A REVIEW OF SUBSTRUCTURE COUPLING METHODS FOR DYNAMIC ANALYSIS*

Roy R. Craig, Jr. and Ching-Jone Chang
The University of Texas at Austin

SUMMARY

This paper assesses the state of the art in substructure coupling for dynamic analysis. A general formulation, which permits all previously described methods to be characterized by a few constituent matrices, is developed. Limited results comparing the accuracy of various methods are presented.

INTRODUCTION

Analysis of the response of a complex structure to dynamic excitation is usually accomplished by analyzing a finite element model of the structure. Since the finite element model may contain thousands of degrees of freedom, and since the structure may consist of several substructures which are designed and fabricated by different organizations, it is desirable to have a method of dynamic analysis which permits the number of degrees of freedom of the dynamic model to be reduced and which also allows as much independence as possible in the design and analysis of substructures. The names substructure coupling and component mode synthesis have been applied to the process of partitioning a structure into substructures, or components, and describing the physical displacements of the substructures in terms of generalized coordinates which are the amplitudes of predetermined substructure modes. A number of substructure coupling methods have been proposed. The goal of most of these has been to permit analytical determination of system natural modes and frequencies from given finite element models of the structure. To a lesser extent, the use of experimentally-determined substructure data to synthesize mathematical models of structures has been considered.

One classification of substructure coupling methods is based on the conditions imposed at the interface between one substructure and the adjoining substructures when mode shapes are determined for the substructure. One class is called fixed-interface methods, and a second is called free-interface methods. Related to the latter is a class which may be called loaded-interface methods. Finally, some consideration has been given to permitting arbitrary interface conditions which may be a combination of the above three types. Such a method may be called a hybrid method.

The following classes of modes are used in defining substructure generalized coordinates: normal modes, constraint modes, attachment modes, and rigid-body modes. These are defined in greater detail in a later section of the paper.

*This work was supported by NASA Grant NSG 1268.

SYMBOLS

The principal defining equations are given in parenthesis after the definition of each symbol.

A interface equilibrium matrix (29)
B displacement compatibility matrix (29)
C combination of A and B (33)
$\underline{f}$ substructure force vector (1)
$\bar{F}$ equivalent force vector (15)
G flexibility matrix (19)
k substructure stiffness matrix (1)
K system stiffness matrix (30, 37, 45)
L Lagrangian (26)
m substructure mass matrix (1)
M system mass matrix (30, 37, 45)
p substructure generalized coordinate vector (22, 25)
q system generalized coordinate vector (31)
R inertia relief matrix (14)
T substructure kinetic energy (21)
T_1 substructure transformation matrix (22)
T_2 system transformation matrix (31, 36)
U substructure potential energy (21)
x substructure physical coordinate vector (1)
η Lagrange multiplier vector (26)
Θ free-interface or loaded-interface normal mode matrix (7)
κ substructure generalized stiffness matrix (24, 25)
λ,Λ substructure eigenvalue, eigenvalue matrix (2, 3)
μ substructure generalized mass matrix (24, 25)
ν Lagrange multiplier vector (26)
ξ generalized coordinate (27)
σ Lagrange multiplier vector (38)
Φ fixed-interface normal mode matrix (4)
$\underline{X}$ modified attachment mode matrix (20)
$\bar{X}$ unmodified attachment mode matrix (13, 17)
Ψ constraint mode matrix (11)

Subscripts and Superscripts:

d dependent coordinates (32)
i non-interface (interior) coordinates (1)
j interface (junction) coordinates (1)
k kept coordinates (18)
ℓ linearly-independent coordinates (32)
r rigid-body modes, temporary constraints (14, 15)
u unrestrained coordinates (15)

HISTORICAL REVIEW

The following is a brief review of the development of a number of substructure coupling methods:

Hurty (refs. 1,2) developed the first substructure coupling method capable of analyzing substructures with redundant interface connection. Fixed interface normal modes, rigid-body modes and redundant constraint modes are used to define substructure generalized coordinates.

Bamford (ref. 3) introduced attachment modes, and developed a hybrid substructure coupling method.

Craig and Bampton (ref. 4) and Bajan and Feng (refs. 5,6) modified Hurty's method by pointing out that it is unnecessary to separate the set of constraint modes into rigid-body modes and redundant constraint modes.

Goldman (ref. 7) and Hou (ref. 8) developed methods which employ free-interface substructure normal modes. They differ in the technique used to effect coupling of the substructures, as will be explained in a subsequent section.

Benfield and Hruda (ref. 9) introduced two new concepts: they employed Guyan reduction (ref. 10) to determine interface loading, and they used a coupling strategy which differs slightly from strategies used by previous authors. These features serve as the basis for four methods described by Benfield and Hruda: free-free, constrained, free-free with interface loading, and constrained with interface loading.

MacNeal (ref. 11) developed a hybrid method which allows some substructure interface coordinates to be constrained while others are free. He also suggested the use of statically derived modes to improve the representation of the substructure motion.

Goldenberg and Shapiro (ref. 12) employed a method similar to Hou's, but provided for arbitrary mass loading of interface points.

Rubin (ref. 13) extended MacNeal's method to include second-order residual effects of modes truncated from the final set free-interface substructure normal modes.

Kuhar and Stahle (ref. 14) introduced a dynamic transformation which approximates the effect of modes which are truncated from the final set of system generalized coordinates.

