FLUID-PLASTICITY OF THIN CYLINDRICAL SHELLS*

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SUMMARY

The paper considers dynamic plastic response of a thin cylindrical shell, immersed in a potential fluid initially at rest, subjected to internal pressure pulse of arbitrary shape and duration. The shell is assumed to respond as a rigid-perfectly plastic material while the fluid is taken as inviscid and incompressible. The fluid back pressure is incorporated into the equation of motion of the shell as an added mass term. Since arbitrary pulses can be reduced to equivalent rectangular pulses, the equation of motion is solved only for a rectangular pulse. The influence of the fluid in reducing the final plastic deformation is demonstrated by a numerical example.

FORMULATION OF THE PROBLEM

Consider a rigid-ideally plastic, thin-walled, circular, cylindrical shell of infinite length. The shell is surrounded by a pool of potential (inviscid and incompressible) fluid infinitely extended in all directions. The shell is subjected to an internal pressure pulse $P(z,t)$, varying both along the axis and with time. $P(z,t)$ is further assumed to be axisymmetric and symmetric in z with respect to $z = 0$ (fig. 1).

This paper examines the influence of the fluid in reducing the plastic (residual) deformation of the shell. It is known that the pressure with which potential fluid resists the motion of a deforming solid can be considered as an increase in the inertia of the solid. Therefore in order to solve the problem it is necessary to establish the so-called effective mass consisting of the actual mass of the shell and the added (virtual) mass reflecting the fluid resistance. Then the problem is reduced to the analysis of a shell deforming in vacuum. For the sake of continuity, we will adopt the notation introduced in reference 1.

GOVERNING EQUATIONS

The equation of motion of the shell is:

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$$\frac{\partial^2 M}{\partial z^2} = P - P_f - \frac{N}{R} - \rho H \frac{\partial V}{\partial t} \tag{1}$$

where M is the axial bending moment, N the circumferential (hoop) normal force, R, H and ρ the radius, the wall thickness and the mass density of the shell respectively, V the radial velocity of the points on the middle surface of the shell, and P(z,t) and $P_f(z,t)$ are the externally applied pressure pulse and the back pressure of the fluid resisting the motion of the shell respectively.

We assume that the yield condition in the M,N space is defined by the limited interaction curve of fig. 2. The implications of this assumption are discussed in detail by Drucker (ref. 2) and Hodge (ref. 3). The yield values M_y and N_y are given by

$$M_y = \frac{1}{4} HRP_o \qquad N_y = RP_o \tag{2}$$

where

$$P_o = \sigma_y \frac{H}{R} \tag{3}$$

with σ_y being the yield stress.

It is known (see, for example, ref. 1) that four different phases (modes) of plastic deformation may occur during the motion. We will consider herein only one of these phases which occurs for all possible types of loading, though this restricts the magnitude of the loading to a certain limit. In the considered phase the deformation is characterized by a stationary plastic hinge circle at z = 0 and two moving hinge circles at $z = \pm \zeta(t)$.

It can be shown (see, for example, Eason and Shield (ref. 4)) that the plastic regimes (see fig. 2) are as follows:

$$\begin{aligned} z = 0:&\quad M = -M_y, &\quad N = N_y &\quad \text{Regime A} \\ z = \zeta:&\quad M = M_y, &\quad N = N_y &\quad \text{Regime B} \\ 0 < z < \zeta:&\quad -M_y < M < M_y, &\quad N = N_y &\quad \text{Regime AB} \end{aligned} \tag{4}$$

Thus, from the normality of the strain-rate vector to the yield surface,

$$\frac{\partial^2 V}{\partial z^2} = 0 \quad , \qquad V > 0 \quad \text{for} \quad 0 < z < \zeta \tag{5}$$

For this deformation mode the velocity, V(z,t), is therefore a linear function of z, i.e.,

$$V(z,t) = \begin{cases} V_o(t) \left\{1 - \frac{z}{\zeta(t)}\right\} & 0 < z \le \zeta \\ 0 & z > \zeta \end{cases} \tag{6}$$

In the above equations and in the sequel because of symmetry it is enough to consider only half of the shell $z \geq 0$.

DETERMINATION OF THE ADDED MASS

Before attempting to solve equation (1), the backpressure $P_f(z,t)$ should be determined as a function of $V(z,t)$ and its derivatives. The equation governing the flow of the potential fluid is in polar coordinates

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} = 0 \quad \text{in} \quad r \geq R \tag{7}$$

where $F(r,z,t)$ is the fluid velocity potential.

As the shell is impermeable, the velocities of the fluid and the shell at the points of contact must be identical, i.e.,

$$\frac{\partial F}{\partial r} = V \quad \text{at} \quad r = R \tag{8}$$

Furthermore, from the radiation principle,

$$V \to 0 \quad , \quad \frac{\partial F}{\partial r} \to 0 \quad , \quad \frac{\partial F}{\partial z} \to 0 \quad \text{as} \quad \max(r,z) \to \infty \tag{9}$$

Once the fluid velocity potential is determined from the Laplace equation (7), subject to the boundary conditions (eqs. (8) and (9)) the pressure exerted by the fluid on the shell can be computed from the Cauchy integral,

$$P_f(z,t) = -\rho_f \frac{\partial F}{\partial t} \quad \text{at} \quad r = R \tag{10}$$

where ρ_f is the mass density of the fluid.

