LARGE DEFLECTIONS OF A SHALLOW CONICAL MEMBRANE

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SUMMARY

This work is concerned with large deflections of a shallow elastic conical membrane fixed at the outer edge and loaded by either uniform or hydrostatic pressure. The governing equations were solved by the method of matched asymptotic expansions and by a finite-difference method. Agreement between the two methods was excellent for the small values of the perturbation parameter.

INTRODUCTION

This paper is concerned with the moderately large axisymmetric deformation of a shallow elastic conical membrane. The purpose of this work is to further investigate the application of the method of matched asymptotic expansions (see Van Dyke, reference 1) to the solution of membrane-shell problems involving large deflections. The success of this method is based on the fact that for small loads the linear membrane solution is a good approximation to the actual solution everywhere except in the immediate vicinity of boundaries. In these regions thin boundary layers exist where the variables undergo rapid changes to accommodate themselves to the boundary conditions that cannot be satisfied by the linear membrane solution. In the method of matched asymptotic expansions separate perturbation expansions are found in the interior and boundary-layer regions and matched in an appropriate way to insure that they join smoothly.

Bromberg and Stoker (ref. 2) initiated this type of analysis of membrane shells when they found one term of both the interior and boundary-layer expansions for a uniformly pressurized shallow spherical shell. The next two terms in the interior and boundary-layer expansions were found by Smith, Peddieson, and Chung (ref. 3) and used by them to investigate the accuracy of finite-difference solutions of the same problem. One term of the interior and boundary-layer expansions for deep membranes of arbitrary shape has been given by Rossettos (ref. 4). This work generalizes the results given in the references listed in reference 4.

In the present paper three terms of the interior and boundary-layer expansions are found for the case of a shallow conical membrane loaded by either uniform or hydrostatic pressure. It is found that complications arise which do not appear in the solution of the corresponding sphere problem. The solution method is modified somewhat to account for this. Numerical results are presented to illustrate some of the interesting features of the solution.

GOVERNING EQUATIONS

Consider a shallow conical membrane (opening upward) with base radius a, thickness h, and initial angle ϕ_o with the horizontal made of a linearly elastic material with modulus of elasticity E and Poisson's ratio ν. The equations governing moderately large axisymmetric deflections of such a structure can be obtained from the work of Reissner (ref. 5). The resulting equations are (in dimensionless form)

$$\psi'' + \psi'/r - \psi/r^2 + (1+\varepsilon^2\beta/2)(\beta/r) = 0$$

$$(1+\varepsilon^2\beta)\psi = rV$$

$$N_r = \psi/r, \quad N_\theta = \psi'$$

$$u = r\psi' - \nu\psi, \quad w' = \beta \qquad (1)$$

where ar is the radial coordinate, $V_o a\psi/\phi_o$ is a stress function (V_o being a characteristic vertical force resultant), V_oV is the vertical force resultant, V_oN_r/ϕ_o is the radial stress resultant, V_oN_θ/ϕ_o is the transverse stress resultant, ε is a load parameter, $a\phi_o^2\varepsilon^2u$ is the horizontal displacement, $a\phi_o\varepsilon^2w$ is the vertical displacement, $\phi_o\varepsilon^2\beta$ is the rotation, and a prime denotes differentiation with respect to r.

In the present paper a uniform pressure p_o and a hydrostatic loading $\gamma_o\phi_o a(1-r)$ are considered. It can be shown by considering the vertical equilibrium of the membrane centered on the vertex and having radius r that

$$V = r/2 - jr^2/3 \qquad (2)$$

where j = 0 for the uniform pressure and j = 1 for the hydrostatic pressure. The characteristic vertical force resultant is given by

$$V_o = \begin{cases} p_o a, & j = 0 \\ \gamma_o a^2\phi_o, & j = 1 \end{cases} \qquad (3)$$

The load parameter ε is defined to be

$$\varepsilon = (V_o/Eh\phi_o^3)^{\frac{1}{2}} \qquad (4)$$

In the present work it is desired to solve equations (1) subject to the boundary conditions

$$u(1) = w(1) = 0 \qquad (5)$$

Special attention will be given to situations where $\varepsilon \ll 1$.

