Adjutant

OPTIMAL DESIGN AGAINST COLLAPSE AFTER BUCKLING

E. F. Masur
University of Illinois at Chicago Circle

SUMMARY

After buckling, statically indeterminate trusses, beams, and other "strictly symmetric" structures may collapse under loads which reach limiting magnitudes. The current paper discusses optimal design for prescribed values of these collapse loads.

INTRODUCTION

The principles and techniques of optimally designing structural elements against buckling have been widely investigated. For example, there exists an extensive literature on the problem of finding the least weight design for a column of prescribed Euler buckling strength (see, for example, ref. 1,2,3), and two recent publications (ref. 4,5) deal with the analogous problem of finding the lightest beam to resist lateral buckling under prescribed loads. The common feature of these problems is the fact that the structures considered are statically determinate in the sense that the prebuckling stresses themselves are independent of the design.

If the structure is indeterminate, and if the prebuckling stresses themselves are therefore affected by design changes, the problem becomes vastly more complicated and no general optimality principles appear to have been developed. On the other hand, it is likely that in cases of this type the buckling load itself does not represent an important design criterion. Some structures buckle under decreasing loads and are therefore imperfection-sensitive. Others may buckle under increasing loads, and their actual strength is again governed by factors other than the critical buckling load.

It has been shown that certain "strictly symmetric" types of structures necessarily buckle under increasing loads, and that these loads often approach limiting values as buckling deformations increase indefinitely. Examples of structures of this kind are statically indeterminate trusses (ref. 6) or beams buckling laterally (ref. 7), and recent numerical (ref. 8,9) and experimental (ref. 10) results have confirmed the general theory (ref. 11). It may therefore be realistic to study the optimal design of such structures as their collapse strength, rather than their buckling strength, is prescribed. The object of this paper is to introduce a general discussion of this problem and to indicate a method of solution.

POSTBUCKLING MODEL

The postbuckling behavior of strictly symmetric structures has been described in total generality in reference 11. It can easily be visualized by means of a simple model consisting of a pin-jointed truss of n (say, $n = 2$) degrees of indeterminacy. If the external loads are increased by increasing a common load parameter λ, then the "critical" load value is reached when the compressive force in one of the bars (say, bar 1) reaches the Euler value for that bar. Nevertheless, the load-carrying capacity of the truss is obviously not yet exhausted. While member 1 buckles under sensibly constant compressive force, λ continues to increase until member 2 similarly starts to buckle. Collapse occurs when member 3 also buckles, and $\lambda = \lambda_c$ then remains constant.

This simple process can be visualized within a format that is applicable to all strictly symmetric structures. Let $\underset{\sim}{S}$, the vector of all bar forces, be of the form

$$\underset{\sim}{S} = \lambda \underset{\sim}{S}_o + \alpha_r \underset{\sim}{S}_r \quad , \tag{1}$$

in which, for simplicity, the self-equilibrated bar force systems $\underset{\sim}{S}_r$ are selected so as to satisfy the orthonormality condition

$$\sum_i \frac{S_r^i S_s^i \ell_i}{A_i E_i} \equiv \underset{\sim}{S}_r \cdot \underset{\sim}{S}_s = \delta_{rs} \qquad (r,s = 1,2) \tag{2}$$

where the summation extends over all the bars and ℓ_i, A_i, E_i represent, respectively, the length, cross-sectional area, and Young's modulus of the i^{th} bar. Moreover, if $\underset{\sim}{S}_o$ is the actual force system in the unbuckled structure, $(\alpha_r = 0)$, then

$$\underset{\sim}{S}_o \cdot \underset{\sim}{S}_r = 0 \qquad (r = 1,2) \ . \tag{3}$$

In the absence of any limitations on the tensile strength of any member, the condition of "statical admissibility" is given by

$$S^i \geqq -N^i \qquad (N^i > 0 = \text{Euler force}) \ , \tag{4}$$

which, in view of equation (1), becomes

$$\alpha_r S_r^i \geqq -N^i - \lambda S_o^i \qquad (i = 1,2,\ldots,n) \tag{5}$$

For given value of λ equations (5) define a statically admissible region in the α_r space, whose convex boundary consists of hyperplanes whose normal vectors are proportional to S_r^i (fig. 1). The region so defined need not be closed. For definiteness we assume $\lambda > 0$ and $S_o^i < 0$ $(i = 1,2,3,\ldots,p \leqq n)$; in that case the region "shrinks" for increasing values of λ.

