DYNAMIC INELASTIC RESPONSE OF THICK SHELLS USING ENDOCHRONIC THEORY AND THE METHOD OF NEARCHARACTERISTICS*

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SUMMARY

The endochronic theory of plasticity originated by Valanis has been applied to study the axially symmetric motion of circular cylindrical thick shells subjected to an arbitrary pressure transient applied at its inner surface. The constitutive equations for the thick shells have been obtained. The governing equations are then solved by means of the nearcharacteristics method.

INTRODUCTION

The problem of dynamic plastic response of shells has received considerable attention in recent years. Most of the published works are based on the flow theory of plasticity and usually limited to isotropic linear work-hardening materials. Theoretically, the flow theory is based on the existence of an initial yield surface coupled with an assumed hardening rule to obtain subsequent yield surfaces; an extensive bookkeeping is necessary to trace the evolution of the yield surface which changes as deformation progresses. The analysis of inelastic responses of the bodies is therefore complicated by path dependence and the yield condition, which introduces different governing equations in the distinct regions - elastic and inelastic. Valanis (ref. 1) presented a new theory of plasticity termed endochronic theory, which completely abandoned the concept of a yield surface and its subsequent hardening rule.

The endochronic theory of plasticity is based on thermodynamic theory of internal variables and conforms to experimentally observed material behavior. The basis of the endochronic theory is the assumption that the current state of stress is a functional of the entire history of deformation. The influence of past deformation on the current stress is measured in terms of a monotonically increasing time scale of strain-defined (ref. 1) or stress-defined (ref. 2) endochronic time. This theory has been applied to give analytic predictions for the quasi-static mechanical response of engineering materials (metallic (ref. 3) and non-metallic (ref. 4)), the dynamic response of a

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thin-walled tube subjected to a combined longitudinal and torsional step loading (refs. 5,6), and the dynamic plastic response of circular cylindrical thin shells (refs. 7, 8). It has been shown that the theory does indeed have the capability of explaining the observed phenomena quantitatively with sufficient accuracy.

In this paper, the endochronic theory is applied to thick axially-symmetric cylindrical shell subjected to dynamic loading. The governing equations are then solved by the method of nearcharacteristics.

FORMULATION OF THE PROBLEM

Consider a circular cylindrical thick shell with mean radius R and thickness H. For the axisymmetric motion of shell, the stress and strain states are

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_x & \sigma_{xr} & 0 \\ \sigma_{xr} & \sigma_r & 0 \\ 0 & 0 & \sigma_\theta \end{pmatrix} \qquad \underset{\sim}{s} = \frac{1}{3}\begin{pmatrix} 2\sigma_x-\sigma_r-\sigma_\theta & 3\sigma_{xr} & 0 \\ 3\sigma_{xr} & 2\sigma_r-\sigma_x-\sigma_\theta & 0 \\ 0 & 0 & 2\sigma_\theta-\sigma_x-\sigma_r \end{pmatrix} \tag{1}$$

$$\underset{\sim}{\varepsilon} = \begin{pmatrix} \varepsilon_x & \varepsilon_{xr} & 0 \\ \varepsilon_{xr} & \varepsilon_r & 0 \\ 0 & 0 & \varepsilon_\theta \end{pmatrix} \qquad \underset{\sim}{e} = \frac{1}{3}\begin{pmatrix} 2\varepsilon_x-\varepsilon_r-\varepsilon_\theta & 3\varepsilon_{xr} & 0 \\ 3\varepsilon_{xr} & 2\varepsilon_r-\varepsilon_x-\varepsilon_\theta & 0 \\ 0 & 0 & 2\varepsilon_\theta-\varepsilon_x-\varepsilon_r \end{pmatrix} \tag{2}$$

where $\underset{\sim}{\sigma}$ is the Cauchy stress tensor, $\underset{\sim}{\varepsilon}$ is small strain tensor, $\underset{\sim}{s}$ and $\underset{\sim}{e}$ are the deviatoric stress and strain tensors, respectively, and subscripts x, r, θ refer to the components in longitudinal, radial, and circumferential directions, respectively. Let U and W denote the displacements in the axial and radial directions respectively at time t of the cross section a distance x from a reference section, and u and w are the corresponding velocity components. The equation of motion in the x and r directions have the following form:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xr}}{\partial r} - \rho \frac{\partial u}{\partial t} = - \frac{\sigma_{xr}}{R} \tag{3}$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{xr}}{\partial x} - \rho \frac{\partial w}{\partial t} = \frac{\sigma_\theta - \sigma_r}{R} \tag{4}$$

where ρ is the density.