The equations (7) and (10) imply the assumption that the perturbations about average values can be neglected.

As a further approximation, we will assume that the functional relation between $P_f(z,t)$ and ζ is not sensitive to the time dependence of ζ, and hence ζ may be treated as a constant for the determination of this relation. Then in view of equations (6), (7) and (8), we may write

$$F(r,z,t) = V_o(t)\, f(r,z) \tag{11}$$

and from equation (10),

$$P_f(z,t) = -\rho_f f(R,z) \frac{dV_o}{dt} \tag{12}$$

where $-\rho_f f(R,z)$ is the added (virtual) mass arising due to the resistance of the fluid being displaced by the shell.

Substituting equations (11) and (6), (with ζ being constant) into equations (7) to (9), it follows

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{13}$$

subject to

$$\frac{\partial f}{\partial r} = \begin{cases} \left(1 - \frac{z}{\zeta}\right) & \text{at} \quad r = R, \quad z \le \zeta \\ 0 & \text{at} \quad r = R, \quad z > \zeta \end{cases} \tag{14}$$

and

$$f \to 0 \quad , \quad \frac{\partial f}{\partial r} \to 0 \quad , \quad \frac{\partial f}{\partial z} \to 0 \quad \text{as} \quad \max(r,z) \to \infty \tag{15}$$

The details of the solution of equation (13) are omitted herein for the sake of brevity. A closed form integral solution is obtained after introducing the Fourier cosine transform. The argument of this integral is rather complicated and the integration is performed in three stages using asymptotic formulae and Filon's method, subject to the restriction, $\zeta \le R$ which is subsequently seen to be not severe. In order to make this numerical solution amenable for substitution into equation (1), the result is subjected to a series of polynomial regression analyses. Finally we obtain

$$f(R,z) = -R\left\{g_0\left(\frac{\zeta}{R}\right) + g_1\left(\frac{\zeta}{R}\right)\frac{z}{\zeta}\right\} \tag{16}$$

where

$$g_0(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 \tag{17}$$

and

$$g_1(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \tag{18}$$

with

$$\begin{aligned} \alpha_0 &= .004994512 & \beta_0 &= .02050149 \\ \alpha_1 &= -.5420473 & \beta_1 &= 1.664447 \\ \alpha_2 &= -.1058701 & \beta_2 &= -1.105309 \\ \alpha_3 &= .1627719 & \beta_3 &= .4096600 \end{aligned} \tag{19}$$

Note α_i and β_i are dimensionless constants that do not depend on the shell/fluid parameters.

PLASTIC DEFORMATION OF THE SHELL

Equation (1) now becomes, in view of equations (2), (6), (12) and (16),

$$\frac{\partial^2 M}{\partial z^2} = P(z,t) - P_o - \left[\left\{\rho H + \rho_f R g_0\left(\frac{\zeta}{R}\right)\right\} - \left\{\rho H - \rho_f R g_1\left(\frac{\zeta}{R}\right)\right\}\frac{z}{\zeta}\right]\frac{dV_o}{dt}$$

$$\text{in } 0 < z < \zeta \tag{20}$$

In equation (20), $\zeta = \zeta(t)$. For arbitrary $P(z,t)$ the above equation may only be solved by numerical methods. As a first step in simplifying the work, the approach introduced by Youngdahl (ref. 1) will be used to approximate a complex loading function by (i.e., correlate it to) a simple rectangular pulse. Since the standard limit analysis of the shell is independent of any surrounding medium, the method given by Youngdahl (ref. 1) to determine the equivalent rectangular pulse does not need any modification in this case. Thus correlated, $P(z,t)$ can be expressed as

$$\begin{aligned} P(z,t) &= P_e \qquad |z| \le L_e \quad \text{and} \quad 0 \le t \le t_e \\ P(z,t) &= 0 \qquad |z| > L_e \quad \text{or} \quad t > t_e \end{aligned} \tag{21}$$

where P_e is the magnitude, t_e the duration and $2L_e$ the length of the loaded area of the equivalent rectangular pulse (see ref. 1 for their derivation).

