OPTIMUM VIBRATING BEAMS WITH STRESS AND DEFLECTION CONSTRAINTS

Manohar P. Kamat
Virginia Polytechnic Institute and State University

SUMMARY

The fundamental frequency of vibration of an Euler-Bernoulli or a Timoshenko beam of a specified constant volume is maximized subject to the constraint that under a prescribed loading the maximum stress or maximum deflection at any point along the beam axis will not exceed a specified value. In contrast with the inequality constraint which controls the minimum cross-section, the present inequality constraints lead to more meaningful designs. The inequality constraint on stresses is as easily implemented as the minimum cross-section constraint but the inequality constraint on deflection uses a treatment which is an extension of the matrix partitioning technique of prescribing displacements in finite-element analysis.

INTRODUCTION

The problem of maximizing the fundamental frequency of vibration of beams of a fixed, prescribed volume and likewise its dual problem have been investigated by a great many investigators (see reference 1). It appears that no consensus has been reached however, on the existence of non-trivial solutions for beams with certain types of boundary conditions. While the numerical experiments do strongly emphasize the existence of such solutions (see refs. 2 and 3), mathematical proofs have been constructed (see ref. 4) to prove otherwise. This situation is rather unique since more often than not it is the dismal failure of the numerical techniques in obtaining a solution, which is only presumed to exist, that calls upon mathematics to establish its existence or non-existence.

The difficulty stems from singularities which result from vanishing stiffness at some points along the beam axis. Although at such points the curvature $w,_{xx}$ assumes an infinite value, the products $I(x)w,_{xx}$ and $I(x)w,^2_{xx}$ are nonetheless finite at such points. Furthermore, the function $I(x)w,^2_{xx}$ is required to be integrable over the length of the beam. Fallacies of the mathematical proofs, if any, could well result from a failure to take proper account of these properties for the functions $I(x)$ and $w(x)$.

Finite-element solutions of reference 3, which incidently emphasize existence even in the absence of any inequality constraints appear to have very limited practical value because the resulting designs are far from being useful as load-carrying members. Controlling the minimum cross section of the beam does not appear to be the answer. The optimized beam must sustain a given loading, presumably the worst loading, without exceeding a prescribed level of stress or a prescribed value for the maximum deflection. In general, the cross section with the least area is not necessarily the critical section in terms of stress nor are the constraints on deflections met in a rational and an expeditious manner simply by controlling the minimum cross-sectional area of the beam.

To generate more practical designs, it is deemed appropriate to require that the optimum beam shall not (i) be stressed to more than a specified multiple of the maximum stress or (ii) deflect more than a specified multiple of the maximum deflection of the corresponding uniform beam of the same volume. The present formulation allows the specification of an arbitrary vector of stresses or of deflections, with those corresponding to the uniform beam case being specializations of the arbitrarily specified vectors.

PROBLEM FORMULATION

The formulation is restricted to discretized finite-element models of beams. Since the case of an Euler-Bernoulli beam can be obtained as a special case of a Timoshenko beam, the latter will be implied in the formulation.

The approach is exactly similar to the one used in ref. 3. It consists of maximizing the minimum value of the Rayleigh quotient, ω^2, for a Timoshenko beam subject to the equality and the inequality constraints. For a discretized finite-element model

$$\omega^2 = \frac{[q][K]\{q\}}{[q][M]\{q\}} \tag{1}$$

where [K] and [M] are, respectively, the assembled stiffness and mass matrices derived on the basis of a uniform cross-section beam element and {q} is the mode shape of free vibration. In the case of a Timoshenko beam the stiffness matrix accounts for the effects of shear deformations and the mass matrix accounts for the effects of rotary inertia. Furthermore, for a general case, the stiffness matrix may include the effect of a specified distribution of axial loading and elastic foundation and likewise the mass matrix may include the effects of a specified distribution of non-structural mass.

