^2 is multiplied by a factor of 100, a maximum of two times in an attempt to render the undecomposed matrix non-singular. If this procedure fails, the problem execution is aborted and the user is informed that the singularities cannot be removed from the stiffness matrix.

6. ERROR BOUNDS ON THE COMPUTED EIGENVALUES

Once the modes of the reduced problem have been extracted, close upper and lower bounds on the eigenvalue errors can be obtained rather economically. This provides the user with an a posteriori check on the number of accurately calculated eigenvalues which, in most cases, will be greater than the number requested.

In carrying out the development, it is convenient to write the eigenproblem in its inverse form

$$[\bar{B}]\{X\} = \Lambda\{X\} \tag{1}$$

where

$$\Lambda = \frac{1}{\lambda_a - \lambda_o} \tag{2}$$

Considering some approximation, $\{\bar{X}_i\}$, to an exact eigenvector, $\{X_i\}$, it can be expanded in terms of the exact modal matrix as follows:

$$\{\bar{X}_i\} = [Z][b_i] \tag{3}$$

where

$$\underset{nxn}{[Z]} = [\{X_1\} \,\vdots\, \{X_2\} \,\vdots\, \ldots \,\vdots\, \{X_n\}] \tag{4}$$

and

$$\{b_i\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} \tag{5}$$

is a vector of scalar coefficients.

If the eigenvalue approximation associated with $\{\bar{X}_i\}$ is denoted by $\bar{\Lambda}_i$, then the residual vector for the ith modal approximation is given by

$$\{R_i\} = [\bar{B} - \bar{\Lambda}_i[I]]\{\bar{X}_i\}$$

$$= [\bar{B} - \bar{\Lambda}_i[I]][Z]\{b_i\} \tag{6}$$

However

$$[\bar{B}][Z] = [Z][\tilde{\Lambda}_d\tilde{}] \tag{7}$$

where $[\tilde{\Lambda}_d\tilde{}]$ is the diagonal matrix of eigenvalues, Λ_i, i = 1, n.

It follows that

$$\{R_i\} = [Z][[\tilde{\Lambda}_d\tilde{}] - \bar{\Lambda}_i[I]]\{b_i\} \tag{8}$$

Denoting a weighted root-mean-square residual by

$$r_i = (\{R_i\}^T[D]\{R_i\})^{1/2}, \tag{9}$$

it can be seen that

$$r_i^2 = \{b_i\}^T[[\tilde{\Lambda}_d\tilde{}] - \bar{\Lambda}_i[I]][Z]^T[D][Z][[\tilde{\Lambda}_d\tilde{}] - \bar{\Lambda}_i[I]]\{b_i\} \tag{10}$$

However, without any loss of generality, the eigenvectors can be orthonormalized so that

$$\{X_j\}^T[D]\{X_i\} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases} \tag{11}$$

The modal matrix is therefore unitary, i.e.,

$$[Z]^T[D][Z] = [I] \tag{12}$$

and

$$r_i^2 = \{b_i\}^T[[\tilde{\Lambda}_d\tilde{}] - \bar{\Lambda}_i[I]]^2\{b_i\}$$

$$= \sum_{j=1}^{n} b_{ji}^2 (\Lambda_j - \bar{\Lambda}_i)^2 \tag{13}$$

where $\{b_{ij}\}$ refers to the jth element of the vector $\{b_i\}$.

At this point it should be noted that NASTRAN also normalizes the eigenvectors $\{y_i\}$ of the reduced tridiagonal problem, so that

$$\{\bar{X}_i\}^T[D]\{\bar{X}_i\} = \{y_i\}^T[V]^T[D][V]\{y_i\} = 1 \tag{14}$$

where

$$[V] = [\{v_1\} \vdots \{v_2\} \vdots \ldots \vdots \{v_m\}] \tag{15}$$

Thus,

$$\{\bar{X}_i\}^T[D]\{\bar{X}_i\} = \sum_{j=1}^{n} b_{ji}^2 = 1 \tag{16}$$

and it follows that

$$r_i^2 \geq \min_{1 \leq j \leq n} (\Lambda_j - \bar{\Lambda}_i)^2 \tag{17}$$

Therefore, when $\bar{\Lambda}_i$ is close to an exact eigenvalue, Λ_i, it can be seen that

$$|\Lambda_i - \bar{\Lambda}_i| \leq |r_i| \tag{18}$$

or

$$\left|\left(\frac{\bar{\lambda}_{ai} - \lambda_o}{\lambda_{ai} - \lambda_o}\right) - 1\right| \leq \left|\frac{r_i}{\bar{\Lambda}_i}\right| \tag{19}$$

Since $\bar{\lambda}_{ai}$ is close to λ_{ai} when $\bar{\Lambda}_i$ is close to Λ_i, Equation (19) implies that

$$\left|1 - \frac{\bar{\lambda}_{ai}}{\lambda_{ai}}\right| \leq \left|\frac{r_i}{\bar{\Lambda}_i(1 + \lambda_o\bar{\Lambda}_i)}\right| , \tag{20}$$

which is a measure of the maximum relative error in the ith physical eigenvalue*.

The residual, r_i, in the above equation can be evaluated quite easily via the following approach.

*This test is obviously invalid when $\lambda_{ai} = 0$, i.e., a rigid body solution. In this event, the computed value of $\bar{\lambda}_{ai}$ is, in itself, a measure of the absolute error in the physical eigenvalue and no further accuracy information is needed. The existence of an eigenvalue as an additional rigid body mode not requested on a SUPØRT card in the NASTRAN Bulk Data Deck is detected by the criterion $|\bar{\lambda}_{ai}| < 10^{-t/3}$, in which case Equation (20) is bypassed.

