RECENT ADVANCES IN SHELL THEORY

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INTRODUCTION

The results to be reviewed are divided into two categories: those that relate two-dimensional shell theory to three-dimensional elasticity theory and those concerned with shell theory _per se_. In the second category I further distinquish between results for general elastic systems that carry over, by specialization or analogy, to shells and results that are unique to shell theory itself. Because of the limitations of space and my interests, I do not mention multilayered or sandwich shells. A good discussion of these with an ample list of references may be found in Librescu's book [1]. Also, in view of the excellent review articles by Stein [2] and Hutchinson and Koiter [3], I have not attempted to review the enormous literature on shell buckling.

TWO APPROACHES TO SHELL THEORY

Most texts derive shell theory by a mixture of two-and three-dimensional considerations. However, a number of recent papers have adopted one of the following two extreme approaches:

A. A shell is idealized as a material surface in three-dimensional Euclidean space capable of transmitting forces and moments. The physical laws for this two-dimensional continuum are postulated in analogy with those for a three-dimensional one. Stress-strain laws and even failure criteria are formulated in terms of two-dimensional variables and may be deduced directly from experiments on the shell material. The papers by Sanders [4], Ericksen and Truesdell [5], Serbin [6], Budiansky [7], Simmonds and Danielson [8], and Reissner [9], to mention but a few, as well as much of the monumental treatise by Naghdi [10] are written in this spirit.

B. No matter how thin, a shell must be regarded as a three-dimensional continuum. However, the governing equations can be enormously simplified by considering various formal asymptotic expansions of the unknowns in terms of appropiate "thinness" parameters. In the interior of the shell (i.e. away from edges, concentrated loads or geometric discontinuities of one sort or another) the leading terms of the expansions satisfy various sets of two-dimensional equations that we call, collectively, the shell equations. Among those who have contributed recently to this second approach are Green [11], Johnson and Reissner [12], Cicala [13], Van der Heijden [14], and especially Goldenveiser (see the references cited in [15].)

The virtue of the first approach is also its shortcoming: there is no way to estimate intrinsically the errors made by neglecting three-dimensional effects. Or, from another viewpoint, there is no systematic way to construct a refined shell theory.

A drawback of the second approach, aside from its tediousness, is that it requires a knowledge at the edges of the shell of the distribution in the thickness direction of the applied stresses or displacements. As Koiter has emphasized [16], we never know these distributions precisely, except at a free edge. Another drawback of the second approach is that, because the thickness of the shell is always incorporated in the expansion parameters, one set of uniformly valid interior (i.e. shell) equations does not emerge. Rather there is one set of equations for a "membrane" state, another for an "inextensional bending" state, another for a "simple edge effect", another for a "degenerate edge effect", and, if one is dealing, for example, with an infinite cylindrical shell subject to self-equilibrating edge loads, still another set of equations is needed to recover the "semi-membrane" theory of Vlasov [17, p. 254].

THE ASYMPTOTIC APPROACH

The goal here is to provide a systematic method of refining the analysis of thin-walled bodies. One important consequence of the asymptotic approach is the verification and refinement of the classical Kirchhoff boundary conditions. Another useful result is that it gives a method for computing the dominant stresses in the immediate vicinity of an edge without the need of solving a full three-dimensional problem. We shall first illustrate the essence of the asymptotic method by means of a simple example drawn from the work of Goldenveiser and Van der Heijden. Then we shall indicate the implication of the results for nonlinear shell theory.

Let (r,θ,z) denote a set of cylindrical coordinates and consider a homogeneous, elastically isotropic plate that occupies the region $0\leqslant r\leqslant R$, $-H\leqslant z\leqslant H$. Let the plate be free of body forces and edge tractions but subject to self-equilibrating normal tractions on its upper and lower faces. The linear equations of elasticity may be expressed as three equilibrium equations for the six independent components $(\sigma_r, \tau, \sigma_\theta, \tau_r, \tau_\theta, \sigma)$ of the symmetric stress tensor plus six stress-strain relations with the strains expressed in terms of the components (u,v,w) of the displacement vector. Let $\rho=r/R$ and $\zeta=z/H$. Then the boundary conditions read

$$\sigma(\rho,\theta, \pm 1)=\pm \tfrac{1}{2}H^2\sigma_o p(\rho,\theta), \quad \tau_r(\rho,\theta,\pm 1)=\tau_\theta(\rho,\theta,\pm 1)=0 \tag{3.1}$$

$$\sigma_r(1,\theta,\zeta)=\tau(1,\theta,\zeta)=\tau_r(1,\theta,\zeta)=0, \tag{3.2}$$

where σ_o is a reference stress chosen so that $|p|\leqslant 1$. The boundary conditions induce a state of pure bending in which $(\sigma_r,\tau,\sigma_\theta,\sigma,u,v)$ are odd in ζ and (τ_r,τ_θ,w) are even.

