FREE VIBRATIONS OF LAMINATED COMPOSITE ELLIPTIC PLATES

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SUMMARY

A study is made of the free vibrations of laminated anisotropic elliptic plates with clamped edges. The analytical formulation is based on a Mindlin-Reissner type plate theory with the effects of transverse shear deformation, rotary inertia, and bending-extensional coupling included. The frequencies and mode shapes are obtained by using the Rayleigh-Ritz technique in conjunction with Hamilton's principle. A computerized symbolic integration approach is used to develop analytic expressions for the stiffness and mass coefficients and is shown to be particularly useful in evaluating the derivatives of the eigenvalues with respect to certain geometric and material parameters. Numerical results are presented for the case of angle-ply composite plates with skew-symmetric lamination.

INTRODUCTION

Although a number of studies have been devoted to the free-vibration analysis of isotropic elliptic plates (refs. 1 to 4), investigations of orthotropic plates are rather limited in extent (refs. 5 and 6), and to the authors' knowledge, no publications exist dealing with the free vibration of laminated anisotropic elliptic plates. The present study focuses on this problem. More specifically, the objectives of this paper are (1) to present a computational procedure based on the use of computerized symbolic integration in conjunction with the Rayleigh-Ritz technique for the free-vibration analysis of laminated anisotropic elliptic plates and (2) to study the effect of variations in the lamination and geometric parameters of the plate on its vibration characteristics.

The analytical formulation is based on a form of the Mindlin-Reissner plate theory with the effects of transverse shear deformation, anisotropic material behavior, rotary inertia, and bending-extensional coupling included. The frequencies and mode shapes are obtained by using the Rayleigh-Ritz technique in conjunction with Hamilton's principle. The stiffness and mass coefficients are developed using the symbolic and algebraic manipulation language MACSYMA (refs. 7 and 8). Computerized algebraic manipulation, in addition to reducing the tedium of the analysis and the likelihood of errors, was shown to be particularly useful in evaluating the derivatives of the eigenvalues with respect to certain geometric and material parameters. Other applications of computerized algebraic manipulation in structural mechanics are reported in references 9 and 10.

SYMBOLS

a_1, a_2	semimajor and semiminor axes of elliptic plate
$C_{\alpha\beta\gamma\rho}, C_{\alpha 3\beta 3}$	extensional and transverse shear stiffnesses of plate, respectively
$D_{\alpha\beta\gamma\rho}$	bending stiffnesses of plate
E_L, E_T	elastic moduli in direction of fibers and normal to fibers, respectively
$F_{\alpha\beta\gamma\rho}$	stiffness interaction coefficients of plate
G_{LT}, G_{TT}	shear moduli in plane of fibers and normal to plane of fibers, respectively
h	plate thickness
$[K]$	element stiffness matrix
K_{ij}	stiffness coefficients
$[M]$	mass matrix
M_{ij}	mass coefficients
m_o, m_1, m_2	density parameters of plate
T	kinetic energy of plate
U	strain energy of plate
u_α, w	displacement components in coordinate directions
θ	fiber orientation angle of individual layers
ν_{LT}	Poisson's ratio measuring strain in T-direction due to uniaxial normal stress in the L-direction
$\Pi(u_\alpha, w, \phi_\alpha)$	functional defined in equation (1)
ρ	material density of the plate

ϕ_α	rotation components
$\{\psi\}$	vector of undetermined parameters
ψ_i	ith component of vector $\{\psi\}$
Ω	plate domain
ω	circular frequency of vibration of the plate
∂_α	$\equiv \frac{\partial}{\partial x_\alpha}$

MATHEMATICAL FORMULATION

The analytical formulation is based on a form of the Mindlin-Reissner plate theory with the effects of transverse shear deformation, anisotropic material behavior, rotary inertia, and bending-extensional coupling included. A displacement formulation is used with the fundamental unknowns consisting of the displacement and rotation components of the middle plane of the plate u_α, w, and ϕ_α. (See fig. 1 for sign convention.) Throughout this paper, the range of the Greek indices is 1,2 and a term in which any Greek index appears twice is to be summed over that index. The fundamental unknowns are assumed to have a sinusoidal variation in time with angular velocity ω (the circular frequency of vibration of the plate). The functional used in the development of the stiffness and mass matrices is given by