In a recent paper Hintz (ref. 15) describes two statically complete interface mode sets which he calls "the method of attachment modes" and "the method of constraint modes." The former set is combined with both free-interface normal modes and with fixed-interface normal modes to form system coordinates. The latter is combined only with fixed-interface normal modes.

In reference 16 Craig and Chang describe three methods for reducing the

number of interface coordinates in the final system equations obtained by the Hurty method or the Craig-Bampton method. In reference 17 Craig and Chang provide examples of substructure coupling based on the methods of MacNeal and Rubin.

The previous references are primarily concerned with the use of substructure coupling methods in the analytical determination of modes and frequencies of complex structures. Several studies, however, explore the use of experimental data as input to coupling procedures. The following studies are of this nature:

Klosterman's thesis (ref. 18) provides a comprehensive study of the experimental determination of modal representations of structures including the use of these models in substructure coupling. In reference 19 Klosterman treats substructure coupling by two methods which he calls "component mode synthesis" and "general impedance method" respectively. The former closely parallels Bamford's work. In reference 20 Klosterman and McClelland introduce "inertia restraint" and outline a coupling procedure that appears to be especially suited to coupling two substructures where one is represented by modes and the second by a finite-element model.

Kana and Huzar (refs. 21,22) developed a semi-empirical energy approach for predicting the damping of a structure in terms of damping of substructures.

Hasselman (ref. 23) employs a perturbation technique to describe substructure damping and discusses, in a general way, coupling of substructures using either free-interface modes or fixed-interface modes.

Two symposia on the topic of substructure coupling have been held (refs. 24,25). Survey papers of particular importance, which were presented at these symposia, are references 26 and 27.

A GENERAL FORMULATION OF SUBSTRUCTURE COUPLING FOR DYNAMIC ANALYSIS

The substructure coupling methods mentioned in the preceding section may be described by a single comprehensive formulation. Differences in the methods result from the use of different mode sets to describe substructure generalized coordinates and different methods of enforcing compatibility of substructure interfaces. We will first define the mode sets used in representing the substructure physical displacements in terms of substructure generalized coordinates. Then, using the Lagrange multiplier method, we will show how enforcement of compatibility at substructure interfaces leads to system equations of motion. Finally, the vectors and matrices which define the various methods are tabulated.

Definition of Mode Sets

The physical displacements of each substructure are represented in terms of substructure generalized coordinates through the use of various "assumed modes," including normal modes of the substructure and certain static deflec-

tion modes.

The equation of motion of a substructure, when connected to other substructures and executing undamped free vibration, may be written in the form

$$\begin{bmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{bmatrix}^{\alpha} \begin{Bmatrix} \ddot{x}_i \\ \ddot{x}_j \end{Bmatrix}^{\alpha} + \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix}^{\alpha} \begin{Bmatrix} x_i \\ x_j \end{Bmatrix}^{\alpha} = \begin{Bmatrix} 0 \\ f_j \end{Bmatrix}^{\alpha} \tag{1}$$

Fixed-Interface Normal Modes

Fixed-interface normal modes are obtained by setting $x_j \equiv 0$ and solving for the free-vibration modes of the substructure. Equation (1) reduces to the eigenvalue problem

$$(k_{ii} - \lambda^2 m_{ii})\, x_i = 0 \tag{2}$$

The resulting substructure eigenvalues (frequencies) form a diagonal matrix

$$\Lambda \equiv \text{diag}\ (\lambda_1^2\ \lambda_2^2\ \ldots\ \lambda_{N_i}^2) \tag{3}$$

and the corresponding normalized eigenvectors (mode shapes) form the modal matrix

$$\Phi \equiv \begin{bmatrix} \phi_{i1} & \phi_{i2} & \cdots & \phi_{iN_i} \\ \hline 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4}$$

where N_i is the total number of substructure interior coordinates.

Free-Interface Normal Modes; Loaded-Interface Normal Modes

Free-interface normal modes are obtained by setting $f_j \equiv 0$ in equation (1) and solving for the resulting modes and frequencies of the substructure. Thus,

$$(k - \lambda^2 m)\, x = 0 \tag{5}$$

The matrix of eigenvalues is

$$\Lambda \equiv \text{diag}\ (\lambda_1^2\ \lambda_2^2\ \ldots\ \lambda_N^2) \tag{6}$$

where $N = N_i + N_j$ is the total number of substructure degrees of freedom. Since the structure may be unrestrained, there may be N_r rigid-body modes. The normalized eigenvectors form the modal matrix

$$\Theta \equiv \begin{bmatrix} \theta_{i1} & \theta_{i2} & \cdots & \theta_{iN} \\ \theta_{j1} & \theta_{j2} & \cdots & \theta_{jN} \end{bmatrix} \tag{7}$$

Several methods (e.g., refs. 9,12) employ loaded-interface normal modes. These are obtained by augmenting the interface mass and/or stiffness in equation (5) to give

$$\left[\begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & (k_{jj} + \bar{k}_{jj}) \end{bmatrix} - \lambda^2 \begin{bmatrix} m_{ii} & m_{ij} \\ m_{ji} & (m_{jj} + \bar{m}_{jj}) \end{bmatrix}\right] \begin{Bmatrix} x_i \\ x_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8}$$

$\bar{k}_{jj}$ and $\bar{m}_{jj}$ are the interface "loading" matrices. The symbol Θ will be used for the modal matrix corresponding to equation (8).