The optimization is to be carried out subject to the equality constraint of a fixed, given total volume V which for a beam with elements each of length ℓ_i and cross-sectional area A_i, $i=1,2...m$, reduces to

$$\sum_{i=1}^{m} A_i \ell_i = V \tag{2}$$

The required relation between the cross-sectional area and the moment of inertia is provided by a consideration of cross-sectional shapes for which

$$I_i = \rho A_i^n \tag{3}$$

$\rho > 0$ and n being appropriate constants depending upon the type of cross section.

Stress Constraint

It is required that for a beam satisfying eqs. (1) through (3), the Rayleigh quotient of eq. (1) be maximized subject to the constraint that

$$\{\sigma\} \leq k_\sigma^2 \{\tilde{\sigma}\} \tag{4}$$

where $\{\sigma\}$ is the vector of nodal stresses for the optimum beam under a prescribed loading and $\{\tilde{\sigma}\}$ is the vector of prescribed stresses. Since stress at an internal node is discontinuous, the vectors $\{\sigma\}$ and $\{\tilde{\sigma}\}$ are assumed to be of size 2m by one.

A beam element with a cubic transverse displacement field has a linear variation of bending moment within an element. Thus, the maximum bending moment within an element can occur only at the two nodes and hence, as in eq. (4), only the nodal stresses need be monitored for the purposes of implementing the stress constraints.

The stress σ_{1i} due to a bending moment M_{1i} at node 1 of element i is

$$|\sigma_{1i}| = \left|\frac{M_{1i} c_i}{I_i}\right| \tag{5}$$

For cross-sections specified by eq. (3), it can be easily verified that

$$\frac{c_i}{c_i^0} = \left(\frac{I_i}{I_i^0}\right)^{\frac{n-1}{2n}} \tag{6}$$

where quantities with superscript 0 pertain to the uniform beam of total volume V. Equations (5) and (6) together imply that

$$\{\sigma\} = \left\{\frac{M}{(I)^{\frac{n+1}{2n}}}\right\} \tag{7}$$

Accordingly, eq. (4) can be written as

$$\left\{\frac{M}{(I)^{\frac{n+1}{2n}}}\right\} \leq k_\sigma^2 \{\tilde{\sigma}\} \tag{8}$$

The inequality constraint, eq. (8), can be transformed into an equivalent equality constraint by Valentine's principle. An auxiliary functional which is the original functional of eq. (1) modified by the two equality constraints with the aid of undetermined Lagrange multipliers is constructed. In terms of non-dimensional quantities this functional can be shown to be

$$(\omega^2)^* = \frac{[q^*][K^*]\{q^*\}}{[q^*][M^*]\{q^*\}} - \lambda_0^*\left(\sum_{i=1}^{m} A_i^* \ell_i^* - 1\right)$$

$$- \sum_{i=1}^{m} \left\langle \lambda_{1i}^*\left[\left(\frac{M_{1i}^*}{(I_i^*)^{\frac{n+1}{2n}}}\right) - k_\sigma^2\,(\tilde{\sigma}_{1i}^*) + \phi_{1i}^{*2}\right]\right. \tag{9}$$

$$\left. + \lambda_{2i}^*\left[\left(\frac{M_{2i}^*}{(I_i^*)^{\frac{n+1}{2n}}}\right) - k_\sigma^2(\tilde{\sigma}_{2i}^*) + \phi_{2i}^{*2}\right]\right\rangle$$

where

$(\omega^2)^*$ = square of the non-dimensional fundamental frequency

$$= \frac{\gamma}{g}\,\frac{A^0 \ell^4 \omega^2}{EI^0}$$

A^* = non-dimensional cross-sectional area

$$= \frac{A}{A^0} = \frac{A\ell}{V} \tag{10}$$

I^* = non-dimensional cross-sectional moment of inertia

$$= \frac{I}{I^0}$$

M^* = non-dimensional bending moment

$$= \frac{M\ell}{EI^0}$$

$\tilde{\sigma}_i^*$ = non-dimensional stress

$$= \frac{\tilde{\sigma}_i \ell}{Ec^0}$$

where ℓ, A^0, I^0, and c^0 are, respectively, the length, the cross-sectional area, moment of inertia and distance of the extreme fiber from the centroidal axis of the cross-section of the equivalent uniform beam of volume V. ϕ_1^* and ϕ_2^* are the non-dimensional auxiliary functions of $\xi=x/\ell$, which transform the inequality constraints into equivalent equality constraints.

