ON THE EXPLICIT FINITE ELEMENT FORMULATION OF THE DYNAMIC CONTACT PROBLEM OF HYPERELASTIC MEMBRANES*

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SUMMARY

Contact-impact problems involving finite deformation axisymmetric membranes are solved by the finite element method with explicit time integration. The formulation of the membrane element and the contact constraint conditions are discussed in this paper. The hyperelastic, compressible Blatz and Ko material is used to model the material properties of the membrane. Two example problems are presented.

INTRODUCTION

The purpose of this paper is to present a method for the dynamic analysis of contact-impact problems involving hyperelastic compressible membranes. A strain energy functional developed by Blatz and Ko (ref. 1) is used to characterize the material of the membrane. This element was added to HONDO (ref. 2), a finite element code that explicitly integrates the equations of motion. The contact-impact algorithm, which was also added to HONDO, was recently developed by Hallquist (ref. 3) and is briefly described here.

Two examples are provided to demonstrate the capability of the method: in the first, a flat circular membrane is inflated by a pressure loading into a thick-walled sphere; and in the second, the sphere is impacted into the membrane.

FORMULATION

Equation of Motion

Since an explicit time integration scheme is being considered, the equation of motion becomes

$$\vec{M}\ddot{\vec{u}} = \vec{P} - \vec{F} \tag{1}$$

where $\vec{M}$ is the diagonal (lumped) mass matrix, $\ddot{\vec{u}}$ is a global vector of nodal accelerations, $\vec{P}$ is the applied load vector, and $\vec{F}$ is the stress divergence vector. This equation is integrated by the velocity-centered central difference method.

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Material Properties

The strain energy density per unit undeformed volume u_s for a compressible hyperelastic material is expressed as

$$u_s = \mu\left[I_1 - 3 + \frac{1 - 2\nu}{\nu}\left(I_3^{-\frac{\nu}{1 - 2\nu}} - 1\right)\right] \tag{2}$$

where μ is the shear modulus, ν is Poisson's ratio, and I_i is the ith strain invariant. These invariants can be expressed in terms of the principal stretch ratios λ_1, λ_2, λ_3 in the meridional, circumferential, and transverse directions, respectively, as

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_3 &= \lambda_1^2\,\lambda_2^2\,\lambda_3^2 \end{aligned} \tag{3}$$

For thin membranes, the stress component normal to the midsurface is assumed to be zero; hence, λ_3 can be expressed in terms of λ_1 and λ_2

$$\lambda_3 = (\lambda_1\lambda_2)^{-\frac{\nu}{1-\nu}} \tag{4}$$

and the strain energy density becomes a function of λ_1 and λ_2.

Membrane Element

An isoparametric axisymmetric membrane element is shown in Figure 1. The R, Z, and meridional coordinates S of the undeformed configuration are related to the natural coordinate L through

$$\begin{aligned} R &= \frac{1}{2}(1 - L)R^i + \frac{1}{2}(1 + L)R^j \\ Z &= \frac{1}{2}(1 - L)Z^i + \frac{1}{2}(1 + L)Z^j \\ S &= \frac{1}{2}(1 - L)S^i + \frac{1}{2}(1 + L)S^j \end{aligned} \tag{5}$$

and similarly for the displacement components u_r and u_z

$$\begin{aligned} u_r &= \frac{1}{2}(1 - L)u_r^i + \frac{1}{2}(1 + L)u_r^j \\ u_z &= \frac{1}{2}(1 - L)u_z^i + \frac{1}{2}(1 + L)u_z^j \end{aligned} \tag{6}$$

In the deformed configuration, the r and z coordinates along the midsurface are given by

$$\begin{aligned} r &= u_r + R \\ z &= u_z + Z \end{aligned} \tag{7}$$

The principal sketch ratios λ_1 and λ_2 can be defined as

$$\lambda_1 = \left[\left(\frac{dr}{dS}\right)^2 + \left(\frac{dz}{dS}\right)^2\right]^{1/2} \qquad \lambda_2 = \frac{r}{R} \tag{8}$$

Substitution of equations (5) and (6) into equation (7), putting the result into equation (8), and applying the chain rule leads to expressions for λ_1 and λ_2 in terms of the nodal point quantities

$$\lambda_1 = \frac{1}{\ell}\left[\left(R^j + u_r^j - R^i - u_r^i\right)^2 + \left(Z^j + u_z^j - Z^i - u_z^i\right)^2\right]^{1/2}$$

$$\lambda_2 = 1 + \frac{(1 - L)u_r^i + (1 - L)u_r^j}{(1 - L)R^i + (1 + L)R^j} \tag{9}$$

where $\ell = S^j - S^i$.