$$\Pi(u_\alpha, w, \phi_\alpha) = T - U \tag{1}$$

where

$$U = \frac{1}{2} \int \Big[C_{\alpha\beta\gamma\rho}\ \partial_\alpha u_\beta\ \partial_\gamma u_\rho + 2F_{\alpha\beta\gamma\rho}\ \partial_\alpha u_\beta\ \partial_\gamma \phi_\rho + D_{\alpha\beta\gamma\rho}\ \partial_\alpha \phi_\beta\ \partial_\gamma \phi_\rho + C_{\alpha 3\beta 3}\ (\partial_\alpha w\ \partial_\beta w + 2\phi_\alpha\ \partial_\beta w + \phi_\alpha \phi_\beta) \Big]\, d\Omega \tag{2}$$

$$T = \frac{1}{2}\, \omega^2 \int \Big[m_o (u_\alpha u_\alpha + ww) + 2m_1 u_\alpha \phi_\alpha + m_2 \phi_\alpha \phi_\alpha \Big]\, d\Omega \tag{3}$$

In equations (2) and (3), $C_{\alpha\beta\gamma\rho}$, $D_{\alpha\beta\gamma\rho}$, and $F_{\alpha\beta\gamma\rho}$ are extensional stiffnesses, bending stiffnesses, and stiffness interaction coefficients of the plate; $C_{\alpha 3\beta 3}$ are transverse shear stiffnesses of the plate; m_o, m_1, and

m_2 are density parameters of the plate; Ω is the plate domain; and $\partial_\alpha \equiv \frac{\partial}{\partial x_\alpha}$.

The displacement and rotation components are approximated by expressions of the form

$$\begin{Bmatrix} u_\alpha \\ w \\ \phi_\alpha \end{Bmatrix} = [N] \{\psi\} \tag{4}$$

where $[N]$ is a matrix of a priori chosen approximation functions and $\{\psi\}$ is a vector of undetermined coefficients. In the present study the functions in the matrix $[N]$ are chosen to be polynomials in x_1 and x_2.

The stiffness and mass matrices of the plate are obtained by first replacing the generalized displacements in equations (2) and (3) by their expressions in terms of the approximation functions and then applying the stationary condition of the functional Π, namely,

$$\delta\Pi = 0 \tag{5}$$

If the undetermined coefficients $\{\psi\}$ are varied independently and simultaneously, one obtains the following set of equations for the plate:

$$[K]\{\psi\} = \omega^2 [M]\{\psi\} \tag{6}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices of the plate, respectively. The matrix $[K]$ is symmetric and positive definite and the matrix $[M]$ is symmetric. The eigenvalues and eigenvectors are obtained by using the technique described in reference 11.

EVALUATION OF STIFFNESS AND MASS COEFFICIENTS

The stiffness and mass coefficients were evaluated using the computerized symbolic and algebraic manipulation system MACSYMA. The MACSYMA program used in evaluating these coefficients is given in the appendix. The major tasks performed on MACSYMA are

(1) Selecting approximation functions for each of the fundamental unknowns with undetermined coefficients $\{\psi\}$ in equation (4) and developing analytic expressions for the strain and kinetic energies as quadratic functions in $\{\psi\}$

(2) Specifying a pattern-matching technique for evaluating the integrals over the elliptic domain (using the function INT(F) (see appendix))

(3) Forming the stiffness and mass coefficients as second derivatives of the strain and kinetic energies with respect to the undetermined coefficients as

$$K_{ij} = \frac{\partial^2 U}{\partial \psi_i \, \partial \psi_j} \qquad \omega^2 M_{ij} = \frac{\partial^2 T}{\partial \psi_i \, \partial \psi_j} \tag{7}$$

In view of the symmetry of K_{ij} and M_{ij}, only the upper triangular portions are formed in a machine readable (LISP) format. These are subsequently converted using the MACSYMA system to a form which closely resembles FORTRAN code (the MACSYMA program used in the conversion is not included in the appendix). Finally, a TECO program (DEC's editor for PDP-10 computers) is executed to produce the final code.