The requirement of the vanishing of the variation of $(\omega^2)^*$ with respect to $\{q^*\}$, λ_j^* and ϕ^* yields the necessary optimality conditions. Based on the work of ref. 3, these conditions can be shown to be the following:

In those portions of the beam where the inequality constraint is not effective, the conditions

$$(nU^*_{bi} + U^*_{si} - T^*_{ti} - nT^*_{ri})/V_i = \text{constant}, \quad i=1,2\ldots m \qquad (11)$$

hold true; while in other portions the stress constraint is effective. In eq. (11) U^*_{bi} and U^*_{si} denote non-dimensional strain energies due to pure bending and shear deformations, respectively; T^*_{ti} and T^*_{ri} denote non-dimensional kinetic energy densities due to translational and rotary inertia, respectively, and V_i denotes the volume of the i-th element.

Implementation of the stress inequality constraint in the optimization procedure proceeds in a manner very similar to the one used for the minimum cross-section inequality constraint of ref. 3. The moments of inertia of elements leading to improved designs are determined by recurrence relations designed to force the specific energy density of eq. (11) to be a constant for all elements assuming initially that none of the elements are governed by any inequality constraint. (See reference 3 for details of these recurrence relations.) In each iteration, however, determining if the stress constraint is effective or not requires a complete static stress analysis of the beam to obtain the vector of nodal stresses. The cross-sectional inertias of those elements which violate the constraint are then set equal to

$$I_i^* = \max\left[\left(\frac{\tilde{M}^*_{1i}}{\tilde{\sigma}^*_{1i}}\right)^{\frac{2n}{n+1}}, \left(\frac{\tilde{M}^*_{2i}}{\tilde{\sigma}^*_{2i}}\right)^{\frac{2n}{n+1}}\right] \qquad (12)$$

The cross-sectional inertias of the other elements which do not violate the inequality constraint are adjusted to meet the volume equality constraint, eq. (2).

Although for statically determinate beams eq. (12) guarantees the satisfaction of the stress constraint in any given iteration of the frequency optimization the same is not true of statically indeterminate beams. For the latter, one could conceivably iterate within the static stress analysis to determine the appropriate element stiffnesses so as to satisfy the stress constraints to within a desired tolerance. However, in view of the iterative nature of the frequency optimization procedure, such additional effort is not warranted especially if stiffness changes in successive iterations are kept small enough.

In view of the equality constraint, eq. (2), it is obvious that the maximum number of elements which may be governed by this constraint is at most m-1 for a consistent constrained optimization.

Deflection Constraint

In this case it is required that for a beam satisfying eqs. (1) through (3), the Rayleigh quotient of eq. (1) be maximized subject to the constraint that

$$\{r\} \leq k_\delta^2 \{\tilde{r}\} \tag{13}$$

where {r} is the vector of nodal displacements for the optimum beam under a prescribed loading and $\{\tilde{r}\}$ is the vector of prescribed displacements. Both vectors are of size (2m+2) by one. As with the stress constraint the maximum number of elements whose cross-sectional moment of inertia can be arbitrarily specified is at most m-1. Hence, under the limiting case of a strict equality in eq. (13), the number of equations which imply prescribed displacements cannot exceed m-1 for a consistent constrained optimization.