STRAIGHTFORWARD SOLUTION

To begin the solution process a straightforward perturbation solution to equations (1) is sought for $\varepsilon \ll 1$. To do this it is convenient to rearrange equations (1a) and (1b) to yield

$$\varepsilon^2(\psi'' + \psi'/r - \psi/r^2) - (1 - (rV/\psi)^2)/(2r) = 0$$

$$\beta = (rV/\psi - 1)/\varepsilon^2 \tag{6}$$

A straightforward perturbation solution for $\varepsilon \ll 1$ has the form

$$\psi_s \sim \psi_{s0} + \varepsilon\psi_{s1} + \varepsilon^2\psi_{s2} + \ldots \tag{7}$$

where the subscript s indicates the straightforward solution. Substituting equation (7) into equation (6a), expanding for $\varepsilon \ll 1$, setting the coefficient of each power of ε equal to zero in the usual way, and solving the resulting algebraic equations yields

$$\begin{aligned}\psi_s \sim (r^2/2 - jr^3/3) &+ \varepsilon^2(r^2/2 - jr^3/3)(3r/2 \\ &- 8jr^2/3) + \varepsilon^4(r^2/2 - jr^3/3)(75r^2/8 \\ &- j(79r^3/2 - 32r^4)) + \ldots\end{aligned} \tag{8}$$

From equations (1) and (6b) it can then be shown that

$$\begin{aligned}N_{rs} \sim (r/2 - jr^2/3) &+ \varepsilon^2(r/2 - jr^2/3)(3r/2 \\ &- 8jr^2/3) + \varepsilon^4(r/2 - jr^2/3)(75r^2/8 \\ &- j(79r^3/2 - 32r^4)) + \ldots\end{aligned}$$

$$\begin{aligned}N_{\theta s} \sim (r - jr^2) &+ \varepsilon^2(9r^2/4 - j(22r^3/3 - 40r^4/9)) \\ &+ \varepsilon^4(75r^3/4 - j(915r^4/8 - 175r^5 \\ &+ 224r^6/3)) + \ldots\end{aligned}$$

$$\begin{aligned}\beta_s \sim -(3r/2 - 8jr^2/3) &- \varepsilon^2(57r^2/8 \\ &- j(63r^3/2 - 224r^4/9)) + \ldots\end{aligned}$$

$$\begin{aligned}u_s \sim (1 - \nu/2)r^2 - j(1 - \nu/3)r^3 &+ \varepsilon^2(3(3 - \nu)r^3/4 \\ &- j(11(4 - \nu)r^4/6 - 8(5 - \nu)r^5/9)) \\ &+ \varepsilon^4(75(4 - \nu)r^4/16 - j(183(5 - \nu)r^5/8 \\ &- 175(6 - \nu)r^6/6 + 32(7 - \nu)r^7/3)) + \ldots\end{aligned}$$

$$w_s \sim -3(r^2 - 1)/4 + j8(r^3 - 1)/9 - \varepsilon^2(19(r^3 - 1)/8$$
$$- j(63(r^4 - 1)/8 - 224(r^4 - 1)/45)) + \ldots \qquad (9)$$

where equation (5b) has been used to determine the constants of integration in equation (9c). By comparison with the results given in Kraus (ref. 6) it can be seen that the first term in each series expansion is the linear membrane solution. It should also be noted that the first terms in equations (9d) and (9e) are due to the second term in equation (8). Thus to obtain β_s and w_s to $O(\varepsilon^2)$ it is necessary to find ψ_s to $O(\varepsilon^4)$. The boundary condition represented by equation (5a) cannot be satisfied by equation (9d). Thus a boundary-layer expansion is needed in the vicinity of $r = 1$.

