SOME CONVERGENCE PROPERTIES OF FINITE ELEMENT APPROXIMATIONS OF PROBLEMS IN NONLINEAR ELASTICITY WITH MULTI-VALUED SOLUTIONS*

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SUMMARY

Some results of studies of convergence and accuracy of finite element approximations of certain nonlinear problems encountered in finite elasticity are presented. A general technique for obtaining error bounds is also described together with an existence theorem. Numerical results obtained by solving a representative problem are also included.

INTRODUCTION

In this note I summarize some recent results obtained on finite element approximations of certain nonlinear elliptic-boundary-value problems in finite elasticity. The results I quote here are given in a more elaborate form elsewhere. In reference 1, Ricardo Nicolau and I reported some results on a class of problems in which bifurcations occur. There we consider cases in which, for a given set of external forces, not only can multiple solutions occur, but a loss of regularity can apparently result on certain solution paths. A complete account of these results is to be published in a lengthier article.

The principal features of this work are (1) a priori error estimates and proofs of convergence of finite element approximations of highly nonlinear elasticity problems (these estimates are optimal), (2) error estimates for multiple solutions of a nonlinear elliptic problem (these estimates are also optimal, but the predicated bounds are different for different solution paths), (3) a discussion of specific numerical results and certain special problems connected with the numerical analysis of this class of problems.

NOTATION AND PRELIMINARIES

We shall employ the following notations and conventions:

$\underset{\sim}{w} = (u,v,w)$ = displacement vector in a material body $\mathcal{B}$, u, v, and w being the cartesian components of displacement in the material directions X, Y, Z.

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∇w = gradient of w

W = strain energy per unit volume of the body in a reference configuration, W being an appropriately invariant twice-continuously differentiable function of ∇w.

$V = V(w,p)$ = potential of the external forces per unit reference volume, p being a real loading parameter.

$\Sigma = \partial W/\partial \nabla w$ = stress tensor $\equiv \Sigma(w)$

$\mathcal{U}$ = space of admissible displacements = $\{ w: \int_\Omega (W + V)dXdYdZ < \infty; \; w = 0 \text{ on } \partial\Omega \}$

(Here Ω is a bounded open set of particles composing the interior of the body $\mathcal{B}$ and $\partial\Omega$ is its boundary)

To indicate various dependences, we also use such notations as $\Sigma(w)$, $\nabla V(w,p)$, etc.

The potential $V(w,p)$ is assumed to be of the form

$$V(w,p) = -(pf,w) + V_o(w,p)$$

where pf is a body force term and $V_o(w,p)$ is nonlinear in w. To simplify notations, we also introduce the operator

$$\langle A(w,p),\eta\rangle = \int_\Omega \left(\Sigma \cdot \nabla\eta - \frac{\partial V_o}{\partial w} \cdot \eta\right)dXdYdZ \tag{1}$$

Then, <u>formally</u>, A is given by

$$A(w,p) = -\operatorname{Div}\Sigma(w) - \frac{\partial V_o(w,p)}{\partial w} \tag{2}$$

We are concerned with nonlinear boundary-value problems of the following type: find $w \in \mathcal{U}$ such that

$$\langle A(w,p),\eta\rangle = (pf,\eta) \qquad \forall\, \eta \in \mathcal{U} \tag{3}$$

We are particularly concerned with Galerkin approximations of (3). We introduce a real parameter h, $0 < h \leq 1$, which, of course, corresponds to the mesh parameter in finite element approximations, and denote $\{\mathcal{U}_h\}_{0<h\leq 1}$ = a family of finite-dimensional subspaces of $\mathcal{U}$ such that $\bigcup_{0<h\leq 1} \mathcal{U}_h$ is dense in $\mathcal{U}$.

The Galerkin approximation of (3) then amounts to resolving the following problem: find $\underset{\sim}{w}_h \in \mathcal{U}_h$ such that

$$\langle A(\underset{\sim}{w}_h, p), \underset{\sim}{\eta}_h \rangle = (p\underset{\sim}{f}, \underset{\sim}{\eta}_h) \qquad \forall\, \underset{\sim}{\eta}_h \in \mathcal{U}_h \tag{4}$$

Upon subtracting (4) from (3) evaluated on $\underset{\sim}{\eta} = \underset{\sim}{\eta}_h$, we obtain the orthogonality condition:

$$\langle A(\underset{\sim}{w}, p) - A(\underset{\sim}{w}_h, p), \underset{\sim}{\eta}_h \rangle = 0 \qquad \forall\, \underset{\sim}{\eta}_h \in \mathcal{U}_h \tag{5}$$

SOME HYPOTEHSES ON THE STRESS AND POTENTIAL OPERATORS

In many problems in finite elasticity, it appears to be justified to make hypotheses of the following type concerning the operator A and the space $\mathcal{U}$:

I. The operator A of (1) maps $\mathcal{U}$ into its topological dual $\mathcal{U}'$; $\mathcal{U}$ is a reflexive Banach space with norm $||\underset{\sim}{w}||_{\mathcal{U}}$.