In this case, the auxiliary functional in terms of non-dimensional quantities is

$$(\omega^2)^* = \frac{\lfloor q^* \rfloor [K^*]\{q^*\}}{\lfloor q^* \rfloor [M^*]\{q^*\}} - \lambda_0^* \left(\sum_{i=1}^{m} A_i^* \ell_i^* - 1\right) - \sum_{i+1}^{m+1} \lambda_i^* [(r_i^*)^2 - k_\delta^2 (\tilde{r}_i^*)^2 + (\psi_i^*)^2] \tag{14}$$

where

$$\begin{aligned} r_i^* &= r_i/\ell \quad \text{for translational degree of freedom} \\ &= r_i \quad \text{for rotational degree of freedom} \end{aligned} \tag{15}$$

Proceeding as before the optimality conditions can be shown to be eq. (11) in those portions of the beam for which the deflection constraint is not effective; while in other portions the deflection constraint is effective. Since the transverse displacement field varies cubically over the length of the element, satisfaction of the constraint at the two nodes of the element does not guarantee that the constraint is not violated in the interior, especially if large changes in curvatures take place within the element. This is circumvented by refining the discretization sufficiently.

Strictly speaking, the implementation of the stress constraint is, in general, an implicit, nonlinear phenomenon which is rendered explicit by the use of a very simple and approximate relation, eq. (12). No such approximations are necessary for the implementation of deflection constraints. The problem in this case reduces to determing element stiffnesses which guarantee prescribed displacements under prescribed loads. Let $[K^*_s]$ denote the assembled matrix of the supported beam and let $\{r^*_\beta\}$ denote those nodal displacements which violate the constraints, eq.(13). The matrix $[K^*_s]$ and the corresponding displacement and load vectors are accordingly partitioned as

$$\begin{bmatrix} K^*_{\alpha\alpha} & K^*_{\alpha\beta} \\ K^*_{\beta\alpha} & K^*_{\beta\beta} \end{bmatrix} \begin{Bmatrix} r^*_\alpha \\ \tilde{r}^*_\beta \end{Bmatrix} = \begin{Bmatrix} \tilde{Q}^*_\alpha \\ \tilde{Q}^*_\beta \end{Bmatrix} \tag{16}$$

where $\{\tilde{Q}^*_\alpha\}$ and $\{\tilde{Q}^*_\beta\}$ are the vectors of externally prescribed loads with the latter being associated with those degrees of freedom which violate the displacement constraints and are accordingly prescribed as being equal to $\{\tilde{r}^*_\beta\}$. Equations (16) yield

$$[K^*_{\alpha\alpha}]\{r^*_\alpha\}+[K^*_{\alpha\beta}]\{\tilde{r}^*_\beta\}= \{\tilde{Q}^*_\alpha\} \tag{17 a}$$

$$[K^*_{\beta\alpha}]\{r^*_\alpha\}+[K^*_{\beta\beta}]\{\tilde{r}^*_\beta\}= \{\tilde{Q}^*_\beta\} \tag{17 b}$$

Simultaneous solution of equations (17 a) and (17 b) yields

$$[K^*_{\beta\beta}]\{\tilde{r}^*_\beta\}=\{\tilde{Q}^*_\beta\}-[K^*_{\beta\alpha}][K^*_{\alpha\alpha}]^{-1}(\{\tilde{Q}^*_\alpha\}-[K^*_{\alpha\beta}]\{\tilde{r}^*_\beta\}) = \{F^*_\beta\} \tag{18}$$

If the elements of the matrix $[K^*_{\beta\beta}]$ are assumed to be functions of moments of inertia of as many beam elements as the number of prescribed displacements $\{\tilde{r}^*_\beta\}$, then the system of equations (18) can be uniquely solved for the unknown moments of inertia which guarantee the satisfaction of the deflection constraint, eq. (13).

Those displacements which violate the constraints are prescribed as being equal to the specified values. Invariably, more than one alternative will exist for the specification of stiffnesses with prescribed displacements. If both the degrees of freedom of a joint are prescribed, then the moments of inertia of both elements common to the joint must be prescribed. However, if a single degree of freedom is prescribed at a joint, then it is not obvious which of the two elements should have a prescribed stiffness. Herein may lie the nonuniqueness of the resulting solution for beams with certain boundary conditions with certain loadings. A rational criterion for making such a decision should be based on the magnitudes of displacements of one joint relative to the other, since such relative displacements are functions of the properties of the element alone. Accordingly, relative displacements of joints, on either side of the joint whose displacement is prescribed, are determined. The element with the joint which has a higher relative displacement is selected for the purposes of prescribing the moment of inertia.