BOUNDARY-LAYER SOLUTION

There are several ways to carry out the boundary-layer analysis in this problem. One is to work in terms of the original stress function ψ. If this is done the differential equation for the first boundary-layer approximation turns out to be nonlinear. Bromberg and Stoker (ref. 2) discovered that a linear equation could be obtained in the first approximation for a spherical membrane by a method which is equivalent to working with a dependent variable which is the difference between the actual and the linear stress functions. This was tried in the present problem but matching difficulties were encountered. These were due to the fact that equations (8) and (9) do not terminate with one term for the cone as the corresponding straightforward expansions do for a sphere. It was, therefore, decided to use the difference between the actual stress function and the straightforward stress function as the dependent variable. This guarantees that the outer expansion for this dependent variable will be zero. Thus it is necessary to find only the inner expansion.

Substituting

$$\psi = \psi_s + \psi_b \qquad (10)$$

(where the subscript b denotes the boundary-layer solution) into equation (6), defining the boundary-layer variables F and ξ by the equations

$$\psi_b = \varepsilon F, \quad r = 1 - \varepsilon\xi, \qquad (11)$$

expanding F as

$$F \sim F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots, \qquad (12)$$

and carrying out the usual perturbation analysis yields

$$\ddot{F}_0 - S^2 F_0 = 0 \qquad (13)$$

and two other equations governing F_1 and F_2 where

$$S = (6/(3 - 2j))^{\frac{1}{2}} \qquad (14)$$

In equation (13) $(\dot{\ }) = d(\ )/d\xi$. Define N_{rb}, $N_{\theta b}$, β_b, u_b, and w_b by the following equations

$$N_r = N_{rs} + \varepsilon N_{rb}, \quad N_\theta = N_{\theta s} + N_{\theta b}$$

$$\beta = \beta_s + \beta_b/\varepsilon, \quad u = u_s + u_b, \quad w = w_s + w_b \tag{15}$$

Now expand as follows

$$A_b \sim A_{b0} + \varepsilon A_{b1} + \varepsilon^2 A_{b2} + \ . \ . \ . \tag{16}$$

where A_b is any one of the boundary-layer variables. Substituting equations (10), (11), (12), (15), and (16) into equations (1), expanding for $\varepsilon << 1$, and equating the coefficients of like powers of ε to zero one obtains

$$N_{rb0} = F_0, \quad N_{\theta b0} = -\dot{F}_0, \quad \beta_{b0} = -S^2 F_0$$

$$u_{b0} = -\dot{F}_0, \quad w_{b0} = S^2 \int_0^\infty F_0 d\xi \tag{17}$$

and two similar sets of equations relating A_{b1} and A_{b2} to F_0, F_1, and F_2. A similar procedure applied to equation (5a) leads to boundary condition

$$\dot{F}_0(0) = 1 - j - (1/2 - j/3)\nu \tag{18}$$

and boundary conditions for $\dot{F}_1(0)$ and $\dot{F}_2(0)$.

To illustrate the solution procedure the first approximation will now be carried out in detail. The solution of equation (13) is easily seen to be

$$F_0 = c_1 \exp(S\xi) + c_2 \exp(-S\xi) \tag{19}$$

Since the outer expansion has been forced to vanish because of equation (10) the matching process (see Van Dyke, reference 1) is equivalent in this case to a statement that positive exponential terms must vanish. Thus

$$c_1 = 0 \tag{20}$$

Substituting equations (19) and (20) into equation (18) yields

$$c_2 = -(1 - j - (1/2 - j/3)\nu)/S \tag{21}$$

Thus

$$F_0 = -(1 - j -(1/2 - j/3)\nu)\exp(-S\xi)/S \tag{22}$$

Substituting equation (22) into equations (17) one obtains

$$N_{rb0} = -(1 - j -(1/2 - j/3)\nu)\exp(-S\xi)/S$$

$$N_{\theta b0} = (1 - j - (1/2 - j/3)\nu)\exp(-S\xi)$$

$$\beta_{b0} = (1 - j - (1/2 - j/3)\nu)S \exp(-S\xi)$$

$$u_{b0} = -(1 - j - (1/2 - j/3)\nu)\exp(-S\xi)$$

$$w_{b0} = -(1 - j - (1/2 - j/3)\nu)(1 - \exp(-S\xi)) \qquad (23)$$

The results for higher approximations are found in a similar way but the calculations are quite lengthy. For the sake of brevity this work is omitted.