II. The displacement field in the body corresponding to a given load p is contained in a space $\hat{\mathcal{U}}$ with stronger topology than $\mathcal{U}$, $\hat{\mathcal{U}}$ being densely and continuously imbedded in $\mathcal{U}$.

III. The operator A is weakly continuous; i.e. if $\{w_n\}$ is any sequence converging weakly to $\underset{\sim}{w}_o$, then $A(\underset{\sim}{w}_n, p)$ converges weakly to $A(\underset{\sim}{w}_o, p)$.

IV. The operator A is coercive; i.e.

$$\lim_{||\underset{\sim}{w}||_{\mathcal{U}} \to +\infty} \frac{\langle A(\underset{\sim}{w}, p), \underset{\sim}{w} \rangle}{||\underset{\sim}{w}||_{\mathcal{U}}} = +\infty \tag{6}$$

V. A sufficient condition that II holds is that A be a potential operator with a Gateaux differential DA such that $\langle DA(\underset{\sim}{w}_o + \theta(\underset{\sim}{w}_n - \underset{\sim}{w}_o)) \cdot \underset{\sim}{\eta}, \underset{\sim}{w}_n - \underset{\sim}{w}_o \rangle = 0$ as $n \to \infty$ for any sequence $\{\underset{\sim}{w}_n\}$ converging weakly to $\underset{\sim}{w}_o$, $\forall\, \underset{\sim}{\eta} \in \mathcal{U}$.

VI. A sufficient condition for coerciveness is that there exists a constant $\mu > 0$ such that

$$\langle A(\underset{\sim}{w}_1, p) - A(\underset{\sim}{w}_2, p), \underset{\sim}{w}_1 - \underset{\sim}{w}_2 \rangle \geq \gamma_o ||\underset{\sim}{w}_1 - \underset{\sim}{w}_2||_{\mathcal{U}}^p - \mu \tag{7}$$

where γ_o is a positive constant and $p > 1$.

VII. There exist functions B: $\mathcal{U} \times \mathcal{U} \to \mathbb{R}$ and C: $\mathcal{U} \times \mathcal{U} \to \mathbb{R}$, B weakly continuous, such that $\forall\ \underset{\sim}{w}_1, \underset{\sim}{w}_2, \underset{\sim}{w}_3 \in \mathcal{U}$,

$$|\langle A(\underset{\sim}{w}_1, p) - A(\underset{\sim}{w}_2, p), \underset{\sim}{w}_3 \rangle| \leq ||\underset{\sim}{w}_3||_{\mathcal{U}} ||\underset{\sim}{w}_1 - \underset{\sim}{w}_2||_{\mathcal{U}} B(\underset{\sim}{w}_1, \underset{\sim}{w}_2) \tag{8}$$

$$|\langle A(\underset{\sim}{w}_1, p) - A(\underset{\sim}{w}_2, p), \underset{\sim}{w}_1 - \underset{\sim}{w}_2 \rangle| \geq \gamma ||\underset{\sim}{w}_1 - \underset{\sim}{w}_2||^p_{\mathcal{U}} \tag{9}$$

where γ is a positive constant and $p > 0$.

Theorem 1 (Existence). Let either of the following hold:

(i) Conditions I, III, and IV above, or

(ii) Conditions I, IV, and V, or

(iii) Conditions I, III, and VI, or

(iv) Conditions I, IV, and VI.

Then there exists at least one vector $\underset{\sim}{w} \in \mathcal{U}$ that satisfies (3) for each $p\underset{\sim}{f} \in \mathcal{U}'$. ■

We emphasize that the operator A is not necessarily monotone.