Constraint Modes

To complement fixed-interface substructure normal modes a set of constraint modes may be employed (e.g., refs. 2,4). A constraint mode is defined by imposing a unit displacement on one physical coordinate and zero displacement on the remainder of a specified subset of the substructure physical coordinates. The procedure employed to obtain constraint modes is equivalent to applying a Guyan reduction to all interior coordinates; i.e., the mass is neglected in the top row-partition of equation (1) and unit displacements are imposed successively on all junction coordinates giving

$$\begin{bmatrix} k_{ii} & k_{ij} \end{bmatrix} \begin{bmatrix} \Psi_{ij} \\ I_{jj} \end{bmatrix} = 0 \tag{9}$$

Thus, the N_j constraint modes which form the columns of the constraint mode matrix Ψ are obtained by solving the (multiple) static deflection problem

$$k_{ii}\, \Psi_{ij} = -k_{ij} \tag{10}$$

Then,

$$\Psi \equiv \begin{bmatrix} \Psi_{ij} \\ I_{jj} \end{bmatrix} \tag{11}$$

If the substructure is unrestrained, Ψ will contain N_r linearly independent rigid-body modes. As noted in reference 4, constraint modes and fixed-interface normal modes are orthogonal with respect to the stiffness matrix k.

Attachment Modes

Attachment modes are "static" modes which may be used to complement free-interface substructure normal modes (e.g., refs. 3,11,15,18). An attachment mode is defined by imposing a unit force on one physical coordinate and zero force on the remainder of a specified subset of substructure physical coordi-

nates. Attachment modes will be described first for restrained structures (for which k is non-singular) and then for unrestrained structures.

Attachment modes for restrained substructures.-Attachment modes for a restrained substructure are obtained by solving the multiple static deflection problem

$$\begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix} \begin{bmatrix} \bar{X}_{ij} \\ \bar{X}_{jj} \end{bmatrix} = \begin{bmatrix} 0 \\ I_{jj} \end{bmatrix} \tag{12}$$

Then the attachment mode matrix is defined by

$$\bar{X} \equiv \begin{bmatrix} \bar{X}_{ij} \\ \bar{X}_{jj} \end{bmatrix} \tag{13}$$

Attachment modes can be expressed as linear combinations of free-interface normal modes. However, in a later section when the normal mode set is truncated, the attachment modes will be modified so that they are orthogonal to the kept normal modes. The modified attachment mode set will be called X.

Attachment modes for unrestrained substructures.-For an unrestrained substructure, attachment modes may be obtained by using rigid-body inertia forces to equilibrate applied forces and by temporarily imposing a set of N_r nonredundant constraints. Let Θ_r be the set of N_r (normalized) rigid-body modes of the substructure and let

$$R = I - m\Theta_r \Theta_r^T \tag{14}$$

be the inertia relief matrix (ref. 15). Then, the attachment modes may be obtained from

$$\begin{bmatrix} k_{rr} & k_{ru} & k_{rj} \\ k_{ur} & k_{uu} & k_{uj} \\ k_{jr} & k_{ju} & k_{jj} \end{bmatrix} \begin{bmatrix} 0 \\ \bar{X}_{uj} \\ \bar{X}_{jj} \end{bmatrix} = R \begin{bmatrix} 0 \\ 0 \\ I_{jj} \end{bmatrix} \equiv \begin{bmatrix} \bar{F}_{rj} \\ \bar{F}_{uj} \\ \bar{F}_{jj} \end{bmatrix} \tag{15}$$

where r stands for the N_r restrained interior coordinates and u stands for the $N_u = N_i - N_r$ unrestrained interior coordinates. From equation (15)

$$\begin{bmatrix} k_{uu} & k_{uj} \\ k_{ju} & k_{jj} \end{bmatrix} \begin{bmatrix} \bar{X}_{uj} \\ \bar{X}_{jj} \end{bmatrix} = \begin{bmatrix} \bar{F}_{uj} \\ \bar{F}_{jj} \end{bmatrix} \tag{16}$$

Finally,

$$\bar{X} \equiv \begin{bmatrix} 0 \\ \bar{X}_{uj} \\ \bar{X}_{jj} \end{bmatrix} \tag{17}$$

Rigid-body modes may be removed from the $\bar{X}$ matrix by premultiplying it by R^T.

Truncation of Mode Sets

One of the most significant features of substructure coupling techniques is that they permit the number of degrees of freedom of a system to be reduced in a systematic manner through truncation of the mode sets which define the generalized coordinates of the system. Hintz (ref. 15) has provided a comprehensive discussion of truncation of mode sets. Although truncation is usually accomplished by elimination of some coordinates associated with substructure normal modes (e.g., ref. 26), truncation may also be associated with other coordinates such as constraint mode coordinates (e.g., ref. 16). Attention will be confined here to the former, i.e., truncation of normal mode coordinates. The subscript k will be used to denote the columns of Φ or Θ which are kept. For example, the N_k modes which are kept form the columns of Θ_k, where

$$\Theta_k \equiv \begin{bmatrix} \Theta_{ik} \\ \Theta_{jk} \end{bmatrix} \tag{18}$$

The diagonal matrix of corresponding eigenvalues will be denoted by Λ_{kk}.

As noted previously, attachment modes can be expressed as linear combinations of the free-interface normal modes. However, when the normal mode set is truncated, the attachment modes can no longer be represented in terms of Θ_k. On the contrary, it is possible to modify the attachment modes so that they are orthogonal to the modes in Θ_k (e.g., see refs. 13,17). This will be illustrated here for attachment modes of a restrained substructure.