The procedure is straightforward from this point onwards. The moments of inertia of the constrained elements which guarantee the satisfaction of the deflection constraints are obtained by the solution of eq. (18). The inertias of the remaining elements initially obtained through the use of energy based recurrence relation of reference 3 are finally adjusted to satisfy the equality volume constraint, eq. (2).

RESULTS AND DISCUSSION

In general, because of the necessity of satisfying the equality constraint, eqs. (12) and (18) do not guarantee the satisfaction of the stress and deflection constraints exactly. This causes the optimization procedure to fail to converge or converge extremely slowly to the optimum solution. This is avoided by modifying the inequality constraints with a multiplicative constraint factor, R^{β}, which tends to unity with convergence to the optimum solution. The parameter R is chosen to be the least of the ratios of the prescribed displacements to the actual displacements in the case of displacement constraints or to be the maximum of the ratios of the actual stress to the prescribed stress in the case of stress constraints. β is chosen to be greater than unity. Increasingly higher values of β imply increasingly stiffer designs.

Figures 1 and 2 portray the effects of the implementation of the stress constraints on the optimum design of vibrating beams with two different support conditions. Figure 3 illustrates the effect of implementing the deflection constraint on the optimum design of a vibrating cantilever beam.

Figure 1 considers the case of a cantilever beam subjected to two different types of loading for the implementation of stress constraints in the optimization of its fundamental frequency of free vibration. In one case the loading consists of a concentrated load at the tip with $k_{\sigma}^2=5$ and $\{\tilde{\sigma}\}=(\sigma_{max})^0_{load}\{1\}$. In the other case the loading consists of a concentrated bending moment at the tip with $k_{\sigma}^2=5$ and $\{\tilde{\sigma}\}=(\sigma_{max})^0_{load}\{1\}$. As expected, the constraint corresponding to the moment loading is much more severe and accordingly leads to a drastic reduction of the optimized fundamental frequency. A comparison of these designs with the optimized beam without these constraints emphasizes the importance of such constraints in optimal design.

Figure 2 considers the case of a clamped-clamped beam subjected to a concentrated load at the center with $\{\tilde{\sigma}\}=(\sigma_{max})^0\{1\}$ for two distinct values of k_{σ}^2. If it were not for the stress constraints, the moment of inertia would approach zero at the center of the beam as in reference 3. Severity of the stress constraints brings about increased quantities of material to be disposed around the center of the beam.

Figure 3 illustrates the material distribution of an optimum cantilever beam subject to the deflection inequality constraint with $k_{\delta}^2=5$ and $\{\tilde{r}\}=\{r\}^0{}_{load}$ under a concentrated load at the free end of the beam. Since no singularity exists with inequality constraints of either the displacement or stress type and since the deflected shape of the beam under a concentrated end load or a moment involves no change of curvature, it can be expected that the solution obtained using only ten elements for the cantilever beam model is a good approximation to the optimum continuous model.

In conclusion, it may be remarked that with only a minor change of the computer logic the formulation extends quite easily to cases wherein both deflection and stress constraints are specified simultaneously.

