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HIGHER-ORDER NUMERICAL METHODS DERIVED FROM THREE-POINT POLYNOMIAL INTERPOLATION

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16. Abstract

Higher-order collocation procedures resulting in tridiagonal matrix systems are derived from polynomial spline interpolation and Hermitian finite-difference discretization. The equations generally apply for both uniform and variable meshes. Hybrid schemes resulting from different polynomial approximations for first and second derivatives lead to the nonuniform mesh extension of the so-called compact or Pade difference techniques. A variety of fourth-order methods are described and this concept is extended to sixth-order. Solutions with these procedures are presented for the similar and non-similar boundary layer equations with and without mass transfer the Burgers equation, and the incompressible viscous flow in a driven cavity. Finally, the interpolation procedure is used to derive higher-order temporal integration schemes and results are shown for the diffusion equation.

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I. INTRODUCTION

Three-point finite-difference discretization has formed the basis for the overwhelming majority of numerical solutions of the equations of fluid mechanics. For uniform meshes these procedures are typically second-order accurate in the mesh width h. A decrease in order of accuracy results for nonuniform grids. A wide variety of temporal or marching integration schemes have been developed and these include explicit (one step or two step methods) or implicit procedures. For the latter, which generally have better stability properties, the primary advantage of the three-point differencing is that the resulting algebraic matrix system is of a block-tridiagonal* form; therefore, an efficient and well developed two-pass algorithm[1] can be applied to invert the matrix operator.

Recently, a number of higher-order numerical methods have been proposed. The obvious extension is to five-point differencing which leads to a fourth-order accurate system. Unfortunately, for implicit integration the matrix system is penta-diagonal and, therefore, the boundaries require special consideration. In addition, the truncation error is considerably larger than that found with the Spline and Hermite methods to be discussed. Graves[2] has proposed a five-point partial implicit procedure that simplifies the inversion process; although this method is inconsistent in the transient it can be useful for time asymptotic solutions.

A second class of collocation procedures which are also fourth-order accurate for uniform meshes and which retain a 2x2 block-tridiagonal form for the governing matrix system have recently been proposed. These Hermite or Spline collocation techniques treat both the functional values and certain derivatives as unknown at the three collocation points. These procedures generally result in a somewhat lower truncation error than that found with a five-point functional discretization and can be derived from appropriate Taylor series expansions (Hermite) or polynomial interpolation (Spline). In the former category

*The blocks are 2x2 for the fourth-order methods and 3x3 for the sixth-order methods.

we have the Pade approximation of Kreiss[3] or so-called compact scheme[4], the Mehrstellung[5] formulation and Hermitian finite-difference development of Peters[6] In the latter group are the spline collocation methods described by Rubin and Graves[7] and Rubin and Khosla[8]. In addition, a spline-on-spline techniques is shown to result from a hybrid formulation.

The purpose of the present analysis is 1) to clarify the relationship between the various Hermite developments, 2) to point out an apparent inconsistency in the Peters' formulation, 3) to derive the Hermite tridiagonal system for a nonuniform mesh, since all previous developments are for uniform meshes, 4) to extend the Hermite philosophy in order to develop a variable mesh sixth-order tridiagonal procedure, 5) to briefly review the spline interpolation method, develop this collocation procedure for several new polynomial forms resulting in block-tridiagonal systems and to demonstrate that, in fact, all of the results obtained by the Hermite Taylor series development can be recovered by appropriate spline polynomial interpolation. Comparative solutions are presented for the boundary layer on a flat plate, boundary layers with uniform and variable mass transfer, the nonlinear Burgers equation, and the viscous incompressible Navier-Stokes equations describing the flow in a driven cavity. Finally, 6) the polynomial interpolation procedure is used to develop higher-order temporal integration schemes. Solutions are obtained for the diffusion equation describing the impulsive motion of a flat plate (Rayleigh problem).

II. POLYNOMIAL SPLINE INTERPOLATION

Consider a mesh with nodal points x_j such that

$$a = x_o < x_1 < \ldots < x_N < x_{N+1} = b$$

Define the mesh width $h_j = x_j - x_{j-1}$, with $\sigma_j = \sigma = h_{j+1}/h_j$. Consider a function $u(x)$ such that at the mesh points x_j, we specify $u(x_j) = u_j$. For the purposes of the present analysis we define the polynomial spline $S(x_j;n,k) \equiv S(n,k)$, such that at the mesh points x_j we prescribe $S(x_j;n,k) = u_j$. $S(n,k)$ is an n^{th} order polynomial defined on any interval [j-1,j] and in the set C^{n-k} [a,b]; k is defined as the deficiency of the polynomial spline; i.e., we are considering an n^{th} order polynomial having n-k continuous derivatives on [a,b].

The so-called simple spline[1] has deficiency k=1. The familiar cubic spline is a cubic polynomial of deficiency one or S(3,1). For a more detailed discussion of the properties of polynomial splines see, for example, Reference (1).

Cubic splines have been widely used for curve fitting and interpolation purposes, but only recently has spline collocation been adapted for the numerical solution of ordinary[9-10] and partial differential equations[7,8,11]. These procedures have been applied to the equations of fluid mechanics by Rubin and Graves[7] and Rubin and Khosla[8]. In these papers the spline collocation technique is described for the basic cubic spline S(3,1) as well as a higher-order accurate quintic spline of deficiency three S(5,3). The former has been termed spline 2 and the latter spline 4. In addition, in Reference (8) it is shown that the basic three-point finite-difference discretization formulae can be obtained by considering the quadratic spline of zero deficiency, i.e., S(2,0).

The general spline interpolation procedure of References (7) and (8) can be summarized as follows: an n^{th} order polynomial is specified on the interval [j-1,j]. The n+1 constants are related to the functional values u_{j-1}, u_j, as well as certain spline derivatives m_{j-1}, m_j, M_{j-1}, M_j. m_j, M_j are the spline derivative approximations to the functional derivatives $u'(x_j)$, $u''(x_j)$ respectively. A similar procedure is considered on the interval $[j,j+1]$. Continuity of derivatives is then specified at x_j. This process results in two coupled equations for m_j, M_j, $j=1, \ldots, N$. Boundary conditions are required at $j=0$ and $j=N+1$. The system is closed by the governing differential equation for $u(x_j)$, where all derivatives are replaced by their spline polynomial approximations m_j, M_j. The details of this procedure for spline 1, 2, 4 are given in References (7) and (8) where a variety of explicit, implicit, two-step, relaxation and ADI methods are explored.

This spline procedure can be applied to other polynomials of other orders and deficiencies and thereby a variety of systems can be derived. Since the equations of fluid mechanics are second-order we restrict our attention to polynomial splines defined solely by the functional values and the spline first and second derivatives at the nodal points. In addition, only those polynomial splines resulting in at most 3x3 block-tridiagonal matrix systems are considered. In this regard, in addition to splines 1, 2 and 4, i.e., $S(2,0)$, $S(3,1)$, $S(5,3)$, which have previously been described, the equations governing the spline derivatives m_j, M_j for $S(4,0)$ and $S(5,1)$ have been evaluated. All of the governing systems for the various procedures are now specified. The spline polynomial on $[j-1,j]$ is also discussed. Consider the polynomial spline on $[j-1,j]$

$$S(x;n,k) = \sum_{i=o}^{n} A_i \, t^i \tag{1}$$

at the nodes specify

$$S(x_{j-1},n,k) = u_{j-1} \; ; \qquad S(x_j;n,k) = u_j \tag{2}$$

In addition we require some or all of the following conditions:

$$S'(x_{j-1};\ n,k) = m_{j-1} \tag{3a}$$

$$S'(x_j;n,k) = m_j \tag{3b}$$

$$S''(x_{j-1};n,k) = M_{j-1} \tag{4a}$$

$$S''(x_j;n,k) = M_j \tag{4b}$$

where m_j, M_j are the spline derivative approximations of $u'(x)$, $u''(x)$, respectively. The interior point spline equations are as follows:

1. <u>S(2,0) or Spline 1</u> - Conditions (2) and either (3a) or (3b) are specified for the constants A_i in (1).

$$S(x;2,0) = u_j t + u_{j-1}\ (1-t) + (u_j - u_{j-1} + h_j m_j)\ t(1-t), \tag{5a}$$

where

$$t = (x-x_{j-1})/h_j.$$

$$m_j = (u_{j+1} + (\sigma^2-1)u_j - \sigma^2 u_{j-1})/(\sigma(1+\sigma)h_j) \tag{5b}$$

$$M_j = 2(u_{j+1}/\sigma - (1+\sigma^{-1})\ u_j + u_{j-1})/(h_j^2\ (1+\sigma)) \tag{5c}$$

2. <u>S(3,1) or Spline 2 and S(5,3) or Spline 4</u> - Conditions (2) and either (3) or (4) determine the constants A_i for S(3,1). For S(5,3) we require

(2) and (3), but in lieu of (4), and in order to increase the accuracy of the spline second-derivative approximation, we specify

$$M_j = K_j + G_j; \quad M_{j-1} = K_{j-1} + G_{j-1} \tag{6a}$$

where

$$G_j = \Delta(K_{j+1} - (1+\sigma)\; K_j + \sigma\; K_{j-1})/6 \tag{6b}$$

and

$$\Delta = (1+\sigma^3)/(\sigma(1+\sigma)^2) \tag{6c}$$

In this formulation S(3,1) is recovered from S(5,3) when $\Delta = 0$ so that $M_j = K_j$.

The polynomial spline is given by

$$\begin{aligned} S(x;5,3) = {} & K_{j-1}\;(1-t)^3\; h_j^2/6 + K_j t^3\; h_j^2/6 + (u_{j-1} - K_{j-1}\; h_j^2/6)\;(1-t) \\ & + (u_j - K_j\; h_j^2/6)\; t + G_j\; t^3\;(1-t)^2/2 + G_{j-1}\; t^2(1-t)^3/2 \end{aligned} \tag{7a}$$

Recall that $S(x_j;\, 5,3) \to S(x_j;\, 3,1)$ as $\Delta \to 0$. In addition, we have for both polynomial approximations

$$\begin{aligned} K_{j-1} + 2(1+\sigma)K_j + \sigma\; K_{j+1} = {} & 6(u_{j+1}/\sigma - (1+\sigma^{-1})u_j \\ & + u_{j-1})/h_j^2 \end{aligned} \tag{7b}$$

$$m_j = h_j\;(K_j + 0.5K_{j-1})/3 + (u_j - u_{j-1})/h_j \tag{7c}$$

and

$$m_j \doteq -h_{j+1} \, (K_j + 0.5K_{j+1})/3 + (u_{j+1} - u_j)/h_{j+1} \tag{7d}$$

Combining (7c) and (7d) we obtain

$$\sigma \, m_{j-1} + 2(1+\sigma) \, m_j + m_{j+1} = 3(u_{j+1}/\sigma + u_j(\sigma-\sigma^{-1}) - \sigma \, u_{j-1})/h_j \tag{7e}$$

As shown in References (7) the relations (7c), (7d) are generally preferable to (7e) as they provide a direct evaluation of m_j and, therefore, m_j can be eliminated in favor of K_j, u_j. Also for nonuniform meshes where (7c), (7d) and (7e) for m_j are third-order accurate spline representations of u'(x), the truncation error is increased if (7e) is applied. For spline 2 the governing matrix system is tridiagonal for M_j. For spline 4 a 2x2 block-tridiagonal system results.

3. S(4,0) - The A_i in (1) are evaluated from (2) and either (3) and (4b) or (4) and (3b). Two distinct polynomial splines are generated. We designate the former as $S^1(4,0)$ and the latter as $S^2(4,0)$. The polynomial splines in each case have been obtained. The important spline equations generated from these polynomials are now presented.

A: $S^1(4,0)$

$$\begin{aligned} S'(x;4,0) = {} & u_{j-1} + h_j m_{j-1} \, t + [6(u_j-u_{j-1})-3h_j(m_j+m_{j-1}) + \frac{h_j^2 M_j}{2}] \, t^2 \\ & - [8(u_j-u_{j-1})-h_j \, (5m_j + 3m_{j-1}) + h_j^2 \, M_j] \, t^3 \\ & + [3(u_j-u_{j-1}) - h_j \, (2 \, m_j+m_{j-1}) + \frac{h_j^2}{2} \, M_j] \, t^4 \end{aligned} \tag{8a}$$

$$\frac{\sigma+2}{12} M_j = - \frac{2m_j+m_{j+1}}{3h_{j+1}} + \frac{\sigma^2}{12(1+\sigma)h_{j+1}} \left(\frac{m_j+1}{\sigma} + \frac{\sigma^2-1}{\sigma} m_j - \sigma m_{j-1}\right) + \frac{u_{j+1}-u_j}{h_{j+1}^2} \tag{8b}$$

$$\frac{1+2\sigma}{12\sigma} M_j = \frac{2m_j+m_{j-1}}{3h_j} + \frac{1}{12\ (1+\sigma)h_j} \left(\frac{m_{j+1}}{\sigma} + \frac{\sigma^2-1}{\sigma} m_j - \sigma m_{j-1}\right) - \frac{u_j-u_{j-1}}{h_j^2} \tag{8c}$$

The continuity of M_j leads to

$$m_{j+1} + (1+\sigma)^2 m_j + \sigma^2 m_{j-1} = \frac{2}{1+\sigma} \frac{1}{h_j} \left[\frac{1+2\sigma}{\sigma} u_{j+1} + \frac{\sigma-1}{\sigma} (1+\sigma)^3 u_j - \sigma^2(2+\sigma) u_{j-1}\right] \tag{8d}$$

For $\sigma=1$ it can be shown that $S^1(4,0)$ is equivalent to $S(5,3)$.

.B : $\underline{S^2(4,0)}$

$$\begin{aligned} S^2(x;\ 4,0) = u_{j-1} &+ \left[2(u_j-u_{j-1}) - h_j m_j + \frac{h_j^2}{6} (M_j-M_{j-1})\right] t \\ &+ \frac{h_j^2}{2} M_{j-1} t^2 - \left[2(u_j-u_{j-1}) - 2h_j m_j + \frac{h_j^2}{2} (M_j+M_{j-1})\right] t^3 \\ &+ \left[(u_j-u_{j-1}) - h_j m_j + \frac{2M_j+M_{j-1}}{6} h_j^2\right] t^4 \end{aligned} \tag{9a}$$

$$m_j = \frac{u_j-u_{j-1}}{h_j} + \frac{h_j}{12} \left[\frac{1+4\sigma}{\sigma} M_j - \frac{1}{\sigma(1+\sigma)} M_{j+1} + \frac{1+2\sigma}{1+\sigma} M_{j-1}\right] \tag{9b}$$

$$m_j = \frac{u_{j+1}-u_j}{h_{j+1}} - \frac{h_{j+1}}{12}\left[(4+\sigma)\; M_j + \frac{2+\sigma}{1+\sigma}\; M_{j+1} - \frac{\sigma^2}{1+\sigma}\; M_{j-1}\right] \tag{9c}$$

The continuity of m_j leads to

$$\frac{\sigma^2+\sigma-1}{12\sigma} M_{j+1} + \frac{\sigma^3+4\sigma^2+4\sigma+1}{12\sigma} M_j + \frac{1+\sigma-\sigma^2}{12} M_{j-1}$$
$$= \frac{1}{h_j^2}\left(\frac{u_{j+1}}{\sigma} - \frac{1+\sigma}{\sigma} u_j + u_{j-1}\right) \tag{9d}$$

It is significant that for all of the polynomial splines considered thus far the governing system consists of a tridiagonal form for m_j or (M_j) and a pair of algebraic relations for the other spline derivative M_j (or m_j). Therefore, at most a 2x2 block-tridiagonal system results. We shall now consider the simple quintic spline S(5,1). For the first time the governing interior point system enlarges to a 3x3 block-tridiagonal in terms of both of the spline derivatives m_j and M_j. The simpler algebraic relations no longer appear and, therefore, the final matrix system will be somewhat more complex. On the other hand, increased order of accuracy is achieved.

4. S(5,1) - The six A_i in (1) are evaluated with all of the conditions (2-4). Continuity of the third and fourth derivative leads to the following interior point equations:

$$\begin{aligned} S(x;5,1) = {} & u_{j-1} + h_j\, m_{j-1}\, t + \frac{h_j^2}{2}\, M_{j-1}\, t^2 \\ & + \left[10(u_j-u_{j-1}) - h_j(4m_j+6m_{j-1}) + \frac{h_j^2}{2}(M_j-3M_{j-1})\right] t^3 \\ & - \left[15(u_j-u_{j-1}) - h_j(7m_j+8m_{j-1}) + \frac{h_j^2}{2}(2M_j-3M_{j-1})\right] t^4 \\ & + \left[6(u_j-u_{j-1}) - 3h_j(m_j+m_{j-1}) + \frac{h_j^2}{2}(M_j-M_{j-1})\right] t^5 \end{aligned} \tag{10a}$$

$$7m_{j+1} + 8(1+\sigma^3)\, m_j + 7\sigma^3\, m_{j-1} = \frac{15}{h_j}\left(\frac{u_{j+1}}{\sigma} + \frac{\sigma^4-1}{\sigma}\, u_j - \sigma^3 u_j\right)$$

$$+ \sigma h_j \left(M_{j+1} + \frac{3}{2}(\sigma^2-1)\, M_j - \sigma^2\, M_{j-1}\right) \tag{10b}$$

and

$$-\frac{1}{2\sigma} M_{j+1} + \frac{3}{2}\frac{1+\sigma}{\sigma} M_j - \frac{1}{2} M_{j-1} = \frac{10}{h_j^2}\left(\frac{u_{j+1}}{\sigma^3} - \frac{1+\sigma^3}{\sigma^3}\, u_j + u_{j-1}\right)$$

$$+ \frac{1}{h_j}\left[-\frac{4}{\sigma^2} m_{j+1} + 6\,\frac{\sigma^2-1}{\sigma^2}\, m_j + 4m_{j-1}\right] \tag{10c}$$

Other polynomial splines can be considered, however, for polynomials of fifth or lower order, the spline formulations presented herein appear to be the most efficient. For higher-order splines, we require that the third or higher-order spline derivatives be specified in the evaluation of the A_i in (1). These formulations are not discussed here, although the tridiagonal sixth-order accurate system for M_j derivable from S(6,0) is presented later in this report.