Set

$$\{\bar{X}_i\} = [V]\{y_i\} \tag{21}$$

in Equation (6). Then

$$\{R_i\} = [[\bar{B}][V] - \bar{\Lambda}_i[V]]\{y_i\}. \tag{22}$$

However, from Equation (26) of Section 2.2, it is seen that

$$[\bar{B}][V] = [V][A] + d_{m+1}[G_{m+1}] \tag{23}$$

where [A] is the tridiagonal reduced matrix and

$$[G_{m+1}] = [\{0\} \vdots \{0\} \vdots \ldots \vdots \{v_{m+1}\}] \ . \tag{24}$$

Therefore,

$$\{R_i\} = [[V][A] + d_{m+1}[G_{m+1}] - \bar{\Lambda}_i[V]]\{y_i\} \ . \tag{25}$$

But

$$[A]\{y_i\} = \bar{\Lambda}_i\{y_i\}, \tag{26}$$

so that

$$\{R_i\} = d_{m+1}[G_{m+1}]\{y_i\}, \tag{27}$$

and

$$r_i^2 = \{R_i\}^T[D]\{R_i\} = d_{m+1}^2 \ \{y_i\}^T[G_{m+1}]^T[D][G_{m+1}]\{y_i\}. \tag{28}$$

It can be seen that

$$[G_{m+1}]^T[D][G_{m+1}] = \begin{bmatrix} 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & 1 \end{bmatrix} \tag{29}$$

so that finally,

$$r_i^2 = d_{m+1}^2 \, y_{mi}^2 \,, \tag{30}$$

where y_{mi} is the last element of the vector $\{y_i\}$.

Therefore, Equation (20) assumes the form

$$\left|1 - \frac{\bar{\lambda}_{ai}}{\lambda_{ai}}\right| \leq \frac{|d_{m+1} \cdot y_{mi}|}{|\bar{\Lambda}_i(1 + \lambda_o \bar{\Lambda}_i)|} \,. \tag{31}$$

Thus, it is seen that the eigenvalue errors are proportional to d_{m+1}, which is the next off-diagonal term that would be generated, had the reduced tridiagonal matrix [A] been increased from order m to order m+1. Equation (31) shows that this term is further modified by a weighting factor y_{mi}, which is the last term in the reduced-system eigenvector associated with $\bar{\lambda}_{ai}$.

The use of the above error bound formula as a criterion for selecting acceptable eigensolutions is described in Section 7.

7. SUMMARY OF COMPUTATIONAL PROCEDURES AND FLOW CHARTS

Flow diagrams illustrating the computational procedures are shown in Figures 1 and 2. The details of each block are summarized below.

(1) Calculate Small Negative Shift Parameters, α^2, (See Section 5)

In the case of unshifted vibration mode problems the negative shift parameter for removing possible singularities is found from:

$$\alpha^2 = \max\ (\alpha^2_{min},\ \alpha^2_o) \tag{1}$$

where

$$\alpha^2_{min} = n(10^{2-t})\left|\frac{K_{ii}}{m_{ii}}\right|_{max}; \quad m_{ii} \neq 0 \tag{2a}$$

and

$$\alpha^2_o = 10^{-t/3}\ \left|\frac{K_{ii}}{m_{ii}}\right|_{min} \tag{2b}$$

K_{ii} and m_{ii} are the diagonal elements of $[K_{aa}]$ and $[M_{aa}]$, respectively, n is the number of $\{u_a\}$ degrees of freedom, and t is the number of decimal digits carried by the computer.

(2) Zero-Out Excessively Small Elements of [M] Matrix (See Section 2.3)

a. Compare the magnitudes of all off-diagonal elements of [M] with the corresponding diagonal elements to determine whether

$$\left|\frac{m_{ij}}{m_{ii}}\right| \leq 10^{-2t/3}; \quad i = j, \quad m_{ii} \neq 0. \tag{3}$$

b. Set $m_{ij} = 0.0$ for every off-diagonal element satisfying the above criterion.

(3) Establish Tentative Reduced Problem Size (See Section 2.3)

a. Count the number, $\bar{n}$, of non-null columns or rows in the above modified [M] matrix and set

$$\bar{r} = \bar{n} - f \tag{4}$$

where f is the number of previously computed eigensolutions.

(b) Calculate a tentative size, m, of the reduced eigenproblem from

$$m = \min[2\bar{q} + 10, \bar{r}] \tag{5a}$$

where

$$\bar{q} = q - f \tag{5b}$$

and q is the total number of accurate eigenvalues requested by the user, including previously computed modes. If $\bar{q} > \bar{r}$, warn user that too many eigensolutions have been requested - program will try to find all existing solutions.

(4) Construct Factors of $[\bar{K}]$ Matrix (See Section 2.1)

a. Set

(i) $[\bar{K}] = [K_{aa} - \omega_o^2 M_{aa}]$ (Shifted Vibration Mode Problems) (6a)

or

(ii) $[\bar{K}] = [K_{aa} + \alpha^2 M_{aa}]$ (Unshifted Vibration Mode Problems) (6b)

or

(iii) $[\bar{K}] = [K_{aa}]$ (Buckling Problems) (6c)

b. Perform a non-square root decomposition:

$$[\bar{K}] = [L][\bar{d}][L]^T \tag{7a}$$

for case (i), or a Cholesky symmetric decomposition:

$$[\bar{K}] = [C][C]^T \tag{7b}$$

for cases (ii) and (iii), using real arithmetic without pivoting. Save the triangular and diagonal factors. If the decomposition for case (i) fails or the decompositions for cases (ii) and (iii) fail after two increases in α^2 by factors of one hundred, then the program is aborted and a fatal error message is issued, indicating that there are unremovable stiffness matrix singularities.