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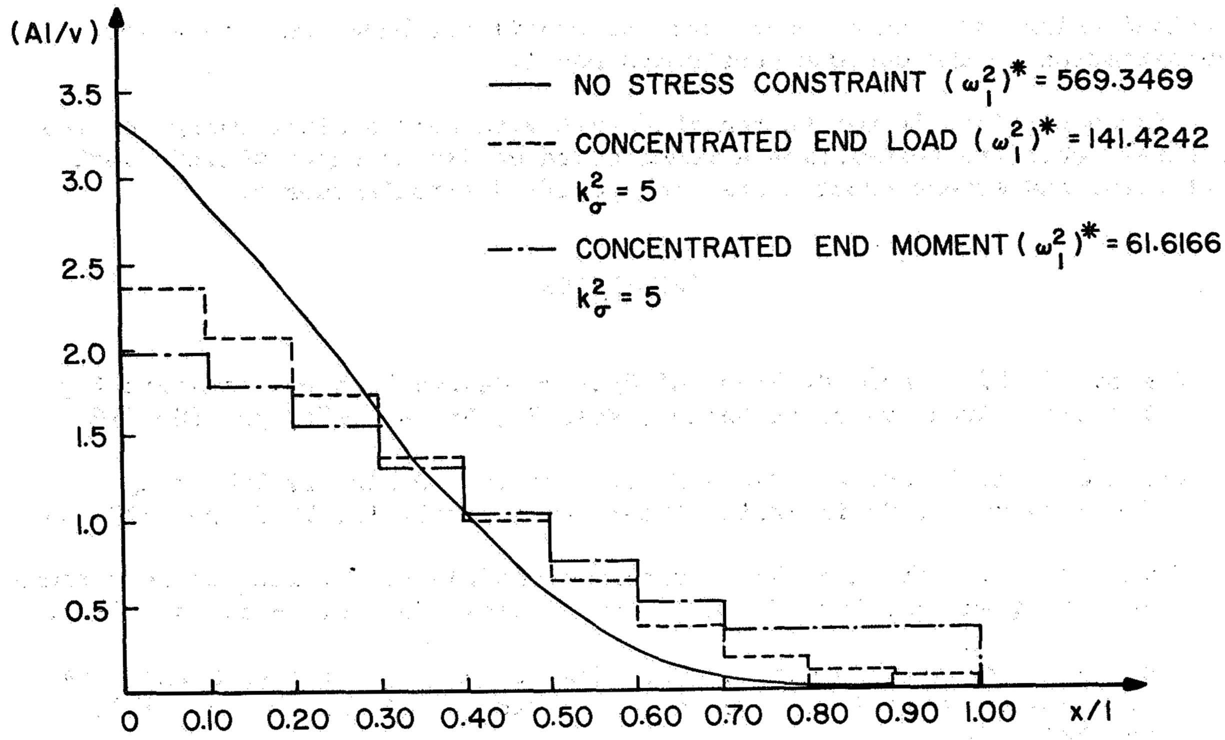

Figure 1.- Optimum area distribution for a beam clamped at x=0 and free at x=ℓ under stress constraints; n=2.