Goldenveiser's approach, following earlier work by Friedrichs and Dressler [18] and Green [11], is to express each unknown as the sum of a "basic" or interior contribution plus <u>two distinct</u> "auxiliary" or edge zone contributions.

The edge zone contributions are expressed in terms of the scaled variable $\zeta=(R-r)/H\equiv\varepsilon^{-1}(1-\rho)$ so that, for example,

$$\sigma_r(\rho,\theta,\zeta;\varepsilon)=\sigma_0[\Sigma_r(\rho,\theta,\zeta;\varepsilon)+\tilde{\sigma}_r(\xi,\theta,\zeta;\varepsilon)+\hat{\sigma}_r(\xi,\theta,\zeta;\varepsilon)]$$

$$\text{and } u(\rho,\theta,\zeta;\varepsilon)+(R/E)[U(\rho,\theta,\zeta;\varepsilon)+\tilde{u}(\xi,\theta,\zeta;\varepsilon)+\hat{u}(\xi,\theta,\zeta;\varepsilon)].$$

For a traction free edge, Goldenveiser [19] assumes the following formal asymptotic expansions

$$(\Sigma_r,T,\Sigma_\theta,T_r,T_\theta,\Sigma,U,V,W) \sim\sum_0^\infty\varepsilon^n(\Sigma_r^n,T^n\Sigma_\theta^n,\varepsilon T_r^n,\varepsilon T_\theta^n,\varepsilon^2\Sigma^n,U^n,V^n,\varepsilon^{-1}W^n) \qquad (3.3)$$

$$(\tilde{\sigma}_r,\ldots,\tilde{W})\sim\sum_0^\infty\varepsilon^n(\varepsilon\tilde{\sigma}_r^n,\tilde{\tau}^n,\varepsilon\tilde{\sigma}_\theta^n,\varepsilon\tilde{\tau}_r^n,\tilde{\tau}_\theta^n,\varepsilon\tilde{\sigma}^n,\varepsilon^2\tilde{u},\varepsilon\tilde{v},\varepsilon^2\tilde{w}) \qquad (3.4)$$

$$(\hat{\sigma}_r,\ldots,\hat{W})\sim\varepsilon\sum_0^\infty\varepsilon^n(\hat{\sigma}_r^n,\varepsilon\hat{\tau}^n,\hat{\sigma}_\theta^n\ \hat{\tau}_r^n,\varepsilon\hat{\tau}_\theta^n,\hat{\sigma}^n,\varepsilon\hat{u}^n,\varepsilon^2\hat{v}^n,\varepsilon\hat{w}^n). \qquad (3.5)$$

The edge zone contributions are assumed to vanish exponentially as $\zeta\to\infty$.

When these representations are substituted into the elasticity equations and their assumed asymptotic character accounted for, there results an infinite sequence of differential equations for each infinite sequence of coefficients $\{\Sigma_r^n,\ldots,W^n\},\{\tilde{\sigma}_r^n,\ldots,\tilde{w}^n\},\{\hat{\sigma}_r^n,\ldots,\hat{w}^n\}$. Furthermore, the boundary conditions (3.1) and (3.2) imply that for $\zeta=\pm1$,

$$\Sigma^0=\pm\tfrac{1}{2}p, \quad (\Sigma^{n+1},T_r^n,T_\theta^n)=0 \qquad (3.6)$$

$$(\tilde{\tau}_\theta,\tilde{\tau}_\theta^1,\tilde{\tau}_\theta^{n+2}+\hat{\tau}_\theta^n,\tilde{\tau}_r^n+\hat{\tau}_r^n,\tilde{\sigma}^n+\hat{\sigma}^n)=0 \qquad (3.7)$$

and that for $\rho=1$ and $\xi=0$:

$$(\Sigma_r^0,T^0+\tilde{\tau}^0,T^1+\tilde{\tau}^1,\Sigma_r^{n+1}+\tilde{\sigma}_r^n+\hat{\sigma}_r^n,T^{n+2}+\tilde{\tau}^{n+2}+\hat{\tau}^n,T_r^n+\tilde{\tau}_r^n+\hat{\tau}_r^n)=0 \qquad (3.8)$$

where $n=0,1,2,\ldots$.