For the polynomial spline formulations presented here, the truncation errors $T(h_j)$ for the various spline derivatives m_j and M_j are depicted in Table I. We recall that

$$m_j = u'(x_j) + T(h_j)$$

$$M_j = u''(x_j) + T(h_j)$$

For completeness the truncation errors $T(h_j)$ are also given for the five point finite-difference discretization with a uniform grid. Note that although these errors are fourth-order, they are somewhat larger than those obtained with any of the fourth-order polynomial spline formulations.

For $\sigma=1$ the minimum truncation errors of the fourth-order methods are obtained with S(5,3) and $S^1(4,0)$. $S^1(4,0)$ and $S^2(4,0)$ retain fourth-order accuracy for

TABLE 1

TRUNCATION ERROR OF SPLINE DERIVATIVES

	Uniform Mesh (σ=1)		Nonuniform Mesh	
	m_j	M_j	m_j	M_j
S(2,0)	$\frac{h^2}{6}(u''')_j$	$\frac{h^2}{12}(u^{iv})_j$	$\frac{\sigma h_j^2}{6}(u''')_j$	$\frac{\sigma-1}{3} h_j (u''')_j + \frac{h_j^2}{12}\frac{1+\sigma^3}{1+\sigma}(u^{iv})_j$
S(3,1)	$\frac{h^4}{180}(u^v)_j$	$\frac{h^2}{12}(u^{iv})_j$	$^*\sigma\frac{(\sigma-1)}{24}h_j^3(u^{iv})_j + \frac{(1-\sigma+\sigma^2)}{180}h_j^4(u^v)_j$	$\frac{1+\sigma^3}{1+\sigma}\frac{h_j^2}{12}(u^{iv})_j$
S(5,3)	$\frac{h^4}{180}(u^v)_j$	$\frac{h^4}{360}(u^{vi})_j$	Same as (3,1)	$\frac{7h_j^3}{180}(1+\sigma^2)(\sigma-1)(u^v)_j + h_j^4[\frac{\sigma^2}{360}+\frac{(\sigma-1)^2}{1080}(7\sigma^2-2\sigma+7)](u^{vi})_j$
$S^1(4,0)$	$\frac{h^4}{180}(u^v)_j$	$\frac{h^4}{360}(u^{vi})_j$	$\frac{\sigma^2}{240}\frac{(1+\sigma)^2}{1+\sigma+\sigma^2}h_j^4(u^v)_j$	$\sigma(\sigma-1)\frac{h_j^3}{120}\frac{3+5\sigma+3\sigma^2}{1+\sigma+\sigma^2}(u^v)_j + \frac{\sigma^4-\sigma^2+1}{\sigma^2+\sigma+1}\frac{\sigma h_j^4}{120}(u^{vi})_j$
$S^2(4,0)$	$\frac{7}{360}h^4(u^v)_j$	$\frac{h^4}{240}(u^{vi})_j$	$\frac{3(\sigma^4+1)-5(1+\sigma+\sigma^3+\sigma^4)}{720}h_j^4(u^v)_j$	$(\sigma-1)(2+5\sigma+2\sigma^2)\frac{h_j^3}{180}(u^v)_j + [3(\sigma^2-1)^2+\sigma(2\sigma^2-\sigma+2)]\frac{h_j^4}{720}(u^{vi})_j$

TABLE I (Concluded)

	Uniform Mesh (σ=1) m_j	M_j	Nonuniform Mesh m_j	M_j
S(5,1)	$\frac{h^6}{5040}(u^{vii})_j$	$\frac{h^4}{720}(u^{vi})_j$	$\frac{\sigma^2(\sigma-1)}{15(1+\sigma^3)}(1+6\sigma+6\sigma^2+\sigma^3)\frac{h_j^5}{1440}(u^{vi}) + \frac{\sigma^3}{5040}h_j^6(u^{vii})_j$	$\frac{\sigma(1+\sigma^3)}{1+\sigma}\frac{h_j^4}{720}(u^{vi})_j$

5 - pt. Finite-Difference: σ=1

$$u_x = \frac{8(u_{j+1}-u_{j-1}) - (u_{j+2}-u_{j-1})}{12h} + \frac{h^4}{30}(u^v)_j$$

$$u_{xx} = \frac{16(u_{j+1}+u_{j-1}-2u_j) - (u_{j+2}-2u_j+u_{j-2})}{12h^2} + \frac{h^4}{90}(u^{vi})_j$$

Hermite 6: σ=1 ; $m_j = u_x + h^6(u^{vii})_j/9450$; $M_j = u_{xx} + h^6(u^{viii})_j/66360.$

*This is obtained from (7c) or (7d). If evaluated from (7c) 24 in the denominator changes to 72.

m_j even with a variable mesh. The other fourth-order polynomial splines lead to third-order accurate formulas for m_j with $\sigma \neq 1$. S(5,1) is sixth-order for m_j with $\sigma=1$ and fifth order with $\sigma \neq 1$. M_j is fourth-order in both cases. A sixth order formulation for M_j is discussed later in this paper. From Table I we see that even with h=1.0 there is a significant reduction in truncation error with the higher-order methods. This is due to smaller numerical coefficients in the error terms.

III. TAYLOR SERIES FORMULATION - HERMITE COLLOCATION

1. Compact Formulation - As discussed previously, higher-order finite-difference equations can be derived from Taylor series expansions. For a uniform grid, fourth-order accuracy is achieved with a five point expansion formula. The resulting system is penta-diagonal with implicit integration procedures. Recently, Orszag and Israeli[3] have reported an idea due to Kreiss in which a compact[4] or Padé approximation transforms the penta-diagonal system for the functional values at the nodes to a 3x3 block-tridiagonal system for the functional values and their derivatives at the nodes.

It has been observed[3] that with

$$m_j = (u_x)_j \;, \qquad M_j = (u_{xx})_j \;,$$

$$D^+u_j = (u_{j+1}-u_j)/h_{j+1} \;, \; D^-u_j = (u_j-u_{j-1})/h_j \;, \tag{11a}$$

$$D^o u_j = 2(u_{j+1}/\sigma - (1+\sigma)\; u_j/\sigma + u_{j-1})/((1+\sigma)h_j^2) \;, \tag{11b}$$

$$D^* u_j = (u_{j+1}/\sigma + (\sigma^2-1)\; u_j/\sigma - \sigma u_{j-1})/(1+\sigma)h_j) \;, \tag{11c}$$

for a uniform mesh, the five-point difference discretization is of the form

$$m_j = (1 - \frac{h^2}{6} D^+D^-)\;(u_{j+1}-u_{j-1})/2h$$

$$M_j = (1 - \frac{h^2}{12} D^+D^-)\;(D^+D^-u_j)$$

The truncation errors are given in Table I. These expressions can be rewritten with a Padé or compact approximation such that

$$m_j = \frac{(u_{j+1}-u_{j-1})/2h}{1 + \frac{h^2}{6} D^+D^-} \tag{12a}$$

$$M_j = \frac{D^+D^-u_j}{1 + \frac{h^2}{12} D^+D^-} \tag{12b}$$

or

$$(1 + \frac{h^2}{6} D^+D^-) \quad m_j = (u_{j+1}-u_{j-1})/2h \tag{13a}$$

$$(1 + \frac{h^2}{12} D^+D^-) \quad M_j = D^+D^-u_j \tag{13b}$$

This results in a fourth-order block-tridiagonal interior point system for the function u_j and the derivatives m_j, M_j. As before the system is closed with the differential equation and appropriate boundary conditions.

The system (13) has appeared in a number of places[3] and is termed compact[4], Mehrstellung[5] and Hermitian[6] differencing. The system (13) is fourth-order with a somewhat smaller truncation error than the five-point difference equations.

The Equations (13) have previously been observed in the spline analysis presented herein. For a uniform mesh, (13a) is equivalent to (7e) in S(3,1) and (13b) corresponds to (9d) in $S^2(4,0)$. Therefore, this compact formulation is the result of two different polynomial spline formulations for m_j, M_j. The derivation (13) does not provide the simpler expressions (7c,7d) or (9b,9c) relating the derivatives m_m, M_j. These expressions are particularly useful in consideration of boundary conditions and in order to eliminate m_j and, thus, reduce the size of the governing matrix system. Moreover, (7e) and (9d) have been obtained for nonuniform meshes. An extension to variable grids of the compact formulation could be quite laborious and will not be considered here. On the other hand, the "compact" formulae (13) can be derived directly with a three-point Hermitian collocation procedure. It is the approach that will be discussed further.

2. <u>Hermitian Collocation</u> - Consider the finite three-point Taylor series expansions

$$u_{j+1} = u_j+h_{j+1}u_x + h_{j+1}^2 u_{xx}/2 + h_{j+1}^3 u_{xxx}/6 + h_{j+1}^4 u_{xxxx}/24 \tag{14a}$$

$$u_{j-1} = u_j-h_j u_x + h_j^2 u_{xx}/2 - h_j^3 u_{xxx}/6 + h_j^4 u_{xxxx}/24 \tag{14b}$$

Let $\quad m_j = u_x,\ M_j = u_{xx}$

A. <u>Hermite</u> 1 - *with the operators* (11) *Equations* (14) *can be rewritten in the form*

$$m_j = (u_j-u_{j-1})/h_j + (h_j/2)\ [1-(h_j/3)D^* + (h_j^2/12)D^o]M_j \tag{15a}$$

$$m_j = (u_{j+1}-u_j)/h_{j+1} - (h_{j+1}/2)\ [1-(h_{j+1}/3)D^* - (h_{j+1}^2/12)D^o]M_j \tag{15b}$$

These equations are identical with (9b) and (9c). Eliminating m_j the tridiagonal relation (9d) is recovered. Therefore, Hermite 1 is identical to $S^2(4,0)$. The tridiagonal Equation (9d) represents the extension to variable meshes of the compact formula (13b), since (9d) $\rightarrow$ (13b) for $\sigma = 1$. Significantly, (9d) can be derived with a compact operator directly from three-point finite-difference expansion. The cumbersome Padé approximation which is not easily extendable to nonuniform meshes or higher order schemes is avoided. This represents an alternate derivation of the polynomial spline equations.

B. <u>Hermite 2</u> - If we consider finite three-point Taylor series expansions

$$m_{j+1} = m_j + h_{j+1}u_{xx} + h_{j+1}^2 u_{xxx}/2 + h_{j+1}^3 u_{xxxx}/6 \tag{16a}$$

$$m_{j-1} = m_j - h_j u_{xx} + h_j^2 u_{xxx}/2 - h_j^3 u_{xxxx}/6 \tag{16b}$$

and combine with (14) to eliminate u_{xxxx}, ($u_{xxx} = D^o m_j$), we recover the Equations (8). Therefore, Hermite 2 is identical with $S^1(4,0)$. The tridiagonal form (8d) is a nonuniform mesh extension of the "compact" formula (13a). This relation is fourth-order accurate for both uniform and variable meshes, see Table I.

C. Hermite 3 - If in (16) we eliminate u_{xxx} and replace u_{xxxx} by $D^o M_j$ we obtain

$$M_{j+1} + 2(1+\sigma)M_j + \sigma M_{j-1} = (3/h_j)\ [m_{j+1}/\sigma + (\sigma^2-1)m_j/\sigma - \sigma m_{j-1}] \tag{17}$$

The block-tridiagonal system obtained with (8d) from Hermite 2 and (17) is termed Hermite 3. This is equivalent to what has been called spline-on-splines[1]. Since u_{xxxx} is treated differently in obtaining the equations for m_j, M_j this formulation does not result from a single polynomial spline interpolation.

D. Hermite 4 - If we consider the block-tridiagonal system obtained with (8d) from Hermite 2 and (9d) from Hermite 1 for m_j, M_j, respectively, we have what is termed Hermite 4. Once again this cannot be derived from a single polynomial spline and for $\sigma = 1$ reduces to (13) or what has been termed the Padé or compact or Mehrstellung formulation.

E. Hermite 6 - Uniform mesh. Consider the finite three-point Taylor series expansions

$$D^o u_j = M_j + \frac{h^2}{12} u^{iv} + \frac{h^4}{360} u^{vi}$$

$$D^o M_j = u^{iv} + \frac{h^2}{12} u^{vi}$$

$$D^* m_j = M_j + \frac{h^2}{6} u^{iv} + \frac{h^4}{120} u^{vi}$$

Eliminating u^{iv} and u^{vi} we obtain

$$-M_{j+1} + 8M_j - M_{j-1} = 18\{\tfrac{4}{3} (u_{j+1}-2u_j+u_{j-1})/h^2 - (m_{j+1}-m_{j-1})/2h\} \tag{18a}$$

From the finite Taylor series

$$D^* u_j = m_j + \frac{h^2}{6} u_{xxx} + \frac{h^4}{120} u^v$$

$$D^o m_j = u_{xxx} + \frac{h^2}{12} u^v$$

$$D^* M_j = u_{xxx} + \frac{h^2}{6} u^v$$

we obtain

$$7m_{j+1} + 16m_j + 7m_{j-1} = 15(u_{j+1}-u_{j-1})/h + h(M_{j+1}-M_{j-1}) \tag{18b}$$

For a variable mesh a similar procedure applies and we obtain

$$\begin{aligned} -\frac{\sigma}{18} M_{j-1} + \frac{38\sigma-11-11\sigma^2}{36\sigma} M_j - \frac{1}{18\sigma} M_{j+1} &= \frac{1}{6h_j^2}\Big[\frac{1+15\sigma}{\sigma^3(1+\sigma)} u_{j+1} \\ &- \Big(\frac{1+15\sigma}{\sigma^3(1+\sigma)} + \frac{\sigma(15+\sigma)}{1+\sigma}\Big) u_j + \frac{\sigma(\sigma+15)}{1+\sigma} u_{j-1}\Big] \\ - \frac{1}{h_j}\Big[\frac{4+5\sigma}{9\sigma^2(1+\sigma)} m_{j+1} &- \frac{(\sigma-1)(15\sigma^2-114\sigma+15)}{54\sigma^2} m_j + \frac{\sigma(5+4\sigma)}{9(1+\sigma)} m_{j-1}\Big] \end{aligned} \tag{18c}$$

$$\begin{aligned} &\frac{14+10\sigma-10\sigma^2}{5\sigma} m_{j+1} + \Big[\frac{32}{5\sigma} + \frac{2(\sigma-1)}{5\sigma}(8\sigma^4+18\sigma^3+13\sigma^2-7\sigma+1)\Big] m_j \\ &+ \frac{14\sigma^2+10\sigma-10}{5} m_{j-1} = -\frac{6}{h_j}\Big[\frac{(\sigma^2-\sigma-1)}{\sigma^2} u_{j+1} - \frac{(\sigma-1)(1+\sigma)^2(1+\sigma^3)}{\sigma^2} u_j \\ &+ \sigma^2(\sigma^2+\sigma-1) u_{j-1}\Big] - \frac{h_j}{5}\big[(\sigma+1)(\sigma-2) M_{j+1} - 3(\sigma-1)(\sigma+1)^3 M_j \\ &+ (2\sigma-1)(\sigma+1) M_{j-1}\big] \end{aligned} \tag{18d}$$

The block-tridiagonal system (18a), (18b) for M_j, m_j is termed Hermite 6. This is a sixth-order accurate formulation when combined with the differential equation and appropriate boundary conditions. Simple relationships such as (7c) between m_i and M_i no longer exist and once again these formulae are not derivable from a single polynomial spline. (18b) is the uniform mesh (σ=1) analogue of the form (10b) for m_j found in S(5,1) for a variable mesh. Although we have not carried out the details, based on previous experience, we expect that (18c) for a nonuniform mesh is derivable from some form of S(6,0). The truncation errors for Hermite 6 (18a,18b) are given in Table I.

Therefore, it is possible to derive the polynomial spline results of Section II with an Hermitian discretization procedure. Moreover, hybrid systems, which represent approximations resulting from multiple spline formulations, can also be conceived. One of these hybrid systems is the variable mesh extension of the so-called Pade or compact differencing scheme. The truncation errors for all possible systems can be obtained from Table I. Finally, the hybrid systems result in a block-tridiagonal form of m_j, M_j. The simpler relations relating m_j directly to M_j found in the polynomial spline formulations are not obtained. This concept has been extended to a sixth-order system in Hermite 6. Higher-order approximations have not been considered.

3. <u>Hermitian Polynomial Interpolation - Peters Method</u>[6] - Recently, Peters has presented an Hermitian differencing procedure for uniform meshes which also leads to the "compact" or Hermite 4 3x3 block-tridiagonal system for u_j, m_j and M_j. Peters has then carried out a reduction process that appears most attractive as a single system for u_j results; however, it can be shown that the resulting system is inconsistent with the differential equation and as such results in an attendant loss of accuracy. A brief description follows.

Consider the interpolation polynomial S(x) on the three-point interval [j-1, j+1].

$$S(x) = u_{j+1}\ (t^2+t)/2 + u_j\ (1-t^2) + u_{j-1}\ (t^2-t)/2$$

$$+ \alpha t\ (1-t^2) + \beta t^2\ (1-t^2)$$

as before $t = (x-x_j)/h$. α and β are two free parameters which are assumed constant on [j-1, j+1].

With $m_j = S'(x_j)$ and $M_j = S''(x_j)$ we obtain

$$m_j = D^*u_j + \alpha/h,\quad m_{j+1} = hD^o u_j + D^* u_j - 2(\alpha+\beta)/h$$
$$m_{j-1} = - hD^o u_j + D^* u_j - 2(\alpha-\beta)/h \tag{19}$$

and

$$M_j = D^o u_j + 2\beta, \ M_{j+1} = D^o u_j - (6\alpha+10\beta)/h^2$$
$$M_{j-1} = D^o u_j + (6\alpha-10\beta)/h^2 \tag{20}$$

A. Eliminating α and β from (19) we obtain

$$m_{j+1} + 4m_j + m_{j-1} = 6D^* u_j \tag{21a}$$

This is precisely (13a) or (7e).

B. Eliminating α and β from (20) we obtain

$$M_{j+1} + 10M_j + M_{j-1} = 12D^o u_j \tag{21b}$$

This is precisely (13b) or (9d) and, therefore, with (21a) and (21b) the compact or Hermite 4 discretization is recovered.

Peters has not considered the tridiagonal Equations (21) but instead has evaluated the differential equation under consideration at the three points x_{j-1}, x_j, x_{j+1}. The derivative approximations at these points are given by (19) and (20). This leads to three equations for the five unknowns α, β, u_{j-1}, u_j, u_{j+1}. α and β are eliminated and a single tridiagonal system for u_j results.

However, α and β can be determined independently from (19) and (20) respectively, and these results effectively imply different polynomials $S(x)$ leading to the tridiagonal system (21). When α and β are evaluated from Peters *tridiagonal equations the resulting values are inconsistent with those leading* to (21a) and (21b). This can be shown in the following simple example.

