ELASTO-PLASTIC IMPACT OF HEMISPHERICAL SHELL

IMPACTING ON HARD RIGID SPHERE

D. D. Raftopoulos
Professor of Mechanical Engineering
The University of Toledo

A. L. Spicer
Research Engineer, New Departure-Hyatt
Division of General Motors Corp.

ABSTRACT

This paper extends an analysis of plastic stress waves, originated by G. I. Taylor in reference 1, for cylindrical metallic projectile in impact to an analysis of a hemispherical shell suffering plastic deformation during the process of impact. In that, it is assumed that the hemispherical shell with a prescribed launch velocity impinges a fixed rigid sphere of diameter equal to the internal diameter of the shell. Particularly this study is directed in order to investigate the dynamic biaxial state of stress present in the shell during deformation.

The results of this analysis are compared with Taylor's reference 1 and it has been found that this analysis is an extension of the one-dimensional analyses of references 1, 2, 3, and 4, to spherical coordinates. It is valuable for studying the state of stress during large plastic deformation of a hemispherical shell.

INTRODUCTION

The object of this paper is to develop an analysis of plastic hemispherical stress-wave propagation and to use this analysis for determining the dynamic biaxial yield stress. The Tresca yield criteria is used as the yield condition. Higher order terms are included in the derivations; thus, this analysis is valid for large deformations.

G. I. Taylor in reference 1 used the governing physical laws and the geometry during plastic deformation of the cylindrical projectile to formulate differential equations which are solved in order to determine the dynamic yield stress in impact. This analysis of a hemispherical shell impacting a fixed rigid sphere, of diameter equal to the internal diameter of the shell, is similar to the analysis of a cylindrical projectile impacting a rigid target of references 1, 2, 3, and 4. In fact, in all these cases during impact it is assumed that when the stress rise exceeds the elastic limit of the material, two waves are generated. The first of these is the elastic wave, which travels with a velocity c. It is followed by the plastic stress wave which travels with a slower velocity v. Through an analysis of the propagation of these two stress waves, a method is formed which can be used to determine the dynamic yield stress of the material of a hemispherical shell. The proper choice of the time increment, dt, simplifies the analysis greatly. The choice is to make the time increment equal to the length of time required for the elastic wave to complete a double passage of the elastic zone. If the difference equations are derived by utilizing this time increment, which is eliminated by combining the derived difference equations, the governing equations which are derived are free of this time increment. This mathematical approach, for the biaxial state of stress of the hemispherical shell, closely parallels Taylor's analysis of the cylindrical projectile.

NOMENCLATURE

A_o	Projection at the elastic-plastic boundary undeformed area at time t
A	Projection at the elastic-plastic boundary deformed area at time t + dt
a	Initial inside radius
b	Initial outside radius
c	Elastic wave velocity
dh	Incremental plastic radius
dr	Incremental elastic radius
dt	Time for a double passage of elastic region by elastic wave
E	Young's modulus
h	Thickness of plastic region at time t
r_o	Initial elastic length of shell in the radial direction = b - a
r_1	Final total length of shell in the radial direction
r	Thickness of shell in the elastic zone at time t
R	Final thickness of the elastic region
S_1	Dyhamic yield stress - calculated by the approximate method
S	Dynamic yield stress - calculated by the exact method
t	Time

U Initial radial velocity due to launch velocity

u Particle velocity in the elastic region

v Absolute velocity of the plastic wave front

Y Yield stress in uniaxial tension

ε_1 Initial radial strain

ε Radial strain at time t

ρ Density

ν Poisson's ratio

ANALYSIS

Problem Description

A hemispherical metallic shell strikes with a prescribed velocity a rigid sphere (of diameter equal to the internal diameter of the shell) which is permanently fixed at a base retaining zero velocity during the process of impact. During this impact, a radial motion is directed from the internal surface of the shell toward the external surface of the same. The radial particle velocity of the internal surface of the shell is initially the same as the impact velocity and is denoted by U. If the biaxial stress exceeds the elastic limit, two waves are generated at the internal surface of the shell. The first wave is the elastic wave which travels with velocity c. The second is the plastic wave which travels with velocity v. The elastic compressive stress wave, which propagates radially outward in the elastic region with velocity c, will reduce the impact velocity U to $U-(S/\rho c)$. During this time, the stress reaches the elastic limit. This elastic wave will reflect at the external surface of the sphere, resulting in an elastic tensile wave being superposed on the compressive elastic wave. The material which has been passed by this reflected elastic wave is stress free and has a velocity equal to $U-(2S/\rho c)$. At the particular time when this wave reaches the elastic-plastic boundary, the shell is in a condition similar to the initial impact, except that its speed is equal to $U-(2S/\rho c)$ and its elastic thickness is less than the original value. At this time, it is assumed that the plastically deformed material will be attached to the sphere and acts on the elastic part of the shell as a rigid material. This continues until the speed of the shell becomes equal to zero.