Since λ_1 and λ_2 are now functions of the natural coordinate L, the total strain energy stored within the membrane element during deformation can be expressed as the integral

$$U = \pi h\ell \int_{-1}^{1} u_s \, RdL \tag{10}$$

in which h is the thickness of the undeformed membrane.

The partial derivatives of U with respect to the nodal displacement components yield nodal point forces that are subsequently accumulated into the stress divergence vector. In the problem under consideration these derivatives can be calculated very easily. For example, the nodal point force acting in the r-direction at the ith node is given by

$$\frac{\partial U}{\partial u_r^i} = \pi h\ell \int_{-1}^{1} \left(T_1 \frac{\partial \lambda_1}{\partial u_r^i} + T_2 \frac{\partial \lambda_2}{\partial u_r^i}\right) RdL \tag{11}$$

where T_1 and T_2 are Lagrange stresses in the meridional and circumferential directions, respectively. A two point Gauss quadrature is used to perform the above integrations.

The lumped masses for each element are found by the addition of the off-diagonal terms of the consistent mass matrix to the diagonal term. Each membrane element yields the following contributions to the nodal point mass at nodes i and j, respectively,

$$m_i = 2\pi\rho\ell h \, (R^i/3 + R^j/6)$$

$$m_j = 2\pi\rho\ell h \, (R^j/3 + R^i/6) \tag{12}$$

where ρ is the mass density of the undeformed membrane.

For stability the time step Δt is restricted such that the inequality

$$\Delta t^2 < \frac{4}{\lambda^2} \tag{13}$$

is satisfied where λ^2 is the maximum eigenvalue of $\vec{M}^{-1}\vec{K}$ in which $\vec{K}$ is the stiffness matrix.

A time step Δt is calculated for every element in the mesh and 90 percent of the smallest value is then used. For the membrane element λ^2 is calculated exactly from

$$\lambda^2 = \frac{1}{M_i}\left(\frac{\partial^2 U}{\partial u_r^{i2}} + \frac{\partial^2 U}{\partial u_z^{i2}}\right) + \frac{1}{M_j}\left(\frac{\partial^2 U}{\partial u_r^{j2}} + \frac{\partial^2 U}{\partial u_z^{j2}}\right) \tag{14}$$

Contact-Constraint Conditions

Two elastic bodies occupying regions B^1 and B^2 in the reference configuration at time $t = 0$ are shown in Figure 2. The boundaries of B^1 and B^2 are denoted by ∂S^1 and ∂S^2, respectively. After deformation at time $t \neq 0$, these bodies occupy regions b^1 and b^2. The boundaries of b^1 and b^2 are denoted by ∂s^1 and ∂s^2. Whenever b^1 and b^2 are in contact, the nodal points on ∂s^1 in the contact region are constrained to slide along line segments connected by nodal points lying on ∂s^2. Separation is permitted when the interface pressure is negative. Impact and release conditions are applied whenever nodal points on ∂s^1 come into contact with ∂s^2. These conditions, which are based on the generalization of those given by Hughes, et al. (ref. 4), conserve linear and angular momentum.

Constraint conditions must be imposed into the equations of motion for each node of ∂s^1 in contact with a segment of ∂s^2. These conditions are imposed through a transformation of displacements which is performed at the beginning of each time step. In this transformation the radial and vertical displacement components of the node on ∂s^1 are transformed into a displacement component tangential to the segment and a relative displacement component normal to the segment of ∂s^2 on which it rests. Since no separation is permitted during the time step the displacement, velocity, and acceleration of this latter component are set to zero. A transformation matrix $\vec{T}$ is constructed which relates the vector of global displacements $\vec{u}$ to a vector $\vec{u}'$ containing the transformed components

$$\vec{u} = \vec{T}\,\vec{u}' \tag{15}$$

Letting $\vec{T}$ remain constant throughout the time step and differentiating equation (15) with respect to time yields

$$\ddot{\vec{u}} = \vec{T}\,\ddot{\vec{u}}' \tag{16}$$

Equation (16) is substituted into equation (1) and the resulting equation is premultiplied by $\vec{T}^t$ in order to obtain the modified equations of motion

$$\vec{M}'\,\ddot{\vec{u}}' = \vec{T}^t(\vec{P} - \vec{F}) \tag{17}$$

which contains the contact constraints. Here $\vec{M}' = \vec{T}^t\,\vec{M}\vec{T}$. Although $\vec{M}$ is diagonal $\vec{M}'$ is not. For computational efficiency the appropriate off-diagonal masses are summed to the diagonal.

