AN OPTIMAL STRUCTURAL DESIGN ALGORITHM USING OPTIMALITY CRITERIA

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SUMMARY

An algorithm for optimal design is given which incorporates several of the desirable features of both mathematical programming and optimality criteria, while avoiding some of the undesirable features. The algorithm proceeds by approaching the optimal solution through the solutions of an associated set of constrained optimal design problems. The solutions of the constrained problems are recognized at each stage through the application of optimality criteria based on energy concepts. Two examples are described in which the optimal member size and layout of a truss is predicted, given the joint locations and loads.

INTRODUCTION

In the field of optimal structural design, two general techniques for finding the optimum design may be distinguished: mathematical programming methods and the use of optimality criteria. In the present paper, an algorithm is given which resembles a technique of mathematical programming in that it proceeds by stages, with an improved design generated at each stage. However, in contrast to most mathematical programming methods, the improved design is identified at each stage by the application of optimality criteria, rather than by a search technique. In this way, the computationally expensive search procedure is avoided, yet the principle of approaching the optimum through a succession of small changes is preserved. The algorithm is explained and illustrated by application to the optimal design of a truss, where member cross-sectional areas are taken as the design variables.

SYMBOLS

A_i	cross-sectional area of truss member i
a_r	slack function
$D(p,S^*)$	trial design corresponding to p and S*

E	elastic modulus
F_j	x and y components of external loads applied at nodes and numbered consecutively
L	augmented function
ℓ_i	length of member i
m	total number of nodes
n	total number of truss members, assuming each node connected to every other node by a member
P	potential energy
S	value of lower bound constraint
V	specified volume of material
δ_j	nodal displacements, numbered corresponding to F_j
ε_i	strain of member i
Λ_i	Lagrange multipliers for area constraints
λ	Lagrange multiplier for volume constraint (also equal to specific strain energy of fully-stressed members)
$\eta_k, \eta_k(p,S)$	specific strain energy of member k, corresponding to fully-stressed set p and constraint value S

ENERGY FORMULATION

Consider the problem of finding the maximum stiffness design of a planar truss, given a specified total volume of material to be allocated to the various members of the truss, and specifying inequality constraints on the truss members' cross-sectional areas. The connectivity of the truss is unrestricted; however, locations of nodes are specified beforehand, and the possibility of member buckling is ignored. Taylor (ref. 1) and Hiley (ref. 2) have shown how a problem of the type just described may be formulated by the use of the potential energy function of the structure. In the present paper a similar energy formulation will be used. The potential energy of the truss may be written

$$P = \sum_{i=1}^{n} \ell_i A_i \eta_i - \sum_{j=1}^{2m} F_j \delta_j \tag{1}$$

(See the list of symbols for definitions of the parameters.)

The specific strain energy η_i is related to the strain ε_i by

$$\eta_i = E\varepsilon_i^2/2 \tag{2}$$

where E is the elastic modulus.

The volume constraint is

$$\sum_{i=1}^{n} A_i \ell_i = V \tag{3}$$

where V is the specified volume of material. The inequality constraints are

$$A_i \geq S \tag{4}$$

where S is the specified lower bound constraint.

It can be shown that the problem of maximum stiffness design is equivalent to that of maximizing the potential energy P (refs. 1,3).

The constraints may be introduced directly into the problem formulation by defining the slack functions a_r by

$$A_r - a_r^2 = S, \qquad r = 1,2,\ldots,n \tag{5}$$

and introducing Lagrange multipliers λ and Λ_i to form the augmented function

$$L = P + \lambda(V - \sum_{i=1}^{n} A_i \ell_i) + \sum_{i=1}^{n} \Lambda_i (S - A_i + a_i^2) \tag{6}$$

Requiring the first derivatives of L with respect to δ_k, A_r, and a_r to vanish gives

$$\sum_{i=1}^{n} \ell_i A_i \frac{\partial \eta_i}{\partial \delta_k} - F_k = 0 \tag{7}$$

$$\eta_r \ell_r - \lambda \ell_r - \Lambda_r = 0 \tag{8}$$

$$\Lambda_r a_r = 0 \tag{9}$$

while application of the Kuhn-Tucker theorem of non-linear programming gives

$$\Lambda_r \leq 0 \tag{10}$$

These equations can be shown to be both necessary and sufficient for optimality (refs. 1,4,5).