The aforementioned computerized algebraic manipulation approach significantly reduced the tedium of the analysis and the likelihood of errors. Moreover, since analytic exact expressions are obtained for both the stiffness and mass coefficients, the derivatives of the eigenvalues with respect to any of the material or geometric parameters can be readily computed by using the following formula (ref. 12):

$$\frac{\partial \left(\omega_i^2\right)}{\partial d} = \{\psi\}_i^T \left[\left[\frac{\partial K}{\partial d}\right] - \omega_i^2 \left[\frac{\partial M}{\partial d}\right]\right] \{\psi\}_i \; . \tag{8}$$

where d refers to any of the material or geometric parameters of the plate and subscript i refers to the ith eigenvalue and eigenvector. In equation (8), the eigenvectors are assumed to be $[M]$ orthonormal, i.e.,

$$\{\psi\}_i^T \, [M] \, \{\psi\}_i = 1 \tag{9}$$

The two matrices $\left[\frac{\partial K}{\partial d}\right]$ and $\left[\frac{\partial M}{\partial d}\right]$ can be easily evaluated using the MACSYMA system.

Equation (8) shows that the derivatives of the eigenvalues with respect to any of the geometric and material parameters of the plate can be calculated with little extra work. These derivatives can be used to obtain an approximate estimate for the eigenvalues corresponding to a modified (new) value of the parameters without having to resolve the eigenvalue problem, equation (6). To accomplish this, a first-order Taylor's series expansion of the eigenvalues in terms of the problem parameter is used (see ref. 12)

$$(\omega_i^*)^2 \simeq \omega_i^2 + (d^* - d) \frac{\partial (\omega_i^2)}{\partial d} \tag{10}$$

where an asterisk refers to a modified (new) value.

NUMERICAL STUDIES

Numerical studies were conducted to investigate the effects of variations in the plate geometry and lamination parameters on the vibration characteristics of elliptic plates with clamped edges. Angle-ply laminates having antisymmetric lamination with respect to the middle plane are considered. The material characteristics of the individual layers were taken to be those typical of high-modulus graphite-epoxy composites, namely,

$$E_L/E_T=40 \qquad G_{LT}/E_T=0.6 \qquad G_{TT}/E_T=0.5 \qquad \nu_{LT}=0.25$$

where subscript L refers to the direction of the fibers, subscript T refers to the transverse direction, and ν_{LT} is the major Poisson's ratio. The fiber orientation was taken to be $+\theta/-\theta/+\theta/-\theta/\ldots$, $(0<\theta\leq 45)$. All numerical studies were obtained using the Rayleigh-Ritz technique with 10-term approximation functions for each of the fundamental unknowns. The special symmetries exhibited by the free-vibration modes of antisymmetric laminates were utilized in the analysis (see refs. 13 and 14). The four combinations of symmetry and antisymmetry with respect to the x_1- and x_2-axis have been considered. Typical results are presented in figures 2 to 4 showing the effects of variations in each of the following parameters on the vibration frequencies: (1) the aspect ratio of the plate a_1/a_2, (2) the number of layers of the plate NL, and (3) the fiber orientation angle θ of the individual layers.

Figure 2 shows that for elliptic plates having the same h/a_2, the frequencies of free vibration decrease with the increase in the aspect ratio a_1/a_2. The differences between the frequency curves for thick and thin plates in figure 2 are mainly attributed to transverse shear deformation. As expected, these differences are more pronounced for the higher modes. Figure 3 shows that the frequencies increase rapidly as the number of layers increases from 2 to 4. Further increase in the number of layers does not have significant effect on the lower frequencies. Figure 4 shows that the minimum frequency associated with each of the four basic symmetric-antisymmetric modes increases with the increase in the fiber orientation angle θ from 5° to 45°. This is not true, in general, for the higher modes.

CONCLUDING REMARKS

The free-vibration response of anisotropic plates with clamped edges is studied. The analytical formulation is based on Mindlin-Reissner type theory with the effects of transverse shear deformation, rotary inertia, and bending-extensional coupling included. The frequencies and mode shapes are obtained by using the Rayleigh-Ritz technique in conjunction with Hamilton's principle. A computerized symbolic integration approach is used to develop analytic exact expressions for the stiffness and mass coefficients and is shown to be particularly useful for evaluating the derivatives of the eigenvalues with respect to certain geometric and material parameters. Numerical results are presented showing the effects of variation in the geometric and material parameters on the free-vibration response of composite elliptic plates with clamped edges.

APPENDIX

MACSYMA PROGRAM FOR ANGLE-PLY COMPOSITE ELLIPTIC PLATE

The MACSYMA program used in evaluating the stiffness and mass coefficients of angle-ply elliptic plates is given herein. The definitions of the different symbols in the program are shown on the right.