(5) Execute Tridiagonal Reduction Algorithm (See flow diagram for this block, Figure 2)

(5.1) Initialize the Recurrence Algorithm (See Section 2.2)

Initialize the vector index to $i = 0$ and set

$$\{v_o\} = \{0\} \tag{8}$$

where $\{v_o\}$ is an (nx1) null vector.

(5.2) Generate a Starting or Restart Vector and set $d_{i+1} = 0.0$ (See Section 3)

(a) Construct an n-element vector $\{w\}$ using a pseudo-random number generator.

(b) Solve for an un-normalized trial vector from the equation

$$\{\bar{v}_{i+1}\} = [\bar{B}]\{w\} \tag{9}$$

where

$$[\bar{B}] = [L^{-1}]^T[\sim d\sim]^{-1}[L^{-1}][M_{aa}] \qquad \text{(case i)} \tag{10a}$$

or

$$[\bar{B}] = [C^{-1}][M_{aa}][C^{-1}]^T \qquad \text{(case ii)} \tag{10b}$$

or

$$[\bar{B}] = [C^{-1}][K^d_{aa}][C^{-1}]^T \qquad \text{(case iii)} \tag{10c}$$

Forward and backward passes are used to perform the above inverse operations.

(c) Normalize the above vector:

$$\{v_{i+1}^{(0)}\} = \left[\frac{1}{\{\bar{v}_{i+1}\}^T[D]\{\bar{v}_{i+1}\}}\right]^{1/2}\{\bar{v}_{i+1}\} \tag{11}$$

where

$$[D] = [M_{aa}] \qquad \text{(case i)} \tag{12a}$$

$$[D] = [I] \qquad \text{(cases ii and iii)} \tag{12b}$$

(d) Set $d_{i+1} = 0.0$ and proceed to block (5.5).

(5.3) Create One Approximate Trial Vector and One Diagonal Coefficient (See Section 2.2)

The recurrence algorithm is:

$$a_{i,i} = \{v_i\}^T[B]\{v_i\} \tag{13}$$

$$\{\bar{v}_{i+1}\} = [\bar{B}]\{v_i\} - a_{i,i}\{v_i\} - d_i\{v_{i-1}\} \tag{14}$$

$$\bar{d}_{i+1} = [\{\bar{v}_{i+1}\}^T[D]\{\bar{v}_{i+1}\}]^{1/2} \tag{15}$$

$$\{v_{i+1}^{(0)}\} = \frac{1}{\bar{d}_{i+1}}\{\bar{v}_{i+1}\} \tag{16}$$

where

$$[B] = [D][\bar{B}] \tag{17}$$

and $\{\bar{v}_{i+1}^{(0)}\}$ is an approximation to the new trial vector.

(5.4) First Normalization Test (See Section 3)

The test is

$$|\bar{d}_{i+1}| > 10^{2-t} |a_{i,i}| \tag{18}$$

Pass: Proceed directly to block (5.5)

Fail: Return to block (5.2), generate a new restart vector for $\{v_{i+1}^{(0)}\}$, and then proceed to block (5.5).

(5.5) Iterate to Obtain Orthogonalized Vector (See Section 4)

Designate $\{\tilde{X}_j\}$, j = 1, f as previously calculated and stored eigenvectors. Perform the iterations,

$$\{v_{i+1}^{(s+1)}\} = \{v_{i+1}^{(s)}\} - \sum_{j=1}^{i} [\{v_j\}^T[D]\{v_{i+1}^{(s)}\}]\{v_j\}$$

$$- \sum_{j=1}^{f} [\tilde{X}_j\}^T[D]\{v_{i+1}^{(s)}\}]\{\tilde{X}_j\}; \quad s = 0,1,2... \tag{19}$$

until

$$\max_{1 \le j \le i} |\{v_j\}^T[D]\{v_{i+1}^{(s)}\}| \le 10^{2-t}$$

and (20)

$$\max_{1 \le j \le f} |\{\tilde{X}_j\}^T[D]\{v_{i+1}^{(s)}\}| \le 10^{2-t}$$

or

$$s = 14.$$

If the orthogonality criterion, Equation (20), is satisfied, proceed to block (5.6). Otherwise, set the problem size, m, equal to i and proceed to Exit.

(5.6) Normalize the Orthogonalized Trial Vector (See Section 4)

Compute

$$\{v_{i+1}\} = \frac{\{v_{i+1}^{(s+1)}\}}{[\{v_{i+1}^{(s+1)}\}[D]\{v_{i+1}^{(s+1)}\}]^{1/2}} \tag{21}$$

This is the new orthogonalized and normalized trial vector.

(5.7) Second Normalization Test and Creation of Off-Diagonal Coefficient (See Section 4)

(a) Compute the next off-diagonal term in the reduced tridiagonal matrix from

$$d_{i+1} = \{v_{i+1}\}^T[B]\{v_i\} \tag{22}$$

(b) Verify whether the following test is met:

$$|d_{i+1}| > 10^{2-t} |a_{i,i}| \tag{23}$$

If it has, set $i = i+1$ and return to block (5.3) for continuation of the recurrence algorithm. If the test fails, set $m = i$ to reduce the problem size and proceed to Exit. Issue message to user that only i modes can be obtained - more than $r - f$ modes may have been requested.