To find the complete solution the boundary-layer expansions must be added to the corresponding straightforward expansions. The first approximations to these expressions are

$$\psi_0 = r^2/2 - jr^3/3 - \varepsilon(1 - j - (1/2 - j/3)\nu)\exp(-S(1 - r)/\varepsilon)/S$$

$$N_{r0} = r/2 - jr^2/3 - \varepsilon(1 - j - (1/2 - j/3)\nu)\exp(-S(1 - r)/\varepsilon)/S$$

$$N_{\theta 0} = r - jr^2 + (1 - j - (1/2 - j/3)\nu)\exp(-S(1 - r)/\varepsilon)$$

$$\beta_0 = (1 - j - (1/2 - j/3)\nu)S \exp(-S(1 - r)/\varepsilon)/\varepsilon$$

$$u_0 = (1 - \nu/2)r^2 - j(1 - \nu/3)r^3 - (1 - j - (1/2 - j/3)\nu)\exp(-S(1 - r)/\varepsilon)$$

$$w_0 = 3(1 - r^2)/4 - 8j(1 - r^3)/9 - (1 - j - (1/2 - j/3)\nu)(1 - \exp(-S(1 - r)/\varepsilon)) \qquad (24)$$

In writing equations (24) the boundary-layer solution was treated as the fundamental expansion. All terms in the straightforward expansion with magnitude equal to or greater than the first term in the boundary-layer expansion were added to this term to form the first approximation. The same method was used to obtain the second and third approximations.

RESULTS AND DISCUSSION

Numerical results were computed for the first, second, and third approximations to the variables ψ, N_r, N_θ, β, u, and w. These calculations were made for a variety of values of the load parameter ε and Poisson's ratio ν. To evaluate the accuracy of the perturbation method, selected cases were compared with numerical solutions to equation (6a) obtained by the finite-difference method discussed by Smith, Peddieson, and Chung (ref. 3). It was found that the third approximation to the perturbation solution agreed with the finite-difference results up to $\varepsilon = 0.1$. It should be pointed out that for small values of ε, the numerical method is difficult to apply because a variable step size must be used near the edge and the optimum arrangement of step sizes can only be approached by trial and error. The explicit formulas obtained in the present work are much easier to use for $\varepsilon \ll 1$.

To illustrate the behavior of the solution some of the computed results are shown in figures 1 - 4. For the sake of brevity, data are presented for only the radial stress resultant N_r, the transverse stress resultant N_θ, and the vertical deflection w. The solid lines represent the three-term perturbation solution while the dashed lines represent the linear membrane solution. The linear membrane solution is shown only when it differs significantly from the perturbation solution.

Figures 1 and 2 present results for uniform pressurization ($j = 0$). Figure 1 shows that thin boundary layers exist for N_θ and w for $\varepsilon = 0.01$ while N_r does not exhibit boundary-layer behavior. As ε increases the boundary layers become wider for all variables. This is illustrated by figure 2. Figures 3 and 4 contain results for hydrostatic loading ($j = 1$). The parametric trends illustrated by these results are identical to those discussed above but the behavior of the solution variables is more complicated. These results illustrate the utility of the perturbation method. Complicated functions of this type can be represented numerically only if extreme care is used.

Results were also computed for several other values of ν. It was found that the qualitative behavior of the solution is not significantly influenced by this parameter.

CONCLUSION

In this paper, the rotationally symmetric moderately large deformation of a linearly elastic shallow conical membrane subjected to either uniform or hydrostatic pressure was investigated. A single differential equation having a stress function as dependent variable was solved by the method of matched asymptotic expansions. The accuracy of the solution was verified by comparison with a finite-difference numerical solution of the governing equation for the stress function. Selected results were presented graphically to illustrate interesting features of the solutions.