Consider the differential equation

$$u_x + u_{xx} = 0 \tag{22}$$

Using (19) and (20) and evaluating (22) at j-1, j, j+1 following Peters, we obtain for h=1

$$(D^{*}+D^{o})\ u_j + 2\beta + \alpha = 0$$

$$(2D^{o}+D^{*})\ u_j - 12\beta - 8\alpha = 0$$

$$D^{*}u_j - 8\beta + 4\alpha = 0$$

From these equations we find

$$8\alpha = -\ (5D^{*}+4D^{o})\ u_j$$

Substituting this expression into (19) we obtain for m_j

$$m_j = (\frac{3}{8}\ D^{*} - \frac{1}{2}\ D^{o})\ u_j$$

It is clear that we do not recover even the leading term in the expansion for $u'(x)$. A similar result is found for M_j.

Numerical experiments with Burgers equation

$$u_t + (u - \frac{1}{2})\ u_x = \nu u_{xx}$$

have shown this inconsistency. Both the viscosity (ν) and convection ($u - \frac{1}{2}$) coefficients are effectively modified. For large values of ν the effect of the inconsistency is diminished. For small ν values, diagonal dominance of the resulting matrix system is lost. Therefore, we believe that the reliability of the Peters reduction procedure is questionable.

IV. EXAMPLES

1. Similar Boundary Layer: Zero Mass Transfer - As a first test of the various polynomial spline or Hermite formulations considered in the previous sections, solutions have been obtained for the similarity equations governing laminar boundary layer behavior[12].

$$u'' + fu' + \beta(1-f'^2) = 0, \quad u = u(\eta), \; f' = u \tag{23}$$

Primes denote differentiation with respect to η, where $\eta = y(Re/2x)^{\frac{1}{2}}$; Re is the Reynolds number; y is measured normal to the surface and x along the surface. The respective velocities are v and u. We approximate the derivatives u_j', u_j'', f' with m_j, M_j and $\bar{m}_j$, respectively, so that the governing Equations (23) become

$$\begin{aligned} &M_j + f_j m_j + \beta(1-\bar{m}_j^2) = 0 \\ &\bar{m}_j = u_j \end{aligned} \tag{24}$$

The additional equations for m_j, M_j, $\bar{m}_j$ are given in Sections II and III for each of the polynomial interpolation procedures. The systems are closed with the boundary conditions at the surface y=0 (j=1) and the edge of the boundary layer $y=y_e$ or j=N.

$$f_1 = u_1 = 0, \; u_N = 1 \tag{25}$$

The additional boundary conditions on m_1, m_N, M_1, M_N are obtained from the Equations (23, 24), the spline formulas typified by (8b) or (8c), or from the Hermite expansions (14) and (15). The boundary conditions used here have truncation errors that parallel those for the interior systems shown on Table I. For spline 2 and spline 4, boundary conditions have been discussed in greater detail in References (7, 8); however, only third-order conditions were used for the spline 4 calculations so that the present results are somewhat more accurate.

The results of the polynomial spline calculations are presented in Table 2 for a variety of uniform and variable meshes. The notation σ=1.5/1 means that

TABLE 2: SIMILAR BOUNDARY LAYER SOLUTION: $f''(0)$

$\beta = 0$

η_{MAX}	h_2	σ	SPLINE 2 S(3,1)	$S^2(4,0)$	HERMITE 4 (COMPACT)	SPLINE 4 S(5,3)	HERMITE 6
6.0	0.1	1.0	0.469634	0.469597	0.469600	0.469600	0.469600
20.0	1.0	1.0	0.475357	0.479359	0.473602	0.467960	0.469690
16.078	0.1	1.5/1	0.464325	0.471666		0.470025	
16.063	0.05	1.8/1	0.462008	0.469926		0.468885	

$f''(0) = 0.469600$ (ROSENHEAD[12])

$\beta = 1$

η_{MAX}	h_2	σ	SPLINE 2 S(3,1)	$S^2(4,0)$	HERMITE 4 (COMPACT)	SPLINE 4 S(5,3)	HERMITE 6
6.0	0.1	1.0	1.23227	1.23260	1.23258	1.23259	1.23259
20.0	1.0	1.0	1.20612	1.20863	1.21260	1.21863	1.23242
9.448	.001	1.8/1	1.23604	1.23301		1.23299	

$f''(0) = 1.23259$ (ROSENHEAD[12])

$h_j = \min\{h_j, 1\}$, and that σ=1.5 for $h_j \leq 1$. The remarkable accuracy of Hermite 6 with the uniform mesh h=1.0 is noteworthy. It is apparent that significant improvements in accuracy are achieved by considering higher-order polynomial splines.

2. Similar Boundary Layer: Mass Transfer - In order to carry out more stringent tests of the polynomial methods, boundary layers with surface mass transfer are considered. In this section, similarity solutions corresponding to mass transfer of the type $V_s \sim x^{-\frac{1}{2}}$ are evaluated; in the following section, uniform injection and suction is specified; i.e., $V_s \sim$ constant so that the boundary layer behavior is non-similar. The subscript s denotes the surface values. With large injection it is possible to blow the boundary layer off of the surface, and with large suction the boundary layer becomes very thin and the shear stresses become quite large. Therefore, these boundary layer profiles are more difficult to approximate numerically, and provide more exacting tests of the spline and Hermite collocation procedures.

The equations governing the similar boundary layer with mass transfer are (23 - 24). The only change is in the boundary conditions (25) for f_1, so that now f_1 = K, where K < 0 for injection and K > 0 for suction.

The results of these calculations are shown on Table 3 and Figures (1) and (2). The figures show velocity profiles for suction and injection, respectively. The flat plate Blasius profile is also included in order to emphasize the extreme thinning of the boundary layer with suction and the blow-off obtained with injection. The polynomial solutions shown on the figures are in excellent agreement with the numerically "exact"* values of Emmons and Leigh[13]. These profiles are coincident with the polynomial solutions obtained with spline 4 or Hermite 6 and, therefore, are not specifically included on the figures. The finite-difference results are not as accurate and exhibit an erroneous overshoot for the suction case (Figure 1). For the suction profile only two points lie with the boundary layer. More exact comparisons are shown

*"Exact" here means six-decimal place accuracy.

TABLE 3: $f''(0)$ - SIMILAR BOUNDARY LAYER WITH NON-UNIFORM MASS TRANSFER

h	σ	K	F.D.	SPLINE 4	HERMITE 6	EXACT	N_δ/N
0.1	1.0	0.5	0.7394	0.7394		0.7394	35/81
0.1	1.5/1.	0.5	0.7842	0.7406			7/21
1.0	1.0	0.5	0.7992	0.7545			3/21
0.1	1.5/1.	10.0	7.8903	6.9817		7.1397	2/21
0.15	1.5/1.	10.0	7.6869	6.8703			1/21
0.3	1.0	10.0	5.2677	7.2178	7.0425		1/21
0.1	1.0	-0.5	0.2326	0.2326		0.2326	48/81
0.1	1.5/1.	-0.5	0.2317	0.2321			9/21
1.0	1.0	-0.5	0.2514	0.2253			5/21
0.1	1.5/1.	-1.2	0.0041	0.0046		0.0047	12/21
1.0	1.0	-1.2	0.0009	0.0045	0.0048		9/21

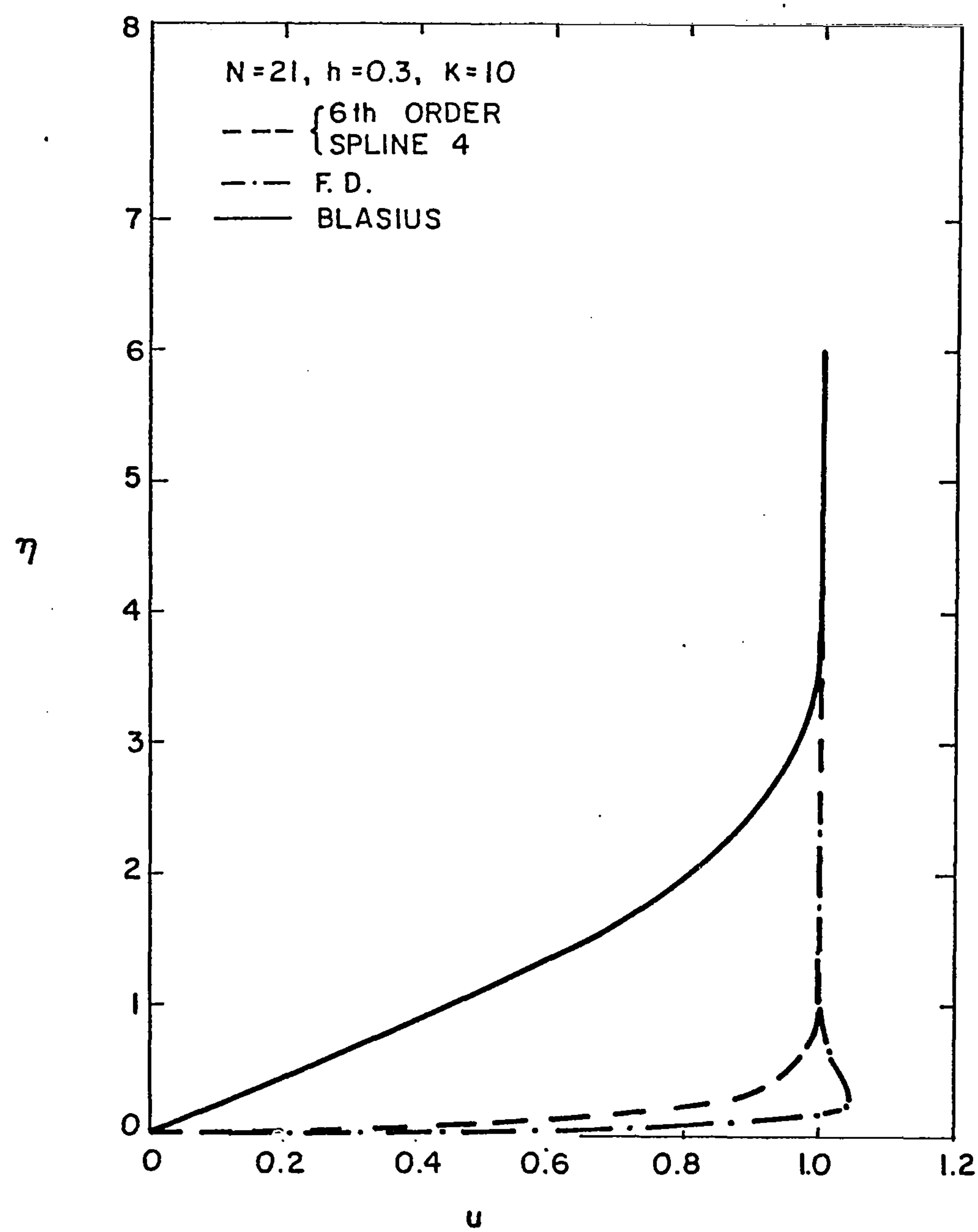

FIG. 1 SIMILAR BOUNDARY LAYER WITH NON-UNIFORM SUCTION

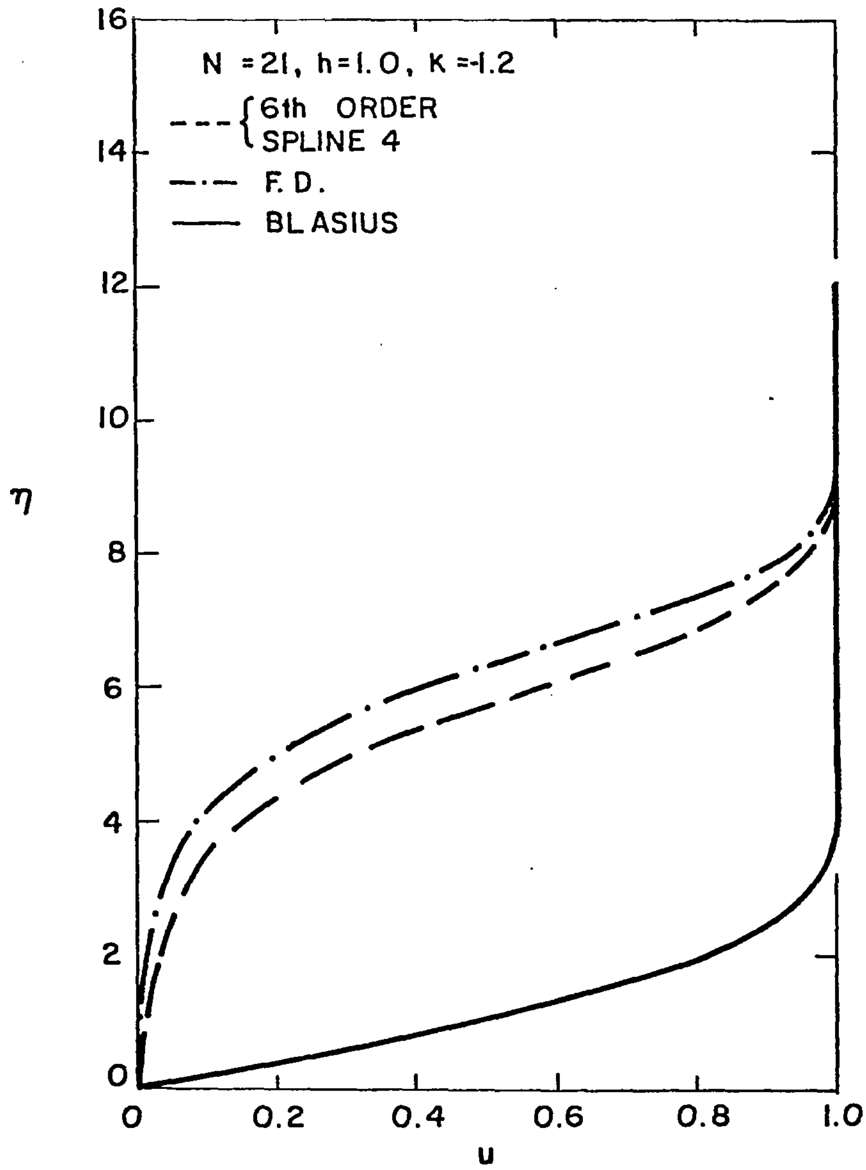

FIG. 2 SIMILAR BOUNDARY LAYER WITH NON-UNIFORM BLOWING

on Table 3. N_δ denotes the number of grid points within the boundary layer. A variety of results for uniform and nonuniform grids are presented. The polynomial solutions retain a high degree of accuracy for both the high shear suction and near separated injection cases. It is generally found that for equal accuracy, spline 4 requires one-quarter the number of mesh points required in finite-difference calculations; e.g., with K = 0.5, $f''(0) = 0.7394$ (81 points with finite-difference) and $f''(0) = 0.7392$ (21 points with spline 4). Similar behavior is found with Burgers equation and the cavity solutions to be discussed in the following sections.

3. Non-Similar Boundary Layer: Uniform Mass Transfer - With uniform injection or suction V_s = constant, and the following coordinate transformation is applied[12]:

$$\xi = V_s\ (xRe/2)^{\frac{1}{2}}\ ;\quad \eta = y\ (Re/2x)^{\frac{1}{2}}$$

$$\psi = (2x/Re)^{\frac{1}{2}}\ f(\xi,\eta);\quad u = \psi_y = f_\eta$$

$$v = -\psi_x = (2Rex)^{-\frac{1}{2}}\ (f + \xi f\xi - \eta f\eta)$$

The governing boundary layer equations become

$$u_{\eta\eta} + fu_\eta + \xi(f_\xi\ u_\eta - uu_\xi) = 0 \tag{26a}$$

and the boundary conditions are,

at $\eta = 0$ $f = \underline{+}\ \xi^*$, $u = 0$, and

$$\lim \eta \to \infty, \quad u \to 1 \tag{26b}$$

The spline equations are

$$M_{ij} + (f_{ij} + \xi_i\ (f_\xi)_{ij})\,m_{ij} = \frac{\xi_i}{2}\ (u_\xi^2)_{ij}$$

*The positive sign denotes suction and the negative sign denotes injection. V_s is positive in both cases.

where $(f_\xi)_{ij} = (f_{ij} - f_{i-1,j})/\Delta\xi$, and with quasi-linearization

$$(u_\xi^2)_{ij} = (2u_{ij}^* u_{ij} - u_{ij}^{*2} - u_{i-1,j}^2)/\Delta\xi$$

Iteration is used for the nonlinear term and the asterisk denotes the values from the previous iteration. The equation for $\bar{m}_j$ is the same as given by (24). Once again the spline derivative boundary conditions are obtained from the governing Equation (26) and the derivative relations obtained with the polynomial interpolation of Sections II and III.

For $\xi >> 1$, with suction the classical[12] asymptotic suction profile will be recovered, i.e.

$$u \sim 1 - \exp(-V_s\ y\ Re)$$

or

$$u \sim 1 - \exp(-2\eta\xi) \qquad (27)$$

For injection, there has been some question[14] as to whether the boundary layer will separate at a finite ξ location. This question will be addressed in the discussion of results which follows.