(6) Solve Reduced-System Eigenvalue Problem (See Section 2.2)

a. The coefficients $a_{11}, a_{22}, \cdots, a_{mm}$ and $d_2, d_3, \ldots, d_m$, computed in block (5) are interpreted as the following symmetric, tridiagonal array:

$$[A] = \begin{bmatrix} a_{11} & d_2 & & & & \\ d_2 & a_{22} & d_3 & & & \\ & d_3 & a_{33} & \ddots & & \\ & & \ddots & \ddots & d_m & \\ & & & d_m & a_{mm} & \end{bmatrix} \tag{24}$$

b. The mth order eigenvalue problem

$$[A]\{y\} = \bar{\Lambda}\{y\} \tag{25}$$

is solved for the eigenvalues, $\bar{\Lambda}_i$, and eigenvectors $\{y_i\}$ using a Q-R algorithm and eigenvector computational procedure similar to that described in the NASTRAN Theoretical Manual (Reference [13]).

c. The reduced system eigenvectors are normalized so that

$$\{y_i\}^T\{y_i\} = 1; \quad i = 1, m \tag{26}$$

(7) Compute Maximum Eigenvalue Errors (See Section 6)

a. The maximum absolute relative errors in the computed physical eigenvalues are obtained from

$$\xi_i = \frac{|(\bar{d}_{m+1})(y_{mi})|}{|\bar{\Lambda}_i(1 + \lambda_o\bar{\Lambda}_i)|}; \quad i = 1, m \tag{27a}$$

where $\bar{d}_{m+1}$ is the last off-diagonal term computed in block (5.3) and y_{mi} is the last element in the vector $\{y_i\}$. If the physical eigenvalue $1/\bar{\Lambda}_i + \lambda_o$, corresponds to a rigid body mode, the above computation is invalid and therefore bypassed. A rigid body mode is assumed to occur whenever

$$\left|\frac{1}{\bar{\Lambda}_i} + \lambda_o\right| \leq 10^{-t/3} \tag{27b}$$

and is denoted by setting the relative error, ξ_i, equal to a flat zero.

b. The eigenvalues are processed in order of increasing distance from the center of range of interest, λ_o, to determine whether their associated ξ_i values meet an acceptable relative error tolerance set by the user on the EIGR or EIGB bulk data card (the default value is .001/n where n is the order of the stiffness matrix). The first eignevalue not meeting the tolerance test,

as well as all subsequent eigenvalues further removed from the center of interest, are considered to lack sufficient accuracy and are therefore rejected.

c. Acceptance eigenvalues obtained in the above manner are reordered in terms of increasing physical value for subsequent processing by NASTRAN.

8. Compute Physical Eigenvalues and Eigenvectors (See Section 2.2)

The mathematical eigenvalues, $\bar{\Lambda}_i$, and eigenvectors, $\{y_i\}$, are converted to physical form as follows:

$$\bar{\lambda}_i = -\frac{1}{\bar{\Lambda}_i} \qquad \text{(Buckling Problems)} \qquad (28a)$$

$$\bar{\omega}_i^2 = \frac{1}{\bar{\Lambda}_i} - \alpha^2 \qquad \text{(Unshifted Vibration Mode Problems)} \qquad (28b)$$

$$\bar{\omega}_i^2 = \frac{1}{\bar{\Lambda}_i} + \omega_o^2 \qquad \text{(Shifted Vibration Mode Problems)} \qquad (28c)$$

$$\{\bar{\phi}_i\} = [C^{-1}]^T[V]\{y_i\} \qquad \text{(Buckling or Unshifted Vibration Mode Problems)} \qquad (29a)$$

$$\{\bar{\phi}_i\} = [V]\{y_i\} \qquad \text{(Shifted Vibration Mode Problems)} \qquad (29b)$$