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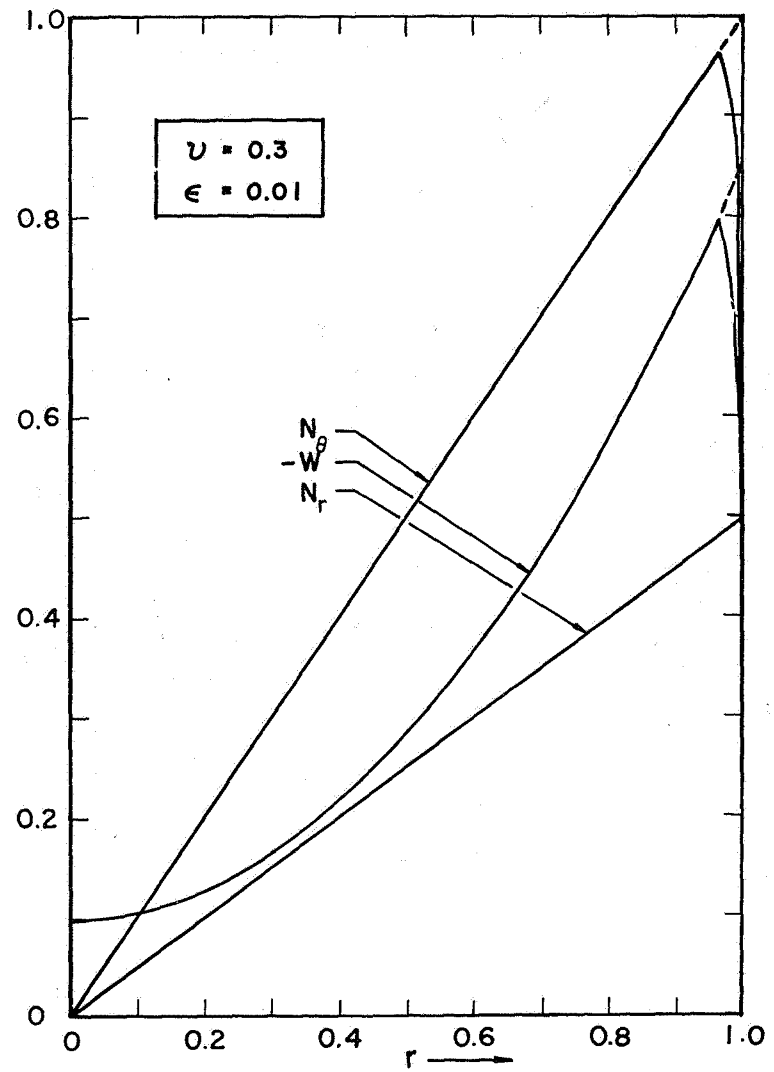

Figure 1.- Stress resultants and axial displacement for uniform pressurization. $\varepsilon = 0.01$.