A basic assumption about the optimal design problem formulated above will now be made. It is assumed that for every value of S in the interval $0<S\leq V/(\sum_{i=1}^{n} \ell_i)$ an optimal design exists. That is, the optimal design is assumed to be a function of S. Furthermore, this function is assumed continuous.

It is of interest to note that at least one optimal design can always be found easily for the value of the lower bound constraint given by

$$S = V/(\sum_{i=1}^{n} \ell_i) \tag{11}$$

For by equation (4) all admissible designs must satisfy

$$A_j \geq S^* \equiv V/(\sum_{i=1}^{n} \ell_i), \qquad j = 1,2,\ldots,n \tag{12}$$

However the strict inequality in equation (12) cannot apply for any j since this would violate the volume constraint in equation (3). Thus the optimal design for the value of S in equation (11) must be the "equally-sized" design

$$A_j = V/(\sum_{i=1}^{n} \ell_i), \qquad j = 1,2,\ldots,n$$

OBSERVATIONS ON GOVERNING EQUATIONS

Inspection of the preceding set of governing equations (3)-(10) leads to several observations of later use in this paper. First note that when a member area A_r in the optimal design is strictly greater than the lower bound constraint value S, then the corresponding slack function $a_r \neq 0$ by equation (5) and $\Lambda_r = 0$ by equation (9), but then equation (8) yields

$$\eta_r = \lambda \tag{13}$$

Thus all members with areas greater than S are stressed to the same level.

Note that by equation (2), equation (13) may be written as a linear equation in the strain ε_r and hence linear in the nodal displacements:

$$\varepsilon_r = \pm\sqrt{2\lambda/E} \tag{14}$$

Next consider a member t in the optimal design which is stressed below the level λ (eqs. (8) and (10) exclude the possibility that an element in the optimal design is stressed <u>above</u> the level λ.):

$$\eta_t < \lambda \tag{15}$$

Then by equation (8) $\Lambda_t \neq 0$ and so equations (9) and (5) imply

$$A_t = S \tag{16}$$

The implication of equations (14) and (16) may be summarized by saying that the members of the optimal design may be divided into two groups: fully-stressed members ($\eta_r = \lambda$ and $A_r > S$) and members at the constraint ($\eta_t < \lambda$ and $A_t = S$). As shall be discussed later in this paper, under certain conditions borderline cases exist where a member is both fully-stressed and at the constraint.

A second observation about the governing equations for the optimal design problem can be made with the help of the fully-stressed condition, equation (14). Introducing equations (14) and (2) into the equilibrium relations (equation 7) yields

$$\sqrt{2\lambda E}\sum_r e_r \ell_r A_r \frac{\partial \varepsilon_r}{\partial \delta_k} + S\sum_t \ell_t \frac{\partial \eta_t}{\partial \delta_k} - F_k = 0 \tag{17}$$

where the first summation is over the set of fully-stressed members, and the second summation is over the set of members at the constraint (hence areas equal S). e_r is the sign associated with member r (compression or tension).

Equations (14) and (17) have been formulated for the problem of maximum stiffness design for a fixed volume of material V. The maximum specific strain energy λ is found as part of the solution. However, this problem may be shown (ref. 6) to be equivalent to the problem of minimum volume design for specified λ. From now on in this paper it will be assumed that a value of λ is specified. The solution corresponding to this value of λ may later be made to correspond to some specified volume of material by multiplying all results by a common factor.