The solutions are shown on Tables 4 and 5 and Figures (3) through (6). With many mesh points all of the methods, including finite-difference, work quite well, see Figures (3a) and (5a). As the mesh size is increased the finite-difference solutions begin to deviate from the polynomial results. This is shown on Figures (3b), (3c) and (5b) but more dramatically on Figures (4) and (6). The surface shear stress, $f''(\xi,0)$, obtained with the finite-difference method becomes very inaccurate for coarse meshes. For the suction calculation, the asymptotic solution (27) gives for $\xi >> 1$

$$f''(0) = 2\xi$$

Therefore, at $\xi = 1.0$, $f''(0) \approx 2$. The spline 4 results very closely approximate this value; these solutions are in all cases more accurate than the Hermite 4 results. Table 4 presents the shear values for both the coarse and fine grids. Also shown is the velocity one grid point away from the surface. The asymptotic

TABLE 4: f"(0) NON-SIMILAR BOUNDARY LAYER WITH UNIFORM SUCTION

ξ	h	σ	N	F. D.		SPLINE 4		HERMITE 4	
				$u(\xi, \Delta\eta)$	$f''(0,\xi)$	$u(\xi, \Delta\eta)$	$f''(0,\xi)$	$u(\xi, \Delta\eta)$	$f''(0,\xi)$
0.09	1.0	1.0	11	0.5321	0.6798	0.5218	0.5829	0.5228	0.5874
0.49				0.7776	1.0539	0.7357	1.1769	0.7369	1.1860
0.79				0.9185	1.3270	0.8249	1.6775	0.8235	1.7202
0.95				0.9833	1.4580	0.8536	1.9751	0.8491	2.0599
1.0				1.0022	1.4970	0.8600	2.0745	0.8544	2.1777
0.09	0.1	1.5	21	0.0628	0.6340	0.0576	0.5817	0.0577	0.5823
0.49				0.1253	1.3119	0.1119	1.1748	0.1120	1.1762
0.79				0.1761	1.8898	0.1556	1.6822	0.1557	1.6828
0.95				0.2038	2.2151	0.1792	1.9678	0.1792	1.9675
1.0				0.2125	2.3184	0.1866	2.0587	0.1866	2.0581
0.09	0.1	1.0	61	0.0575	0.5807	0.0575	0.5807	0.0575	0.5807
0.49				0.0979	1.0167	0.1122	1.1781	0.1122	1.1780
0.79				0.1566	1.6804	0.1563	1.6902	0.1563	1.6900
0.95				0.1806	1.9629	0.1802	1.9790	0.1817	1.9970
1.0				0.1882	2.0526	0.1877	2.0709	0.1877	2.0707

TABLE 5: $f''(0)$ NON-SIMILAR BOUNDARY LAYER WITH UNIFORM BLOWING

ξ	h	σ	N	F. D.		HERMITE 4		SPLINE 4	
				$u(\xi,\Delta\eta)$	$f''(0,\xi)$	$u(\xi,\Delta\eta)$	$f''(0,\xi)$	$u(\xi,\Delta\eta)$	$f''(0,\xi)$
0.09	1.0	1.0	31	0.4004	0.4101	0.3859	0.3587	0.3865	0.3618
0.29				0.2429	0.1866	0.2192	0.1619	0.2185	0.1585
0.59				0.0367	-.0065	0.0128	0.0070	0.0127	0.0069
0.79								1.7×10^{-7}	8.8×10^{-9}
0.84									
0.09	0.1	1.03	81	0.0364	0.3610	0.0364	0.3607	0.0364	0.3607
0.29				0.0172	0.1672	0.0172	0.1670	0.0172	0.1670
0.59				6.8×10^{-4}	6.3×10^{-3}	6.7×10^{-4}	6.4×10^{-3}	6.7×10^{-4}	6.4×10^{-3}
0.79				4.3×10^{-9}	3.9×10^{-8}	5.5×10^{-9}	5.1×10^{-8}	5.5×10^{-9}	5.1×10^{-8}
0.84				2.7×10^{-11}	2.4×10^{-10}	4.4×10^{-11}	4.0×10^{-10}	4.4×10^{-11}	4.0×10^{-10}