The strain-displacement relations and the corresponding compatibility conditions are

$$\varepsilon_x = \frac{\partial U}{\partial x} \qquad \frac{\partial \varepsilon_x}{\partial t} = \frac{\partial u}{\partial x} \tag{5}$$

$$\varepsilon_\theta = \frac{W}{r} \qquad \frac{\partial \varepsilon_\theta}{\partial t} = \frac{w}{r} \tag{6}$$

$$\varepsilon_r = \frac{\partial W}{\partial r} \qquad \frac{\partial \varepsilon_r}{\partial t} = \frac{\partial w}{\partial r} \tag{7}$$

$$\varepsilon_{xr} = \frac{1}{2}\left(\frac{\partial U}{\partial r} + \frac{\partial W}{\partial x}\right) \qquad \frac{\partial \varepsilon_{xr}}{\partial t} = \frac{1}{2}\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial x}\right) \tag{8}$$

For isotropic material under isothermal condition with the assumption of elastic hydrostatic response, the constitutive equations in the endochronic theory can be found from reference 1 as follows:

$$2\mu \frac{d\underset{\sim}{e}}{d\zeta} = \frac{\alpha \underset{\sim}{s}}{1+\beta\zeta} + \frac{d\underset{\sim}{s}}{d\zeta} \tag{9}$$

$$\sigma_{kk} = 3K\,\varepsilon_{kk} \tag{10}$$

$$d\zeta^2 = K_1 d\varepsilon_{kk} d\varepsilon_{\ell\ell} + K_2 d\varepsilon_{ij} d\varepsilon_{ij} \tag{11}$$

where α, β, K_1, K_2 are the material parameters, μ is shear modulus, K is bulk modulus, kk, $\ell\ell$, and ij are subscripts denoting coordinates, $d\zeta$ is the endochronic time measure with the restriction that $K_1 + K_2/3 \geq 0$, $K_2 \geq 0$, and K_1 and K_2 may not both be zero. From the definition of $\underset{\sim}{s}$ and $\underset{\sim}{e}$ considered in this problem, it is possible to express the time measure approximately as

$$\beta d\zeta = \pm \beta_1 \left[1 + \left(\frac{d\varepsilon_x}{d\varepsilon_\theta}\right)^2 + \left(\frac{d\varepsilon_r}{d\varepsilon_\theta}\right)^2\right]^{1/2} d\varepsilon_\theta \tag{12}$$

where $\beta_1 = E_t/\sigma_o$, E_t is the asymptotic slope of the uniaxial stress-strain curve for large strain, σ_o is the intercept of this slope with the stress axis, and the positive sign holds for straining while the negative sign is for unstraining of $d\varepsilon_\theta$. Using (12), and equations (9), (10) and the compatibility conditions (5) to (8) results in the following:

$$\frac{\partial \sigma_x}{\partial t} - \nu \frac{\partial \sigma_r}{\partial t} - \nu \frac{\partial \sigma_\theta}{\partial t} - E\frac{\partial u}{\partial x} = a_1 \tag{13}$$

$$-\nu \frac{\partial \sigma_x}{\partial t} - \nu \frac{\partial \sigma_r}{\partial t} + \frac{\partial \sigma_\theta}{\partial t} = a_2 \tag{14}$$

$$\frac{\partial\sigma_x}{\partial t} + \frac{\partial\sigma_r}{\partial t} + \frac{\partial\sigma_\theta}{\partial t} - 3K\frac{\partial u}{\partial x} - 3K\frac{\partial w}{\partial r} = a_3 \tag{15}$$

$$\frac{\partial\sigma_{xr}}{\partial t} - \mu\frac{\partial u}{\partial r} - \mu\frac{\partial w}{\partial x} = a^4 \tag{16}$$