FINITE ELEMENT APPROXIMATIONS AND ERROR BOUNDS

The subspaces $\mathcal{U}_h$ in (4) are assumed to be constructed using finite element methods. Thus, the solution domain Ω is partitioned into E subdomains Ω_e over which $\underset{\sim}{w}$ is approximated by piecewise polynomials of degree $\leq$ k. If $\underset{\sim}{w} \in \hat{\mathcal{U}} \cap \mathcal{U}$ and $\tilde{\underset{\sim}{w}}_h$ is its projection into $\mathcal{U}_h$, it is well known that the subspace $\mathcal{U}_h$ can be designed so that the following hold:

(i)

$$||\underset{\sim}{w} - \tilde{\underset{\sim}{w}}_h||_{\mathcal{U}} \leq C_o h^{\sigma} ||\underset{\sim}{w}||_{\hat{\mathcal{U}}} \tag{10}$$

h being the mesh parameter and σ a positive number.

(ii)

$$\frac{||\underset{\sim}{w}_h||^2_{\mathcal{U}}}{||\underset{\sim}{w}_h||^p_{\mathcal{U}}} \leq C_1 h^{\nu}, \qquad \nu \geq 0 \tag{11}$$

In (10) and (11), C_o and C_1 are constants independent of h.

We proceed to determine error bounds as follows:

1. The approximation error is $\underset{\sim}{e} = \underset{\sim}{w} - \underset{\sim}{w}_h$:

$$||\underset{\sim}{e}||_{\mathcal{U}} \leq ||\underset{\sim}{w} - \tilde{\underset{\sim}{w}}_h||_{\mathcal{U}} + ||\underset{\sim}{w}_h - \tilde{\underset{\sim}{w}}_h||_{\mathcal{U}} \qquad \text{(by the triangle inequality)}$$

$$\leq C_o h^{\sigma} ||\underset{\sim}{w}||_{\mathcal{U}} + ||\underset{\sim}{w}_h - \underset{\sim}{w}_h||_{\mathcal{U}} \qquad \text{(by (10))}$$

2.

$$||\underset{\sim}{w}_h - \tilde{\underset{\sim}{w}}_h||^2_{\mathcal{U}} \leq C_1 h^{\nu} ||\underset{\sim}{w}_h - \tilde{\underset{\sim}{w}}_h||^p_{\mathcal{U}} \qquad \text{(by (11))}$$

$$\leq C_1 h^{\nu}\, 1/\gamma |\langle A(\underset{\sim}{w}_h,p) - A(\tilde{\underset{\sim}{w}}_h,p), \underset{\sim}{w}_h - \tilde{\underset{\sim}{w}}_h\rangle| \qquad \text{(by (9))}$$

$$= C_1\, 1/\gamma\, h^{\nu} |\langle A(\underset{\sim}{w},p) - A(\tilde{\underset{\sim}{w}}_h;p), \underset{\sim}{w}_h - \tilde{\underset{\sim}{w}}_h\rangle| \qquad \text{(by (5))}$$

$$\leq \frac{C_1}{\gamma} B(\underset{\sim}{w},\tilde{\underset{\sim}{w}}_h) ||\underset{\sim}{w}_h - \tilde{\underset{\sim}{w}}_h||_{\mathcal{U}} ||\underset{\sim}{w} - \underset{\sim}{w}_h||_{\mathcal{U}}\, h^{\nu} \qquad \text{(by (8))}$$

3. For sufficiently small h, we assume that

$$B(\underset{\sim}{w},\tilde{\underset{\sim}{w}}_h) = B(\underset{\sim}{w},\underset{\sim}{w}_h - \underset{\sim}{w} + \underset{\sim}{w})$$

$$= B(\underset{\sim}{w},\underset{\sim}{w}) + O(h^{\mu}) \qquad \mu > 0 \tag{12}$$

owing to the continuity of $B(\cdot,\cdot)$. Thus

$$||\underset{\sim}{w}_h - \tilde{\underset{\sim}{w}}_h||_{\mathcal{U}} \leq \left(\frac{C_1 C_o}{\gamma}\right) h^{\sigma+\nu} ||\underset{\sim}{w}||_{\hat{\mathcal{U}}} B(\underset{\sim}{w}) \tag{13}$$

by virtue of (10), wherein $B(\underset{\sim}{w}) = B(\underset{\sim}{w},\underset{\sim}{w})$.

4. Combining the result 1 with (13), we see that as $h \to 0$, a positive constant C_2 exists such that

$$||\underset{\sim}{e}|| \leq C_2 ||\underset{\sim}{w}||_{\hat{\mathcal{U}}} (h^{\sigma} + h^{\sigma+\nu} B(\underset{\sim}{w})) \tag{14}$$

Thus, for sufficiently smooth w, we obtain the optimal rate of convergence for the nonlinear problem so long as $\nu \geq 0$.