Approximation Functions (Doubly-Symmetric Vibration Modes)

```
AAA[I]:=AA[I,1]
        +AA[I,2]*X^2+AA[I,3]*Y^2
        +AA[I,4]*X^2*Y^2+AA[I,5]*X^4+AA[I,6]*Y^4
        +AA[I,7]*X^4*Y^2+AA[I,8]*X^2*Y^4+AA[I,9]*X^6+AA[I,10]*Y^6
W:(1-R^2)*AAA[1];
F1:X*(1-R^2)*AAA[2];
F2:Y*(1-R^2)*AAA[3];
U:Y*(1-R^2)*AAA[4];
V:X*(1-R^2)*AAA[5];

GRADEF(R,X,COS(TH)/A)$
GRADEF(R,Y,SIN(TH)/B)$
GRADEF(TH,X,-Y/(A*B*R^2))$
GRADEF(TH,Y,X/(A*B*R^2))$
```

X	≡	x_1
Y	≡	x_2
U	≡	u_1
V	≡	u_2
W	≡	w
F_1	≡	ϕ_1
F_2	≡	ϕ_2
R^2	≡	$(x_1/a_1)^2 + (x_2/a_2)^2$
A	≡	a_1
B	≡	a_2
AA[I,J]	≡	$\{\psi\}$

Plate Stiffness Matrices

```
CCC:MATRIX([CC11,CC12,0],[CC12,CC22,0],[0,0,CC66]);
DDD:MATRIX([DD11,DD12,0],[DD12,DD22,0],[0,0,DD66]);
FFF:MATRIX([0,0,FF16],[0,0,FF26],[FF16,FF26,0]);
CCCC:MATRIX([CC55,0],[0,CC44]);
```

Strain-Displacement Relationships

```
UVDERIV:MATRIX([DIFF(U,X)],[DIFF(V,Y)],[DIFF(U,Y)+DIFF(V,X)])$
PHIDERIV:MATRIX([DIFF(F1,X)],[DIFF(F2,Y)],[DIFF(F1,Y)+DIFF(F2,X)])$
WDERIV:MATRIX([DIFF(W,X)+F1],[DIFF(W,Y)+F2])$
```

Strain Energy

```
UU: (1/2)*TRANSPOSE(UVDERIV).(CCC.UVDERIV + FFF.PHIDERIV)
    +(1/2)*TRANSPOSE(PHIDERIV).(FFF.UVDERIV + DDD.PHIDERIV)
    +(1/2)*TRANSPOSE(WDERIV).CCCC.WDERIV$
```

UU ≡ U

Kinetic Energy

```
T: (1/2)*(U^2+V^2+W^2+(H^2/12)*(F1^2+F2^2))$
KILL(U,V,W,F1,F2,CCC,FFF,DDD,CCCC,UVDERIV,PHIDERIV,WDERIV,AAA,LABELS)$
SUB: [X=A*R*COS(TH),Y=B*R*SIN(TH)];
UU:SUBST(SUB,UU)$
TT:SUBST(SUB,TT)$
```

TT ≡ $T/(\rho h\omega^2)$

Integration Over Elliptic Domain

```
II[0,0]:2$
II[MMM,NNN]:=IF NNN = 0
   THEN (2*MMM-1)/(2*MMM)*II[MMM-1,0]
   ELSE (2*NNN-1)/(2*(MMM+NNN))*II[MMM,NNN-1];
NONZERO(XX):=IS(NOT(XX=0));
CPRED(XXX):=IS(FREEOF(SIN,XXX) AND FREEOF(COS,XXX) AND FREEOF(R,XXX));
MATCHDECLARE(COEF,CPRED)$
MATCHDECLARE([EE1,EE2,EE3],NONZERO)$
DEFRULE(RULE1,COEF*R^EE1*COS(TH)^EE2*SIN(TH)^EE3,
   IF REMAINDER(EE2-1,2)=0 AND REMAINDER(EE3-1,2)=0
      THEN COEF/(EE1+1)*II[(EE2-1)/2,(EE3-1)/2]
      ELSE 0);
INT(XXX):=APPLY1(EXPAND(R*COS(TH)*SIN(TH)*XXX),RULE1);
```