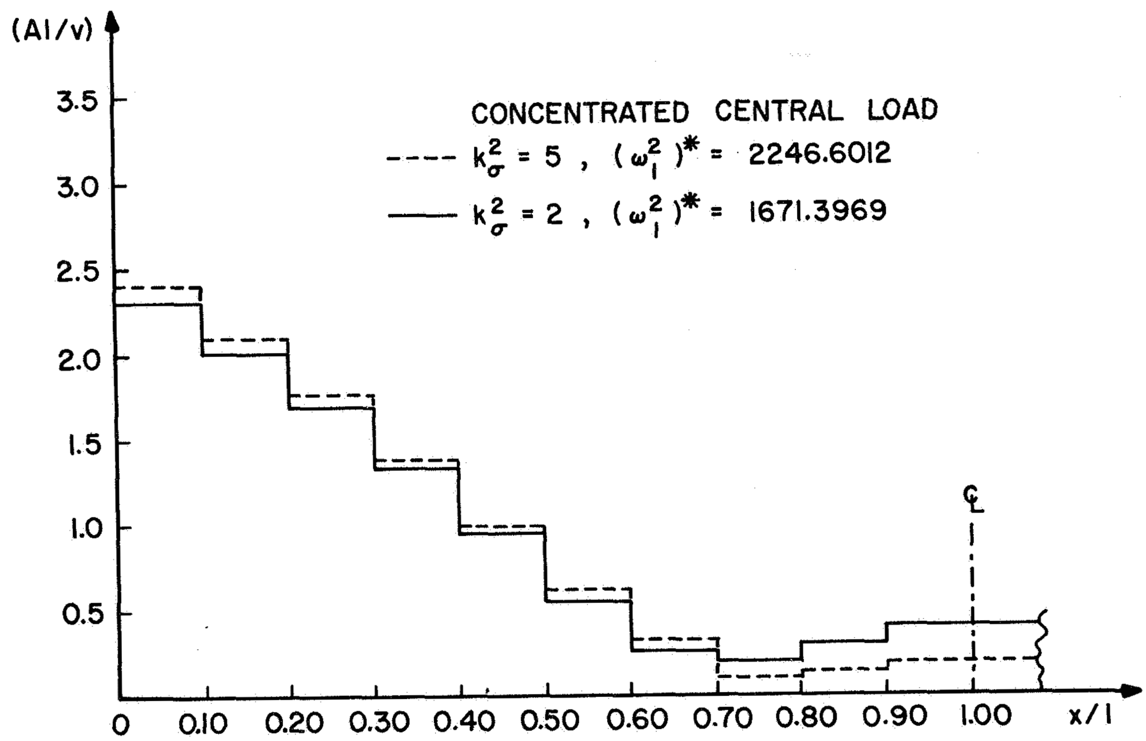

Figure 2.- Optimum area distribution for a beam clamped at both ends under stress constraints; n=2.

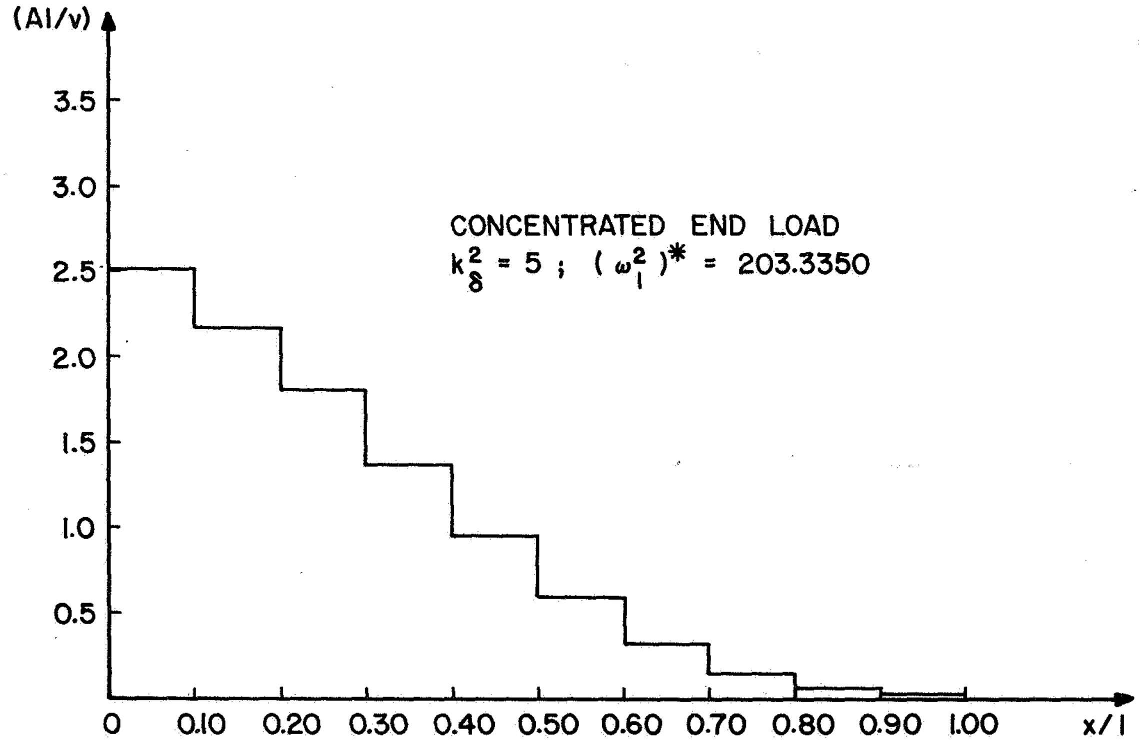

Figure 3.- Optimum area distribution for a beam clamped at x=0 and free at x=ℓ under a deflection constraint; n=2.