For the sake of brevity we rule out the possibility of multiple buckling modes; then the critical value $\lambda = \lambda_1$ is reached when

$$\lambda_1 S_o^1 = -N^1 \; ; \qquad \lambda_1 S_o^i > -N^i \qquad (i = 2,3,\ldots,n) \; . \tag{6}$$

As bar 1 buckles under constant compressive Euler force the first $(i = 1)$ of equations (5), in view of equation (6), becomes

$$\alpha_r S_r^1 = -(\lambda - \lambda_1) S_o^1 \; . \tag{7}$$

At the same time the changes in the bar chord lengths are given by

$$\begin{aligned} \delta_1 &= \frac{S^1 \ell_1}{A_1 E_1} - \delta_1' \\ \delta_i &= \frac{S^i \ell_i}{A_i E_i} \qquad (i = 2,3,\ldots,n) \end{aligned} \tag{8}$$

in which $\delta_1' > 0$ represents the nonlinear effect of the curvature. Hence

$$\sum_i S_r^i \delta_i = \underset{\sim}{S}_r \cdot \underset{\sim}{S} - S_r^1 \delta_1' = 0 \qquad (r = 1,2) \; , \tag{9}$$

or, with equations (1), (2), and (3),

$$\alpha_r = S_r^1 \delta_1' \qquad (r = 1,2) \; . \tag{10}$$

Finally, when equation (10) is substituted into equation (7),

$$\lambda - \lambda_1 = -\frac{\sum_r \left(S_r^i\right)^2}{S_o^1} \delta_1' > 0 \; , \tag{11}$$

confirming, once again, that strictly symmetric structures have stable points of bifurcation.

For $\lambda < \lambda_1$ the origin 0 of the coordinate system in figure 1 is in the statically admissible region and therefore represents the actual stress point. At bifurcation $(\lambda = \lambda_1)$ the hyperplane B_1 passes through the origin and, for increasing values of λ, the origin moves outside of the statically admissible region, while the stress point P moves with B_1. According to equation (10) the vector $\overline{OP}$ is parallel to the normal to B_1 and, because of the convexity of the stable region, P is therefore closer to 0 than any other statically admissible point.

After bar 2 also buckles, point P lies on the intersection of two hyperplanes, and

$$\alpha_r = S_r^1 \delta_1' + S_r^2 \delta_2' \; . \tag{12}$$

Finally, collapse is reached, for $\lambda = \lambda_c$, when the statically admissible region has shrunk to the point P_c representing the intersection of three hyperplanes. In that case the constant values of α_r are given by

$$\alpha_r^c = S_r^1 \delta_1' + S_r^2 \delta_2' + S_r^3 \delta_3' \qquad (r = 1,2) \; , \tag{13}$$

and as collapse proceeds according to

$$\delta_i' = \omega \delta_i^c \qquad (\omega \to \infty) \; , \tag{14}$$

the collapse mechanism satisfies

$$S_r^1 \delta_1^c + S_r^2 \delta_2^c + S_r^3 \delta_3^c = 0 \qquad (r = 1,2) \; . \tag{15}$$

We also note that, in general, this mode as well as the value of λ_c is independent of initial imperfections.

OPTIMALITY

For the more general case we may identify the major state of stress by means of

$$\underset{\sim}{\sigma} = \lambda \underset{\sim}{\sigma}_o + \alpha_r \underset{\sim}{\sigma}_r \; . \tag{16}$$

The equations of compatibility are given by

$$\underset{\sim}{\sigma}_r^T \left[\underset{\sim}{C} \, \underset{\sim}{\sigma} - \frac{1}{2} \, \underset{\sim}{\ell}_2(\underline{v}) \right] = 0 \qquad (r = 1,2,\ldots,n) \tag{17}$$

in which $\underset{\sim}{C}$ is the compliance density with respect to σ, $\underset{\sim}{\ell}_2$ is the quadratic contribution to the major strain associated with the buckling mode $\underline{v}$, and the notation implies an integral or a summation over the entire structure.