Note, in equation (12), that the columns of $\bar{X}$ correspond to columns of the flexibility matrix k^{-1}. The contribution of the kept normal modes to this flexibility matrix is given by (see ref. 17)

$$G_k = \Theta_k \Lambda_{kk}^{-1} \Theta_k^T \tag{19}$$

The contribution of the modes in Θ_k to $\bar{X}$ can be removed from $\bar{X}$ leaving

$$X = \bar{X} - \Theta_k \Lambda_{kk}^{-1} \Theta_{jk}^T \tag{20}$$

Energy Expressions for Substructures; Coordinate Transformation

The derivation of system equations of motion will be based on Lagrange's equations of motion with undetermined multipliers. Expressions for kinetic energy and strain energy of the substructures are required. These will be given first for substructure physical coordinates and then in terms of substructure generalized coordinates.

The kinetic energy and potential energy of a substructure are given by

$$T = \frac{1}{2} \dot{x}^T m \dot{x} \quad , \quad U = \frac{1}{2} x^T k x \tag{21}$$

respectively. The substructure physical coordinates, x, may be expressed in terms of substructure generalized coordinates, p, by the coordinate transformation

$$x = T_1 p \tag{22}$$

When the above coordinate transformation is inserted into equations (21), the substructure generalized mass and stiffness matrices are obtained. Thus,

$$T = \frac{1}{2} \dot{p}^T \mu \dot{p} \quad , \quad U = \frac{1}{2} p^T \kappa p \tag{23}$$

where

$$\mu = T_1^T m T_1 \quad , \quad \kappa = T_1^T k T_1 \tag{24}$$

Substructure Coupling; System Equations of Motion

To illustrate coupling of substructures to form a system, two substructures, α and β, will be employed. Let

$$p \equiv \begin{Bmatrix} p^\alpha \\ p^\beta \end{Bmatrix} , \quad \mu \equiv \begin{bmatrix} \mu^\alpha & 0 \\ 0 & \mu^\beta \end{bmatrix} , \quad \kappa \equiv \begin{bmatrix} \kappa^\alpha & 0 \\ 0 & \kappa^\beta \end{bmatrix} \tag{25}$$

The substructure generalized coordinates are not all independent but are related by force equilibrium and displacement compatibility at substructure interfaces. These relationships may be expressed by the equations

$$A p = 0 \quad , \quad B p = 0$$

respectively. Then, a Lagrangian may be formed as follows:

$$L = \frac{1}{2}\dot{p}^T \mu \dot{p} - \frac{1}{2} p^T \kappa p + \eta^T A p + \nu^T B p \tag{26}$$

The system equations may be obtained by applying Lagrange's equation in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_n}\right) - \frac{\partial L}{\partial \xi_n} = 0 \tag{27}$$

where ξ_n can refer to p_n, η_n or ν_n. Then equations (26) and (27) may be combined to give

$$\mu \ddot{p} + \kappa p = A^T \eta + B^T \nu \tag{28}$$

together with the constraint equations

$$A p = 0 \quad , \quad B p = 0 \tag{29}$$

In the works cited previously, two basic approaches have been employed for solving the coupled equations contained in equations (28) and (29). Both lead to system equations of the form

$$M \ddot{q} + K q = 0 \tag{30}$$

The method used by most authors will be referred to as the implicit method. It involves the use of a coordinate transformation T_2 to replace the set of dependent coordinates, p, by a set of linearly independent coordinates q. Thus,

$$p = T_2 q \tag{31}$$

Let p be partitioned into dependent coordinates, p_d, and linearly independent coordinates, p_ℓ, as follows:

$$p \equiv \begin{Bmatrix} p_d \\ p_\ell \end{Bmatrix} \tag{32}$$

and let the constraint matrices A and B be combined to form the matrix C, i.e.,

$$C p \equiv \begin{bmatrix} A \\ B \end{bmatrix} p = 0 \tag{33}$$

Since C will have fewer rows than columns, equations (32) and (33) may be combined and written in the form

$$[C_{dd} \; C_{d\ell}] \begin{bmatrix} p_d \\ p_\ell \end{bmatrix} = 0 \tag{34}$$

where C_{dd} is a non-singular square submatrix of C. Then

$$\begin{Bmatrix} p_d \\ p_\ell \end{Bmatrix} = \begin{bmatrix} -C_{dd}^{-1} C_{d\ell} \\ I_{\ell\ell} \end{bmatrix} p_\ell \tag{35}$$

Let $q \equiv p_\ell$. Then equations (31) and (35) give

$$T_2 \equiv \begin{bmatrix} -C_{dd}^{-1} C_{d\ell} \\ I_{\ell\ell} \end{bmatrix} \tag{36}$$

as the general expression for transformation matrix T_2. The matrices M and K in equation (30) are given by

$$M = T_2^T \mu T_2 \quad , \quad K = T_2^T \kappa T_2 \tag{37}$$

Goldman (ref. 7) solved equations (28) and (29) by an approach which will be referred to as the explicit method. Let

$$\sigma \equiv \begin{Bmatrix} \eta \\ \nu \end{Bmatrix} \tag{38}$$

Then equation (28) may be written

$$\mu \ddot{p} + \kappa p = C^T \sigma \tag{39}$$

σ may be related to p by multiplying equation (39) by $C \mu^{-1}$ and incorporating equation (33). Then equation (39) may be written in the form

$$\mu \ddot{p} + [I - C^T (C \mu^{-1} C^T)^{-1} C \mu^{-1}] \kappa p = 0 \tag{40}$$

Goldman's final system equations are obtained by letting

$$p = \kappa^{-1/2} q \tag{41}$$

Then equation (40) can be reduced to the form of equation (30) with

$$M = I \ , \ K = \kappa^{1/2} \mu^{-1} [I - C^T (C \mu^{-1} C^T)^{-1} C \mu^{-1}] \kappa^{1/2} \tag{42}$$

Since equation (41) implies no reduction in number of coordinates, equation (30) leads to some extraneous frequencies and modes in the Goldman method.