where

$$[V] = [\{v_1\}\{v_2\} \ldots\ldots \{v_m\}]. \qquad (30)$$

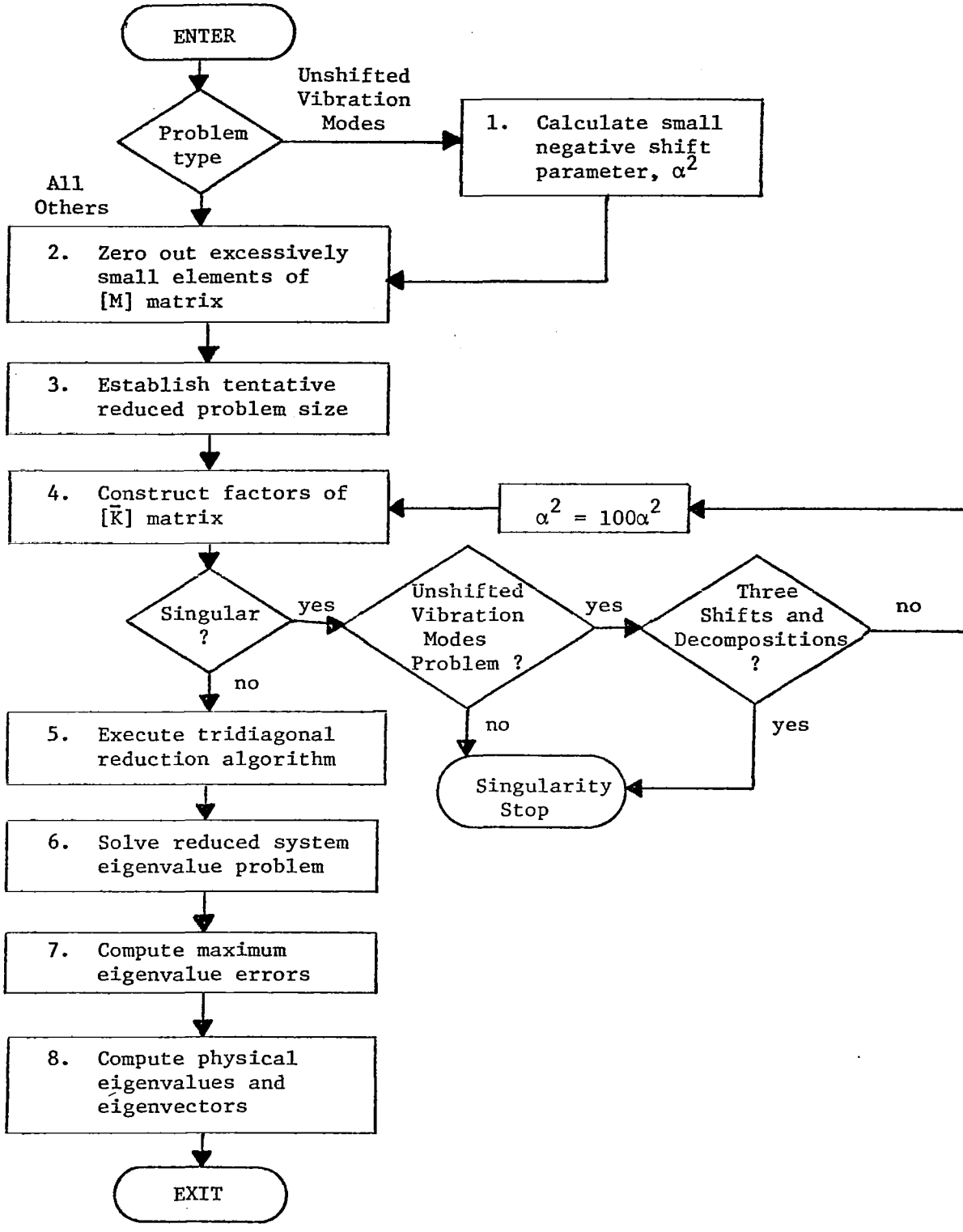

Figure 1. Overall flow diagram for tridiagonal reduction method

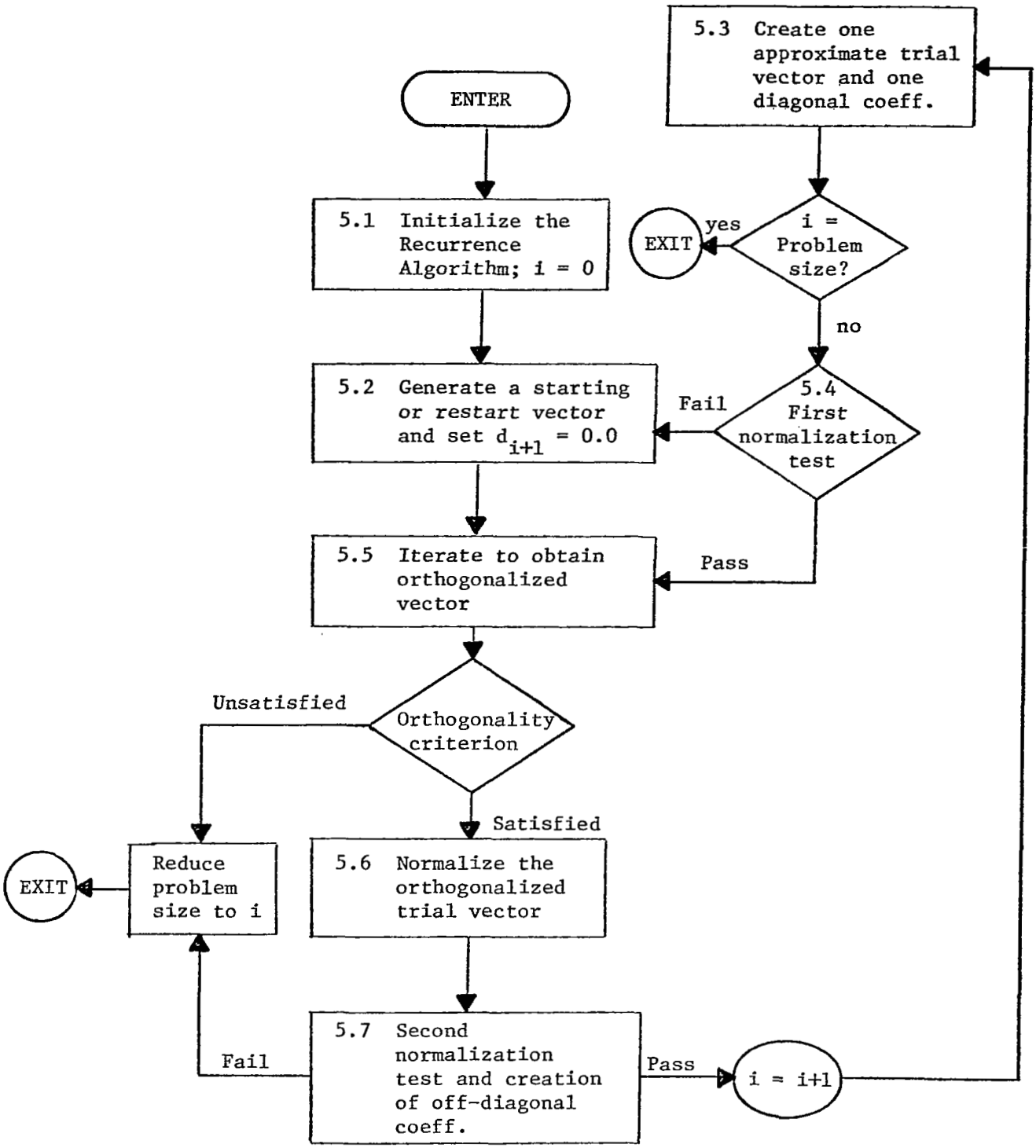

Figure 2. Flow diagram for block 5, execute tridiagonal reduction algorithm

8. NASTRAN USER'S INSTRUCTIONS

The following pages show modifications of the EIGR and EIGB cards in the NASTRAN Bulk Data deck which accommodate user implementation of the tridiagonal reduction method for real eigenvalue analysis. The modifications are constituted of additions to the standard user instructions and are underscored for ease in identification.