Assumptions

In order to work out the mathematical analysis of this problem, several basic assumptions are needed.

First, for axially symmetrical analysis, the shell must be symmetric with respect to its axis of symmetry and maintain this symmetry during the process of impact. The second assumption is that the elastic strain is negligible. This assumption is valid if the plastic strain is large, thus making the elastic strain very small in comparison with the plastic strain. Along the same lines as the previous assumption, the third assumption is that the material is taken to be perfectly plastic. Although no material behaves exactly in a perfectly plastic manner, some materials approach this type of behavior at high strain rates. This dynamic-plastic stress-wave analysis is for extremely high strain rates. Thus it is possible to assume that the material is perfectly plastic without the loss of much generality in the solution. The fourth assumption, which is usually made in plasticity problems, is that the density of the shell material remains constant. The fifth and final assumption is that the material in the plastic region, after being deformed, does not possess elastic properties; thus it behaves as a rigid material with zero velocity.

Physical Laws

By considering the problem description, and assumptions, the governing physical laws can be formulated.

Choose the time increment, dt, to be equal to the time required for a complete double passage of the elastic wave through the elastic region. Since the length of material in the elastic zone is defined as r and the elastic wave velocity is c, it follows that

$$dt = 2r/c \tag{1}$$

where

$$c = \{E\,(1 - \nu)/\,\{\rho(1 + \nu)(1 - 2\nu)\}\}^{1/2}$$

$$dh = v\,(2r/c) \tag{2}$$

$$dr = -(u + v)\,(2r/c) \tag{3}$$

$$du = -\,2S/(\rho c) \tag{4}$$

Using equation (1) to eliminate c from equations (2), (3), and (4) results in

$$\frac{dh}{dt} = v \tag{5}$$

$$\frac{dr}{dt} = -(u + v) \tag{6}$$

$$\frac{du}{dt} = - S/(\rho r) \tag{7}$$

for conservation of mass

$$A\ v = (u + v)\ A_o \tag{8}$$

The momentum equation reduces to

$$S\ (A - A_o) = 1/2(A + A_o)\ (u + v)\ u\ \rho \tag{9}$$

The radial strain is defined, at the plastic boundary, by

$$\varepsilon = 1 - A_o/A \tag{10}$$

Combining equations (8), (9), and (10)

$$\rho u^2/S = 2\ \varepsilon^2/(2 - \varepsilon) \tag{11}$$

Combining equations (6), (7), (8), and (10)

$$\frac{dr}{du} = \rho ur/(S\varepsilon) \tag{12}$$

Integrating equation (12)

$$\begin{aligned} Log_e(r^2) &= \int 1/\varepsilon\ d\{\varepsilon^2/(1 - \varepsilon/2)\} \\ &= 4/(2 - \varepsilon) - 2\ Log_e\ (1 - \varepsilon/2) + Constant \end{aligned} \tag{13}$$

At time $t = 0$, $u = U$, $r = r_o$, and $\varepsilon = \varepsilon_1$; thus equation (11) and equation (13) become, respectively

$$\rho U^2/S = 2\ \varepsilon_1^2\ /(2 - \varepsilon_1) \tag{14}$$

$$\begin{aligned} Log_e(r/r_o)^2 = 4/(2 - \varepsilon) - 2\ Log_e\ (1 - \varepsilon/2) - 4/(2 - \varepsilon_1) \\ + 2\ Log_e\ (1 - \varepsilon_1/2) \end{aligned} \tag{15}$$

When all motion has ceased, $r = R$, and $\varepsilon = 0$, and R can be measured.

$$Log_e(R/r_o)^2 = 2 - 4/(2 - \varepsilon_1) + 2\ Log_e(1 - \varepsilon_1/2) \tag{16}$$

Combining equations (5), (6), (8) and (10)

$$h = \int dh = - \int_{r_o}^{r} (1 - \varepsilon).\ dr \tag{17}$$

Combining equations (7), (11), and (14)

$$Ut/r_o = \varepsilon_1(1 - \varepsilon_1/2)^{-1/2}\int_{\varepsilon}^{\varepsilon_1} r/r_o\ (1 - \varepsilon/4)/(1 - \varepsilon/2)^{3/2}d\varepsilon \quad (18)$$

If uniformly spaced values of ε_1 are placed in equations (14) and (16), $\rho U^2/S$ vs R/r_o can be plotted. Evaluation of equation (17) for h is accomplished by Simpson's rule integration. Results of these calculations are plotted in Figure 1.