The equations for the interior coefficients may be integrated systematically with respect to ζ. Application of the face boundary conditions (3.6) leads, in the first instance, to the classical equation of plate bending

$$(2/3)\Delta\Delta W^0=(1-\nu^2)p, \quad \Delta W^0=W^0_{,\xi\xi}+W^0_{,\zeta\zeta}. \qquad (3.9)$$

All of the remaining lowest order interior coefficients are expressible in terms of W^0; in particular

$$\Sigma_r^0=-(1-\nu^2)^{-1}\zeta[W^0_{,\rho\rho}+\nu(\rho^{-1}W^0_{,\rho}+\rho^{-2}W^0_{,\theta\theta}]\equiv2/3\zeta M_r^0(\rho,\theta) \qquad (3.10)$$

$$T^{o}=(1+\nu)^{-1}\zeta[\rho^{-2}W^{o}_{,\theta}-\rho^{-1}W^{o}_{,\rho\theta}]\equiv 2/3\zeta H^{o}(\rho,\theta) \tag{3.11}$$

$$T^{o}_{r}=-\frac{1}{2}(1-\nu^{2})^{-1}(1-\zeta^{2})(\Delta W^{o})_{,\rho}\equiv 3/4\ (1-\zeta^{2})Q^{o}_{r}(\rho,\theta). \tag{3.12}$$

The first of the edge boundary conditions in (3.8), namely $\Sigma^{o}_{r}=0$, yields only one of the two boundary conditions needed for W^{o}. To obtain the second, the edge zone solutions must be considered.

The infinite sequence of differential equations for the set of edge zone coefficients $(\tilde{\sigma}_{r},\ldots,\tilde{w})$ can be grouped into sets which resemble the nonhomogeneous St.-Vĕnant equations for the torsion of a prism whose cross-section is the semi-infinite strip $\xi\geqslant 0, |\zeta|\leqslant 1$. Likewise the differential equations for the coefficients $(\hat{\sigma}_{r},\ldots,\hat{w})$ can be grouped into sets which resemble the non-homogeneous equations of plane strain for the same semi-infinite strip. The solutions of the torsion and plane strain problems are coupled through the non-homogeneous terms in the differential equations as well as through the boundary conditions (3.7) and (3.8) which also link these solutions with the interior solutions. It should be noted that in the edge zone differential equations, θ appears only as a parameter.

In order that the edge zone solutions decay as $\xi\to\infty$, it is necessary that the forces and moments applied to the boundary of the semi-infinite strip be equilibrated by the non-homogeneous terms in the torsion and plane strain equilibrium equations. These integral conditions yield, ultimately, the additional boundary conditions needed for the various interior solutions. For example, the Kirchhoff boundary condition that relates the shear stress resultant Q^{o}_{r} and the derivative along the edge of the twisting stress couple H^{o} is obtained as follows.

The solution of the lowest order torsion problem may be expressed in terms of a stress function ψ^{o}, where

$$\Delta\psi^{o}=0, \quad \psi_{,\zeta}(\xi,\pm 1)=0, \quad \psi_{,\xi}(0,\zeta)=-\zeta \tag{3.13}$$

and

$$\tilde{\tau}^{o}= -2/3\zeta H^{o}(1,\theta)\psi^{o}_{,\xi}, \quad \tilde{\tau}^{o}_{\theta}= -2/3\zeta H^{o}(1,\theta)\psi^{o}_{,\zeta} \tag{3.14}$$

The lowest order equation for equilibrium in the ζ-direction for the plane strain problem is

$$(\hat{\tau}^{o}_{r}+\tilde{\tau}^{o}_{r})_{,\xi} + (\hat{\sigma}^{o}+\tilde{\sigma}^{o})_{,\zeta}= -\tilde{\tau}^{o}_{\theta,\theta}\equiv R^{o}_{\zeta}. \tag{3.15}$$