When the tridiagonal reduction method is invoked, the F2 or L2 parameter on these cards represents the maximum allowable value of the computed relative error in a physical eigenvalue (see Section 6). If this value is exceeded, the associated eigensolution is not accepted for further processing by NASTRAN. A detailed list of the maximum relative errors in the computed eigenvalues can be obtained by requesting DIAG 16 in the NASTRAN Executive Control Deck (see Section 9).

Input Data Card EIGB Buckling Analysis Data

Description: Defines data needed to perform buckling analysis.

Format and Example:

1	2	3	4	5	6	7	8	9	10
EIGB	SID	METHØD	L1	L2	NEP	NDP	NDN	E	+abc
EIGB	13	DET	0.1	2.5	2	1	1	0.0	ABC

+abc	NØRM	G	C						
+BC	MAX								

Field	Contents
SID	Set identification number (Unique integer > 0)
METHØD	Method of eigenvalue extraction, one of the BCD values "INV", "DET", "FEER", "UINV", or "UDET" INV - Inverse power method, symmetric matrix operations DET - Determinant method, symmetric matrix operations FEER - Tridiagonal reduction method, symmetric matrix operations UINV - Inverse power method, unsymmetric matrix operations UDET - Determinant method, unsymmetric matrix operations
L1, L2	Eigenvalue range of interest (Real; L1 < L2 > 0.0) For METHØD = "FEER", L1 is ignored and L2 is the acceptable relative error tolerance on eigenvalues, in percent (Default is .001/n where n is the order of the stiffness matrix) (Real > 0.0)
NEP	Estimate of number of roots in positive range. Desired number of eigenvalues of smallest magnitude for METHØD = "FEER" (Default is automatically calculated to extract at lease one accurate mode) (Integer > 0)
NDP, NDN	Desired number of positive and negative roots (Default = 3 NEP) (Integer >0). Ignored for METHØD = "FEER"

E — Convergence criteria (optional) (Real > 0.0)

NØRM — Method for normalizing eigenvectors, one of the BCD values "MAX" or "PØINT"

MAX - Normalize to unit value of the largest component in the analysis set

PØINT - Normalize to unit value of the component defined in fields 3 and 4 defaults to "MAX" if defined component. is zero

G — Grid or scalar point identification number (Integer > 0) (Required if and only if NØRM = "PØINT")

C — Component number (One of the integers 1-6) (Required if and only if NØRM = "PØINT" and G is a geometric grid point)

Remarks

1. Buckling analysis root extraction data sets must be selected in the Case Control Deck (METHØD = SID) to be used by NASTRAN.

2. The quantities L1 and L2 are dimensionless and specify a range in which the eigenvalues are to be found. An eigenvalue is a factor by which the prebuckling state of stress (first subcase) is multiplied to produce buckling. If METHØD = "FEER", L1 is ignored and L2 represents the maximum upper bound, in percent, on $|\lambda_{FEER}/\lambda_{EXACT} - 1|$ for acceptance of a computed eigensolution.

3. The continuation card is required.

4. See Sections 10.3.6 and 10.4.2.2 of the Theoretical Manual for a discussion of convergence criteria.

5. If METHØD = "DET", L1 must be greater than or equal to 0.0.

6. If NØRM = "MAX", components that are not in the analysis set may have values larger than unity.

7. If NØRM = "PØINT", the selected component must be in the analysis set.

Input Data Card EIGR

Description: Defines data needed to perform real eigenvalue analysis.

Format and Example:

EIGR	SID	METHØD	F1	F2	NE	ND	NZ	E	+abc
EIGR	13	DET	1.9	15.6	10	12	0	1.-3	ABC

+abc	NØRM	G	C						
+BC	PØINT	32	4						

Field	Content
SID	Set identification number (Unique integer > 0)
METHØD	Method of eigenvalue extraction, one of the BCD values "INV", "DET", "GIV", "FEER", "UINV", or "UDET".
	INV Inverse power method, symmetric matrix operations.
	DET Determinant method, symmetric matrix operations.
	GIV Givens method of tridiagonalization.
	FEER Tridiagonal reduction method, symmetric matrix operations.
	UINV Inverse power method, unsymmetric matrix operations.
	UDET Determinant method, unsymmetric matrix operations.
F1, F2	Frequency range of interest (Required for METHØD = "DET", "INV", "UDET", or "UINV") (Real $\geq$ 0.0; F1 < F2). Frequency range over which eigenvectors are desired for METHØD = "GIV". The frequency range is ignored if ND > 0, in which case the eigenvectors for the first ND positive roots are found. (Real F1 < F2). If METHØD = "FEER", F1 is the center of range of interest (Default is F1 = 0.0) (Real $\geq$ 0.0), and F2 is the acceptable relative error tolerance on frequency-squared, in percent (Default is .001/n where n is the order of the stiffness matrix) (Real > 0.0)
NE	Estimate of number of roots in range (Required for METHØD = "DET", "INV", "UDET", or "UINV", ignored for METHØD = "FEER") (Integer > 0)

ND — Desired number of roots for METHØD = "DET", "INV", "UDET", or "UINV", (Default is 3 NE) (Integer > 0). Desired number of eigenvectors for METHØD = "GIV" (Default is zero) (Integer > 0). Desired number of roots and eigenvectors for METHØD = "FEER" (Default is automatically calculated to extract at least one accurate mode) (Integer > 0)

NZ — Number of free body modes (Optional - used only if METHØD = "DET" or "UDET") (Integer ≥ 0)

E — Mass orthogonality test parameter (Default is 0.0 which means no test will be made) (Real ≥ 0.0).