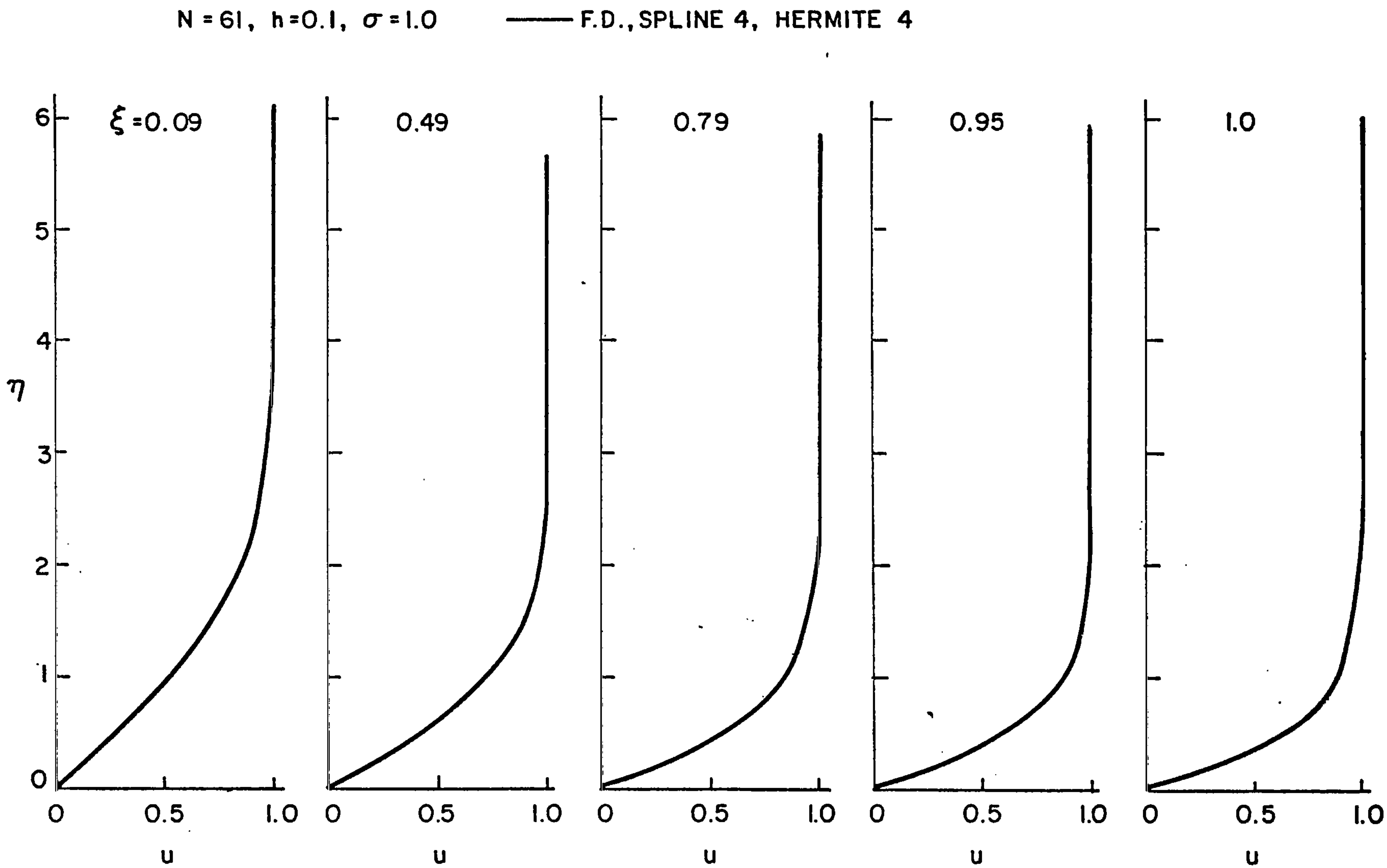

FIG. 3a VELOCITY PROFILES WITH UNIFORM SUCTION

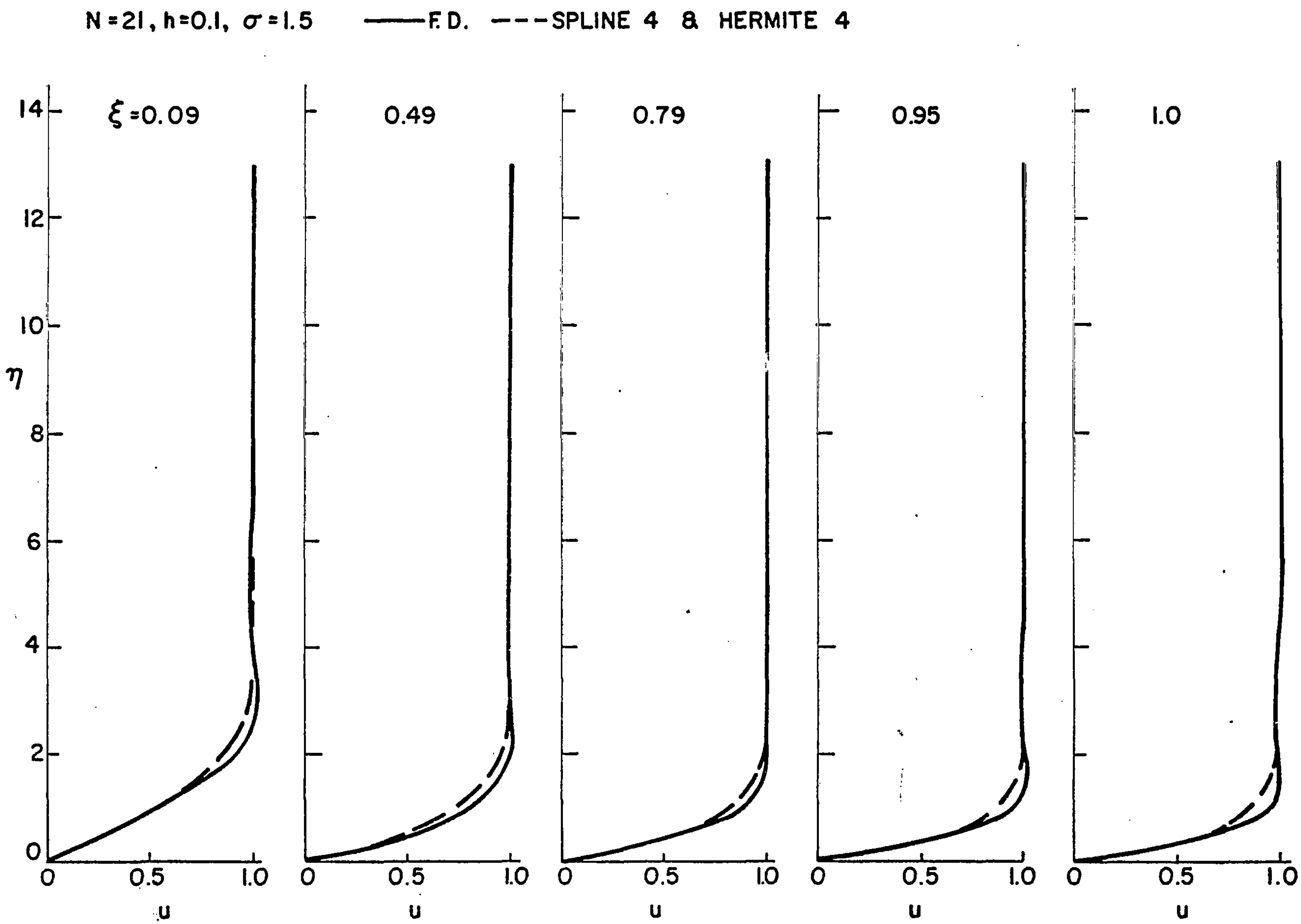

FIG. 3b VELOCITY PROFILES WITH UNIFORM SUCTION

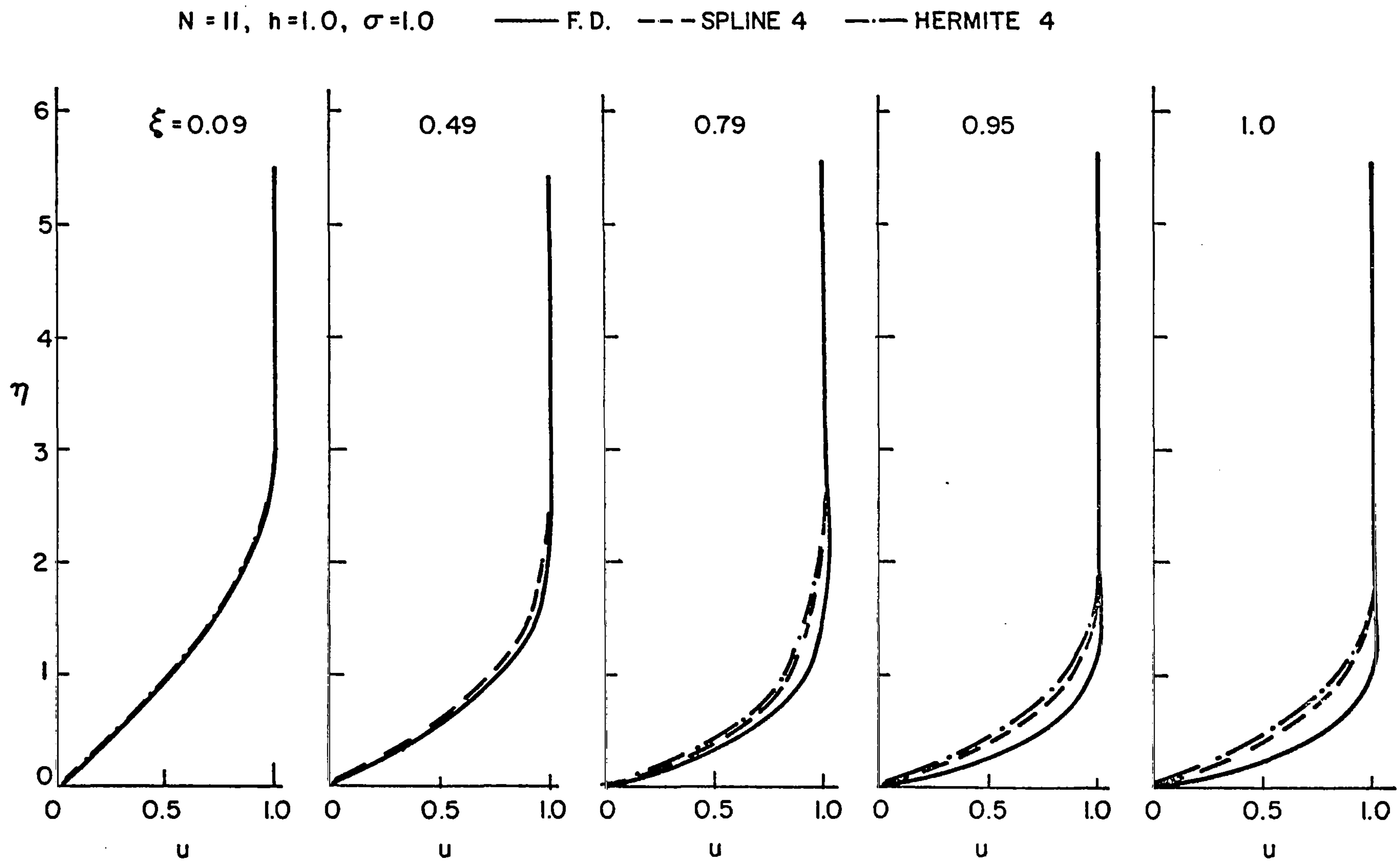

FIG. 3c VELOCITY PROFILES WITH UNIFORM SUCTION

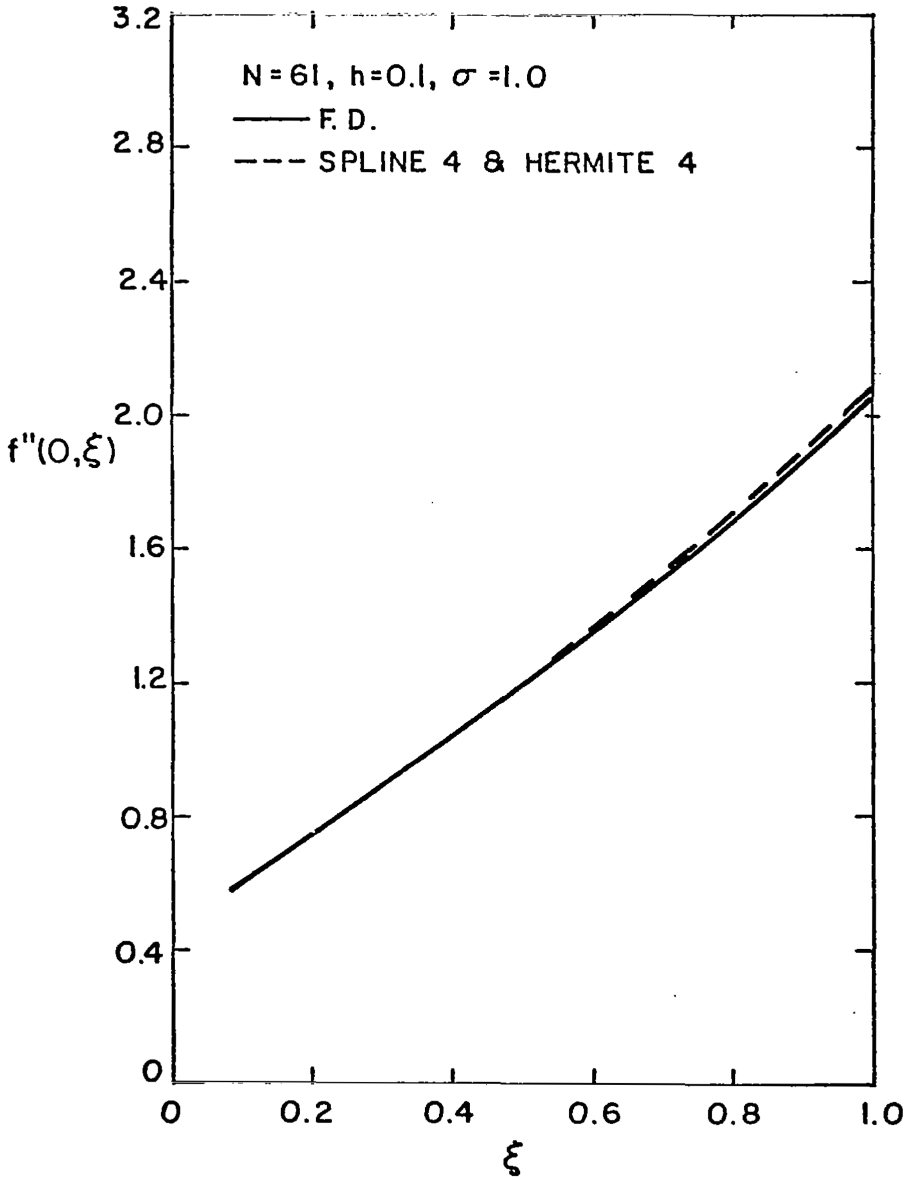

FIG. 4a SHEAR WITH UNIFORM SUCTION

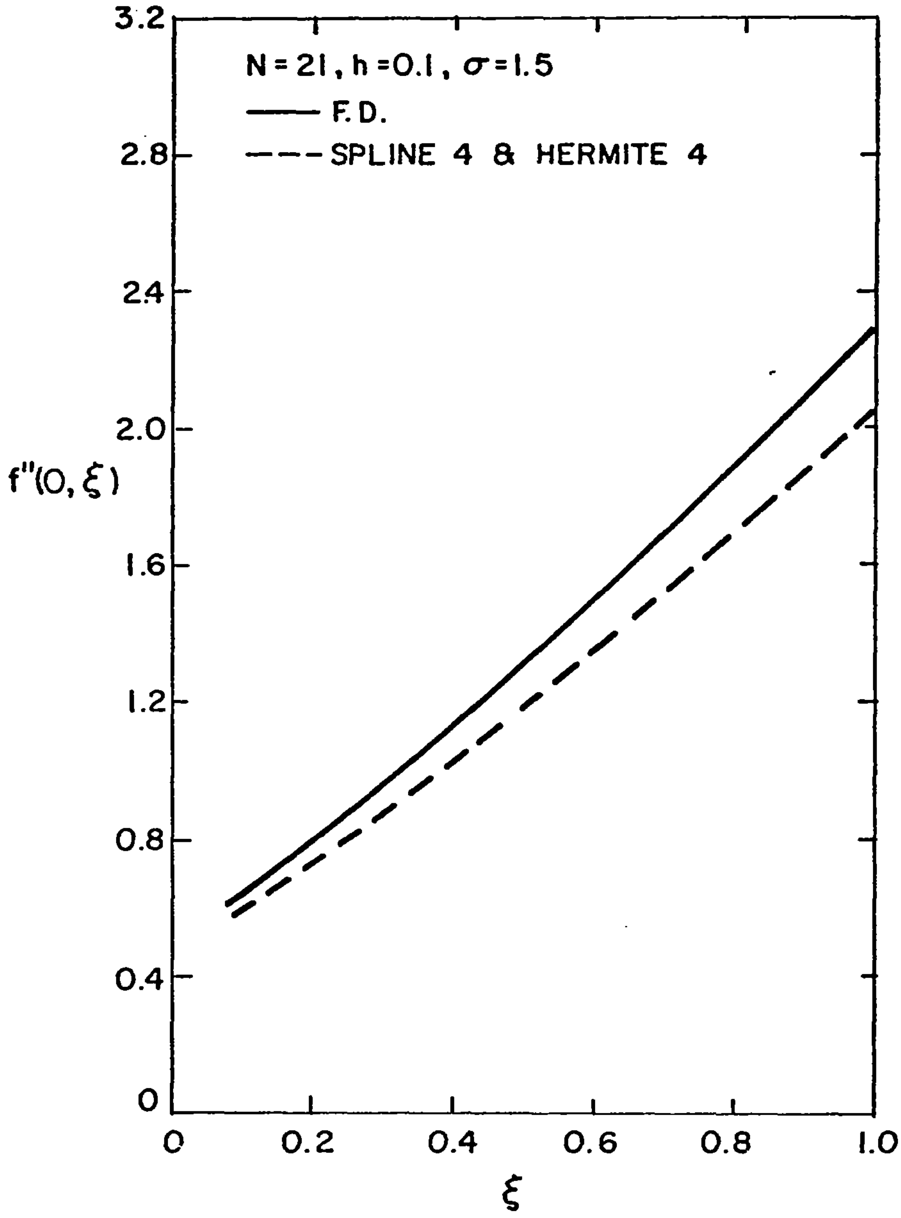

FIG. 4b SHEAR WITH UNIFORM SUCTION

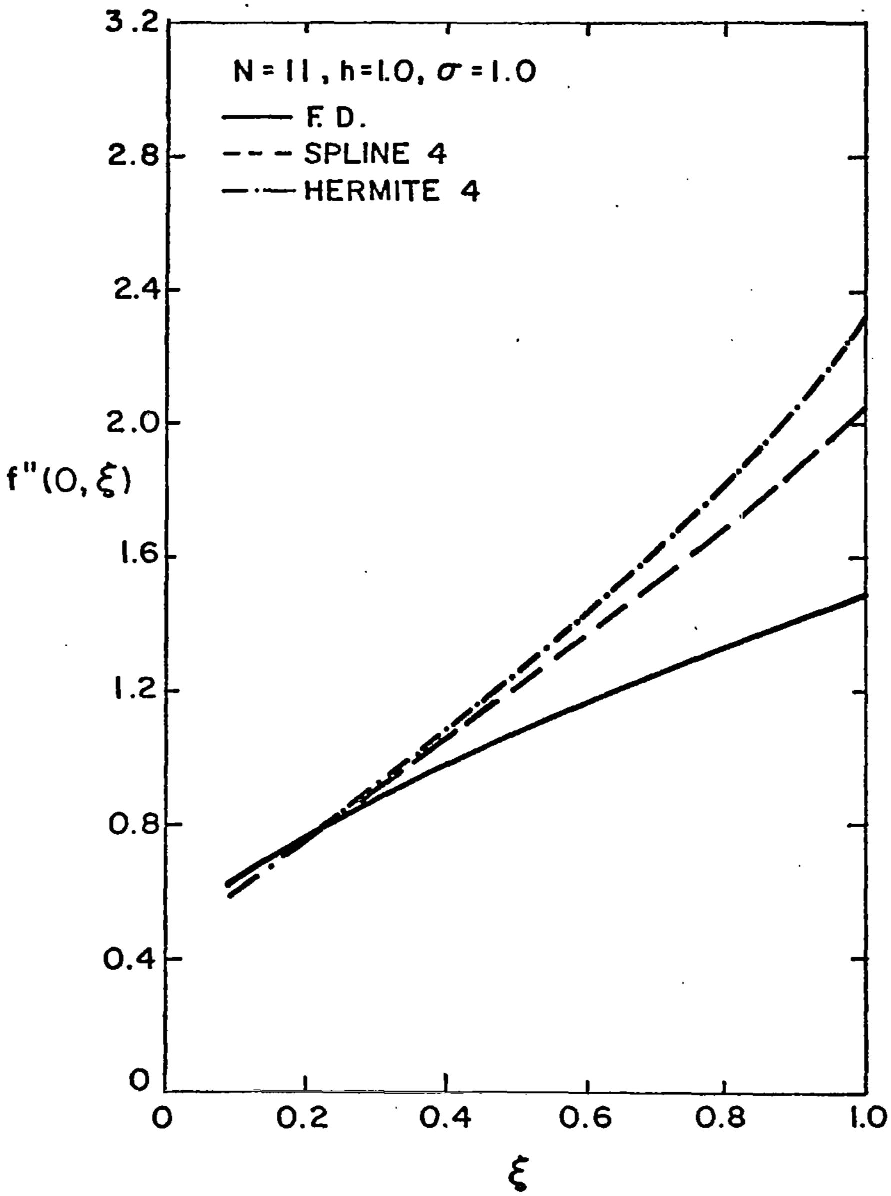

FIG. 4c SHEAR WITH UNIFORM SUCTION

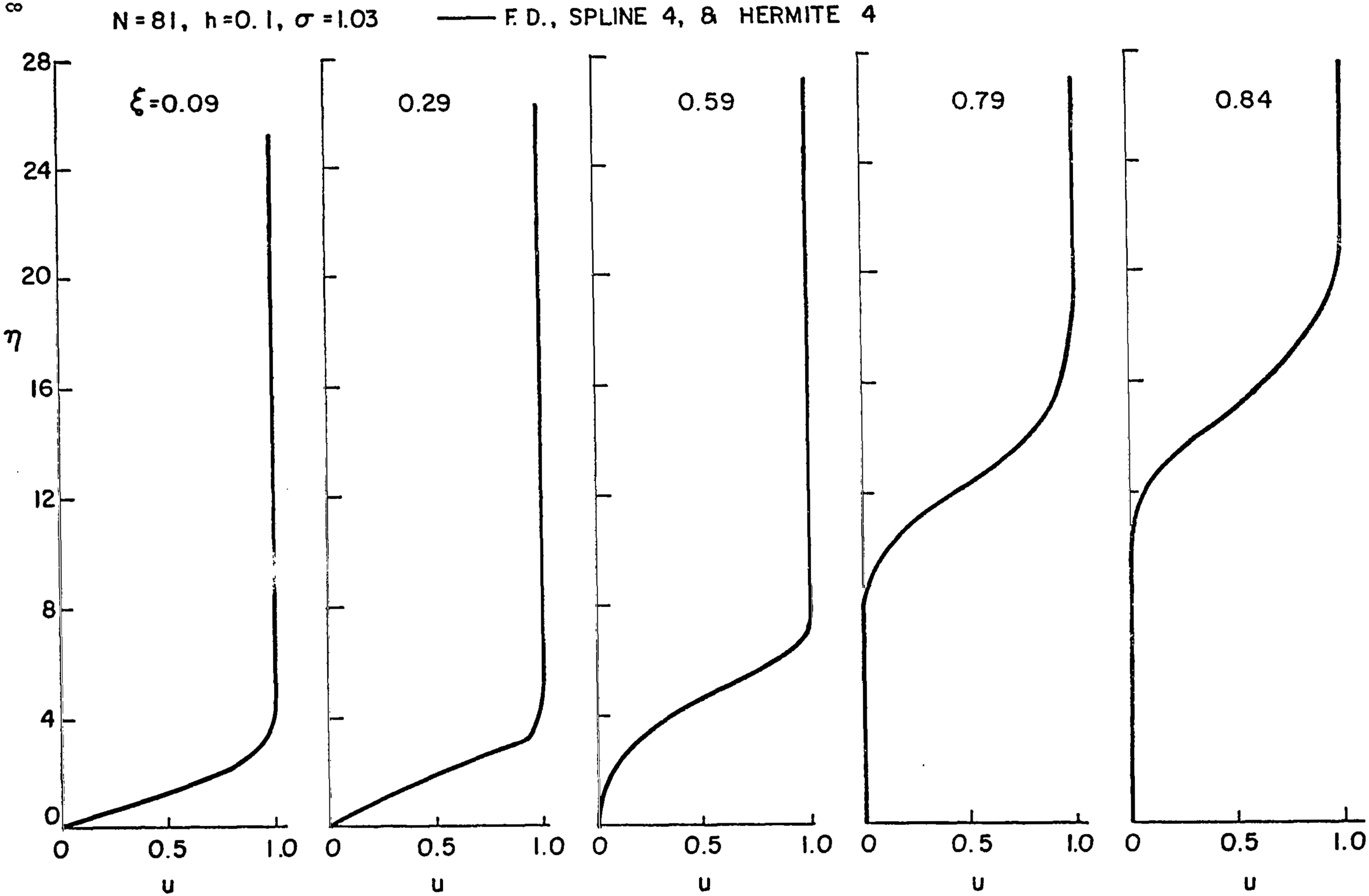

FIG. 5a VELOCITY PROFILES WITH UNIFORM BLOWING

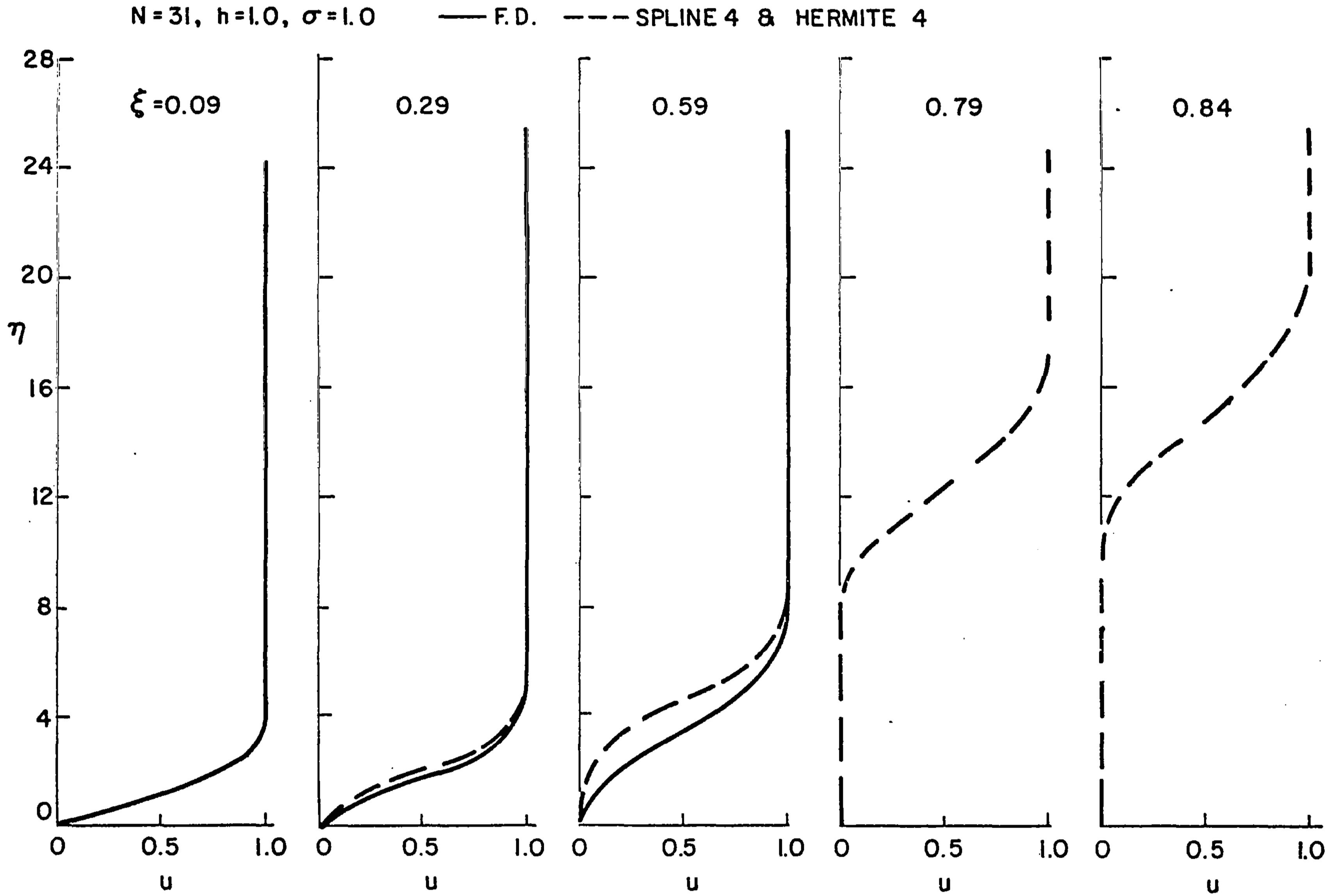

FIG.5b VELOCITY PROFILES WITH UNIFORM BLOWING

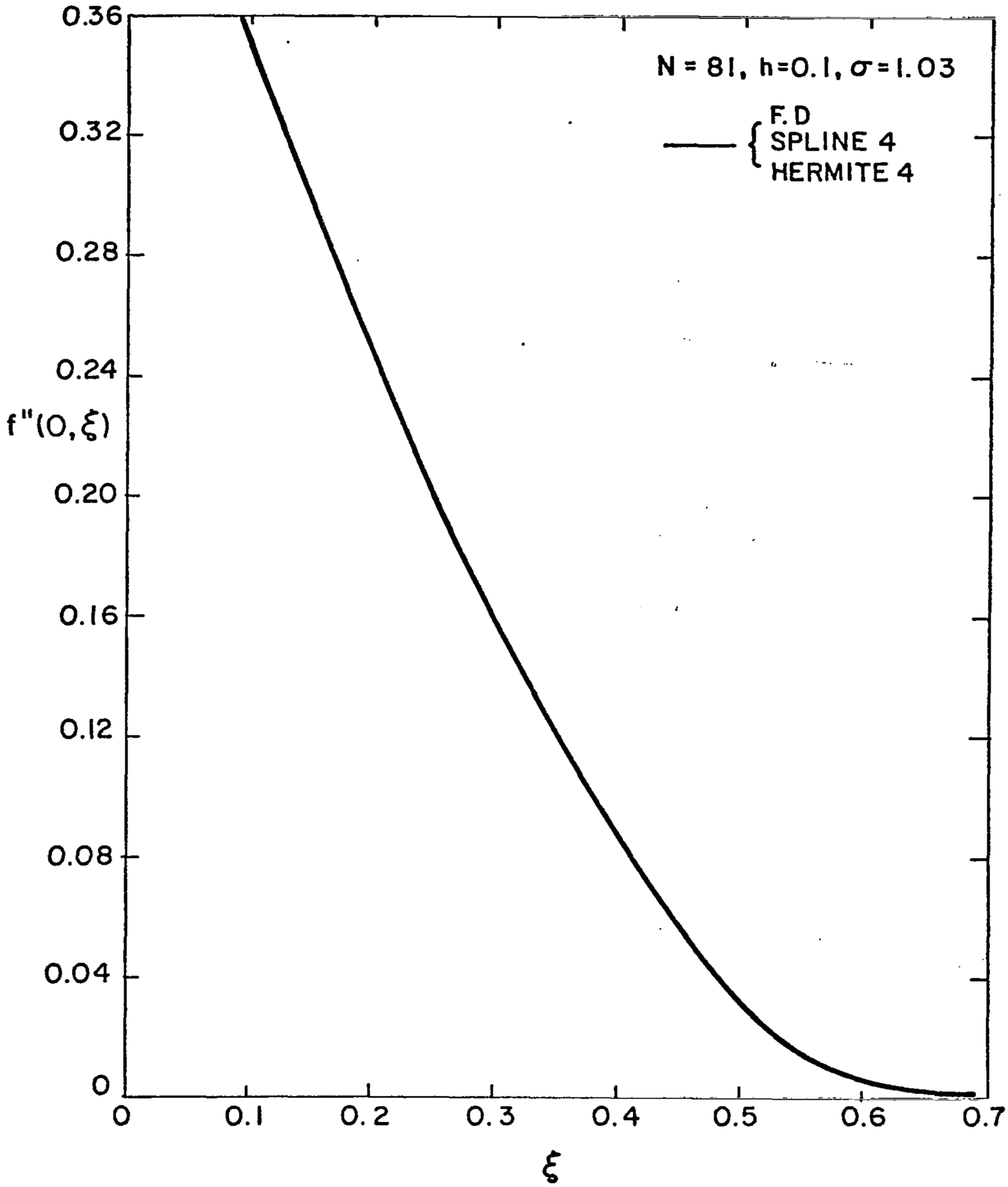

FIG. 6a SHEAR WITH UNIFORM BLOWING

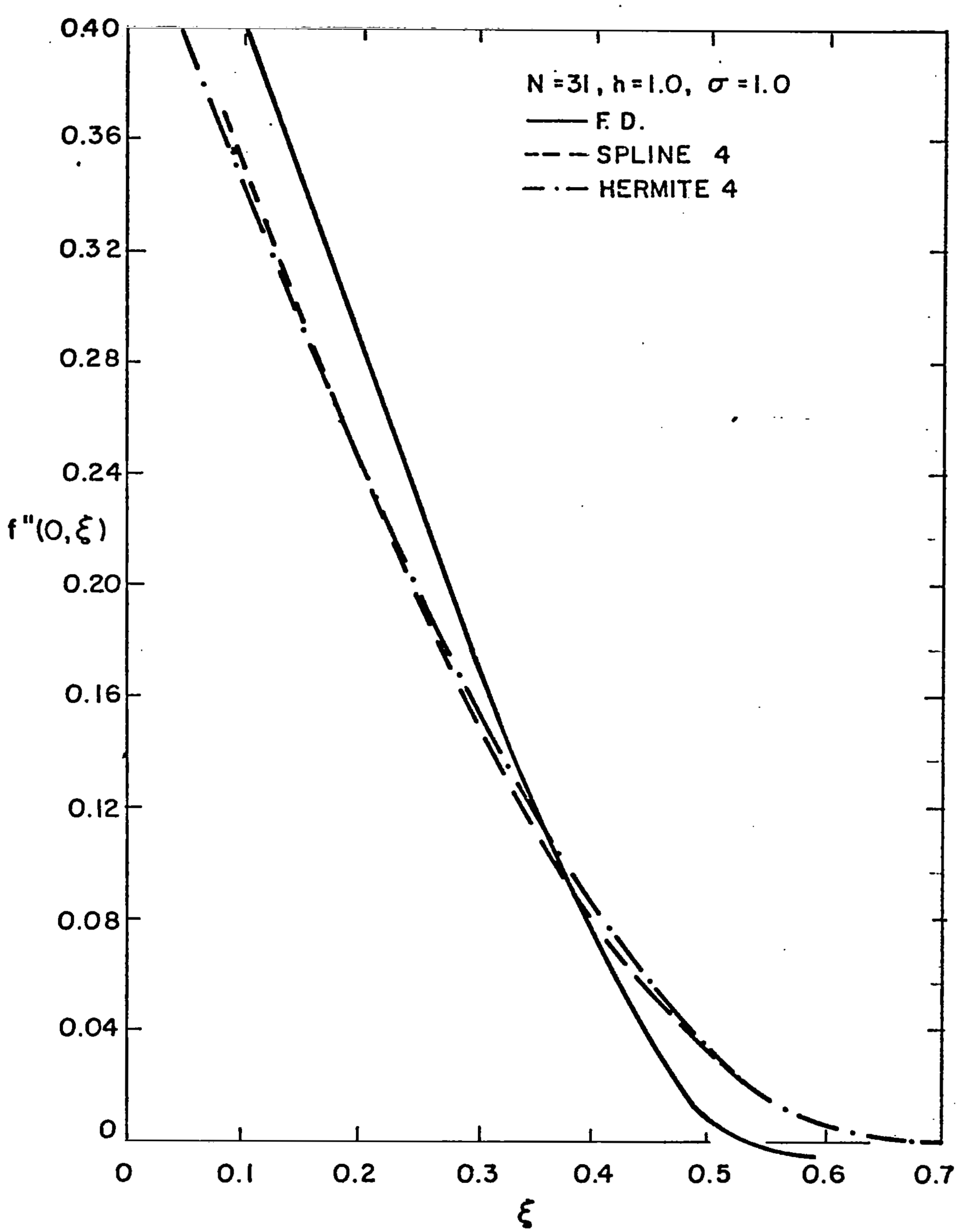

FIG. 6b SHEAR WITH UNIFORM BLOWING

solution (27) gives at $\xi = 1.0$

$$u(0.1) = 0.1812$$

or

$$u(1.0) = 0.8647$$

Once again the spline 4 results are best.

Detailed injection solutions are shown on Table 5. For the very fine grid, there was no indication of separation as inferred in Reference (14). This was true for all claculations. The shear decreased but never vanished. For the coarser grid the finite-difference solution did lead to a separation point, but the polynomial solutions still did not separate. This behavior is also depicted on Figure (6b). The conclusion obtained from these results would appear to be that separation does not occur with uniform injection; instead, the shear decreases asymptotically to zero for large values of ξ.

4. Burgers Equation - Since many results have been obtained in earlier studies[4,7,8] of the nonlinear Burgers equation, only a brief discussion of the higher-order solutions is presented here. The governing differential and spline equations are, respectively,

$$u_t + (u - 0.5)\, u_x = \nu u_{xx}$$

and

$$(u_j^{n+1} - u_j^n)/\Delta t + (u_j^n - 0.5)\, m_j = \nu M_j \tag{28}$$

The boundary conditions are

$$u_1 = 1.0 \; ; \quad u_N = 0.0$$

The boundary conditions on m_j, M_j are obtained from (28) and/or the spline derivative relations. More details on these boundary conditions are given in References (4), (7) and (8).

Typical solutions are shown on Tables 6 and 7 for $\nu = 1/8$ and $\nu = 1/16$, respectively. In both cases the fourth-order methods represent an improvement

TABLE 6:
SOLUTION OF BURGERS EQUATION, $\nu = 1/8$, $\sigma = 1.0$,
31 EQUALLY SPACED POINTS

X \ U	F. D.	SPLINE 2	HERMITE 4	SPLINE 4	EXACT
0	0.5000	0.5000	0.5000	0.5000	0.5000
-0.2	0.6999	0.6860	0.6903	0.6900	0.6900
-0.4	0.8447	0.8290	0.8322	0.8322	0.8320
-0.6	0.9269	0.9160	0.9169	0.9170	0.9170
-0.8	0.9673	0.9620	0.9608	0.9609	0.9610
-1.0	0.9857	0.9830	0.9820	0.9820	0.9820
-1.2	0.9938	0.9930	0.9918	0.9918	0.9920
-1.4	0.9973	0.9970	0.9963	0.9963	0.9960
-1.6	0.9988	0.9990	0.9983	0.9983	0.9980
-1.8	0.9995	0.9990	0.9993	0.9993	0.9990

TABLE 7:
SOLUTION OF BURGERS EQUATION, $\nu = 1/16$, $\sigma = 1.0$
19 EQUALLY SPACED POINTS

X \ U	F.D.	SPLINE 2	HERMITE 4	SPLINE 4	EXACT
0	0.5000	0.5000	0.5000	0.5000	0.5000
- 0.2	0.9000	0.8231	0.8383	0.8356	0.8320
- 0.4	0.9878	0.9641	0.9593	0.9617	0.9608
- 0.6	0.9986	0.9952	0.9922	0.9916	0.9918
- 0.8	0.9998	0.9995	0.9981	0.9982	0.9983
- 1.0	1.0	0.9999	0.9997	0.9996	0.9997
- 1.2	1.0	1.0	0.9999	0.9999	0.9999

over the second-order finite-difference and mixed-order spline 2 procedures; however, as with the previous examples, the spline 4 solutions are most accurate. Similar results were obtained with nonuniform grids[8]. As a general rule of thumb, it was found that the spline 4 solutions required one-quarter the number of mesh points of the finite-difference calculations in order to achieve equal accuracy; (e.g., with $\nu = 1/8$ and $x = -0.4$, $u = 0.8351$ with 101 points (finite-difference) and $u = 0.8356$ with 27 points (spline 4)). This fact was discussed earlier for the similar suction boundary layer and will be demonstrated in more detail in the following section describing the flow in a driven cavity.