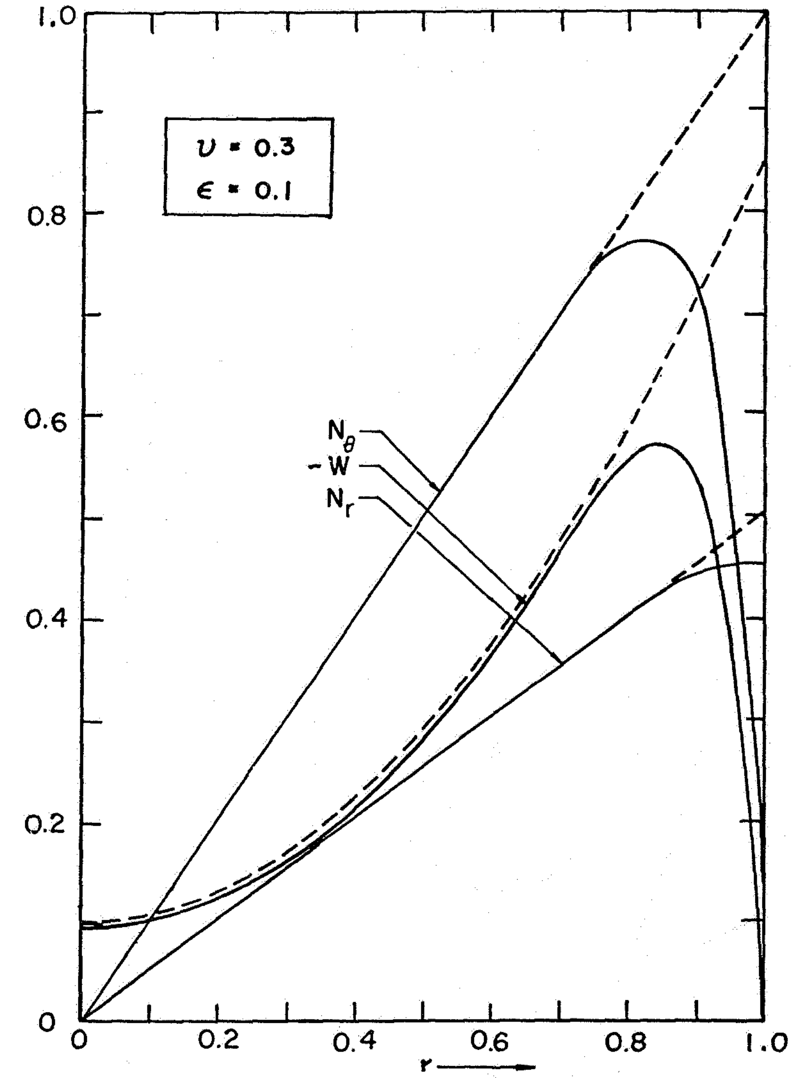

Figure 2.- Stress resultants and axial displacement for uniform pressurization. $\varepsilon = 0.1$.