NØRM — Method for normalizing eigenvectors, one of the BCD values "MASS", "MAX" or "PØINT"

MASS - Normalize to unit value of the generalized mass

MAX - Normalize to unit value of the largest component in the analysis set

PØINT - Normalize to unit value of the component defined in fields 3 and 4 - defaults to "MAX" if defined component is zero

G — Grid or scalar point identification number (Required if and only if NØRM = "PØINT") (Integer ≥ 0)

C — Component number (One of the integers 1-6) (Required if and only if NØRM = "PØINT" and G is a geometric grid point)

Remarks:

1. Real eigenvalue extraction data sets must be selected in the Case Control Deck (METHØD = SID) to be used by NASTRAN.

2. The units of F1 and F2 are cycles per unit time. If METHØD = "FEER", F2 represents the maximum upper bound, in percent, on $|\omega^2_{FEER}/\omega^2_{EXACT} - 1|$ for acceptance of a computed eigensolution.

3. The continuation card is required.

4. If METHØD = "GIV", all eigenvalues are found.

5. If METHØD = "GIV", the mass matrix for the analysis set must be positive definite. This means that all degrees of freedom, including rotations, must have mass properties. ØMIT cards may be used to remove massless degrees of freedom.

6. A nonzero value of E in field 9 also modifies the convergence criteria. See Sections 10.3.6 and 10.4.2.2 of the Theoretical Manual for a discussion of convergence criteria.

7. If NØRM = "MAX", components that are not in the analysis set may have values larger than unity.

8. If NØRM = "PØINT", the selected component must be in the analysis set.

9. If METHØD = "GIV" and rigid body modes are present, F1 should be set to a small negative number rather than zero if the rigid body eigenvectors are desired.

10. The desired number of roots (ND) includes all roots previously found, such as rigid body modes determined with the use of the SUPØRT card, or the number of roots found on the previous run when restarting and APPENDing the eigenvector file.

9. USER MESSAGES AND OPTIONAL DIAGNOSTICS

9.1 NASTRAN Functional Module User Messages for the Tridiagonal Reduction Method

9.1.1 General

The following is a description of the NASTRAN user messages which may be generated by NASTRAN during the execution of the Tridiagonal Reduction method and which are unique to this method. Explanatory information is provided following the text of each message and, in the case of a fatal message, corrective action is indicated. Refer to the NASTRAN Users' Manual, Section 6 for a complete listing of other system and user messages.

Fatal messages cause the termination of the execution following the printing of the message text. These messages will always appear at the end of the NASTRAN output. Warning and information messages will appear at various places in the output stream. Such messages convey only warnings or information to the user. Consequently, the execution continues in a normal manner following the printing of the message text.

9.1.2 List of User Messages

2385*** USER WARNING MESSAGE 2385, DESIRED NUMBER ØF EIGENVALUES EXCEED THE EXISTING NUMBER, ALL EIGENSØLUTIØNS WILL BE SØUGHT.

The desired number of eigenvalues specified on the EIGR card (NEP) or the EIGR card (ND) exceeds the rank of the $[K^{d}_{aa}]$ or $[M_{aa}]$ matrix, which is the maximum number of existing eigenvalues.

2386*** USER FATAL MESSAGE 2386, STIFFNESS MATRIX SINGULARITY CANNØT BE REMØVED BY SHIFTING.

Check the specification of masses on CØNM1, CØNM2, CMASSi, material defintion and element property cards to ensure that the degrees-of-freedom in the analysis set are not all massless.

2387*** USER WARNING MESSAGE 2387, PRØBLEM SIZE REDUCED TØ **** DUE TØ ØRTHØGØNALITY DRIFT ØR NULL TRIAL VECTOR.
ALL EXISTING MODES MAY HAVE BEEN ØBTAINED. USE DIAG 16 TØ DETERMINE ERRØR BØUNDS.

The Tridiagonal Reduction Method cannot generate a reduced problem size of the order prescribed in Section 10.6.2.3 of the Theoretical Manual. However, the desired number of accurate eigenvalues specified on the EIGB card (NEP) or the EIGR card (ND) may have been obtained. A detailed list of the computed error bounds can be obtained by requesting DIAG 16 in the EXECUTIVE CØNTRØL DECK.

2388*** USER WARNING MESSAGE 2388, USER SPECIFIED RANGE NØT USED FØR FEER BUCKLING, THE RØØTS ØF LØWEST MANGITUDE ARE ØBTAINED.

The value of L1 specified on the EIGB card is ignored for buckling analysis by the Tridiagonal Reduction (FEER) method.

2389*** USER WARNING MESSAGE 2389, PRØBLEM SIZE REDUCED. NØ MØRE TRIAL VECTØRS CAN BE ØBTAINED.

The desired number of eigenvalues specified on the EIGB card (NEP) or the EIGR card (ND) exceeds the number that can be calculated by the Tridiagonal Reduction (FEER) method. Check whether the requested number of eigenvalues exceeds the rank of the $[K^d_{aa}]$ or $[M_{aa}]$ matrix, which equals the number of existing eigenvalues.

2390*** USER WARNING MESSAGE 2390, **** FEWER ACCURATE EIGENSØLUTIØNS THAN THE **** REQUESTED HAVE BEEN FOUND. USE DIAG 16 TØ DETERMINE ERRØR BØUNDS.

The number of eigenvalues passing the eigenvalue relative-error test is less than the number requested on the EIGB or EIGR card. The maximum allowable error is specified in field 5 on the above cards. A detailed list of the computed error bounds can be obtained by requesting DIAG 16 in the EXECUTIVE CØNTROL DECK. A checkpoint and restart should be employed to obtain additional accurate eigensolutions.

2391*** USER FATAL MESSAGE 2391, PRØGRAM LØGIC ERROR IN FEER.