The condition of equilibrium is given in variational form by

$$\underset{\sim}{k}^T(\underline{v}) \, \underset{\sim}{K} \, \underset{\sim}{k}(\delta \underline{v}) - \underset{\sim}{\sigma}^T \underset{\sim}{\ell}_{11}(\underline{v} \, \delta \underline{v}) = 0 \; , \tag{18}$$

where $\underset{\sim}{k}$ is the linear buckling strain tensor and $\underset{\sim}{K}$ the stiffness density with respect to $\underset{\sim}{k}$. We note that both $\underset{\sim}{K}$ and $\underset{\sim}{C}$ are, in general, functions of the design variable h.

For optimality we vary the design by replacing h by $h + \dot{h}$, subject to the condition of constant volume

$$\dot{V} = \frac{dA}{dh} \dot{h} = 0 . \tag{19}$$

Since the load is prescribed it follows that $\dot{\lambda} = 0$; nevertheless, the major stress system (identified by α_r) and the buckling mode $\underline{v}$ may change. Variation of equations (17) and (18) then leads to

$$\underset{\sim}{\sigma}_r^T \left[\frac{d\underset{\sim}{C}}{dh} \underset{\sim}{\sigma}\dot{h} + \underset{\sim}{C}\,\underset{\sim}{\sigma}_s \dot{\alpha}_s - \underset{\sim}{\ell}_{11}(\underline{v}\,\dot{\underline{v}})\right] = 0 \qquad (r = 1,2,\ldots,n) \tag{20}$$

$$\underset{\sim}{k}^T(\dot{\underline{v}})\, \underset{\sim}{K}\, \underset{\sim}{k}(\delta\underline{v}) - \underset{\sim}{\sigma}^T \underset{\sim}{\ell}_{11}(\dot{\underline{v}}\,\delta\underline{v}) = \dot{\alpha}_s \underset{\sim}{\sigma}_s^T \underset{\sim}{\ell}_{11}(\underline{v}\,\delta\underline{v}) - \left[\underset{\sim}{k}^T(\underline{v}) \frac{d\underset{\sim}{K}}{dh} \underset{\sim}{k}(\delta\underline{v}) - \Lambda^2 \frac{dA}{dh}\right]\dot{h} \tag{21}$$

in which Λ^2 has been introduced as Lagrangian multiplier to account for equation (19). Equation (18) represents a homogeneous eigenvalue problem, and equation (21) has therefore no solution unless the condition of integrability

$$\dot{\alpha}_s \underset{\sim}{\sigma}_s^T \underset{\sim}{\ell}_2(\underline{v}) - \left[\underset{\sim}{k}^T(\underline{v}) \frac{d\underset{\sim}{K}}{dh} \underset{\sim}{k}(\underline{v}) - \Lambda^2 \frac{dA}{dh}\right]\dot{h} = 0 \tag{22}$$

is satisfied. We note that equations (20) and (22) are similar to the equations derived for the initial buckling case in reference 4, except for the last term in equation (20) representing the contribution of the postbuckling condition.

Letting once again

$$\underline{v} = \omega \underline{v}_c \qquad \Lambda = \omega\Lambda_c \qquad (\omega \to \infty) \tag{23}$$

and assuming collapse under finite load and stress conditions we obtain

$$\underset{\sim}{\sigma}_r^T \underset{\sim}{\ell}_2(\underline{v}_c) = 0 \qquad (r = 1,2,\ldots,n) \tag{24}$$

$$\underset{\sim}{k}^T(\underline{v}_c)\, \underset{\sim}{K}\, \underset{\sim}{k}(\delta\underline{v}) - \underset{\sim}{\sigma}_c^T \underset{\sim}{\ell}_{11}(\underline{v}_c\,\delta\underline{v}) = 0 \tag{25}$$

$$\underset{\sim}{k}^T(\underline{v}_c) \frac{d\underset{\sim}{K}}{dh} \underset{\sim}{k}(\underline{v}_c) = \Lambda^2 \frac{dA}{dh} \tag{26}$$

of which the first two equations represent the collapse condition, and the last constitutes the condition of optimality. It is noted that once again this optimality condition requires constant strain energy density in the design fibers. It is also noted that for collapse (in contrast to initial buckling) the direct effect of a design change on the collapse mode via the compatibility conditions has disappeared. In other words, we see once again a parallel behavior pattern between collapse through buckling and collapse through perfect plasticity.