Description of Various Coupling Methods

Table I shows the constituent vectors and matrices (i.e., T_1, p, T_2, etc.) of a representative selection of the substructure coupling methods named earlier in the historical review. In all cases the methods fit into the general formulation just described. However, in a few cases the notation has been simplified by employing a partitioning of C (or B) different from that indicated in equations (34) and (36).

CONVERGENCE PROPERTIES

Desirable characteristics for substructure coupling methods include (e.g., see refs. 13,15): computational efficiency, interchangeability, component flexibility, synthesis flexibility, static completeness, and test compatibility. Although it is not within the scope of this paper to make a detailed comparison of coupling techniques on the basis of the above criteria, a few results concerning computational efficiency, i.e., convergence, will be presented. Several authors have previously discussed convergence of system frequencies (e.g., refs. 13,16,26,27). Rubin (ref. 13) also considered convergence of mode shapes and shear and moment in beam elements.

Figure 1 shows frequency and RMS bending moment convergence properties of mode 3 of a clamped-clamped uniform beam.

CONCLUDING REMARKS

A general formulation has been presented which permits substructure coupling methods to be defined in terms of a few constituent matrices. Although a detailed comparison of various substructure coupling methods has not been within the scope of this paper, it is hoped that the presentation of this general formulation will facilitate future studies of substructure coupling methods. At the present time the use of substructure coupling as an analysis tool seems to be a well-developed subject. On the contrary, much remains to be learned about effective ways to use substructure coupling in conjunction with experimental studies. It is hoped that this topic will receive increased attention in the future.

REFERENCES

1. Hurty, W.C.: Vibrations of Structural Systems by Component Mode Synthesis. Proc. ASCE, vol. 85, no. EM4, Aug. 1960, pp. 51-69.

2. Hurty, W.C.: Dynamic Analysis of Structural Systems Using Component Modes. AIAA Journal, vol. 3, no. 4, Apr. 1965, pp. 678-685.

3. Bamford, R.M.: A Modal Combination Program for Dynamic Analysis of Structures. Tech. Memo. 33-290, Jet Propulsion Laboratory, Pasadena, CA, July 1967.

4. Craig, R.R., Jr.and Bampton, M.C.C.: Coupling of Substructures for Dynamic Analyses. AIAA Journal, vol. 6, no. 7, July 1968, pp. 1313-1319.

5. Bajan, R.L. and Feng, C.C.: Free Vibration Analysis by the Modal Substitution Method. American Astronautical Society Paper No. 68-8-1, AAS Symposium on Space Projections from the Rocky Mountain Region, Denver, CO, July 1968.

6. Bajan, R.L.; Feng, C.C.; and Jaszlics, I.J.: Vibration Analysis of Complex Structural Systems by Modal Substitution. Proc. 39th Shock and Vibration Symposium, Monterey, CA, Oct. 1968.

7. Goldman, R.L.: Vibration Analysis by Dynamic Partitioning. AIAA Journal, vol. 7, no. 6, June 1969, pp. 1152-1154.

8. Hou, S.: Review of Modal Synthesis Techniques and a New Approach. The Shock and Vibration Bulletin, no. 40, pt. 4, Naval Research Laboratory, Washington, D.C., Dec. 1969, pp. 25-39.

9. Benfield, W.A. and Hruda, R.F.: Vibration Analysis of Structures by Component Mode Substitution. AIAA Journal, vol. 9, no. 7, July 1971, pp. 1255-1261.

10. Guyan, R.J.: Reduction of Stiffness and Mass Matrices. AIAA Journal, vol. 3, no. 2, Feb. 1965, p. 380.

11. MacNeal, R.H.: A Hybrid Method of Component Mode Synthesis. J. of Computers and Structures, vol. 1, no. 4, Dec. 1971, pp. 581-601.

12. Goldenberg, S. and Shapiro, M.: A Study of Modal Coupling Procedures. (Grumman Aerospace Corp.; NASA Contract No. NAS-10635-8.) NASA CR-112252, 1972.

13. Rubin, S.: An Improved Component-Mode Representation. AIAA Paper No. 74-386, AIAA/ASME/SAE 15th Structures, Structural Dynamics and Materials Conference, Las Vegas, Nevada, April 1974.

14. Kuhar, E.J. and Stahle, C.V.: A Dynamic Transformation Method for Modal Synthesis. AIAA Journal, vol. 12, no. 5, May 1974, pp. 672-678.

15. Hintz, R.M.: Analytical Methods in Component Modal Synthesis. AIAA Journal, vol. 13, no. 8, Aug. 1975, pp. 1007-1016.

16. Craig, R.R., Jr. and Chang, C-J.: Substructure Coupling with Reduction of Interface Coordinates--Fixed-Interface Methods. TICOM Rept. No. 75-1, The University of Texas at Austin, Austin, TX, Jan. 1975.