Theorem 2. Let (8), (9), and (13) hold and let there exist solutions to the nonlinear boundary-value problem (3). Let $w_h \in \mathcal{U}_h$ be a finite element approximation of w in a subspace $\mathcal{U}_h$ in possessing properties (10) and (11). Then the approximation error $\underset{\sim}{e} = \underset{\sim}{w} - \underset{\sim}{w}_h$ satisfies the bound (14) as $h \to 0$. Moreover, if $\nu \geq 0$ and w is sufficiently smooth, the optimal rate of convergence is obtained for the nonlinear problem.

AN EXAMPLE AND NUMERICAL EXPERIMENTS

The following example is described in [1]:

$$W = -E_o \ell n\lambda + E_1(\lambda^2 + v'^2 - 1) + E_2(\lambda^2 + v'^2 - 1)^2 + E_3(\lambda^2 + v'^2 - 2) + E_4(\lambda - 1) \tag{15}$$

$$V = -pu + \frac{1}{4} K_o p v^3 \tag{16}$$

where $\lambda = 1 + u'$ ($u = u(x)$, $v = v(x)$), $E_o, \cdots, E_4$, K_o are constants, and $p \geq 0$, In this case,

1. $\mathcal{U} = \{(u,v): \int_0^L (W + V)dX < \infty\} \cap \overset{\circ}{\underset{\sim}{W}}{}^1_4(I)$

$\overset{\circ}{\underset{\sim}{W}}{}^1_4(I)$ = Reflexive Sobolev space = $\{(u,v): \int_0^L (|u'|^4 + |v'|^4)dx < \infty,$

$$u(0) = u(L) = v(0) = v(L) = 0\}.$$

$$|||\underset{\sim}{w}||| = ||u||_{\overset{\circ}{W}{}^1_4(I)} + ||v||_{\overset{\circ}{N}{}^1_4(I)} = \{\int_0^1 |u'|^4 \, dx\}^{\frac{1}{4}} + \{\int_0^L |v'|^4 \, dx\}^{\frac{1}{4}}$$

$$||\underset{\sim}{w}||_{\mathcal{U}} = [||u||^4_{\overset{\circ}{W}{}^1_4(I)} + ||v||^4_{\overset{\circ}{W}{}^1_4(I)}]^{\frac{1}{4}}$$

3. $\hat{\mathcal{U}} = W^{\ell}_4(I) \cap \overset{\circ}{W}{}^1_4(I) \qquad I = (0,L)$

4. $p = 4$, $\sigma = \min(k, \ell-1)$, $\nu = 3/2$

The functions $B(\underset{\sim}{w},\underset{\sim}{w})$ and $C(\underset{\sim}{w},\underset{\sim}{w})$ are complicated functions of the components u and v and are given in [1]. In this case, the operator A is not monotone.

Test problems were solved using piecewise linear finite element ($k = 1$). The problem does not have unique solutions for $p > p_{cr}$. Figure 1 shows the computed solutions for various values of p for the case $L = 10$, $E_1 = 1$, $E_2 = 0.8$, $E_3 = 0.5$, $E_3 = -0.1$, $E_4 = -0.2$, $K_o = 1.0$. Observe that a bifurcation is reached at $p = 0.5$.

Figure 2 shows the rate of convergence actually obtained in the analysis computed by comparing the solution for coarse meshes with that obtained for 100 elements. As predicted, the rate of convergence is

$$O(h^{\sigma} + h^{\sigma+\nu}) = O(h + h^{5/2}) = O(h)$$

REFERENCES

1. Oden, J. T. and Nicolau, R., "Analysis of Finite Element Approximations of a Boundary-Value Problem in Finite Elasticity," Formulations and Computational Algorithms in Finite Element Analysis, Edited by J. Bathe, J. T. Oden, and W. Wunderlich, M.I.T. Press, Cambridge, Mass., 1976.