After equation (17) is solved for $\ddot{\vec{u}}'$, the normal accelerations of the nodes of ∂s^1 on ∂s^2 relative to ∂s^2 are set to zero. The global accelerations then follow directly from equation (16).

EXAMPLES

In the following examples, all physical quantities are given in nondimensional form. Any consistent units may be assumed without altering the results.

Inflation of a Membrane into a Thick-Walled Sphere

A flat unstretched circular membrane with a thickness of 0.01 and a radius of 2.0 is positioned beneath a thick-walled sphere having an inner radius of 0.40 and an outer radius of 0.60. In the undeformed configuration, the distance measured perpendicularly from the center of the membrane to the center of the sphere is 1.20. The hyperelastic material described by equation (2) is used to model the material of both the membrane and the sphere with ν and μ set to 0.463 and 150. Densities of 1.5 and 0.15 were assumed for the material of the membrane and sphere, respectively.

The membrane is inflated by a pressure p defined by

$$\begin{aligned} 0 \le t \le 0.11 \qquad & p = 1.250 \\ 0.11 < t \le 0.15 \qquad & p = 1.250 - 1.125\left(\frac{t - 0.11}{0.40}\right) \qquad (18) \\ t > 0.15 \qquad & p = 0.125 \end{aligned}$$

and is brought into contact with the sphere.

In Figure 3 the deformed shapes at various times throughout the deformation time history are shown. At late times some wrinkling occurs (for example, note the last frame) and the calculation ceases to be meaningful. A total of eighty elements were used in the calculation. Forty elements were of the membrane type.

Thick-Walled Sphere Impacting a Membrane

In this example the thick-walled sphere impacts the flat circular membrane with an initial velocity of 1.0. The dimensions and material properties of the membrane and sphere are identical to those of the preceding example. In Figure 4 the deformed shapes at various times are shown. Maximum deflection occurs at the center of the membrane at approximately t = 0.90 after which rebound begins. Separation of the sphere and membrane occurs at approximately t = 1.94.

In the above examples the stress at the center of the membrane increases significantly after the initial contact thereby providing evidence that a large amount of sliding occurs during contact.

REFERENCES

1. Blatz, P. J.; and Ko, W. L.: Application of Finite Elastic Theory to the Determination of Rubbery Materials. Trans. of the Soc. of Rheology, vol. 6, 1962, pp. 223-251.

2. Key, S. W.: HONDO - A Finite Element Computer Program for the Large Deformation Dynamic Response of Axisymmetric Solids. Sandia Laboratories, SLA-74-39, Albuquerque, New Mexico, 1974.

3. Hallquist, J. O.: A Procedure for the Solution of Finite Deformation Contact-Impact Problems by the Finite Element Method. Lawrence Livermore Laboratory, UCRL-52066, Livermore, California, 1976.

4. Hughes, T. J.; Taylor, R. L.; Sackman, J. L.; Curnier, A.; and Kanoknukulchai, W.: A Finite Element Method for a Class of Contact-Impact Problems. J. of Computer Methods in Applied Mechanics and Engineering. (To be published.)

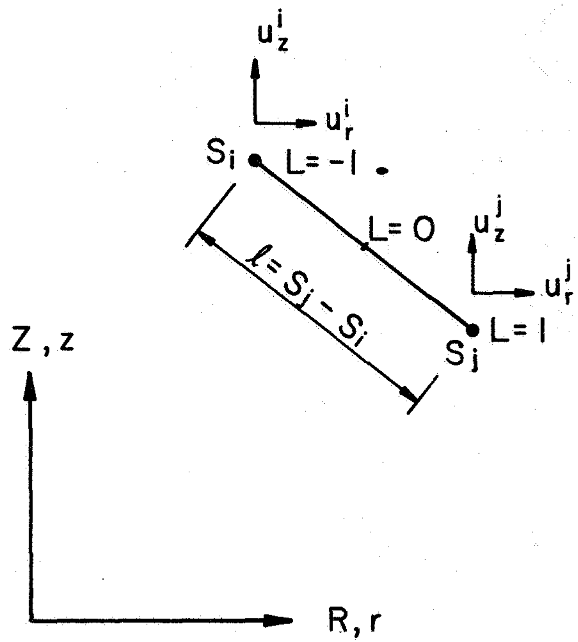

Figure 1.- Definition of membrane element.