where

$$a_1 = \mp\frac{1}{2}\,\frac{\alpha_1(2\sigma_x-\sigma_r-\sigma_\theta)}{1+\beta\zeta}\,\frac{w}{r}$$

$$a_2 = \left[E \mp \frac{1}{2}\,\frac{\alpha_1(-\sigma_x-\sigma_r+2\sigma_\theta)}{1+\beta\zeta}\right]\frac{w}{r}$$

$$a_3 = 3K\,\frac{w}{r}$$

$$a_4 = \mp\frac{\alpha_2\sigma_{xr}}{1+\beta\zeta}\,\frac{w}{r}$$

$$\alpha_1 = \left[1 + \left(\frac{d\varepsilon_x}{d\varepsilon_\theta}\right)^2 + \left(\frac{d\varepsilon_r}{d\varepsilon_\theta}\right)^2\right]^{1/2} \beta_1\alpha$$

$$\alpha_2 = \beta_1\alpha$$

and E is elastic modulus, ν is Poisson's ratio. Equations (3), (4), and (13) to (16) are the fundamental equations of the problem considered here.

NEARCHARACTERISTIC SOLUTION

The governing equations presented in the previous section together with the auxiliary equations can now be written in matrix form as follows:

$$[A]\{X\} = \{B\} \tag{17}$$

where

$$[A] = \left[\begin{array}{cccccc|cccccc|cccccc}
 & & & & -3K & & & & & & & -3K & 1 & 1 & 1 & & & \\
 & & & & -E & & & & & & & & 1 & -\nu & -\nu & & & \\
 & & & & & & & & & & & & -\nu & -\nu & 1 & & & \\
 & & & & & -\mu & & & & & -\mu & & & & & 1 & & \\
1 & & & & & & & & & 1 & & & & & & & -\rho & \\
 & & & 1 & & & & 1 & & & & & & & & & & -\rho \\
\hline
dx & & & & & & dr & & & & & & dt & & & & & \\
 & dx & & & & & & dr & & & & & & dt & & & & \\
 & & dx & & & & & & dr & & & & & & dt & & & \\
 & & & dx & & & & & & dr & & & & & & dt & & \\
 & & & & dx & & & & & & dr & & & & & & dt & \\
 & & & & & dx & & & & & & dr & & & & & & dt
\end{array}\right]$$

$$\{X\} = \left\{ \frac{\partial\sigma_x}{\partial x} \frac{\partial\sigma_r}{\partial x} \frac{\partial\sigma_\theta}{\partial x} \frac{\partial\sigma_{xr}}{\partial x} \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} \,\vdots\, \frac{\partial\sigma_x}{\partial r} \frac{\partial\sigma_r}{\partial r} \frac{\partial\sigma_\theta}{\partial r} \frac{\partial\sigma_{xr}}{\partial r} \frac{\partial u}{\partial r} \frac{\partial w}{\partial r} \,\vdots\, \frac{\partial\sigma_x}{\partial t} \frac{\partial\sigma_r}{\partial t} \frac{\partial\sigma_\theta}{\partial t} \frac{\partial\sigma_{xr}}{\partial t} \frac{\partial u}{\partial t} \frac{\partial w}{\partial t} \right\}^T$$

and

$$\{B\} = \left\{ a_3 a_1 a_2 a_4 \; -\frac{\sigma_{xr}}{R} \; \frac{\sigma_\theta - \sigma_r}{R} \,\vdots\, d\sigma_x d\sigma_r d\sigma_\theta d\sigma_{xr} du dw \right\}^T$$

The above set of equations is of hyperbolic type; the conventional bicharacteristic method would be very tedious for six dependent variables. Using the method of nearcharacteristics first proposed by Sauer (ref. 9), we look for characteristic-like lines in the coordinate planes along which the solution can be extended. (Sauer called these lines nearcharacteristics.) The formulation and numerical technique in the nearcharacteristics resembles the one-dimensional approach except that those partial derivatives which do not lie in the plane of interest are considered of zeroth order in that particular calculation. For example, when the bicharacteristics in the x-t plane are of interest, then those terms in [A] containing partial derivative in r-direction are combined with terms in {B} in equation (17). Now following the same procedures as described in reference 8 for one-dimensional case, the nearcharacteristics in the x-t and r-t planes, respectively, are obtained as follows:

$$dx = dr = 0,0 \tag{18}$$

$$C_D = \frac{dx}{dt} = \frac{dr}{dt} = \pm\sqrt{\frac{(1-\nu)}{(1+\nu)(1-2\nu)}\frac{E}{\rho}} \tag{19}$$

$$C_S = \frac{dx}{dt} = \frac{dr}{dt} = \pm\sqrt{\frac{\mu}{\rho}} \tag{20}$$

The nearcharacteristics obtained here indicate that there are two characteristic cones existing in the present analysis; one of them (eq. (19)) corresponds to the longitudinal wave propagation while the other (eq. (20)) corresponds to shear wave. They are right circular cones with their center lines perpendicular to the x-r plane as shown in figure 1. This is an expected result, because the governing equations have constant coefficients for the highest order terms. There are no convected terms appearing in the present analysis. The compatibility equations along the nearcharacteristics can be found in the same way as in the one-dimensional case. In the x-t plane, we have:

$$d\sigma_x = \pm \rho C_D du + C_1 dx + C_2 dt \qquad \text{along } \frac{dx}{dt} = \pm C_D \tag{21}$$

$$d\sigma_{xr} = \pm \rho C_S dw + C_3 dx + C_4 dt \qquad \text{along } \frac{dx}{dt} = \pm C_S \tag{22}$$

$$d\sigma_r = \frac{\nu}{1-\nu} d\sigma_x + C_5 dt \qquad \text{along } dx = 0 \tag{23}$$

$$d\sigma_\theta = \frac{\nu}{1-\nu} d\sigma_x + C_6 dt \qquad \text{along } dx = 0 \tag{24}$$

where

$$C_1 = -\frac{\sigma_{xr}}{R} - \frac{\partial\sigma_{xr}}{\partial r}$$

$$C_2 = \frac{1}{1+\nu}\left[\frac{E\nu}{1-2\nu}\left(\frac{w}{r}+\frac{\partial w}{\partial r}\right) \mp \frac{\alpha_1}{2}\frac{(2\sigma_x-\sigma_r-\sigma_\theta)}{1+\beta\zeta}\frac{w}{r}\right]$$

$$C_3 = \frac{\sigma_\theta - \sigma_r}{R} - \frac{\partial\sigma_r}{\partial r}$$

$$C_4 = \mp \frac{\alpha_2\sigma_{xr}}{1+\beta\zeta}\frac{w}{r} + \mu\frac{\partial u}{\partial r}$$

$$C_5 = \frac{1}{1-\nu^2}\left\{E\left(\nu\frac{w}{r}+\frac{\partial w}{\partial r}\right) \pm \frac{\alpha_1}{2}\frac{[(1+\nu)\sigma_x-(2-\nu)\sigma_r+(1-2\nu)\sigma_\theta]}{1+\beta\zeta}\frac{w}{r}\right\}$$

$$C_6 = \frac{1}{1-\nu^2}\left\{E\left(\frac{w}{r}+\nu\frac{\partial w}{\partial r}\right) \pm \frac{\alpha_1}{2}\frac{[(1+\nu)\sigma_x+(1-2\nu)\sigma_r-(2-\nu)\sigma_\theta]}{1+\beta\zeta}\frac{w}{r}\right\}$$

Similarly in the r-t plane, we have:

$$d\sigma_r = \pm\rho C_D dw + C_7 dr + C_8 dt \qquad \text{along } \frac{dr}{dt} = \pm C_D \tag{25}$$

$$d\sigma_{xr} = \pm\rho C_S du + C_9 dr + C_{10} dt \qquad \text{along } \frac{dr}{dt} = \pm C_S \tag{26}$$

$$d\sigma_x = \frac{\nu}{1-\nu}d\sigma_r + C_{11}dt \qquad \text{along } dr = 0 \tag{27}$$

$$d\sigma_\theta = \frac{\nu}{1-\nu}d\sigma_r + C_{12}dt \qquad \text{along } dr = 0 \tag{28}$$

where

$$C_7 = \frac{\sigma_\theta - \sigma_r}{R} - \frac{\partial\sigma_{xr}}{\partial x}$$