5. Incompressible Flow in a Cavity - As a final test problem the laminar incompressible flow in a driven cavity is considered. This problem has been studied extensively by many investigators, see Reference (15). The governing equations in terms of a vorticity and stream-function system are:

$$\psi_{xx} + \psi_{yy} = \zeta \tag{29}$$

$$\zeta_t + u\zeta_x + v\zeta_y = \frac{1}{Re} (\zeta_{xx} + \zeta_{yy}) \tag{30}$$

where ψ is the stream function, ζ is the vorticity; $u = \psi_y$ and $v = -\psi_x$ are the velocities in the x- and y-directions, respectively. The boundary conditions and geometry are shown in Figure (7).

Solutions of (30) are obtained with a predictor-corrector procedure described in References (16)-(17). For the Poisson Equation (29) a modified version of Buneman's direct solver[18] is used. The spline approximations of (29, 30) in non-divergence form are

$$\begin{aligned} &L^{\psi}_{ij} + M^{\psi}_{ij} = \zeta_{ij} \\ &\frac{\zeta^{n+1}_{ij} - \zeta^{n}_{ij}}{\Delta t} + u_{ij} \, (m^{\zeta}_{ij})^{n+1} + v_{ij} \, (\ell^{\zeta}_{ij})^{n+1} \\ &\qquad = \frac{1}{Re} \, [L^{\zeta}_{ij} + M^{\zeta}_{ij}]^{n+1} \end{aligned} \tag{31}$$

where ℓ_{ij} and L_{ij} denote the polynomial approximations of $(\ )_y$ and $(\ )_{yy}$

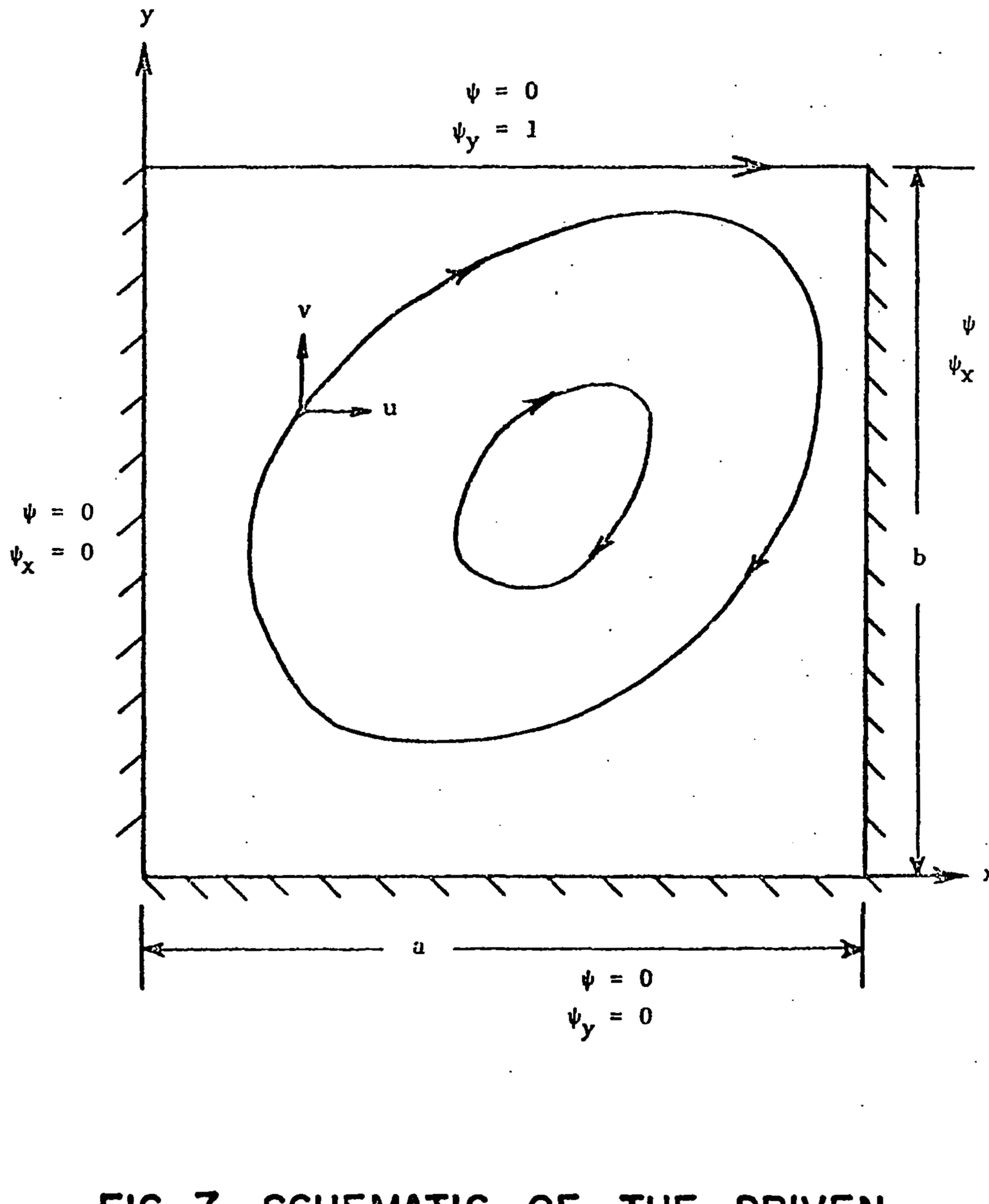

FIG. 7 SCHEMATIC OF THE DRIVEN CAVITY

respectively. The superscripts denote the specific function. The spline boundary conditions are obtained by satisfying (31) at the walls. The details of this procedure have been previously described in References (4), (7) and (8).

Solutions have been obtained for a square cavity with Re = 100. A 17x17 point mesh has been considered. The results are shown in Tables 8 through 12 and Figure (8). Table 8 compares the Hermite 4 and spline 4 solutions with results obtained in previous studies[7]. The maximum value of the stream function, the corner point velocity u and the vorticity at the mid-point of the upper moving wall are presented. It is significant that with a 17x17 mesh the spline 4 solutions parallel those obtained with a 57x57 point finite-difference discretization. Moreover, the spline 4 results are very close to the extrapolated values as projected for the non-divergence equations. This behavior carries over to the velocity profile shown on Figure (8). The complete computer solutions are given on Tables (9) through (12).

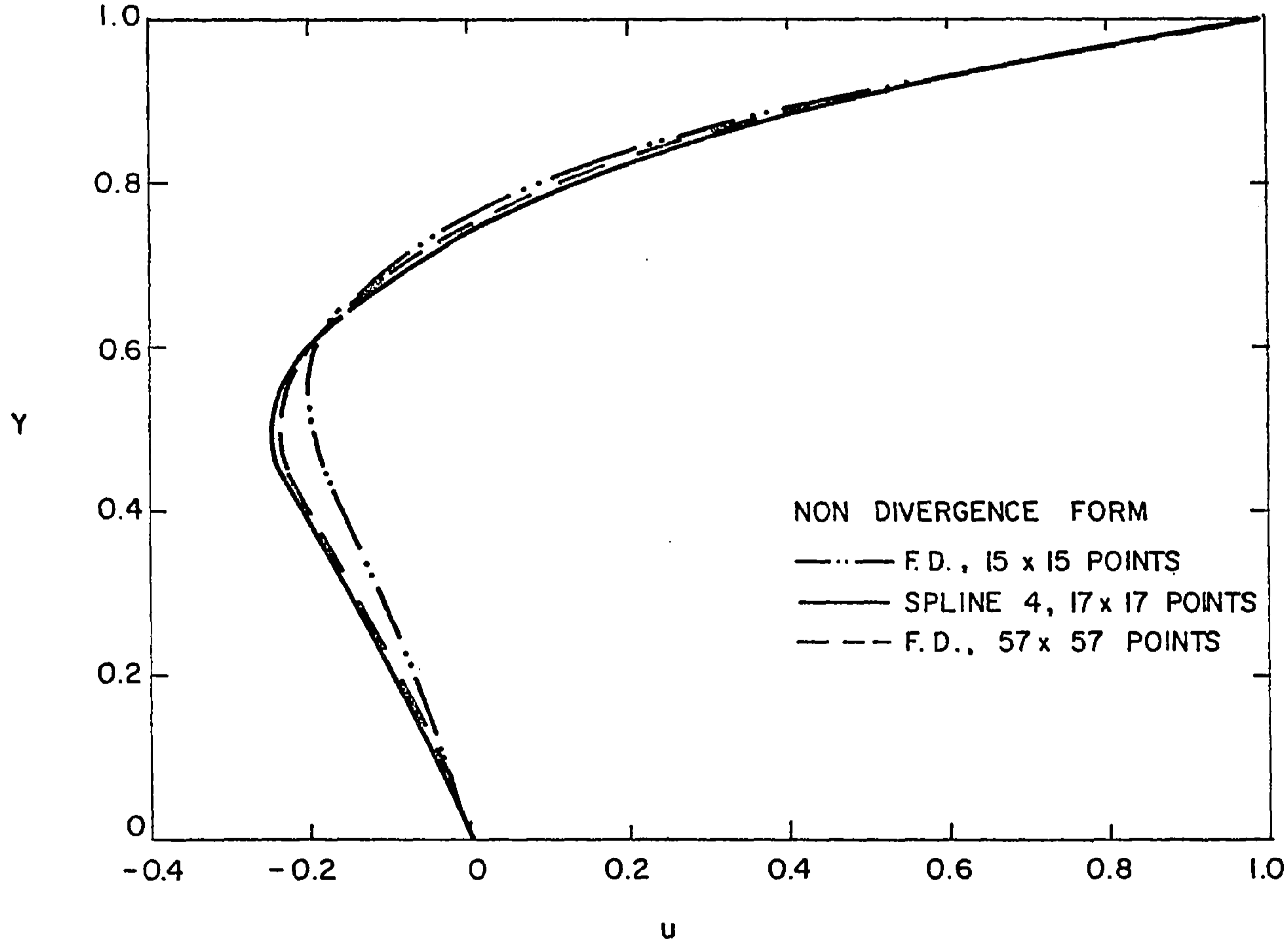

FIG. 8 COMPARISON OF CALCULATED VELOCITY u THROUGH POINT OF MAXIMUM ψ FOR $R_e = 100$

TABLE 8: COMPARISON OF RESULTS FOR THE SQUARE CAVITY, Re = 100

Calculation Method	Number of Points	Moving Surface Center Point Vorticity
Spline 2	15 x 15	7.1376
Spline 2	29 x 29	6.6876
Extrapolated Spline	———	6.5376
Finite Difference	15 x 15	8.9160
Finite Difference	57 x 57	6.6960
Extrapolated Finite Difference	———	6.5480
Finite Difference *	17 x 17	7.3755
Finite Difference *	65 x 65	6.6091
HERMITE 4	15 x 15	6.927
SPLINE 4	17 x 17	6.6104
* Divergence Form		

(a) Vorticity at the center of the moving surface.

Calculation Method	Number of Points	Maximum Stream Function
Spline 2	15 x 15	-.10529
Spline 2	29 x 29	-.10432
Extrapolated Spline	———	-.10399
Finite Difference	15 x 15	-.08742
Finite Difference	57 x 57	-.10128
Extrapolated Finite Difference	———	-.10220
Finite Difference *	17 x 17	-.09867
Finite Difference *	65 x 65	-.10318
HERMITE 4	15 x 15	-.1014
SPLINE 4	17 x 17	-.1023
* Divergence Form		

(b) Maximum stream function.