Formation of Upper Triangular Terms of Stiffness and Mass Matrices

```
ARG(J):=[['ELLIPS,8000+J],'MT,'MU,'TT,'UU,'ARG];
FOR J:50 STEP -1 THRU 1 DO[
   J1:1+REMAINDER(J-1,5),
   J2:ENTIER((J+4)/5),
   GT:DIFF(TT,AA[J1,J2]),
   GU:DIFF(UU,AA[J1,J2]),
   FOR I THRU J DO[
      I1:1+REMAINDER(I-1,5),
      I2:ENTIER((I+4)/5),
      HT:DIFF(GT,AA[I1,I2]),
      HU:DIFF(GU,AA[I1,I2]),
      MT[I,J]:IF HT=0 THEN 0 ELSE INT(HT),
      MU[I,J]:IF HU=0 THEN 0 ELSE INT(HU)   ],
   TT:EV(TT,AA[J1,J2]=0),
   UU:EV(UU,AA[J1,J2]=0),
   IF J=27 OR J=21 OR J=1
      THEN (APPLY(SAVE,ARG(J)),   KILL(MT,MU))    ]$
```

$[UU] \equiv [K]$

$[TT] \equiv [M]/\rho h$

MACSYMA Functions

$$DIFF(F,X) = \frac{\partial F}{\partial X}$$

$$INT(F) = \left[\int_{-\pi}^{\pi} \int_{0}^{1} F(R,\theta)\ R\ dR\ d\theta \right] / \pi\ a_1\ a_2$$

GRADE F functions specify that the derivative of the first argument with respect to the second argument is given by the third argument.

The KILL Command erases from memory expressions which are no longer needed.

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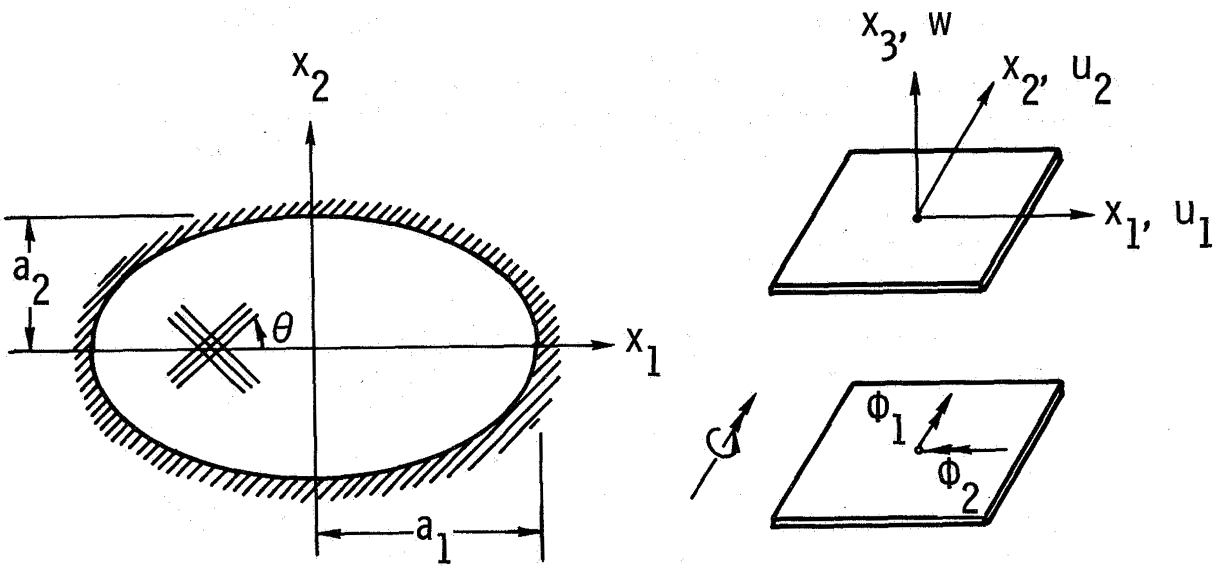
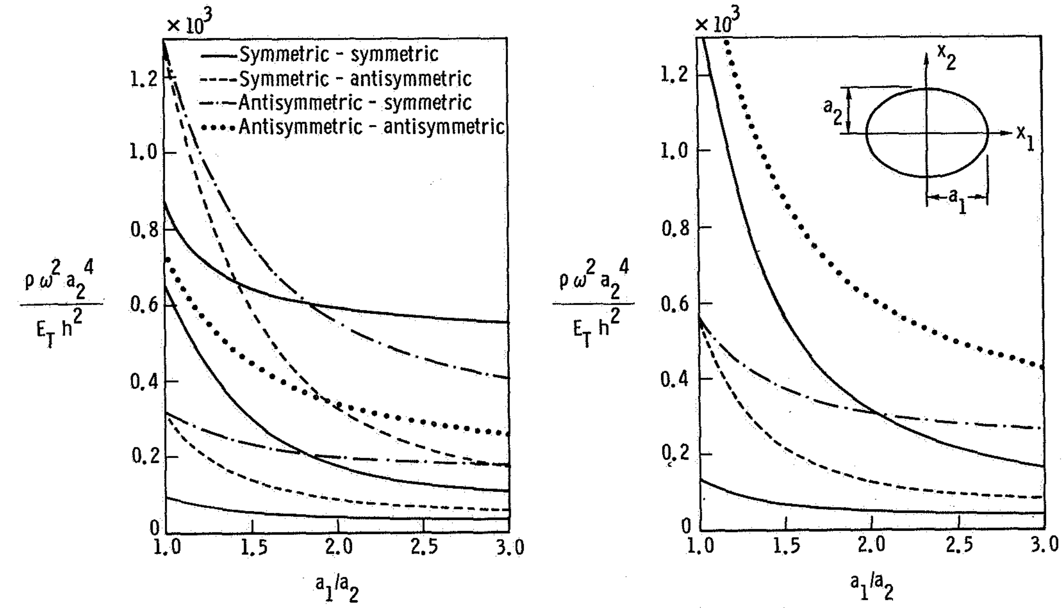