17. Craig, R.R., Jr. and Chang, C-J.: Free-Interface Methods of Substructure Coupling for Dynamic Analysis. AIAA Journal (accepted for publication).

18. Klosterman, A.L.: On the Experimental Determination and Use of Modal Representations of Dynamic Characteristics. Ph.D. dissertation, University of Cincinnati, 1971.

19. Klosterman, A.L.: A Combined Experimental and Analytical Procedure for Improving Automotive System Dynamics. Paper No. 720093, Society of Automotive Engineers, Automotive Engineering Congress, Detroit, MI, Jan. 1972.

20. Klosterman, A.L. and McClelland, W.A.: Combining Experimental and Analytical Techniques for Dynamic System Analysis. presented at 1973 Tokyo Seminar on Finite Element Analysis, Nov. 1973.

21. Kana, D.D. and Huzar, S.: Synthesis of Shuttle Vehicle Damping Using Substructure Test Results. Interim Report for NASA Contract No. NAS8-27569, Southwest Research Institute, San Antonio, TX, June 1972.

22. Kana, D.D. and Huzar, S.: Synthesis of Shuttle Vehicle Damping Using Substructure Test Results. J. Spacecraft and Rockets, vol. 10, no. 12, Dec. 1973, pp. 790-797.

23. Hasselman, T.K.: Damping Synthesis from Substructure Tests. AIAA Paper No. 74-387, presented at AIAA/ASME/SAE 15th Structures, Structural Dynamics and Materials Conference, Las Vegas, NV, Apr. 1974.

24. Neubert, V.H. and Raney, J.P., eds.: Synthesis of Vibrating Systems. The American Society of Mechanical Engineers, Nov. 1971.

25. Ryan, R.S., ed.: Symposium on Substructure Testing and Synthesis. NASA Marshall Space Flight Center, Aug. 1972.

26. Hurty, W.C.: Introduction to Modal Synthesis Techniques. Synthesis of Vibrating Systems, ASME, Nov. 1971.

27. Benfield, W.A.; Bodley, C.S.; and Morosow, G.: Modal Synthesis Methods. Symposium on Substructure Testing and Synthesis, NASA, Aug. 1972.