With λ specified, equations (14) and (17) become linear equations in the remaining unknowns δ_k and A_r. Thus once it has been determined which members are to be fully-stressed in the optimal design, the areas and nodal displacements may be calculated by solving a linear system of equations.

FULLY-STRESSED SET AND TRIAL DESIGN

Suppose that a subset of the n members of the truss have specific strain energy λ, as well as specified signs, and do not violate nodal displacement compatibility. These members will be called a "fully-stressed set".

Suppose that a fully-stressed set p has been designated and a value of the lower bound constraint specified, $S = S^*$. In general, it is not known beforehand if p corresponds to an optimal design for $S = S^*$. However, knowing p and S^*, we can nevertheless determine a corresponding set of areas and displacements by writing equations (17) and (14) for the fully-stressed set p and then solving these equations.

The set of areas and displacements found in this way will be written $D(p,S^*)$ and will be called the "trial design corresponding to p and S^*." Note that by assumption the trial design is a continuous function of the lower bound constraint, for fixed p.

Once a trial design $D(p,S^*)$ has been calculated, equations (10) and (4) may be used to determine if the trial design is also an optimal design. If $D(p,S^*)$ is optimal, then p will be called the "optimal fully-stressed set corresponding to S^*."

BASIS FOR ALGORITHM

Using the definitions just introduced, we can now discuss the basis for an algorithm for finding the optimal design.

Starting with a fully-stressed set r and a value of $S = S^*$ such that $D(r,S^*)$ is optimal (finding such a starting design presents no difficulties, as was observed earlier), S is repeatedly reduced and $D(r,S)$ recalculated until a value of S is found for which $D(r,S)$ is non-optimal. Since the cause of the non-optimality must lie in the incorrect choice of fully-stressed members, a method is needed for identifying those members which must be added to or deleted from the optimal fully-stressed set as S decreases. Such a method may be derived from a close examination of the optimal designs in the neighborhood of a point where the optimal fully-stressed set changes.

Consider the particular case where a single member, for example, j, is to be added to the optimal fully-stressed set. In figure 1, $S = S_c$ is the value of the lower bound constraint for which η_j first equals the constraint value λ as S is decreased from a value S_2 slightly above S_c to a value S_1 slightly below S_c. Note that, for $S = S_c$, member j is an example of a "borderline" case referred to earlier ($A_j = S_c$ and $\eta_j = \lambda$).

If p denotes the fully-stressed set for which D(p,S) is optimal for $S_2 \geq S \geq S_c$, then D(p,S) is non-optimal for $S_c > S \geq S_1$, since by hypothesis p lacks the fully-stressed member j.

Denote by q the fully-stressed set obtained from p by adding member j and consider a member, for example, k, which belongs to neither p nor q. By hypothesis,

$$\eta_k(p,S_c) = \eta_k(q,S_c) < \lambda$$

Furthermore since $\eta_k(p,S)$ and $\eta_k(q,S)$ are continuous functions of S, it follows that

$$\eta_k(p,S) < \lambda \text{ and } \eta_k(q,S) < \lambda$$

for $S_1 \leq S < S_c$. For the same range of S, it must also be true that

$$\eta_j(p,S) > \lambda$$

since D(p,S) has been assumed to be non-optimal. Thus the member to be added to the fully-stressed set p to form the optimal fully-stressed set q (for $S_1 \leq S < S_c$) may be determined by examining the non-optimal design $D(p,S_1)$ - the member to be added is that member with specific strain energy exceeding λ. The sign associated with the member j to be added is identical to the sign of member j in $D(q,S_1)$, as may be established by a continuity argument similar to that given above.

The preceding discussion dealt with the procedure for identifying the member to be added to the optimal fully-stressed set as S decreases. An analogous procedure can be developed for identifying the member to be deleted from the optimal fully-stressed set. Proceeding as in the previous paragraphs, it can be shown that the members of the optimal fully-stressed set can be identified by inspection of a non-optimal design $D(p,S_1)$ - the criterion being that the member in p whose area is less than S_1, is to be deleted from p to form the optimal fully-stressed set.