Calculation Method	Number of Points	u Velocity at Corner*
Spline 2	15 x 15	-.13230
Spline 2	29 x 29	-.10036
Extrapolated Spline	———	-.08971
Finite Difference	15 x 15	.05730
Finite Difference	57 x 57	-.06615
Extrapolated Finite Difference	———	-.07438
Finite Difference ** (Interpolated)	17 x 17	.02079
Finite Difference ** (Interpolated)	65 x 65	-.07560
HERMITE 4	15 x 15	-.118
SPLINE 4	17 x 17	-.086
** Divergence Form		

(c) Corner point u velocity.

TABLE 9

NON-DIVERGENCE FORM SPLINE 4 CALCULATED STREAM FUNCTION FOR 17 X 17 POINTS, Re = 100

Y \ X	.0625	.1250	.1815	.2500	.3125	.3750	.4375	.5000	.5625	.6250	.6875	.7500	.8125	.8750	.9375
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.0625	-.0000	-.0001	-.0004	-.0006	-.0009	-.0011	-.0013	-.0013	-.0013	-.0011	-.0008	-.0005	-.0002	-.0000	.0000
.1250	-.0001	-.0007	-.0015	-.0025	-.0034	-.0042	-.0047	-.0049	-.0047	-.0040	-.0031	-.0020	-.0010	-.0003	-.0000
.1815	-.0004	-.0017	-.0034	-.0054	-.0074	-.0090	-.0102	-.0106	-.0101	-.0088	-.0069	-.0047	-.0026	-.0010	-.0002
.2500	-.0008	-.0030	-.0061	-.0094	-.0127	-.0155	-.0174	-.0182	-.0175	-.0156	-.0124	-.0087	-.0050	-.0021	-.0004
.3125	-.0013	-.0047	-.0093	-.0143	-.0192	-.0234	-.0264	-.0277	-.0271	-.0244	-.0199	-.0143	-.0085	-.0038	-.0008
.3750	-.0019	-.0067	-.0130	-.0199	-.0266	-.0324	-.0368	-.0390	-.0387	-.0355	-.0297	-.0218	-.0134	-.0061	-.0015
.4375	-.0025	-.0087	-.0169	-.0258	-.0345	-.0422	-.0482	-.0517	-.0521	-.0488	-.0418	-.0316	-.0201	-.0095	-.0023
.5000	-.0031	-.0108	-.0208	-.0317	-.0424	-.0521	-.0598	-.0649	-.0664	-.0635	-.0558	-.0437	-.0287	-.0141	-.0036
.5625	-.0037	-.0128	-.0245	-.0372	-.0497	-.0611	-.0706	-.0773	-.0802	-.0782	-.0707	-.0573	-.0393	-.0201	-.0054
.6250	-.0043	-.0145	-.0274	-.0420	-.0558	-.0686	-.0793	-.0873	-.0916	-.0910	-.0844	-.0709	-.0509	-.0275	-.0076
.6875	-.0048	-.0162	-.0306	-.0457	-.0603	-.0736	-.0849	-.0936	-.0989	-.0997	-.0946	-.0824	-.0622	-.0358	-.0106
.7500	-.0055	-.0181	-.0332	-.0465	-.0628	-.0756	-.0865	-.0950	-.1006	-.1023	-.0989	-.0890	-.0708	-.0441	-.0143
.8125	-.0067	-.0207	-.0359	-.0502	-.0629	-.0738	-.0830	-.0902	-.0951	-.0969	-.0949	-.0878	-.0736	-.0500	-.0188
.8750	-.0102	-.0245	-.0379	-.0491	-.0584	-.0660	-.0721	-.0768	-.0800	-.0812	-.0801	-.0758	-.0669	-.0503	-.0245
.9375	-.0123	-.0249	-.0339	-.0396	-.0435	-.0463	-.0484	-.0499	-.0507	-.0509	-.0502	-.0482	-.0448	-.0365	-.0235
1.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

TABLE 10

NON-DIVERGENCE FORM SPLINE 4 CALCULATED u-VELOCITY FOR 17 X 17 POINTS, Re =100

Y \ X	.0625	.1250	.1815	.2500	.3125	.3750	.4375	.5000	.5625	.6250	.6875	.7500	.8125	.8750	.9375
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.0625	-.0019	-.0052	-.0119	-.0198	-.0277	-.0344	-.0389	-.0405	-.0384	-.0330	-.0250	-.0159	-.0078	-.0022	.0000
.1250	-.0033	-.0122	-.0247	-.0387	-.0524	-.0639	-.0717	-.0743	-.0710	-.0621	-.0487	-.0332	-.0186	-.0075	-.0014
.1815	-.0055	-.0189	-.0367	-.0562	-.0749	-.0907	-.1016	-.1060	-.1026	-.0916	-.0740	-.0526	-.0312	-.0139	-.0032
.2500	-.0074	-.0249	-.0474	-.0716	-.0950	-.1151	-.1300	-.1373	-.1356	-.1240	-.1031	-.0758	-.0467	-.0216	-.0053
.3125	-.0088	-.0294	-.0559	-.0842	-.1118	-.1365	-.1559	-.1678	-.1698	-.1598	-.1373	-.1044	-.0665	-.0318	-.0080
.3750	-.0097	-.0324	-.0614	-.0927	-.1236	-.1522	-.1765	-.1940	-.2017	-.1963	-.1752	-.1387	-.0919	-.0454	-.0118
.4375	-.0100	-.0334	-.0634	-.0957	-.1281	-.1591	-.1670	-.2098	-.2244	-.2265	-.2112	-.1756	-.1223	-.0633	-.0170
.5000	-.0096	-.0323	-.0613	-.0924	-.1235	-.1537	-.1823	-.2081	-.2287	-.2399	-.2352	-.2076	-.1543	-.0848	-.0238
.5625	-.0088	-.0298	-.0558	-.0828	-.1090	-.1343	-.1590	-.1834	-.2065	-.2256	-.2345	-.2229	-.1803	-.1077	-.0321
.6250	-.0083	-.0272	-.0486	-.0584	-.0657	-.1014	-.1171	-.1344	-.1546	-.1771	-.1981	-.2079	-.1889	-.1274	-.0415
.6875	-.0089	-.0272	-.0427	-.0521	-.0562	-.0577	-.0594	-.0645	-.0757	-.0950	-.1223	-.1520	-.1666	-.1363	-.0521
.7500	-.0155	-.0343	-.0416	-.0362	-.0224	-.0050	.0114	.0230	.0259	.0163	-.0093	-.0512	-.1002	-.1219	-.0669
.8125	-.0278	-.0510	-.0448	-.0155	.0233	.0647	.1032	.1347	.1549	.1591	.1415	.0965	.0196	-.0620	-.0842
.8750	-.0669	-.0690	-.0225	.0482	.1227	.1927	.2541	.3039	.3389	.3550	.3456	.3036	.2125	.0845	-.0688
.9375	-.1190	.0611	.2149	.3323	.4243	.4972	.5546	.5979	.6261	.6364	.6248	.5824	.4992	.3291	.0563
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE II

NON-DIVERGENCE FORM SPLINE 4 CALCULATED V-VELOCITY FOR 17 X 17 POINTS, Re =100

Y \ X	.0625	.1250	.1815	.2500	.3125	.3750	.4375	.5000	.5625	.6250	.6875	.7500	.8125	.8750	.9375
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.0625	.0010	.0028	.0040	.0043	.0040	.0031	.0018	-.0001	-.0020	-.0038	-.0048	-.0048	-.0037	-.0018	-.0000
.1250	.0054	.0112	.0146	.0155	.0142	.0110	.0060	-.0003	-.0071	-.0131	-.0168	-.0171	-.0138	-.0081	-.0021
.1815	.0137	.0248	.0310	.0324	.0295	.0228	.0126	-.0003	-.0140	-.0263	-.0342	-.0358	-.0303	-.0195	-.0072
.2500	.0256	.0433	.0525	.0542	.0494	.0385	.0221	.0014	-.0211	-.0420	-.0567	-.0613	-.0542	-.0371	-.0160
.3125	.0403	.0659	.0785	.0805	.0737	.0586	.0358	.0064	-.0266	-.0586	-.0833	-.0941	-.0868	-.0627	-.0295
.3750	.0572	.0916	.1079	.1106	.1021	.0833	.0545	.0165	-.0279	-.0736	-.1124	-.1340	-.1299	-.0986	-.0492
.4375	.0751	.1189	.1393	.1428	.1333	.1117	.0782	.0328	-.0227	-.0835	-.1402	-.1788	-.1837	-.1472	-.0776
.5000	.0930	.1463	.1706	.1751	.1649	.1417	.1052	.0546	-.0097	-.0844	-.1611	-.2229	-.2459	-.2100	-.1172
.5625	.1099	.1720	.1995	.2044	.1935	.1694	.1317	.0740	.0100	-.0744	-.1689	-.2575	-.3093	-.2854	-.1703
.6250	.1255	.1949	.2237	.2273	.2149	.1900	.1525	.1008	.0323	-.0548	-.1596	-.2722	-.3612	-.3671	-.2378
.6875	.1407	.2144	.2407	.2402	.2246	.1986	.1626	.1145	.0512	-.0308	-.1348	-.2597	-.3853	-.4412	-.3176
.7500	.1598	.2307	.2476	.2386	.2180	.1908	.1571	.1148	.0606	-.0097	-.1015	-.2204	-.3665	-.4846	-.4041
.8125	.1863	.2445	.2394	.2163	.1895	.1617	.1319	.0976	.0555	.0018	-.0689	-.1639	-.3001	-.4673	-.4790
.8750	.2370	.2299	.1969	.1632	.1342	.1096	.0868	.0632	.0360	.0026	-.0409	-.0996	-.1905	-.3520	-.5180
.9375	.2563	.1772	.1116	.0747	.0523	.0385	.0283	.0188	.0084	-.0039	-.0199	-.0418	-.0783	-.1630	-.4089
1.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

TABLE 12

NON-DIVERGENCE FORM SPLINE 4 CALCULATED VORTICITY FOR 17 X 17 POINTS, Re = 100

Y \ X	.0625	.1250	.1815	.2500	.3125	.3750	.4375	.5000	.5625	.6250	.6875	.7500	.8125	.8750	.9375
0	.0058	-.0491	-.1737	-.3198	-.4765	-.6038	-.7017	-.7310	-.6990	-.5850	-.4260	-.2382	-.0839	.0133	.0214
.0625	-.0547	-.1315	-.2106	-.3135	-.4041	-.4899	-.5344	-.5541	-.5217	-.4658	-.3723	-.2783	-.1779	-.1028	-.0313
.1250	-.1394	-.1829	-.2375	-.2851	-.3443	-.3777	-.4088	-.4070	-.4057	-.3786	-.3566	-.3109	-.2649	-.1906	-.1046
.1815	-.2375	-.2435	-.2351	-.2561	-.2602	-.2783	-.2746	-.2908	-.2959	-.3285	-.3471	-.3722	-.3549	-.3090	-.1976
.2500	-.3826	-.2945	-.2433	-.1948	-.1769	-.1448	-.1438	-.1413	-.1898	-.2491	-.3497	-.4275	-.4852	-.4576	-.3583
.3125	-.5420	-.3657	-.2214	-.1359	-.0509	-.0035	.0468	.0371	-.0016	-.1256	-.2813	-.4775	-.6164	-.6736	-.5792
.3750	-.7395	-.4223	-.2112	-.0389	.0776	.2000	.2714	.3229	.2743	.1496	-.1082	-.4205	-.7436	-.9224	-.9088
.4375	-.9304	-.5012	-.1740	.0472	.2537	.4156	.5840	.6795	.7152	.5853	.2954	-.1995	-.7533	-1.2067	-1.3347
.5000	-1.1440	-.5622	-.1600	.1636	.4145	.6806	.9086	1.1376	1.2469	1.2297	.9319	.3298	-.5639	-1.4237	-1.8989
.5625	-1.3322	-.6490	-.1270	.2469	.5961	.9080	1.2533	1.5568	1.8320	1.9399	1.7964	1.1685	-.0003	-1.4826	-2.5515
.6250	-1.5369	-.7163	-.1288	.3423	.7228	1.1236	1.5061	1.9252	2.2931	2.6021	2.6644	2.2633	.9822	-1.1582	-3.2686
.6875	-1.7226	-.8007	-.1045	.3980	.8515	1.2579	1.7053	2.1406	2.6123	3.0316	3.3650	3.3212	2.3520	-.3030	-3.8875
.7500	-1.9648	-.8244	-.0534	.5289	.9656	1.4387	1.8645	2.3362	2.8008	3.3064	3.7703	4.1207	3.7937	1.3153	-4.1298
.8125	-2.1339	-.6050	.3227	.9488	1.4679	1.9000	2.3353	2.7409	3.1831	3.6175	4.1003	4.5857	4.9630	3.6245	-3.1817
.8750	-1.9409	.7402	1.9733	2.7138	3.1692	3.4497	3.7283	3.9464	4.1378	4.3667	4.6368	5.0262	5.7403	6.1700	.5152
.9375	3.8289	7.7369	8.2772	8.0023	7.4688	6.8898	6.3867	5.9522	5.6486	5.4494	5.4888	5.7386	6.7041	8.2484	8.2117
1.0	40.0252	27.1943	18.7167	13.8511	10.8360	8.6614	7.4909	6.6104	6.1568	6.2505	6.8945	8.5219	10.9989	17.9490	28.9730

V. HIGHER-ORDER TEMPORAL INTEGRATION

From the preceding sections it is evident that higher-order spatial accuracy can be achieved by using polynomial interpolation. However, in these calculations, the temporal accuracy is quite low; either first-order for a fully implicit scheme or second-order for the Crank-Nicholson method. In order to achieve over all fourth or higher-order accuracy, the temporal accuracy should also be improved. Various multi-step methods in conjunction with 5 point finite-differences for space dimensions have been proposed in the literature[19,20]. These schemes are explicit and have severe stability restriction on the allowable time-step.

Some of these explicit methods can be directly extended for the present spline collocation procedures. However, there are many drawbacks of such higher-order time integration schemes when applied in conjunction with the spatial spline procedures; e.g., (1) the stability restriction on the allowable time-step is even more restrictive than with finite-differences; (2) due to the large number of spline curve fits required at each time step, the overall computational times increase. For example, for a fourth-order Runge-Kutta method, at least four curve fits of each function per time-step must be evaluated and stored. In view of these restrictions only implicit higher-order methods are investigated in the present study. Some of the resulting methods can be found in the literature on numerical studies of ordinary differential equations[21]. One such formulation has been recently described by Watanabe and Flood[22] and leads to an overall fourth-order unconditionally stable scheme. For the following analysis polynomial interpolation is used to derive several time integration procedures of third and fourth-order accuracy. Although no attempt has been made here to extend the procedure to higher-orders, the method of derivation is quite general. The utility of these higher-order temporal methods is counterbalanced by their sensitivity to initial conditions, possible "stability" limitations and increased computer time and storage.

A. Polynomial Interpolation

1) Crank-Nicholson Scheme - The following interpolation polynomial $S(\tau)$, on the interval $t_{i-1} \leq t \leq t_i$, satisfies the three conditions given earlier as (2) and (3b).

$$S(\tau) = u_i\tau + u_{i-1}\ (1-\tau) + (u_i-u_{i-1}-\Delta t\ m_i)\ \tau\ (1-\tau)$$

where

$$\tau = \frac{t-t_{i-1}}{\Delta t}\ , \qquad m_i = \left(\frac{\partial u}{\partial t}\right)_{t=t_i}$$

and

$$\Delta t = t_i - t_{i-1}$$

Differentiating, we obtain

$$\Delta t\ m_{i-1} = S'(\tau{=}0) = 2(u_i - u_{i-1}) - \Delta t\ \ m_i$$

or

$$\frac{u_i - u_{i-1}}{\Delta t} = \frac{1}{2}\ (m_i + m_{i-1})$$

This is the well-known Crank-Nicholson scheme which is second-order accurate and is unconditionally stable[23].

ii) Third-Order Accurate Method - The following cubic polynomial satisfies (2) and (3):

$$S(\tau) = u_{i-1} + \tau\ m_{i-1} + \tau^2\ [\Delta t\ \ (m_i + m_{i-1}) - 2(u_i - u_{i-1})] + \tau^3\ [3(u_i - u_{i-1}) - \Delta t\ \ (m_i + 2m_{i-1})] \tag{32}$$

As specified, the polynomial is completely determined by the values of the function and its first derivative at two points i-1 and i; considering (32) on $[t_{i-1}, t_{i+1}]$, we obtain at t_{i+1}

$$\begin{aligned} u_{i+1} &= 5u_{i-1} + 2\Delta t\ m_{i-1} - 4u_i + 4\Delta t\ m_i \\ m_{i+1} &= 12u_{i-1} + 5\Delta t\ m_{i-1} - 12u_i + \Delta t\ m_i \end{aligned} \tag{33}$$

Eliminating u_{i-1}, we get

$$\frac{u_{i+1} - u_i}{\Delta t} = \frac{1}{12}\ \ (5m_{i+1} + 8m_i - m_{i-1}) \tag{34}$$

From (33) other procedures can be developed; e.g., if we eliminate u_i, we obtain

$$\frac{u_{i+1} - u_{i-1}}{\Delta t} = \frac{1}{3}(m_{i-1} + 4m_i + m_{i+1}) \tag{35}$$

Equations (34) and (35) are typical of the Adams-Bashforth methods for time integration (see Reference 21). It should be noted that both of these procedures require one more storage location than the Crank-Nicholson method. The stability of these and other such schemes is discussed in Reference (21). Unlike the C-N method, (34) and (35) are conditionally stable. The temporal accuracy of both is $O(\Delta t^3)$. The coefficients in the polynomial (32) can also be evaluated to include the values at three node points (i-1, i, i+1). The results are unchanged.

3.) Fourth-Order Method - If the cubic polynomial (32) is considered on [i-1, i] and we evaluate the function and the first derivative at the midpoint $t_{i-\frac{1}{2}} = (t_{i-1} + t_i)/2$, we find that

$$\frac{u_i - u_{i-1}}{\Delta t} = \frac{1}{6}(m_{i-1} + 4m_{i-\frac{1}{2}} + m_i) \tag{36}$$

and

$$u_{i-\frac{1}{2}} = \frac{u_i + u_{i-1}}{2} - \frac{\Delta t}{8}(m_i - m_{i-1}) \tag{37}$$

Equations (36) and (37) constitute a two-step procedure, which has recently been suggested by Watanabe and Flood[22]. Once again the increase in storage is minimal and only one additional curve fit is required at the half time step. The method has been shown to be unconditionally stable and of fourth-order accuracy (see Reference 22). Unfortunately, there are convergence restrictions associated with the required iterative method of solution.