EXAMPLE

As an example to illustrate the theory, we consider a beam of length ℓ which is fixed in its own major plane at the right end and subjected to a bending moment λ at the simply supported left end. Collapse occurs when

$$\sigma_c = \lambda_c\left(1 - \frac{3}{2}\frac{x}{\ell}\right) + \alpha_c \frac{x}{\ell} , \tag{27}$$

while the equations of equilibrium (25) assume the form

$$K_1 u_c'' - \sigma_c \beta_c = 0 \qquad (K_2 \beta_c')' + \sigma_c u_c'' = 0 \qquad (0 \leqq x \leqq \ell) \tag{28}$$

where u and β represent the lateral displacement and rotation, respectively, with associated bending and torsional stiffnesses K_1 and K_2. In the development of equations (28), it is assumed that $u = u'' = \beta = 0$ at both ends and that the effect of warping can be neglected. In terms of β alone equations (28) reduce to

$$(K_2 \beta_c')' + \frac{\sigma_c^2}{K_1} \beta_c = 0 \qquad (0 \leqq x \leqq \ell) \tag{29}$$

The collapse condition equation (24) becomes

$$\int_0^\ell x\, u_c'' \beta_c \, dx = \int_0^\ell \frac{x}{K_1} \sigma_c \beta_c^2 \, dx = 0 , \tag{30}$$

while the optimality criterion equation (26) assumes the form

$$\frac{dK_1}{dh}\left(\frac{\sigma_c}{K_1}\right)^2 \beta_c^2 + \frac{dK_2}{dh} \beta_c'^2 = \Lambda_c^2 \frac{dA}{dh} \qquad (0 \leqq x \leqq \ell) . \tag{31}$$

For the specific case of a thin rectangular beam, in which $K_1 = b^3h/12$, $K_2 = b^3h/3$, and $A = bh$, and in view of equation (29), equation (31) can be written in the form

$$h = -\frac{b^2}{3\Lambda_c^2}\beta_c^2\left(\frac{h\beta_c'}{\beta_c}\right)' \quad (32)$$

which lends itself well to an iterative solution scheme. It is also interesting to note that equation (32) is satisfied for constant value of h provided $\beta = \sin \pi x/\ell$; this confirms the curious conclusion arrived at recently by Popelar (ref. 4) that the prismatic design represents an optimum for simply supported beams under constant bending moment.

Numerical results covering equations (29), (30) and (32) for the case under consideration are currently being developed. Because of the variation in the major bending moment, it is expected that in this case the prismatic beam is not optimum, and that optimal design for collapse may lead to a noticeable reduction in weight.

REFERENCES

1. Keller, J. B.: The Shape of the Strongest Column. Arch. Rat. Mech. Anal., Vol. 5, 1960, pp. 275-285.

2. Prager, W. and Taylor, J. E.: Problems of Optimal Structural Design. J. Appl. Mech., Vol. 35, 1968, pp. 102-106.

3. Masur, E. F.: Optimal Placement of Available Sections in Structural Eigenvalue Problems. J. Optim. Theory Appl., Vol. 15, 1975, pp. 69-84.

4. Popelar, C. H.: Optimal Design of Beams against Buckling: A Potential Energy Approach. J. Struc. Mech., Vol. 4, 1976, pp. 181-196.

5. Popelar, C. H.: Optimal Design of Structures against Buckling: A Complementary Energy Approach. J. Struc. Mech., Vol. 5, 1977.

6. Masur, E. F.: Post-Buckling Strength of Redundant Trusses. Trans. ASCE, Vol. 119, 1954, pp. 699-716.

7. Masur, E. F. and Milbradt, K. P.: Collapse Strength of Redundant Beams after Lateral Buckling. J. Appl. Mech., Vol. 24, 1957, pp. 283-288.

8. Bazant, Z. P. and Nimeiri, M. E.: Large Deflection Spatial Buckling of Thin-Walled Beams and Frames. J. Engr. Mech. Div., ASCE, Vol. 99, 1973, pp. 1259-1282.