Figure 1.- Elliptic plate and sign convention.



(a) $h/a_2 = 0.1$.   (b) $h/a_2 = 0.01$.

Figure 2.- Effect of a_1/a_2 on the frequencies of clamped elliptic plates with antisymmetric lamination. Eight-layered plates with fiber orientation $45^o/-45^o/45^o/-45^o/45^o/-45^o/45^o/-45^o$.

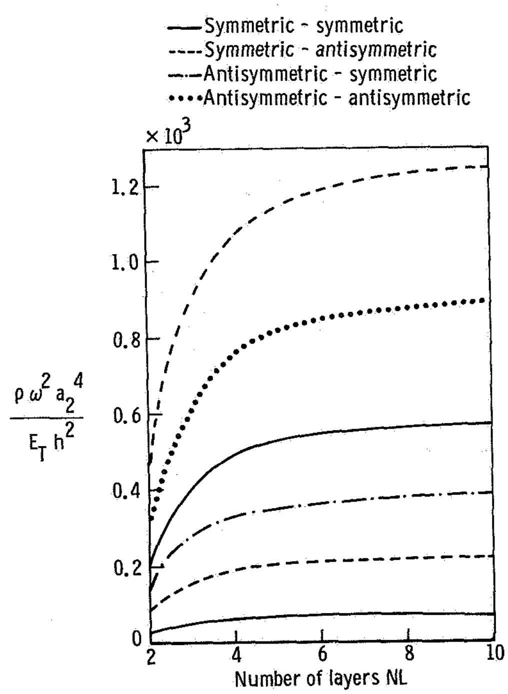


Figure 3.- Effect of number of layers on the frequencies of clamped elliptic plates with antisymmetric lamination. $h/a_2 = 0.01$; $a_1/a_2 = 1.5$; fiber orientation $45^o/-45^o/\ldots$

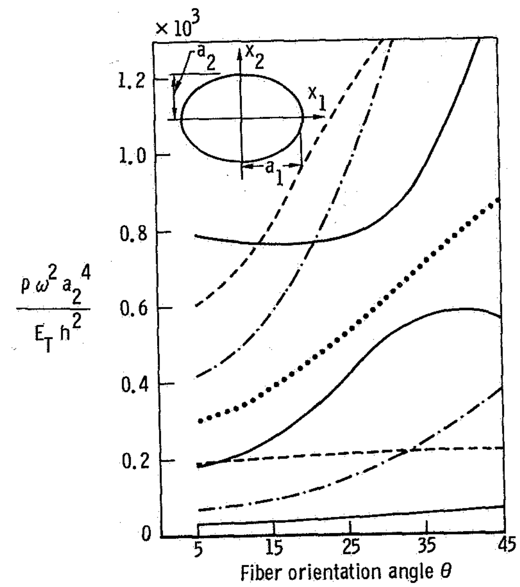


Figure 4.- Effect of fiber orientation θ on the frequencies of clamped elliptic plates with antisymmetric lamination. Eight-layered plates; $h/a_2 = 0.01$; $a_1/a_2 = 1.5$; fiber orientation $\theta/-\theta/\theta/-\theta/\theta/-\theta/\theta/-\theta$.