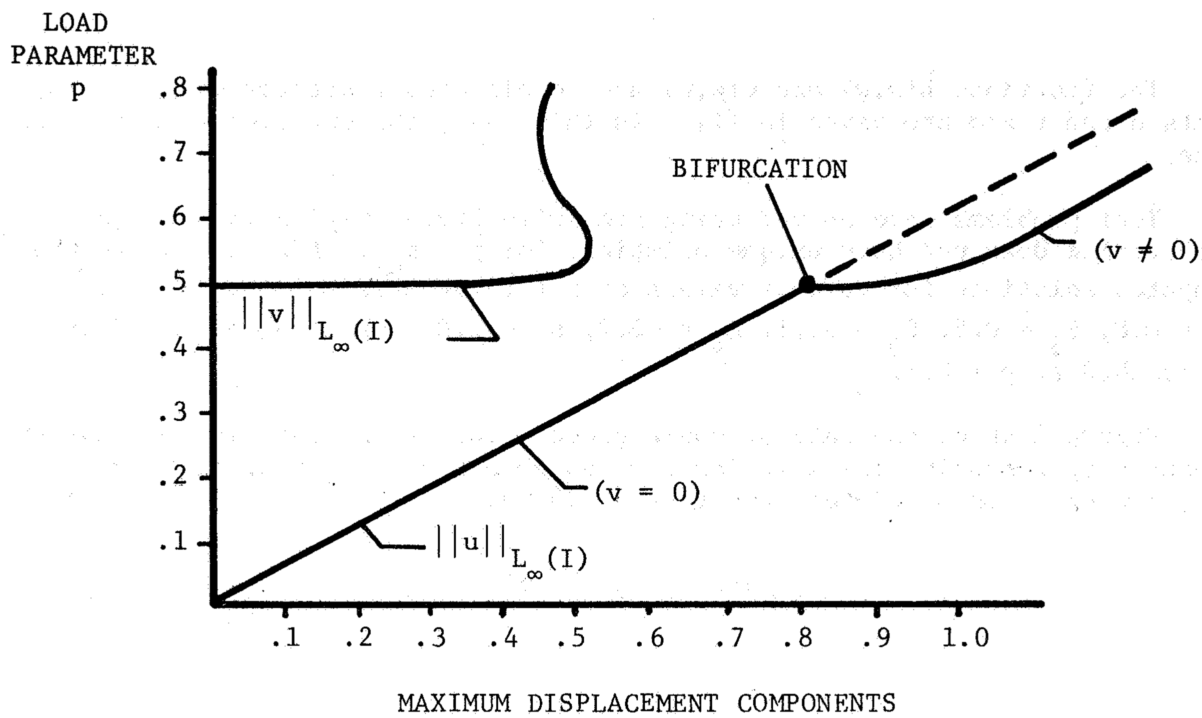

Figure 1.- Computed equilibrium paths.

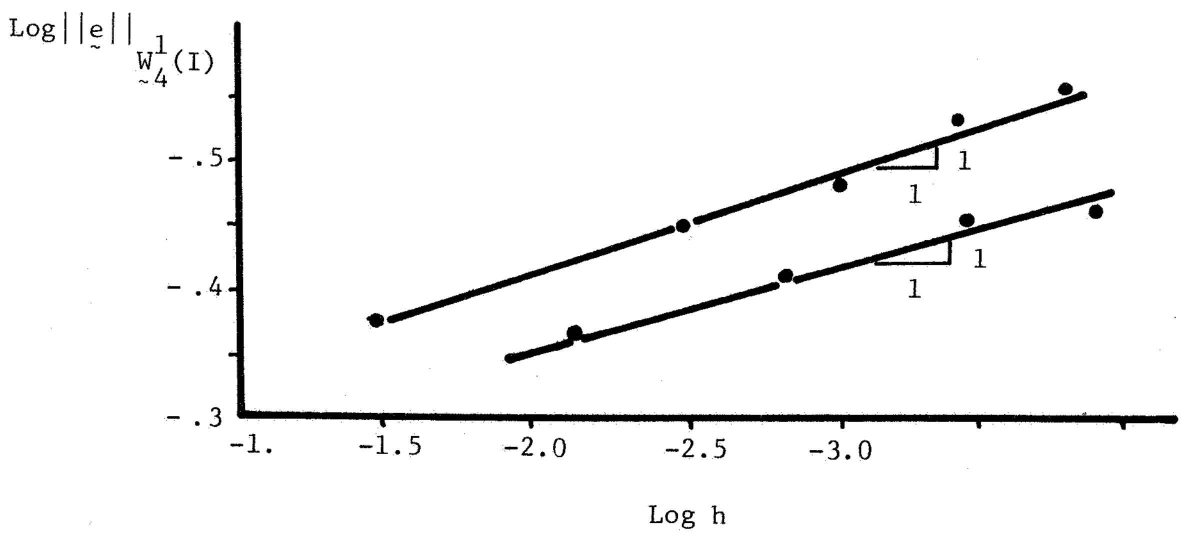

Figure 2.- Computed rates of convergence.