9. Woolcock, S. T. and Trahair, N. S.: Post-Buckling of Redundant Rectangular Beams. J. Engr. Mech. Div., ASCE, Vol. 101, 1975, pp. 301-316.

10. Trahair, N. S.: Elastic Stability of Continuous Beams. J. Struc. Div., ASCE, Vol. 95, 1969, pp. 1295-1312.

11. Masur, E. F.: Buckling, Post-Buckling and Limit Analysis of Completely Symmetric Structures. Int. J. Sol. Struc., Vol. 6, 1970, pp. 587-604.

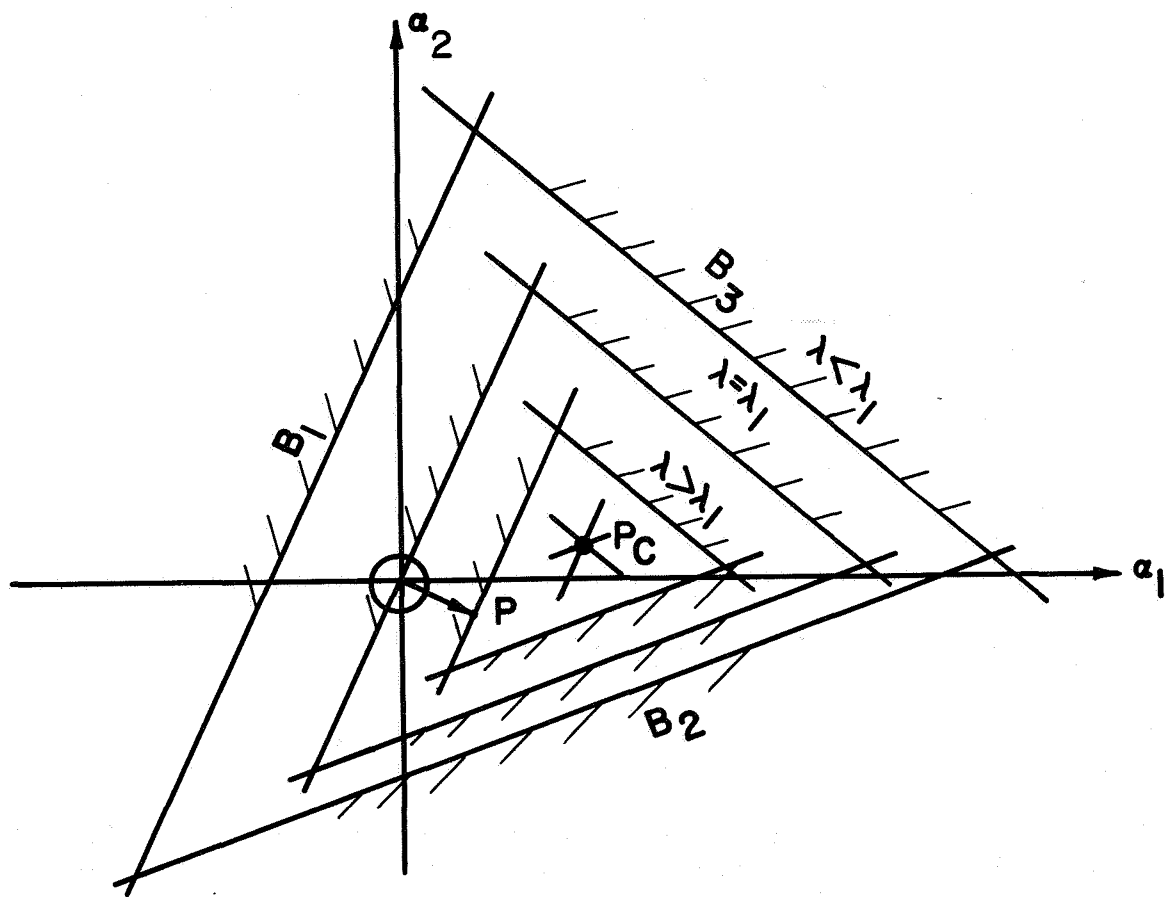

Figure 1.- Redundant stress space.