From the last of the boundary conditions (3.7) and (3.8), the condition that the net forces in the ζ-direction add to zero, to lowest order, is

$$-\int_{-1}^{1}[T^{o}_{r}(1,\theta,\zeta) + \int_{0}^{\infty} R^{o}_{\zeta}(\xi,\theta,\zeta)d\xi]d\zeta=0. \tag{3.16}$$

With the aid of (3.12) and (3.13) to (3.15), (3.16) reduces to

$$Q^{o}_{r} + H^{o}_{,\theta}= 0 \text{ at } \rho=1, \tag{3.17}$$

which is the second boundary condition for W^o. It is important to note that one never needs to actually solve for ψ^o to obtain this result.

GOLDENVEISER'S EXTENSION AND KOITER'S SIMPLIFICATION OF THE PRECEDING RESULTS

The solution for $(\Sigma_r^1, \ldots, W^1)$ reduces to the solution of a biharmonic equation for W^1. To obtain boundary conditions for W^1 one again considers the integral conditions of overall equilibrium necessary to guarantee decaying edge zone solutions. To evaluate these, one must solve explicitly for ψ^o (which is easily done) but needs only to consider the form of the solution of the lowest order plane strain problem. After a straightforward but tedious analysis, there results the refined boundary conditions of Goldenveiser [19]:

$$M_r^1 = AH,_\theta^o, \quad Q_r^1 + H,_\theta^1 + AH,_\theta^o = 0, \tag{4.1}$$

where A=1.260...is computed from the solution for ψ^o. The details of the calculations leading to (4.1) may be found in a report by Van der Heijden [20].

Goldenveiser's results may be restated in the following useful way. Consider a plate of radius R and thickness 2H subject to a self-equilibrated normal pressure p but otherwise free of surface and edge tractions. Solve the classical equation of plate bending subject to the refined boundary conditions.

$$M_r - A(H/R)H,_\theta = 0, \quad Q_r + H,_\theta + A(H/R)H,_\theta = 0 \text{ at } \rho=1. \tag{4.2}$$

Then the stresses in the interior of the plate, to within a relative error of $O(H^2/R^2)$, are given by the formulas for Σ_r^o, T^o, etc. but with W^o replaced by W. Moreover, in the edge zone of the plate, the dominant stresses, to within a relative error of O(H/R), are given by these same formulas except that T^o is replaced by $T^o+\tilde{\tau}^o$, and T_θ^o is replaced by $\tilde{\tau}_\theta^o$, where $\tilde{\tau}^o$ and $\tilde{\tau}_\theta^o$ are given by (3.14)

These results are simple and satisfying. Though derived for, perhaps, the simplest, non-trivial problem imaginable, their qualitative implications for shells with free edges undergoing large deformations is clear, namely 1), the most importnat refinement of the classical shell equations are in the boundary conditions and 2), the dominant stresses near a free edge can be inferred from the solution of the shell equations and the solution of a torsion problem for a semi-infinite strip. To give these statements a quantitative form via an asymptotic analysis would seem to be a formidable task.

The problem of refining the Kirchhoff boundary conditions at a free edge has, fortunately, been solved by Koiter [15] in an alternate way, using an ingenious energy argument. As Danielson [21], and Koiter [22] have shown, the three-dimensional tangential shear stress predicted by shell theory at a free edge does not vanish, even though the Kirchhoff boundary conditions are satisfied exactly. Thus the conventional strain energy expression of shell theory overestimates the torsional energy in the neighborhood of a free edge. To assess this error, Koiter considers the torsional rigidity of a flat strip whose thickness is equal to that of the shell. By comparing this expression with that given by classical plate theory he is able to identify an edge zone correction factor which is proportional to the twist per unit length of the edge of the strip. The torsional energy associated with this term is therefore expressible as a line integral. For an arbitrary shell with a smooth edge curve Koiter argues that one merely needs to insert an appropiate expression for the edge twisting per unit length for the shell into this line integral and then subtract this expression from the conventional surface integral for the shell energy.