TABLE I.-VECTORS AND MATRICES DEFINING SUBSTRUCTURE COUPLING METHODS

Method (Type) / Matrix, etc.	Goldman (Free-interface, Explicit)	Hou (Free-interface, Implicit)	Goldenberg & Shapiro (Loaded-interface, Implicit)	Craig & Bampton Bajan & Feng* (Fixed-interface, Implicit)	Benfield & Hruda** (Free-interface, Implicit)	Hintz† (Free-interface, Implicit)	Craig & Chang†† (Free-interface, Implicit)
T_1^α	$\begin{bmatrix} \Theta_{ik}^\alpha \\ \Theta_{jk}^\alpha \end{bmatrix}$	$\begin{bmatrix} \Theta_{ik}^\alpha \\ \Theta_{jk}^\alpha \end{bmatrix} \equiv \left[\begin{array}{c\|c} \Theta_{id}^\alpha & \Theta_{i\ell}^\alpha \\ \Theta_{jd}^\alpha & \Theta_{j\ell}^\alpha \end{array}\right]$	$\begin{bmatrix} \Theta_{ik}^\alpha \\ \Theta_{jk}^\alpha \end{bmatrix} \equiv \left[\begin{array}{c\|c} \Theta_{id}^\alpha & \Theta_{i\ell}^\alpha \\ \Theta_{jd}^\alpha & \Theta_{j\ell}^\alpha \end{array}\right]$	$\left[\begin{array}{c\|c} \psi_{ij}^\alpha & \phi_{ik}^\alpha \\ I_{jj}^\alpha & 0 \end{array}\right]$	$\begin{bmatrix} \Theta_{ik}^\alpha \\ \Theta_{jk}^\alpha \end{bmatrix}$	$\left[\begin{array}{c\|c} X_{ij}^\alpha & \Theta_{ik}^\alpha \\ X_{jj}^\alpha & \Theta_{jk}^\alpha \end{array}\right]$	$\left[\begin{array}{c\|c} X_{ij}^\alpha & \Theta_{ik}^\alpha \\ X_{jj}^\alpha & \Theta_{jk}^\alpha \end{array}\right]$
T_1^β	$\begin{bmatrix} \Theta_{ik}^\beta \\ \Theta_{jk}^\beta \end{bmatrix}$	$\begin{bmatrix} \Theta_{ik}^\beta \\ \Theta_{jk}^\beta \end{bmatrix}$	$\begin{bmatrix} \Theta_{ik}^\beta \\ \Theta_{jk}^\beta \end{bmatrix} \equiv \left[\begin{array}{c\|c} \Theta_{id}^\beta & \Theta_{i\ell}^\beta \\ \Theta_{jd}^\beta & \Theta_{j\ell}^\beta \end{array}\right]$	$\left[\begin{array}{c\|c} \psi_{ij}^\beta & \phi_{ik}^\beta \\ I_{jj}^\beta & 0 \end{array}\right]$	$\left[\begin{array}{c\|c} 0 & \Theta_{ik}^\beta \\ I_{jj}^\beta & 0 \end{array}\right]$	$\left[\begin{array}{c\|c} X_{ij}^\beta & \Theta_{ik}^\beta \\ X_{jj}^\beta & \Theta_{jk}^\beta \end{array}\right]$	$\left[\begin{array}{c\|c} X_{ij}^\beta & \Theta_{ik}^\beta \\ X_{jj}^\beta & \Theta_{jk}^\beta \end{array}\right]$
p	$\begin{Bmatrix} p_k^\alpha \\ p_k^\beta \end{Bmatrix}$	$\begin{Bmatrix} p_d^\alpha \\ p_\ell^\alpha \\ p_k^\beta \end{Bmatrix}$	$\begin{Bmatrix} p_d \\ p_\ell \end{Bmatrix} \equiv \begin{Bmatrix} p_d^\alpha \\ p_d^\beta \\ p_\ell^\alpha \\ p_\ell^\beta \end{Bmatrix}$	$\begin{Bmatrix} x_j^\alpha \\ p_k^\alpha \\ x_j^\beta \\ p_k^\beta \end{Bmatrix}$	$\begin{Bmatrix} p_k^\alpha \\ x_j^\beta \\ p_k^\beta \end{Bmatrix}$	$\begin{Bmatrix} p_j^\alpha \\ p_k^\alpha \\ p_j^\beta \\ p_k^\beta \end{Bmatrix}$	$\begin{Bmatrix} f_j^\alpha \\ p_k^\alpha \\ f_j^\beta \\ p_k^\beta \end{Bmatrix}$
A	not applicable	not applicable	not applicable	not applicable	not applicable	not applicable	$\left[\begin{array}{c\|c\|c\|c} I_{jj}^\alpha & 0 & I_{jj}^\beta & 0 \end{array}\right]$
B	$\left[\begin{array}{c\|c} \Theta_{jk}^\alpha & -\Theta_{jk}^\beta \end{array}\right]$	$\left[\begin{array}{c\|c\|c} \Theta_{jd}^\alpha & \Theta_{j\ell}^\alpha & -\Theta_{jk}^\beta \end{array}\right]$	$\left[\begin{array}{c\|c\|c\|c} \Theta_{jd} & \Theta_{j\ell} \end{array}\right] \equiv \left[\begin{array}{c\|c\|c\|c} \Theta_{jd}^\alpha & -\Theta_{jd}^\beta & \Theta_{j\ell}^\alpha & -\Theta_{j\ell}^\beta \end{array}\right]$	$\left[\begin{array}{c\|c\|c\|c} I_{jj}^\alpha & 0 & -I_{jj}^\beta & 0 \end{array}\right]$	$\left[\begin{array}{c\|c\|c} \Theta_{jk}^\alpha & -I_{jj}^\beta & 0 \end{array}\right]$	$\left[\begin{array}{c\|c\|c\|c} X_{jj}^\alpha & \Theta_{jk}^\alpha & -X_{jj}^\beta & -\Theta_{jk}^\beta \end{array}\right]$	$\left[\begin{array}{c\|c\|c\|c} X_{jj}^\alpha & \Theta_{jk}^\alpha & -X_{jj}^\beta & -\Theta_{jk}^\beta \end{array}\right]$
T_2	$\kappa^{-1/2}$	$\left[\begin{array}{c\|c\|c} -[\Theta_{jd}^\alpha]^{-1}\Theta_{j\ell}^\alpha & [\Theta_{jd}^\alpha]^{-1}\Theta_{jk}^\beta \\ I_{\ell\ell}^\alpha & 0 \\ 0 & I_{kk}^\beta \end{array}\right]$	$\begin{bmatrix} -\Theta_{jd}^{-1}\Theta_{j\ell} \\ I_{\ell\ell} \end{bmatrix}$	$\left[\begin{array}{c\|c\|c} 0 & 0 & I_{jj}^\alpha \\ I_{kk}^\alpha & 0 & 0 \\ 0 & 0 & I_{jj}^\beta \\ 0 & I_{kk}^\beta & 0 \end{array}\right]$	$\left[\begin{array}{c\|c} I_{kk}^\alpha & 0 \\ \Theta_{jk}^\alpha & 0 \\ 0 & I_{kk}^\beta \end{array}\right]$	$\left[\begin{array}{c\|c\|c} 0 & 0 & I_{jj}^\alpha \\ I_{kk}^\alpha & 0 & 0 \\ [X_{jj}^\beta]^{-1}\Theta_{jk}^\alpha & -[X_{jj}^\beta]^{-1}\Theta_{jk}^\beta & [X_{jj}^\beta]^{-1}X_{jj}^\alpha \\ 0 & I_{kk}^\beta & 0 \end{array}\right]$	$\left[\begin{array}{c\|c\|c} -[X_{jj}^\alpha+X_{jj}^\beta]^{-1}\Theta_{jk}^\alpha & [X_{jj}^\alpha+X_{jj}^\beta]^{-1}\Theta_{jk}^\beta \\ I_{kk}^\alpha & 0 \\ [X_{jj}^\alpha+X_{jj}^\beta]^{-1}\Theta_{jk}^\alpha & -[X_{jj}^\alpha+X_{jj}^\beta]^{-1}\Theta_{jk}^\beta \\ 0 & I_{kk}^\beta \end{array}\right]$
q	$\begin{Bmatrix} q_k^\alpha \\ q_k^\beta \end{Bmatrix}$	$\begin{Bmatrix} q_\ell^\alpha \\ q_k^\beta \end{Bmatrix} \equiv \begin{Bmatrix} p_\ell^\alpha \\ p_k^\beta \end{Bmatrix}$	$\{q_\ell\} \equiv \{p_\ell\}$	$\begin{Bmatrix} q_k^\alpha \\ q_k^\beta \\ q_j \end{Bmatrix} \equiv \begin{Bmatrix} p_k^\alpha \\ p_k^\beta \\ x_j^\alpha \end{Bmatrix}$	$\begin{Bmatrix} q_k^\alpha \\ q_k^\beta \end{Bmatrix} \equiv \begin{Bmatrix} p_k^\alpha \\ p_k^\beta \end{Bmatrix}$	$\begin{Bmatrix} q_k^\alpha \\ q_k^\beta \\ q_j \end{Bmatrix} \equiv \begin{Bmatrix} p_k^\alpha \\ p_k^\beta \\ p_j^\alpha \end{Bmatrix}$	$\begin{Bmatrix} q_k^\alpha \\ q_k^\beta \end{Bmatrix} \equiv \begin{Bmatrix} p_k^\alpha \\ p_k^\beta \end{Bmatrix}$
N_t = total d.o.f.	$N_k^\alpha + N_k^\beta$	$N_k^\alpha + N_k^\beta - N_j$	$N_k^\alpha + N_k^\beta - N_j$	$N_k^\alpha + N_k^\beta + N_j$	$N_k^\alpha + N_k^\beta$	$N_k^\alpha + N_k^\beta + N_j$	$N_k^\alpha + N_k^\beta$