$$C_8 = \frac{1}{1+\nu}\left[\frac{E\nu}{1-2\nu}\left(\frac{w}{r}+\frac{\partial u}{\partial x}\right) \pm \frac{\alpha_1}{2}\frac{(\sigma_x-2\sigma_r+\sigma_\theta)}{1+\beta\zeta}\frac{w}{r}\right]$$

$$C_9 = -\frac{\sigma_{xr}}{R} - \frac{\partial\sigma_{xr}}{\partial x}$$

$$C_{10} = \mp\frac{\alpha_2\sigma_{xr}}{1+\beta\zeta}\frac{w}{r} + \mu\frac{\partial w}{\partial x}$$

$$C_{11} = \frac{1}{1-\nu^2}\left\{E\left(\nu\frac{w}{r}+\frac{\partial u}{\partial x}\right) \mp \frac{\alpha_1}{2}\frac{[(2-\nu)\sigma_x-(1+\nu)\sigma_r-(1-2\nu)\sigma_\theta]}{1+\beta\zeta}\frac{w}{r}\right\}$$

$$C_{12} = \frac{1}{1-\nu^2}\left\{E\left(\frac{w}{r}+\nu\frac{\partial u}{\partial x}\right) \mp \frac{\alpha_1}{2}\frac{[(2\nu-1)\sigma_x-(1+\nu)\sigma_r+(2-\nu)\sigma_\theta]}{1+\beta\zeta}\frac{w}{r}\right\}$$

Note that each set of the above characteristics lies entirely in planes parallel to one of coordinate planes. Equations (21) to (28) have the appearance of a one-dimensional method of characteristics formulation except that they contain the partial derivative terms in the other coordinate direction. The nearcharacteristics equation derived here can be solved numerically by the one-dimensional technique. Two independent solutions can be obtained, each corresponding to one of the coordinate planes.

NUMERICAL EXAMPLE

Consider a central segment of the Clinch River Breeder Reactor steam generator flow shroud with length $2\ell = 1.0668$ m, mean radius 0.47 m, and thickness 0.0127 m. The material is 2.25 Cr-1 Mo at 756 K. The pressure input function was generated by the hydrodynamics module (ref. 10). A constant volume, step pressure pulse of 13.79 MPa was taken as the source pressure p at the center. This is typical of the maxima observed in large sodium-water reaction experiments during the transient period. Since the pressure loading was supposed to be symmetric with respect to the mid-span, only half-length of the shell needed to be considered here. The boundary conditions for the example are shown in figure 2 as follows:

$$\left.\begin{array}{llll} u = 0 & \text{and} & \sigma_{xr} = 0 & \text{at } x = 0 \\ u = 0 & \text{and} & w = 0 & \text{at } x = \ell \\ \sigma_{xr} = 0 & \text{and} & \sigma_r = -p(x,t) & \text{at } r = 0 \\ \sigma_{xr} = 0 & \text{and} & \sigma_r = 0 & \text{at } r = H \end{array}\right\} \qquad (29)$$

It has been shown in reference 11 that the two independent solutions, each based on one coordinate plane, are numerically unstable while a calculation method obtained by averaging the above mentioned independent solution yields a stable solution. In view of equations (21), (22) and the boundary conditions (29), it appears that the nearcharacteristics equations in x-t plane are not a proper choice at $r = 0$ and $r = H$ because σ_r are being prescribed there. Therefore a combination technique is proposed here: on the boundaries $r = 0$ and $r = H$ the solutions are obtained from r-t plane nearcharacteristics equations while at other points the solutions are obtained from the x-t plane. The numerical results here show that this leads to a stable solution. The advantage of this technique over the averaging method is a tremendous saving in computation time. The resulting pressure history at the midspan ($x = 0$) of the middle surface of the shell is shown in figure 3. The resultant dynamic response of radial displacement and velocity as a function of time for the same center point of the shell is also shown in the figure. In figure 4, shell displacement profiles are shown for several times.

CONCLUDING REMARKS

The endochronic theory of plasticity originated by Valanis has been applied to study the axially symmetric motion of circular cylindrical thick shells subjected to an arbitrary pressure transient applied at its inner surface. The constitutive equations for the thick shells have been obtained. The governing equations are then solved by means of the nearcharacteristics method. It has been shown that a stable solution can be obtained by treating the radial boundaries in one coordinate plane while at other points the solutions obtain from the other coordinate plane.