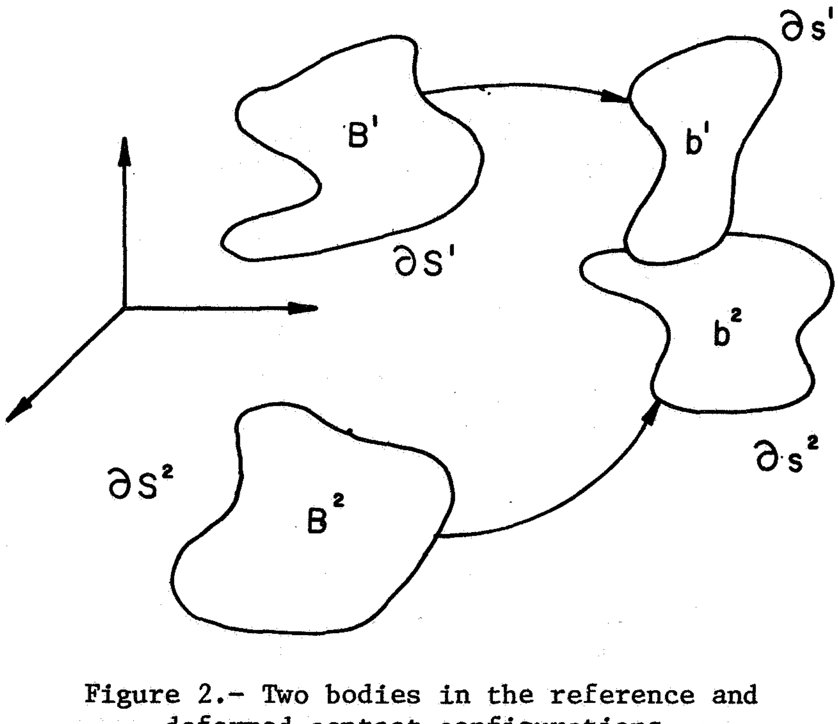

Figure 2.- Two bodies in the reference and deformed-contact configurations.

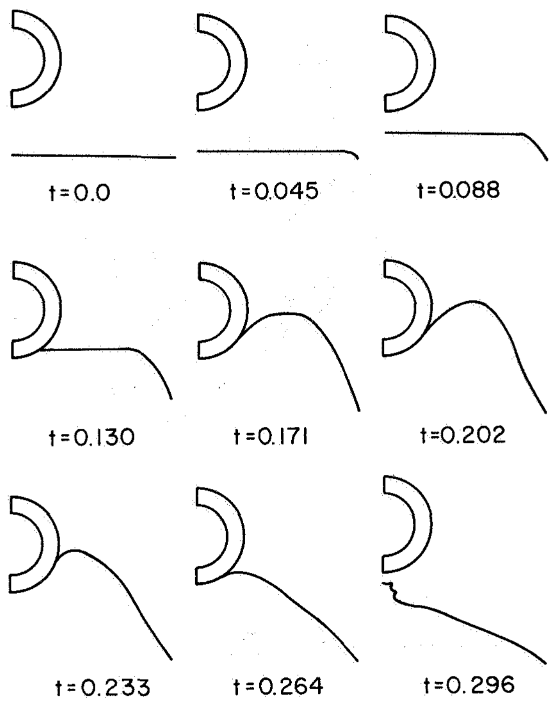

Figure 3.- Inflation of circular membrane into thick-walled sphere.

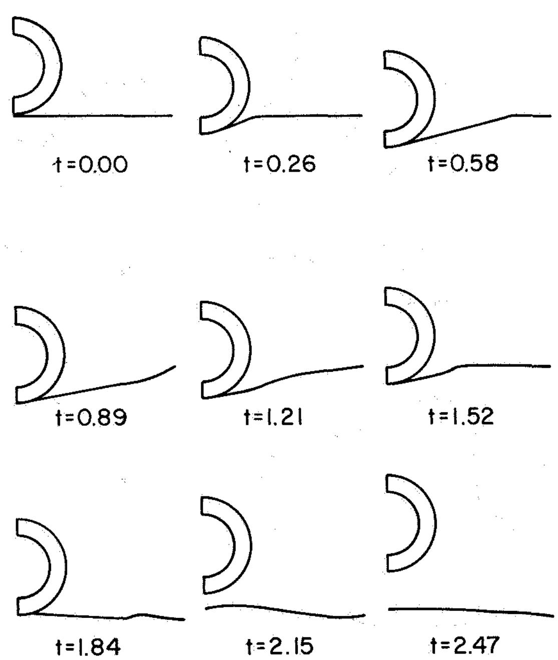

Figure 4.- Impact of thick-walled sphere into circular membrane.