Koiter's result may be of limited practical value. If the shell has other edges that are not free of stress, it is most likely that the associated shell boundary conditions cannot be refined because the corresponding boundary conditions of elasticity theory cannot be determined precisely. The shell equations are elliptic, hence the influence of boundary conditions extend everywhere, and it would be inconsistent to use refined boundary conditions at a free edge but unrefined ones at another edge.

The results of this section also imply that so-called thick shell theories are meaningless if applied to homogeneous shells with edges. We should note, however, that Van der Heijden has shown that Reissner's latest thick plate theory [23] does give fairly good <u>numerical results</u> for stress concentration factors for circular holes in infinite plates.

THE DIRECT APPROACH TO SHELL THEORY

Here and in the following section we mention briefly — space limitations permit no more — some recent work concerning different formulations, implications, simplifications and the reduction of certain problems of the now generally accepted equations of first-approximation shell theory.

Formulations of the Nonlinear Theory

A strictly mechanical theory of shells may be expressed entirely in terms of the midsurface displacement components [5]. If dynamic effects are excluded, alternate formulations are possible in terms of the components of a stress function and rotation vector [8], or in terms of stress resultants and bending strains [15]. In the last case, any displacement boundary conditions need to be reformulated in terms of strains [24,25]. This in itself has advantages, for it automatically leads to the boundary conditions for inextensional deformation and, in the linear theory, it gives boundary conditions that are the geometric analogues of the Kirchhoff conditions.

Thermodynamic Considerations

These are important for at least three reasons. 1) heating a shell may cause it to fail, buckle, or vibrate; 2) the best justification of the static approach to stability for a continuous body is a thermodynamic one; and 3) the coupling of mechanical and thermal effects produces damping.

There is a plethora of papers on 1) that we shall not attempt to review; a few texts give a discussion of the underlying ideas. The thermodynamic aspects of stability in general elastic systems are discussed in [26,27,28]. These results are directly transferable to shell theory. The specific form and role of the laws of thermodynamics in shell theory are discussed in [10]. The effect of thermal damping on the free vibrations of shells is considered in [29] where it is also shown that, because the damping is light, perturbation methods may be used to advantage.

Variational Principles

A problem of long standing in nonlinear elasticity has been to formulate a principle of complementary energy. Recent work [30,31,32] has established conditions under which this is possible. In particular, in [33] and [34], these results have been applied to the nonlinear von Karman plate equations and Marguerre shallow shell equations to obtain upper and lower bounds on an associated energy functional.

SOME NEW RESULTS IN LINEAR SHELL THEORY

Shells As Beams

For general cylindrical shells and shells of revolution, one may consider special classes of solutions that, in a St. Venant sense, correspond to the stretching, bending, twisting, and flexure of a beam. In many cases the resulting equations can be solved explicitly. See [35,36].

Reduction of the Governing Equations

The shell equations constitute a system of eighth order. For analytical purposes, especially for the application of perturbation methods, it is often convenient to attempt to express these equations as two coupled fourth order equations. (A single eighth order equation destroys the very useful static-geometric duality). Such reductions have been found for spherical, general cylindrical, and minimal shells as well as for shells of revolution. A reduction for arbitrary, non-developable shells is also possible, but does involve some loss of accuracy. See [37] where other references are cited.

Membrane Theory

It is well known that shells with the proper shape and boundary support can be analyzed with good accuracy by membrane theory. The details of such an approach are spelled out in a very general but useful way in [38].

Cracks and Cutouts

Shells may contain cutouts by design and cracks by accident. In practice the dimensions of these cracks and cutouts is apt to be small compared to some characteristic geometric dimension of the shell, permitting shallow shell theory to be applied. The calculation of the stresses has been reduced to the solution of coupled singular integral equations [39] that have been solved numerically for several important problems. See [40] and the references cited there.

Pointwise Estimates For Approximate Solutions

The Prager-Synge hypercircle method is useful for constructing approximate solutions to linear shell problems, and provides mean square error estimates for the approximate stress field. More desirable are pointwise estimates for both the approximate stress field and the approximate displacement field. For recent work on this problem see [41] and the references cited therein.

Wave Propagation, Asymptotics, and St-Venant's Principle

These are three additional areas in which there has been significant recent progress but which cannot be reviewed for lack of space.

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