Two different methods of integration were employed to evaluate the integral equations (17) and (18). The first method used was a Simpson's rule integration. Results for various ε_1 are plotted in Figure 2.

The second method of integration was using the asymptotic expansion of the integrals. References 5, 6, and 7 provide information on asymptotic power-series expansions. Values obtained by asymptotic expansion agreed well with those obtained by Simpson's rule integration.

To develop a simple formula for calculating the dynamic yield stress from measurements made before and after the impact, it will be additionally assumed that the plastic boundary propagates at a constant velocity from the inside radius a to its final position. The velocity of the plastic boundary equals C.

Combining equations (6) and (7)

$$\frac{du}{dr} = S/\{\rho r(u + C)\} \quad (19)$$

Integrating equation (19) results in

$$S/\rho \operatorname{Log}_e (r/r_o) = 1/2u^2 + C\,u - 1/2U^2 - C\,U \quad (20)$$

When u = 0, r = R and equation (20) becomes

$$S/\rho \operatorname{Log}_e (R/r_o) = - 1/2U^2 - C\,U \quad (21)$$

At time t = 0, u = U. Assuming u decreases to zero uniformly with time, in a time equal to T

$$T = (r_1 - R)/C = 2(r_o - r_1)/U$$

Rearranging $\quad C/U = 1/2\ (r_1 - R)/(r_o - r_1)$

Therefore, equation (21) becomes

$$S_1/\rho U^2 = (r_o - R)/[2(r_o - r_1)\operatorname{Log}_e (r_o/R)] \quad (22)$$

The fact that the decrease in u is not uniform results in an error which can be calculated. Combining equations (3) and (20)

$$\left[\frac{dr}{dt}\right]^2 = 2S/\rho \ \mathrm{Log}_e \ (r/r_o) + (U + C)^2 \qquad (23)$$

When all motion has **ceased,** $u = 0$, $r = R$, and $\varepsilon = 0$. Therefore, equation (21) becomes

$$2S/\rho \ \mathrm{Log}_e \ (R/r_o) = C^2 - (U + C)^2 \qquad (24)$$

Letting

$$2S/\rho = \alpha^2$$
$$K = (U + C)/\alpha$$
$$R_1 = r/r_o$$
$$t_1 = \alpha t/r_o$$
$$T_1 = \alpha T/r_o$$

where T is the time from the initial impact until the plastic zone velocity equals zero

$$\frac{dR_1}{dt_1} = (K^2 + \mathrm{Log}_e \ R_1)^{1/2}$$

so that

$$T_1 = \int_{\exp[(\frac{C}{\alpha})^2 - K^2]}^{1} (K^2 + \mathrm{Log}_e \ R_1)^{-1/2} dR_1 \qquad (25)$$

Letting

$$Z^2 = K^2 + \mathrm{Log}_e \ R_1$$

results in

$$T_1 = 2 \ e^{-K^2} \int_{C/\alpha}^{K} e^{Z^2} \ dZ \qquad (26)$$

Values of $F(K) = e^{-K^2} \int_0^K e^{Z^2} dZ$ have been tabulated in **references** 5 **and** 6. Equation (26) can be expanded using this function, F(K), to give

$$T_1 = 2 \ \{F(K) - \mathrm{Exp} \ (-K^2 + (C/\alpha)^2) \ F(C/\alpha)\} \qquad (27)$$

Previously it was assumed that the plastic boundary moves with a constant velocity C, or

$$C \ T = r_1 - R$$

or in dimensionless form

$$C/\alpha \; T_1 = r_1/r_o - R/r_o \tag{28}$$

Rearranging, equation (24) becomes

$$\text{Log}_e \, (r_o/R) = K^2 - (C/\alpha)^2 \tag{29}$$

Since R/r_o and r_1/r_o can be measured, C/α, U/α, K and T can be evaluated from equations (27), (28), (29), and

$$K = U/\alpha + C/\alpha \tag{30}$$

Combining equations (27), (28), and (29)

$$r_1/r_o = 2 \; C/\alpha \; F(K) - \{2 \; C/\alpha \; F(C/\alpha) - 1\} \; R/r_o \tag{31}$$