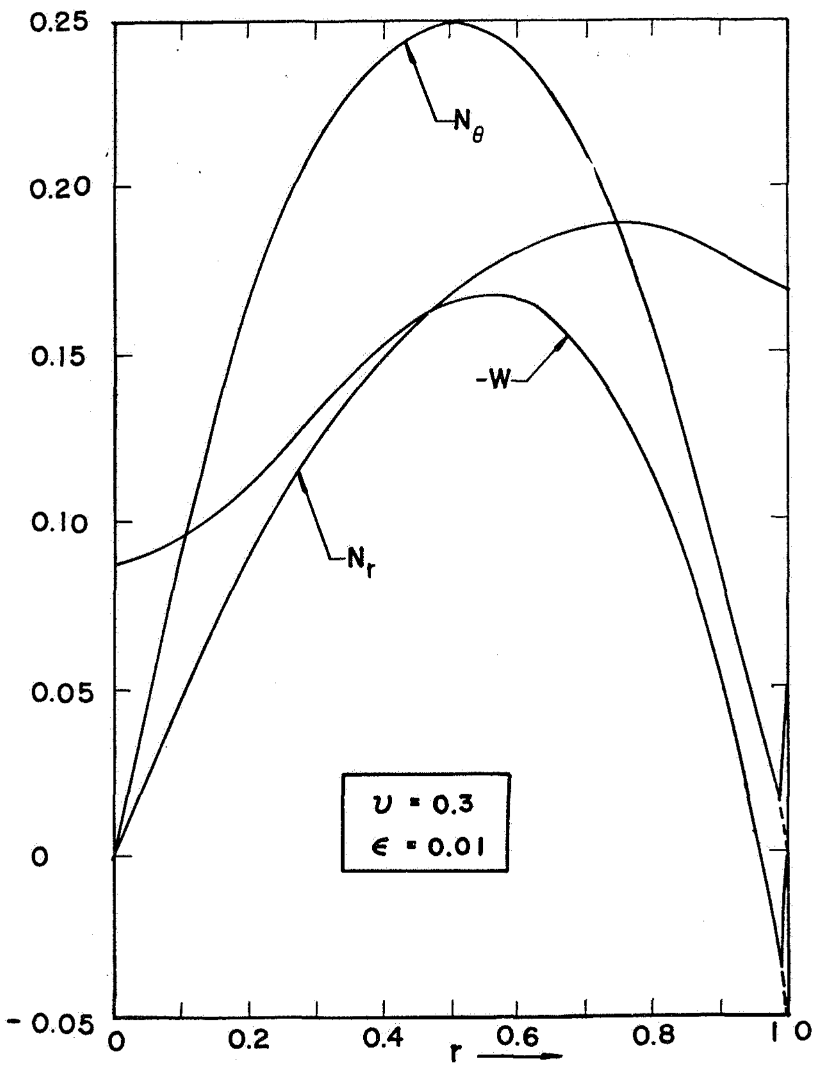

Figure 3.- Stress resultants and axial displacement for hydrostatic loading. $\varepsilon = 0.01$.

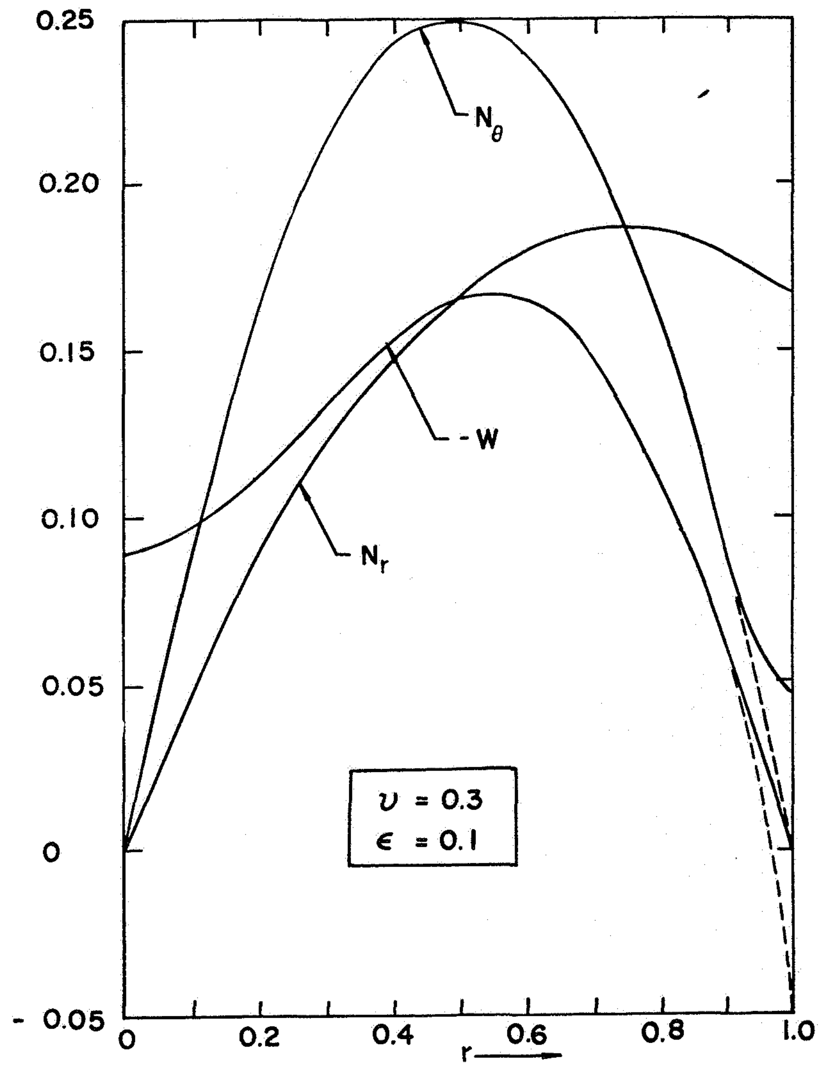

Figure 4.- Stress resultants and axial displacement for hydrostatic loading. $\varepsilon = 0.1$.