For the simplest iterative procedure, where $m_{i-\frac{1}{2}}$ is treated explicitly, it has been shown in Reference (22) that the solution converges only for $\Delta t = O(\Delta x^2)$. This is quite restrictive. By utilizing specialized iterative procedures, the method has been shown to converge for $\Delta t = O(\Delta x)$. In the present investigation a novel iterative procedure has been developed from the polynomial interpolation formula (32). If we add and subtract $2(m_i + m_{i-1})$ to (36) and treat $[m_{i-\frac{1}{2}} - (m_i + m_{i-1})/2]$ explicitly, the procedure also converges for $\Delta t = O(\Delta x)$. The use of this factor is suggested by the following relation, obtained from (32):

$$m_{i-\frac{1}{2}} = \frac{m_i + m_{i-1}}{2} - \frac{\Delta t}{8}(M_i - M_{i-1}) \tag{38}$$

where

$$M_i = (u_{tt})_{t=t_i}$$

The present iterative procedure is equivalent to an explicit treatment of the second derivatives in (38) and, therefore, has improved convergence and "stability" properties.

B. Quadrature Methods

An alternative approach to the derivation of these methods is through the use of quadratures. Let

$$u_t = f(\vec{x}, \vec{u}, \nabla\vec{u}, \nabla^2\vec{u})$$

represent a general partial differential equation. Integrating over the interval $[t_{i-1}, t_i]$, we obtain

$$u_i - u_{i-1} = \Delta t \int_0^1 f \, d\tau \tag{39}$$

Polynomial interpolation is now used for f.

1) Linear:

$$f = f_i \tau + f_{i-1}(1-\tau)$$

Equation (39) becomes

$$\frac{u_i - u_{i-1}}{\Delta t} = \frac{f_i + f_{i-1}}{2}$$

which is the C-N formula.

2) Quadratic - $f = f_i\,\tau + f_{i-1}\,(1-\tau) + [f_i - f_{i-1} - \Delta t\,(f_t)_i]\,\tau\,(1-\tau)$
where $(f_t)_i$ is evaluated from 3 point extrapolation as:

$$(f_t)_i = \frac{3f_i - 4f_{i-1} + f_{i-2}}{2\Delta t}$$

(39) now becomes

$$\frac{u_i - u_{i-1}}{\Delta t} = \frac{1}{12}\,(5f_i + 8f_{i-1} - f_{i-2})$$

which is the Adams-Bashforth formula.

3) Quadratic:

$$f = f_i\,\tau + f_{i-1}\,(1-\tau) + C\,\tau(1-\tau) \tag{40}$$

Therefore,

$$\Delta t\ mf_i = (f_t)_i = f_i - f_{i-1} - C$$

$$\Delta t\ mf_{i-1} = (f_t)_{i-1} = f_i - f_{i-1} + C, \text{ where}$$

C is given by

$$f_{i-\frac{1}{2}} = \frac{f_i + f_{i-1}}{2} + \frac{C}{4}$$

With (40), Equation (39) becomes

$$\frac{u_i - u_{i-1}}{\Delta t} = \frac{1}{6}\,(f_{i-1} + 4f_{i-\frac{1}{2}} + f_i) \tag{41a}$$

and

$$f_{i-\frac{1}{2}} = \frac{f_i + f_{i-1}}{2} - \frac{\Delta t}{8}\,(mf_i - mf_{i-1}) \tag{41b}$$

Since $f_i = (u_t)_i = m_i$ and $mf_i = (u_{tt})_i = M_i$, Equation (41-a) is exactly the one given in (38). Also (41-b) can be written as:

$$\frac{\partial}{\partial t}\left[u_{i-\frac{1}{2}} - \frac{u_i + u_{i-1}}{2} + \frac{\Delta t}{8}\,(f_i - f_{i-1})\right] = 0$$

or

$$u_{i-\frac{1}{2}} = \frac{u_i + u_{i-1}}{2} - \frac{\Delta t}{8}\,(f_i - f_{i-1}) \tag{42}$$

(41-a) and (42) are the fourth-order expressions given by Watanabe and Flood and derived earlier.

As model problems we have considered the diffusion equation describing the impulsive motion of a flat plate (Rayleigh) and the nonlinear Burgers equation discussed earlier. Transient and large time solutions have been obtained. The Rayleigh problem is described by the following equations

$$u_t = \nu u_{yy} \quad , \qquad u = u(t,y)$$

with boundary conditions

$$u(t,0) = 1 \quad , \qquad u(t,\infty) = 0$$

and initial condition

$$u(0,y) = 0$$

The exact solution is

$$u(t,y) = \text{erfc} \ \frac{y}{2\sqrt{\nu t}}$$

Typical solutions for the Rayleigh problem are presented in Tables 13 through 17. Due to the large gradients found for small times, the higher-order methods do not lead to a significant improvement in accuracy; in fact, at certain times, the lower-order solutions are in better agreement with the analytic values. If the exact values are used as initial conditions at $t = 1.0$, the large gradients are avoided and the results are shown on Tables 14, 15 and 16. For small Δt, it is difficult to discern any difference with the various methods. For larger Δt values, some improvement can be observed; however, as Δt is further increased the higher-order methods exhibit some form of instability. For the third-order method this is apparently the numerical instability discussed earlier. For the fourth-order method, the difficulty is associated with the iterative method of solution. Finally, on Table 17 large time solutions are shown. No significant conclusion can be drawn with respect to the advantage of the higher-order methods. For the Burgers equation similar behavior was found; moreover, there was no significant reduction in the number of steps required to attain the steady state. Generally speaking, for the examples considered here, there was

TABLE 13: RAYLEIGH PROBLEM t =0, Δt = 0.1

t	y	C - N	3rd ORDER	EXACT
0.2	0.	.10000E+01	.10000E+01	.10000E+01
	.10000E+00	.49480E+00	.53552E+00	.52709E+00
	.20000E+00	.14655E+00	.15476E+00	.20590E+00
	.30000E+00	.33927E-01	.33234E-01	.57780E-01
	.40000E+00	.76978E-02	.72006E-02	.11412E-01
	.50000E+00	.15759E-02	.13609E-02	.15654E-02
	.60000E+00	.32543E-03	.26803E-03	.14806E-03
	.70000E+00	.63459E-04	.48405E-04	.96707E-05
	.80000E+00	.12481E-04	.90825E-05	.44236E-06
	.90000E+00	.23711E-05	.16062E-05	.14596E-07
	.10000E+01	.45282E-06	.29281E-06	.36336E-09
0.3	0.	.10000E+01	.10000E+01	.10000E+01
	.10000E+00	.58130E+00	.57145E+00	.60558E+00
	.20000E+00	.27156E+00	.29442E+00	.30170E+00
	.30000E+00	.92667E-01	.99480E-01	.12134E+00
	.40000E+00	.26355E-01	.26415E-01	.38867E-01
	.50000E+00	.68659E-02	.64722E-02	.98231E-02
	.60000E+00	.16473E-02	.14198E-02	.19457E-02
	.70000E+00	.38028E-03	.30463E-03	.30083E-03
	.80000E+00	.83881E-04	.61232E-04	.36295E-04
	.90000E+00	.18100E-04	.12236E-04	.34330E-05
	.10000E+01	.37972E-05	.23418E-05	.25745E-06

TABLE 14: RAYLEIGH PROBLEM $t_0=1.0$, $\Delta t=0.4$

t	y	1st ORDER	4th ORDER	EXACT
0.1	.0	1.00000E+00	1.00000E+00	1.00000E+00
	1.00000E-01	7.86842E-01	7.87405E-01	7.87406E-01
	2.00000E-01	5.88706E-01	5.89641E-01	5.89638E-01
	3.00000E-01	4.17469E-01	4.18494E-01	4.18492E-01
	4.00000E-01	2.79864E-01	2.80714E-01	2.80713E-01
	5.00000E-01	1.77012E-01	1.77530E-01	1.77530E-01
	6.00000E-01	1.05478E-01	1.05645E-01	1.05645E-01
	7.00000E-01	5.91578E-02	5.90573E-02	5.90585E-02
	8.00000E-01	3.12138E-02	3.09706E-02	3.09716E-02
	9.00000E-01	1.54937E-02	1.52186E-02	1.52192E-02
	1.00000E+00	7.23834E-03	7.00062E-03	7.00069E-03
0.4	.0	1.00000E+00	1.00000E+00	1.00000E+00
	1.00000E-01	8.09765E-01	8.11071E-01	8.11070E-01
	2.00000E-01	6.30337E-01	6.32589E-01	6.32585E-01
	3.00000E-01	4.70661E-01	4.73294E-01	4.73290E-01
	4.00000E-01	3.36556E-01	3.38983E-01	3.38980E-01
	5.00000E-01	2.30208E-01	2.31999E-01	2.31998E-01
	6.00000E-01	1.50519E-01	1.51493E-01	1.51494E-01
	7.00000E-01	9.40487E-02	9.42623E-02	9.42643E-02
	8.00000E-01	5.61632E-02	5.58269E-02	5.58295E-02
	9.00000E-01	3.20700E-02	3.14417E-02	3.14439E-02
	1.00000E+00	1.75247E-02	1.68261E-02	1.68274E-02
0.6	.0	1.00000E+00	1.00000E+00	1.00000E+00
	1.00000E-01	8.21629E-01	8.23064E-01	8.23063E-01
	2.00000E-01	6.52203E-01	6.54724E-01	6.54721E-01
	3.00000E-01	4.99300E-01	5.02339E-01	5.02335E-01
	4.00000E-01	3.68156E-01	3.71097E-01	3.71094E-01
	5.00000E-01	2.61207E-01	2.63555E-01	2.63552E-01
	6.00000E-01	1.78226E-01	1.79713E-01	1.79712E-01
	7.00000E-01	1.16920E-01	1.17523E-01	1.17525E-01
	8.00000E-01	7.37519E-02	7.36359E-02	7.36384E-02
	9.00000E-01	4.47490E-02	4.41689E-02	4.41716E-02
	1.00000E+00	2.61333E-02	2.53455E-02	2.53475E-02

TABLE 15: RAYLEGH PROBLEM $t_0=1.0$, $\Delta t=0.4$

y	1st ORDER	C - N	3rd ORDER	4th ORDER	EXACT
t = 0.4					
.0	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00
1.00000E-01	8.06701E-01	8.11818E-01	8.10241E-01	8.11062E-01	8.11070E-01
2.00000E-01	6.24995E-01	6.33786E-01	6.31453E-01	6.32468E-01	6.32585E-01
3.00000E-01	4.64329E-01	4.74488E-01	4.72488E-01	4.73384E-01	4.73290E-01
4.00000E-01	3.30585E-01	3.39792E-01	3.38833E-01	3.38872E-01	3.38980E-01
5.00000E-01	2.25617E-01	2.32258E-01	2.32390E-01	2.32093E-01	2.31998E-01
6.00000E-01	1.47779E-01	1.51275E-01	1.52066E-01	1.51447E-01	1.51494E-01
7.00000E-01	9.30980E-02	9.37803E-02	9.46986E-02	9.43083E-02	9.42643E-02
8.00000E-01	5.65815E-02	5.53107E-02	5.60055E-02	5.58098E-02	5.58295E-02
9.00000E-01	3.33039E-02	3.10448E-02	3.14127E-02	3.14501E-02	3.14439E-02
1.00000E+00	1.90706E-02	1.66016E-02	1.67046E-02	1.68168E-02	1.68274E-02
t = 1.2					
.0	1.00000E+00	1.00000E+00	1.00000E+00		1.00000E+00
1.00000E-01	8.43748E-01	8.49181E-01	8.48520E-01		8.48767E-01
2.00000E-01	6.93796E-01	7.03648E-01	7.02554E-01		7.02917E-01
3.00000E-01	5.55627E-01	5.68145E-01	5.66947E-01		5.67269E-01
4.00000E-01	4.33302E-01	4.46431E-01	4.45404E-01		4.45601E-01
5.00000E-01	3.29126E-01	3.40981E-01	3.40286E-01		3.40356E-01
6.00000E-01	2.43657E-01	2.52890E-01	2.52585E-01		2.52559E-01
7.00000E-01	1.75983E-01	1.81954E-01	1.82022E-01	UNSTABLE	1.81926E-01
8.00000E-01	1.24163E-01	1.26910E-01	1.27267E-01		1.27124E-01
9.00000E-01	8.57025E-02	8.57619E-02	8.62730E-02		8.61195E-02
1.00000E+00	5.79671E-02	5.61360E-02	5.66541E-02		5.65305E-02

TABLE 16: RAYLEIGH PROBLEM $t_0=1.0$, $\Delta t=0.8$

t	y	C – N	3rd ORDER	EXACT
0.8	.0	1.00000E+00	1.00000E+00	1.00000E+00
	1.00000E-01	8.36116E-01	8.16124E-01	8.33029E-01
	2.00000E-01	6.78382E-01	6.54357E-01	6.73290E-01
	3.00000E-01	5.32510E-01	5.18349E-01	5.27089E-01
	4.00000E-01	4.03269E-01	4.01081E-01	3.99075E-01
	5.00000E-01	2.93935E-01	2.98699E-01	2.91841E-01
	6.00000E-01	2.05875E-01	2.12406E-01	2.05903E-01
	7.00000E-01	1.38485E-01	1.43931E-01	1.40016E-01
	8.00000E-01	8.95146E-02	9.30037E-02	9.16903E-02
	9.00000E-01	5.57005E-02	5.74196E-02	5.77798E-02
	1.00000E+00	3.34646E-02	3.39737E-02	3.50152E-02
2.4	.0	1.00000E+00	1.00000E+00	1.00000E+00
	1.00000E-01	8.78848E-01	8.67418E-01	8.78089E-01
	2.00000E-01	7.60384E-01	7.50677E-01	7.59006E-01
	3.00000E-01	6.47145E-01	6.46011E-01	6.45388E-01
	4.00000E-01	5.41369E-01	5.44235E-01	5.39498E-01
	5.00000E-01	4.44847E-01	4.45952E-01	4.43103E-01
	6.00000E-01	3.58817E-01	3.57136E-01	3.57386E-01
	7.00000E-01	2.83924E-01	2.81284E-01	2.82934E-01
	8.00000E-01	2.20252E-01	2.18423E-01	2.19768E-01
	9.00000E-01	1.67402E-01	1.67061E-01	1.67421E-01
	1.00000E+00	1.24599E-01	1.25529E-01	1.25047E-01

TABLE 17: RAYLEIGH PROBLEM $\Delta t = 0.1$, $t = 24.9$

y	C-N	3rd ORDER	EXACT
0.	.10000E+01	1.00000E+00	.10000E+01
.10000E+00	.95476E+00	9.54769E-01	.95480E+00
.20000E+00	.90967E+00	9.09682E-01	.90974E+00
.30000E+00	.86486E+00	8.64884E-01	.86497E+00
.40000E+00	.82049E+00	8.20516E-01	.82064E+00
.50000E+00	.77668E+00	7.76714E-01	.77686E+00
.60000E+00	.73357E+00	7.33609E-01	.73379E+00
.70000E+00	.69128E+00	6.91326E-01	.69153E+00
.80000E+00	.64993E+00	6.49983E-01	.65022E+00
.90000E+00	.60963E+00	6.09686E-01	.60995E+00
.10000E+01	.57047E+00	5.70536E-01	.57084E+00

no real advantage of the higher-order techniques. Further study may resolve some of the difficulties found here.

VI. RÉSUMÉ

Polynomial spline interpolation has been used to develop a variety of higher-order collocation methods. Only those polynomials resulting in tridiagonal, or at worst 3x3 block-tridiagonal, matrix systems have been evaluated. The governing systems are obtained directly in terms of the functional values and certain derivatives of the functional values at the specified nodal points. The development is generally given for a nonuniform mesh, for which a high degree of accuracy is maintained.

Recently for a uniform mesh a so-called Padé, compact, Mehrstellung or Hermitian finite-difference procedure, which is 3x3 block-tridiagonal, has been proposed. It is shown that this formulation is a hybrid method resulting from two different polynomial splines. However, the Padé approximation is derived from a five-point discretization formulae and might be difficult to extend to nonuniform mesh systems. The hybrid spline results apply to variable meshes. Also, the compact system of equations does not include certain simple relations between the first and second derivative approximations that are obtained from the polynomial spline interpolation formula. These relations are useful for reducing the size of the matrix system and thereby the computer time; in certain instances, boundary conditions can more easily be satisfied with these equations.

Hermitian polynomial interpolation has been considered by Peters and it is shown herein that the compact differencing system is derivable with this procedure. On the other hand, Peters is able to derive a tridiagonal matrix system involving only the functional nodal values u_j. This would appear to be a significant improvement over the 3x3 compact system. However, the final system appears to be inconsistent, with an attendant loss of accuracy.

Finally, from three-point Taylor series expansions and Hermitian discretization of the functionals and their derivatives at the nodal points, an alternate derivation of the compact differencing scheme is presented. As only three nodal points are considered here, this procedure is less cumbersome than the Padé formulation and has been considered for nonuniform meshes and to develop systems with even higher-order truncation errors. Significantly, all of the polynomial spline block-tridiagonal systems can be recovered with this formulation. Moreover, a sixth-order (hybrid polynomial spline) 3x3 block-tridiagonal scheme has been devised. There does not appear to be any particular advantage of the polynomial spline formulation over the Hermitian discretization derivation.

The truncation errors of all the procedures are presented in tabular form, and results are shown for a variety of viscous flow problems. Of the fourth-order methods, spline $4^{(7,8)}$ has the smallest truncation error. From the solutions to the model problems, the increase in accuracy with decrease in truncation error is apparent. The sixth-order Hermite formulation leads to extraordinary accuracy even with very coarse grids. An important conclusion of the present study is that for equal accuracy the spline 4 procedure requires one-fourth as many points as a finite-difference calculation. This means less computer time and storage.

Finally, polynomial interpolation has been used to develop higher-order implicit temporal integration schemes, which have previously been developed by Hermite collocation. The present formulation leads to additional features of these methods, which are not obtained with the Hermite procedures. For time-asymptotic (steady state) calculations there does not appear to be any advantage of these higher-order methods. For transient analyses, there are "stability" limitations not found with the lower-order techniques, and the integration is sensitive to small errors in the initial conditions. Therefore, for the examples considered herein, the advantages associated with higher-order time integration are minimal.

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