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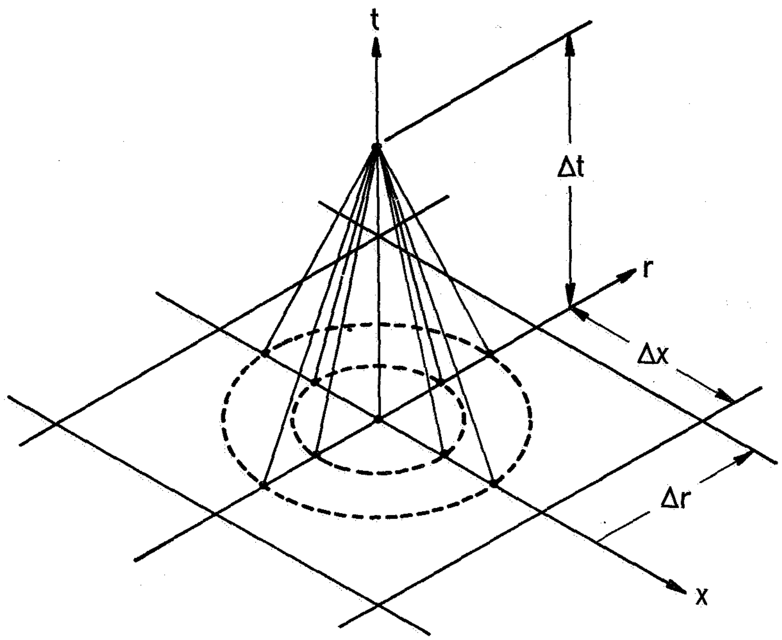

Figure 1.- Nearcharacteristics lines.

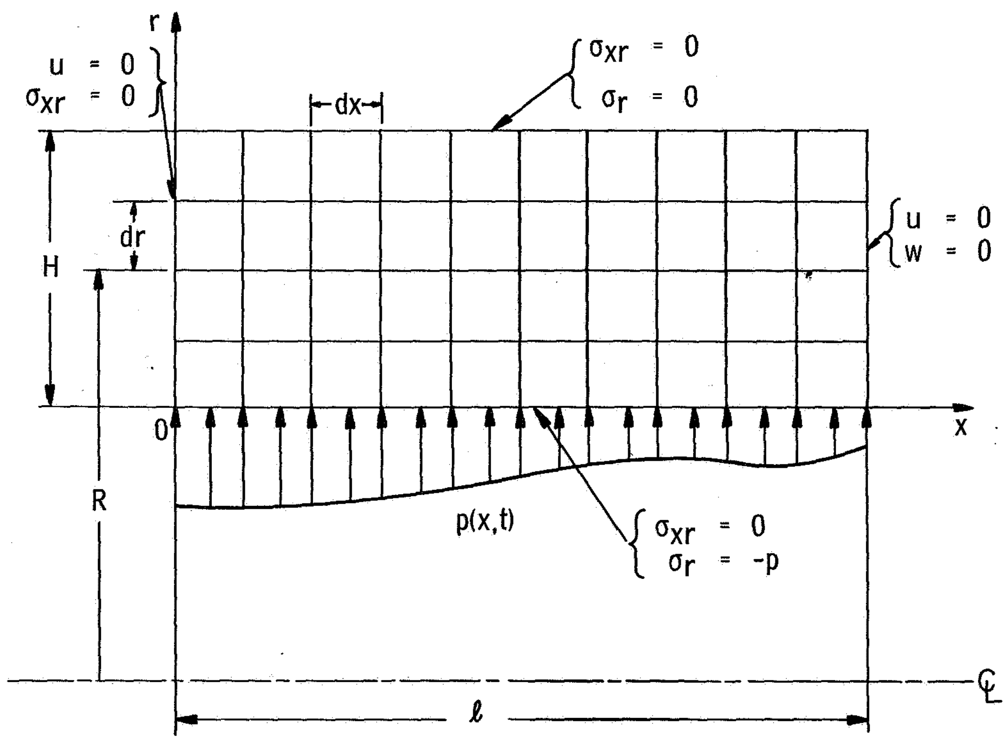

Figure 2.- Boundary conditions.