Since

$$2S/\rho U^2 = \alpha^2/U^2 = 1/(K - C/\alpha)^2$$

dividing this equation by equation (22) therefore results in

$$S/S_1 = (r_o - r_1)/(r_o - R) \; [\text{Log}_e \, (r_o/R)/(K - C/\alpha)^2] \tag{32}$$

Due to the complexity of these equations, the correction factor, S/S_1, cannot be determined directly. To determine S/S_1 given R/r_o and r_1/r_o it is easiest to first form the curves of S/S_1 vs r_1/r_o with contours of equal h/r_o. Values for this curve can be obtained by taking a value of R/r_o and values of C/α which cover the desired range. Therefore, using equation (29), equations (31) and (32) can be evaluated.

The asymptotic expansion of F(K) is

$$F(K) = 1/(2K) + 1/(4K^3) + 3/(8K^5) + 15/(16K^7) + \ldots \tag{33}$$

Through some complex manipulations, it can be shown, although it will not be presented here, that as $C/\alpha \to \infty$

$$r_1/r_o \to 1.0 \tag{34}$$

and

$$S/S_1 = 2 \; \{1/(1 - R/r_o) - 1/(\text{Log}_e(r_o/R))\} \tag{35}$$

From this equation, the limiting values of S/S_1 can be determined as $r_1/r_o \to 1.0$.

This completes the analysis of the problem. Thus, if values of r_o, r_1, and h are given, the dynamic yield stress can be calculated for the hemispherical shell.

DISCUSSION AND CONCLUDING REMARKS

In this paper, a method is developed to investigate the propagation of plastic stress waves in a hemispherical shell. In particular, this study investigates the dynamic yield stress due to the impulsive loading initiated at the interface of the shell. This mathematical approach, for determining the biaxial state of stress of the hemispherical shell, closely parallels Taylor's analysis of the cylindrical projectile. It is interesting to note that if all higher order terms were dropped from this analysis, the results would be the same as those of reference 1 except that the components are defined differently. Graphs which are drawn from this analysis in Figures 1 and 2 are similar to Figures 2 and 3 of reference 1. In addition, comparison between the results of this analysis and the analysis of reference 1 is possible. In fact, when these two analyses are compared, one can observe that the results of the present work parallel the experimental data more closely than the results of reference 1. This is due to the fact that one-dimensional analysis may not possibly explain the spreading out of the projectile near the target. This phenomenon requires taking into account the inertia in the radial direction.

The derivation of the yield stress correction factor is almost identical with the results of reference 1 on page 297. Singularities were observed which were not discussed in reference 1. The discontinuities occurred just before $r_1/r_o \rightarrow 1.0$. If the discontinuity is ignored, the results are similar to those of reference 1.

A method has been presented by which the dynamic yield stress can be calculated, using the Tresca yield criteria, from the radial expansion of a hemispherical shell. The approximate yield stress can be calculated from equation (22), if the initial conditions, final conditions, U, and ρ are specified. The dynamic yield stress could also be calculated from Figure 1. Thus, the dynamic yield stress can be determined if certain initial and final experimental conditions are specified, including the launch velocity, density, and geometrical considerations of the shell. The motion of the plastic boundary, as shown in Figure 2, is similar to the results obtained in Figure 4 of reference 8. Their choice of coordinates is different, which accounts for many of the differences between the shape of their curve and of Figure 2.

REFERENCES

1. Taylor, G. I., "The Use of Flat-ended Projectiles for Determining Dynamic Yield Stress", I: Theoretical Considerations, Proceedings of the Royal Society, London, England, Series A., Vol. 194.

2. Raftopoulos, D. and Davids N., "Elasto-Plastic Impact on Rigid Targets" AIAA Journal, July 5, 1967, pp. 2254-2260.

3. Raftopoulos, D. "Longitudinal Impact of Two Mutually Plastically-Deformable Missiles", Int. Journal of Solids and Structures - Vol. 5, No. 4, pp. 399-412, 1969.

4. Raftopoulos, D. and Al-Salihi, M. "Direct Analysis of Elasto-Plastic Wave Interaction in Impact", Proceeding of the 11th Midwstern Mechanics Conference, Vol. 5, 1969.

5. Jeffreys, Harold and Jeffreys, Bertha Swirles, Methods of Mathematical Physics, University Press, Cambridge, 1950, pp. 498-528.

6. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, edited by Milton Abramowitz and Irene A. Stegun, United States Department of Commerce, 1964, pp. 319-320.

7. Wosow, Wolfgang, Asymptotic Expansions for Ordinary Differential Equations, New York - John Wiley & Sons, Inc., 1965, p. 31.

8. Davids, N., Mehta, P. K., and Johnson, O. T., "Spherical Elasto-Plastic Waves in Materials". Colloquium on "Behavior of Materials Under Dynamic Loading", ASME Winter Annual Meeting, Chicago, November 9, 1965, pp. 125-137.

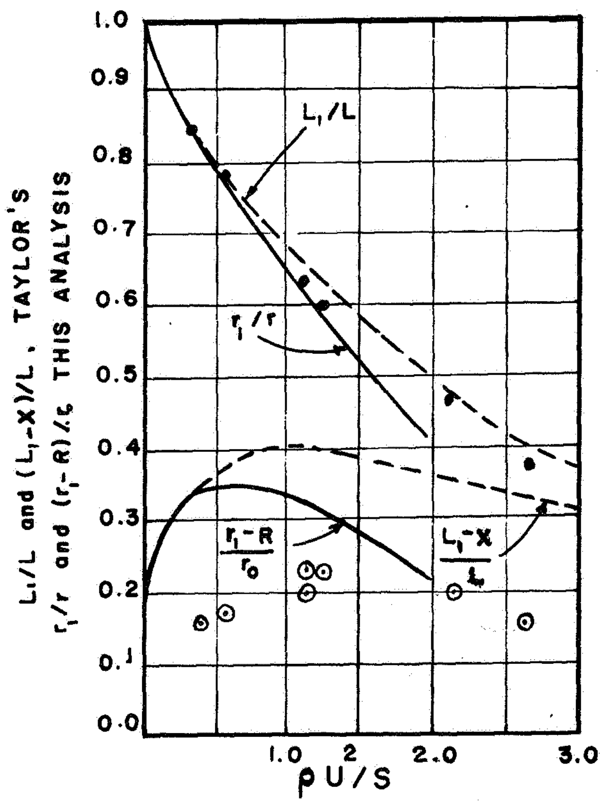

Figure 1.- Calculated results compared with Taylor's theoretical and experimental data.

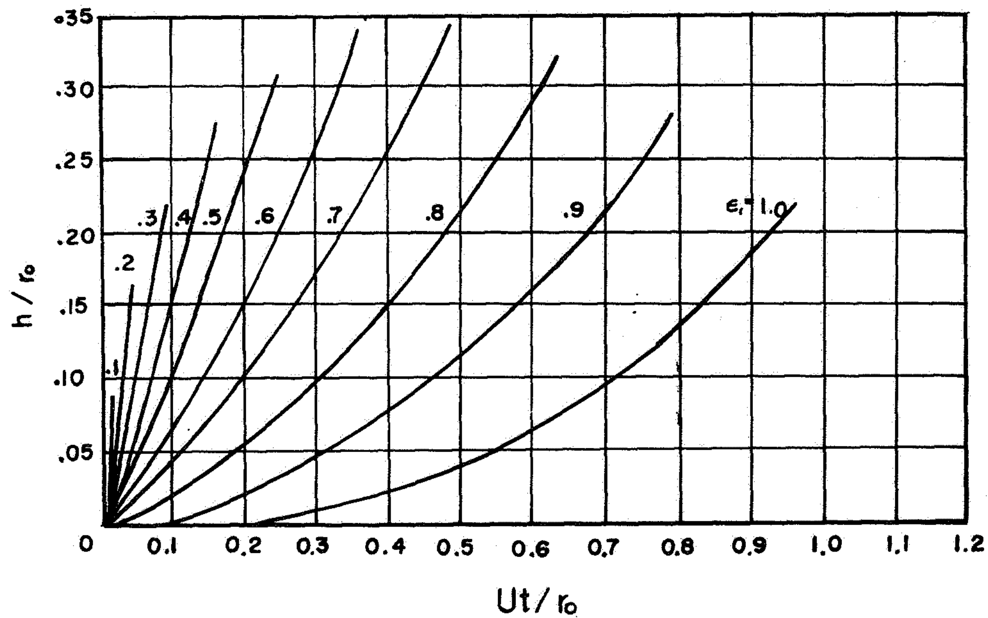

Figure 2.- Propagation of plastic boundary.