A final remark on the algorithm should be added here. In developing the method for adding or deleting fully-stressed members, the assumption was made that only one element at a time could be both fully-stressed and have area equal to the constraint value. In certain problems, especially where a high degree of symmetry is present, this assumption may be violated. The argument presented above for identifying additions or deletions to the optimal fully-stressed set is no longer generally valid. In the examples considered in the course of this study, several instances were observed where more than one member was fully-stressed and also at the constraint for the same value of S. However, the algorithm had no difficulty in these instances and found the optimal fully-stressed set. The information gained by examining the non-optimal design in the vicinity of a change in the fully-stressed set was a reliable guide in determining the elements to be added or deleted. Thus the lack of theoretical justification for the algorithm in this situation does not appear to be serious.

EXAMPLE PROBLEMS

In figure 2 an example is presented, involving sixteen interior nodes loaded as shown and also two support nodes located far from the interior nodes and not shown in the figure. The optimal design (shown in the figure) is self-equilibrated. In this example, the algorithm was able to select the appropriate sixteen members comprising the optimal design from among all possible members. In achieving this result, no advantage was taken of the symmetry of the problem.

In figure 3, seven internal and four support nodes are specified, and a single applied load is to be carried by the truss. The optimum design is found to contain ten members and is reminiscent of a Michell truss.

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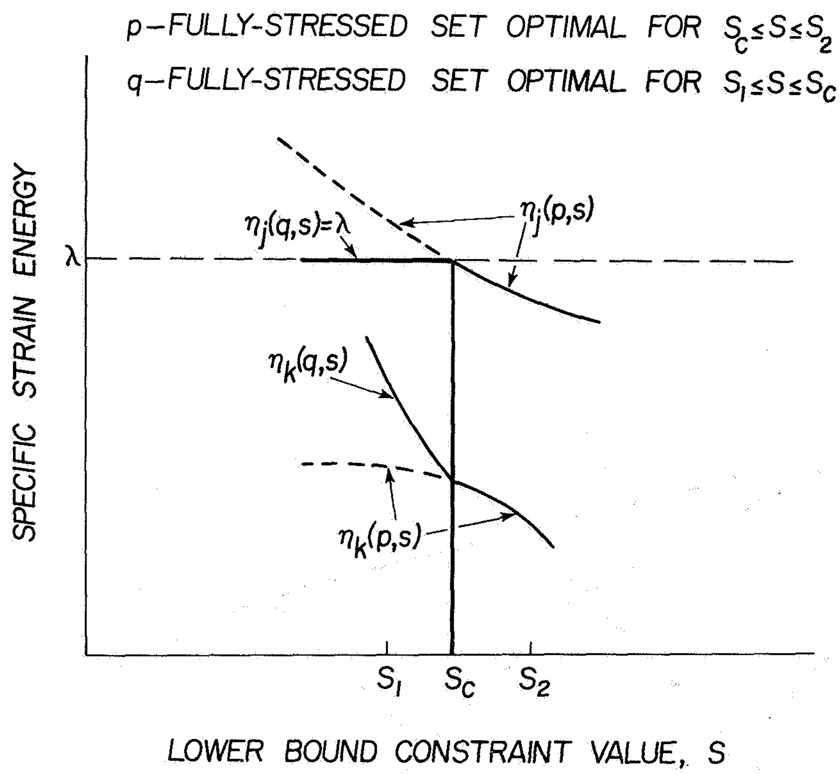

Figure 1.- Specific energies near point where member j is to be added to optimal fully-stressed set.