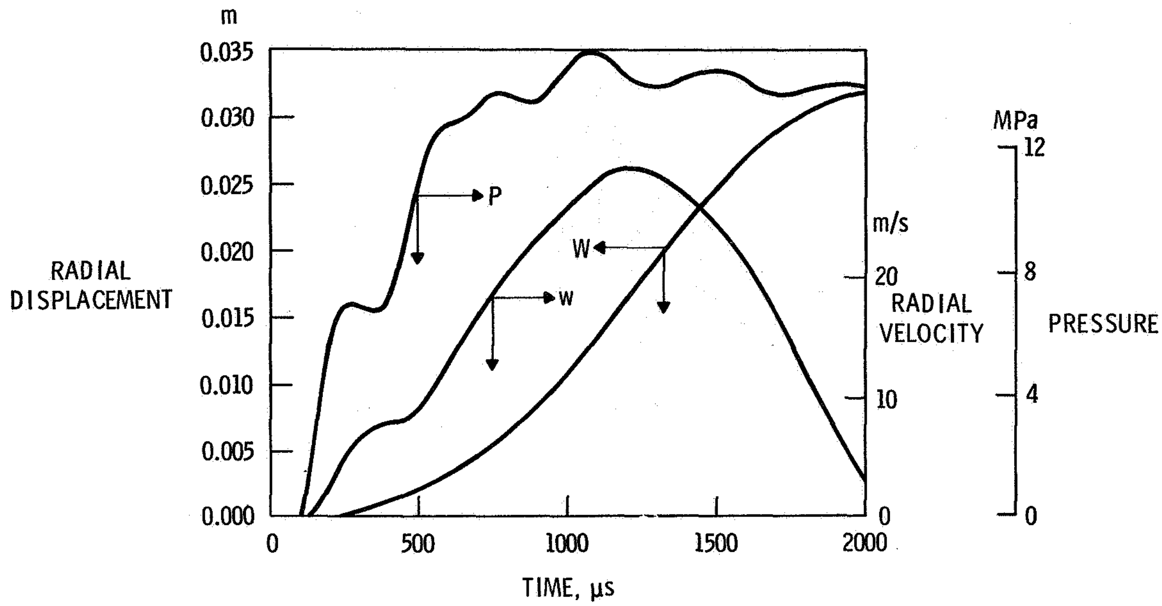

Figure 3.- Radial displacement velocity, pressure history at $x = 0$.

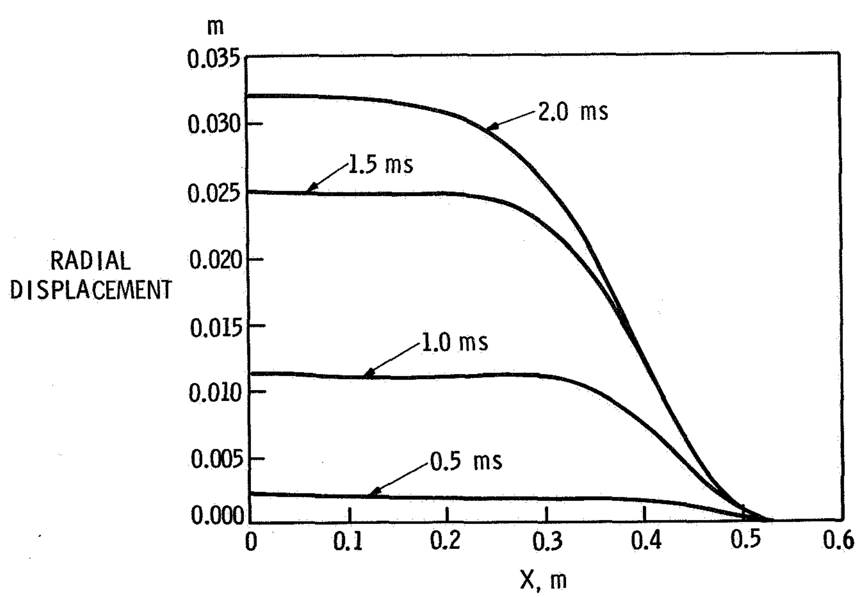

Figure 4.- Radial displacement profiles.