*Hurty's method is equivalent to this method.
**The other methods of Benfield & Hruda may be similarly described.
†The other methods of Hintz may be similarly described.

††A complete description of this method, together with its relationship to the methods of MacNeal and Rubin, is the subject of a forthcoming paper.

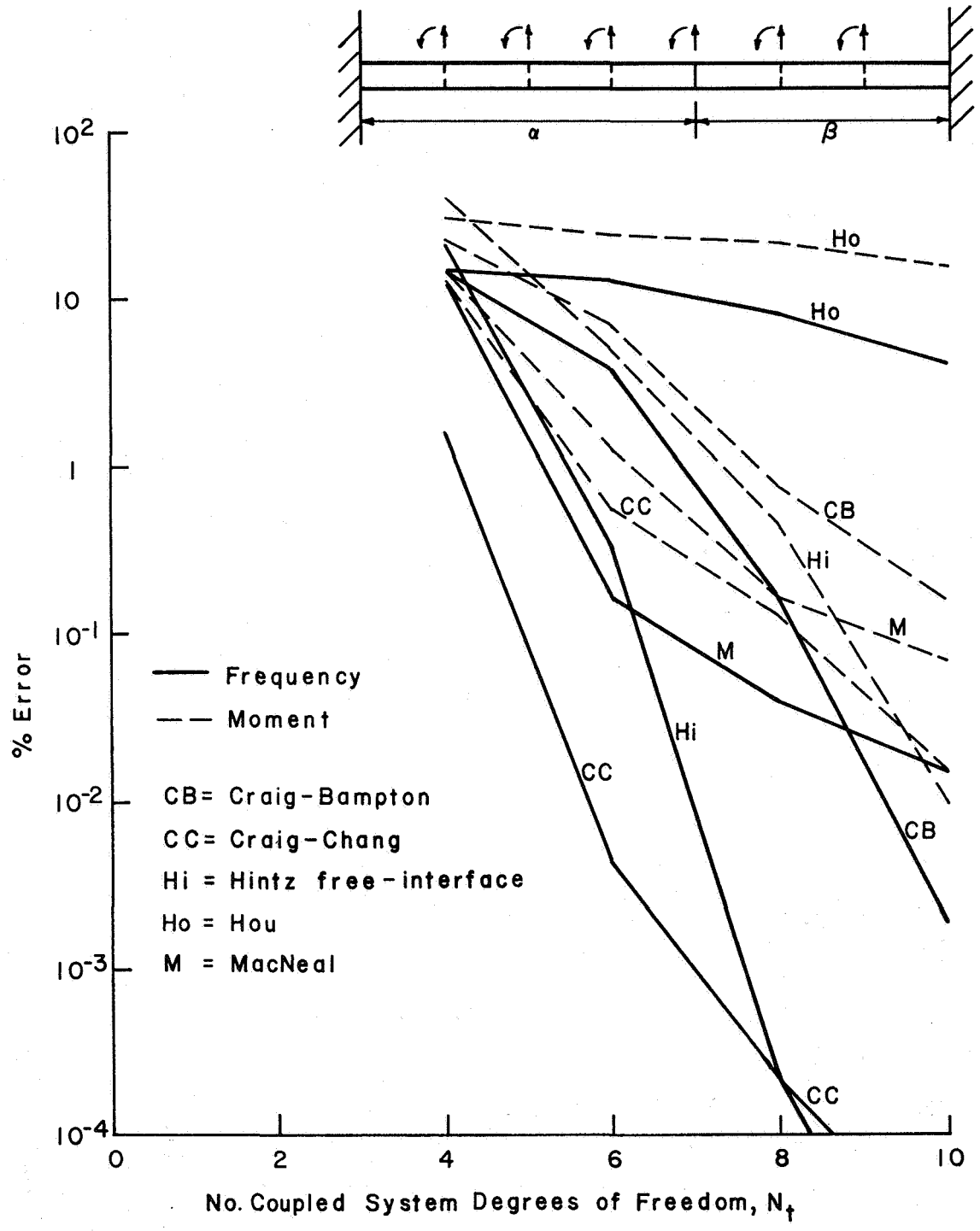

Figure 1.-Frequency and RMS Moment Convergence for Mode 3 of Clamped-Clamped Beam.