2392*** USER INFØRMATIØN MESSAGE 2392, **** MØRE ACCURATE EIGENSØLUTIØNS THAN THE **** REQUESTED HAVE BEEN FOUND. USE DIAG 16 TØ DETERMINE ERRØR BØUNDS.

The number of eigenvalues passing the eigenvalue relative-error test is greater than the number requested on the EIGB or EIGR card. The maximum allowable error is specified in field 5 on the above cards. A detailed list of the computed error bounds can be obtained by requesting DIAG 16 in the EXECUTIVE CØNTRØL DECK.

2393*** USER WARNING MESSAGE 2393, THE REDUCED-SYSTEM EIGENVECTØR CØRRESPØNDING TØ EIGENVALUE **** DØES NØT MEET CØNVERGENCE CRITERIØN. ABSØLUTE RELATIVE ERRØR BETWEEN SUCCESSIVE ITERATES IS ****.

The accuracy of the corresponding physical eigenvector is in doubt. Refer to the Eigenvalue Summary Table for the largest error in the generalized mass matrix.

9.2 The Eigenvalue Summary Table

The following summary of the eigenvalue analysis performed is automatically printed when rigid formats using the Tridiagonal Reduction (FEER) method are invoked:

1. Number of eigenvalues extracted.

2. Number of starting points used.

   This corresponds to the total number of random starting and restart vectors used by the FEER process.

3. Number of starting point moves.

   Not used in FEER (set equal to zero).

4. Number of triangular decompositions.

   Always equal to one, except for unshifted vibration problems (roots starting from the lowest requested). In this case a maximum of three shifts and three decompositions are employed to remove possible stiffness matrix singularities.

5. Total number of vector iterations.

   The total number of reorthogonalizations of all the trial vectors employed.

6. Reason for termination.

   (0) Normal termination

   (1) Fewer than the requested number of eigenvalues and eigenvectors have been extracted.

   (3) The problem size has been reduced. However, the desired number of accurate eigensolutions specified on the EIGB or EIGR card may have been obtained. A detailed list of the computed error bounds can be obtained by requesting DIAG 16 in the EXECUTIVE CØNTRØL DECK.

7. Largest off-diagonal modal mass term and the number failing the mass orthogonality criterion.

9.3 Optional Diagnostic Output

The user can obtain special detailed information relating to the generation of the reduced problem size, the elements of the reduced tridiagonal matrix, computed error bounds and other numerical tests by requesting DIAG 16 in the NASTRAN Executive Control Deck.

The meaning of this information is explained below in the order in which it appears in the DIAG 16 output.

ØRDER - The order of the unreduced problem (size of the $[K_{aa}]$ matrix)

MAX RANK - The maximum number of existing finite eigensolutions as initially detected by FEER

RED ØRDER - The order of the reduced eigenproblem which will be solved to obtain the number of accurate solutions requested by the user

ØRTH VCT - The number of previously computed accurate eigenvectors on the eigenvector file which were generated prior to a restart or by the NASTRAN rigid body mode generator

USER SHIFT - Used only in frequency problems. The user specified shift after conversion from cycles to radians - squared

INTERNAL SHIFT- Used only in frequency problems. A small positive value automatically computed to remove singularities if the user has specified a zero shift. Otherwise, the negative of the user shift

SINGULARITY CHECK - PASS: the shifted stiffness matrix is non-singular
****: the number of internal shifts needed to remove stiffness matrix singularities

TRIDIAGØNAL ELEMENTS RØW j, **, ***, **** - Lists the computed tridiagonal elements of the reduced eigenmatrix:

j - Matrix row
** - Diagonal element
*** - Off-diagonal element
**** - First estimate of off-diagonal element in the next row

ØRTH ITER - The number of times a reorthogonalization of a trial vector has been performed.

MAX PRØJ - The maximum projection of the above trial vector on the previously computed accurate trial vectors (prior to the current reorthogonalization)

NØRMAL FACT - The normalization factor for the reorthogonalized trial vector.

ØPEN CØRE NØT USED *** FEER3 - open core not used by Subroutine FEER3, in single-precision words

FEER QRW ELEMENT *, ITER **, ***, RATIO ****, PRØJ *****:

* - The internal eigenvalue number in the order of its extraction by FEER

** - The number of inverse power iterations performed to extract the associated eigenvector of the reduced system (this is not a physical eigenvector)

*** - If a multiple root has been detected, the number of times that the previous multiple-root, reduced-system eigenvectors have been projected out of the current multiple-root eigenvector before repeating the inverse power iterations

**** - The absolute ratio of maximum, reduced-system eigenvector elements for successive inverse power iterations

***** - The maximum projection of a current multiple-root eigenvector on previously computed eigenvectors for the same root.

PHYSICAL EIGENVALUE *, **, THEØR ERRØR *** PERCENT, PASS OR FAIL:

* - The internal eigenvalue number in the order of its extraction by FEER

** - The associated physical eigenvalue (λ for buckling problems, ω^2 for frequency problems)

*** - Theoretical upper bound on the relative eigenvalue error, in percent

PASS - The computed error is less than or equal to the allowable specified on the EIGB or EIGR bulk data card (default is .001% where n is the order of the stiffness matrix)

FAIL - The computed error is greater than the allowable and this mode is not accepted for further processing

ØPEN CØRE NØT USED *** FEER4 - open core not used by Subroutine FEER4, in single precision words

FEER CØMPLETE *, **, ***, ****

* - The remaining CPU time available following decomposition of the shifted stiffness matrix, in seconds (the total time is specified on the TIME card in the Executive Control Deck)

** - The remaining CPU time, in seconds after completing Subroutine FEER3

*** - The remaining CPU time, in seconds after completing Subroutine FEER4

**** - The total operation count for FEER after decomposition of the shifted stiffness matrix. One operation is considered to be a multiplication or division followed by an addition

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