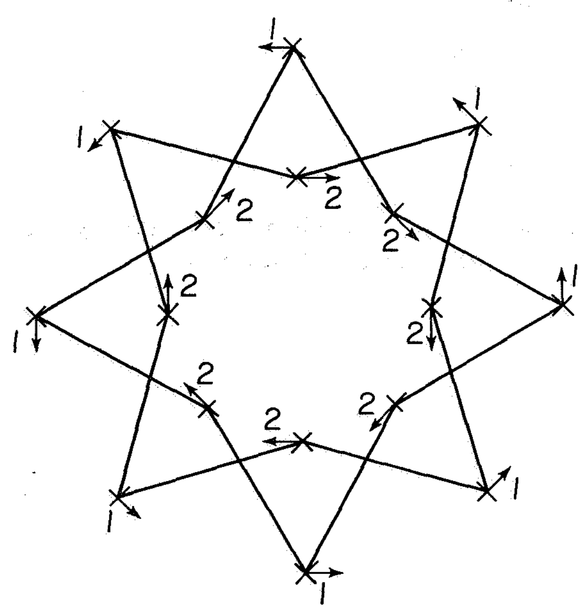

Figure 2.- Optimal truss, with sixteen interior nodes.

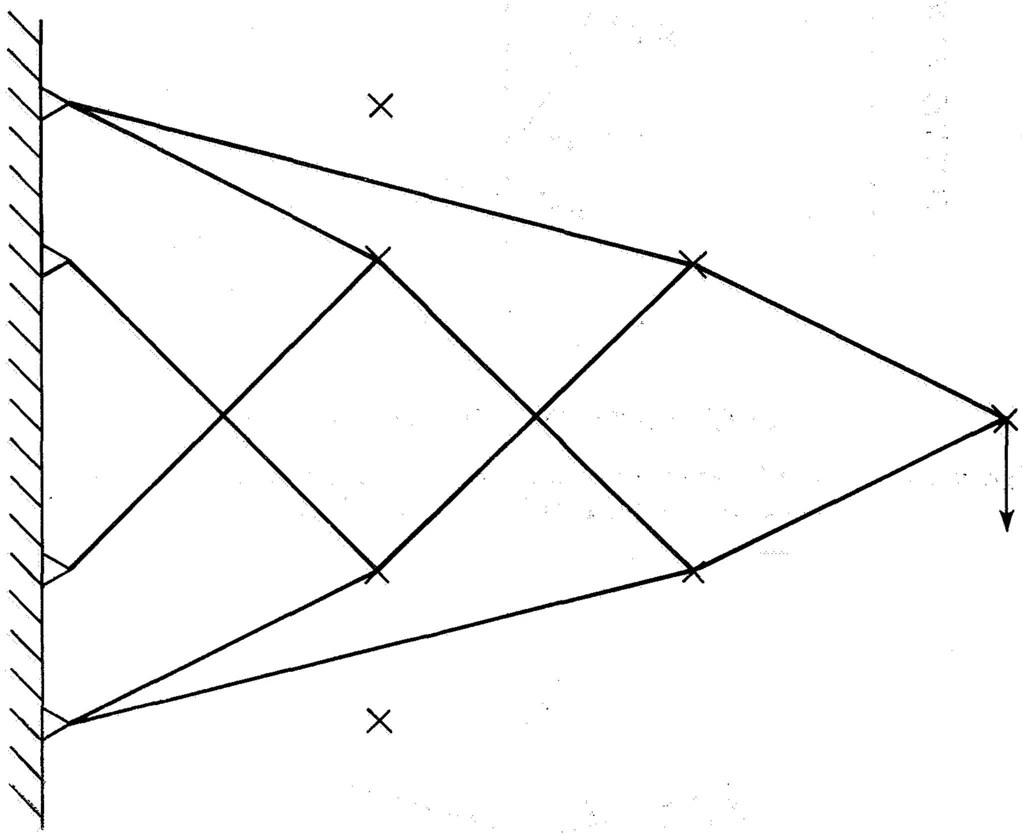

Figure 3.- Optimal truss, with seven interior nodes and four support nodes.