For plastic deformation to occur, P_e must be greater than the limit load. This condition is expressed by (see ref. 1)

$$P_e > \frac{P_o}{2}\left(1 + \sqrt{1 + \frac{4RH}{L_e^2}}\right) \tag{22}$$

For the deformation to take place in the assumed phase, the following boundary conditions must be satisfied

$$\begin{aligned} M &= -M_y \quad , \quad \frac{\partial M}{\partial z} = 0 \quad \text{at} \quad z = 0 \\ M &= M_y \quad , \quad \frac{\partial M}{\partial z} = 0 \quad \text{at} \quad z = \zeta \end{aligned} \tag{23}$$

Further, the condition that the bending moment does not exceed M_y at the hinge circle at $z = \zeta$ implies

$$\frac{\partial^2 M}{\partial z^2} < 0 \quad \text{at} \quad z = \zeta \tag{24}$$

In addition the condition that the bending moment cannot be less than $-M_y$ at the hinge circle at $z = 0$ implies

$$\frac{\partial^2 M}{\partial z^2} > 0 \quad \text{at} \quad z = 0 \tag{25}$$

For the interval $0 \le t \le t_e$, a trial solution as in the case of a shell deforming in vacuum is assumed. This is taken in the form

$$\left.\begin{aligned} \zeta(t) &= z_1 \\ \text{and} \quad V_o(t) &= \frac{K_1}{\rho H}\, t \end{aligned}\right\} \quad 0 \le t \le t_e \tag{26}$$

where z_1 and K_1 are constants. Substituting equation (26) into equation (20) and integrating twice subject to the boundary conditions (23) yields in the end two equations for z_1 and K_1. These two can be reduced to:

$$\alpha(z_1)P_o z_1^2 + \{6 - 4\alpha(z_1)\}P_e L_e z_1 - \{6 - 3\alpha(z_1)\}(P_e L_e^2 + P_o RH) = 0 \tag{27}$$

and

$$K_1 = \frac{6\rho H(P_e L_e z_1 - P_e L_e^2 - P_o RH)}{z_1^2 \left\{\rho H - \rho_f R g_1 \left(\frac{z_1}{R}\right)\right\}} \tag{28}$$

where

$$\alpha(z_1) = \frac{\rho H - \rho_f R g_1 \left(\frac{z_1}{R}\right)}{\rho H + \rho_f R g_0 \left(\frac{z_1}{R}\right)} \tag{29}$$

The inequalities (24) and (25) can be written in the form

$$P_o z_1^2 + K_1 \frac{\rho_f}{\rho} \frac{R}{H} \left\{g_0 \left(\frac{z_1}{R}\right) + g_1 \left(\frac{z_1}{R}\right)\right\} z_1^2 > 0 \tag{30}$$

and

$$4P_e L_e z_1 - P_e z_1^2 - 3P_e L_e^2 - 3P_o RH < 0 \tag{31}$$

From equations (17), (18) and (19) it is easily verified that inequality (30) is always satisfied. If inequality (31) is not satisfied motion cannot start in the phase assumed. When the inequality becomes an equality, P_e takes the bounding value. To determine the bounding value of P_e, the non-linear equations (27) and (31) should be solved simultaneously. This is done numerically. Figure 3 shows the range of values of P_e that satisfies inequalities (22) and (31) and hence gives rise to deformation in the assumed mode. For a P_e belonging to this range, the non-linear equation (27) may be solved numerically. Also it may be verified that the restriction $z_1 < R$ is always satisfied if $L_e < R$. Thus, the solution discussed in this paper is valid for $L_e < R$ and P_e satisfying inequalities (22) and (31).

For $t > t_e$, there is no internal pressure. Letting $P(z,t) = 0$ in equation (20), integrating it twice with respect to z and substituting the result into the boundary conditions (23), we arrive at the following equations

$$\frac{dV_o}{dt} = \frac{-P_o(\zeta^2 + 3RH)}{\zeta^2 \left\{\rho H + \rho_f R g_0 \left(\frac{\zeta}{R}\right)\right\}} \equiv \phi(\zeta) \tag{32}$$

$$\frac{d}{dt}\left(\frac{V_o}{\zeta}\right) = \frac{-6P_o RH}{\zeta^3 \left\{\rho H - \rho_f R g_1 \left(\frac{\zeta}{R}\right)\right\}} \equiv \psi(\zeta) \tag{33}$$

Equations (32) and (33) constitute a system of non-linear first order differential equations for V_o and ζ, the initial conditions being given by

$$\zeta(t_e) = z_1 \tag{34}$$

$$V_o(t_e) = \frac{K_1 t_e}{\rho H} \tag{35}$$

The above differential equations are valid only for $t_e \le t \le t_f$, where t_f is defined by

$$V_o(t_f) = 0 \tag{36}$$

We will denote

$$\zeta_f \equiv \zeta(t_f) \tag{37}$$

From equations (32) and (33), we can express V_o as,

$$V_o(t) = \frac{P_o}{\zeta\left(\frac{d\zeta}{dt}\right)}\left\{\frac{6RH}{\rho H - \rho_f R g_1\left(\frac{\zeta}{R}\right)} - \frac{3RH + \zeta^2}{\rho H + \rho_f R g_0\left(\frac{\zeta}{R}\right)}\right\} \tag{38}$$

From (36) and (37), we see that ζ_f can be obtained from the equation,

$$\frac{6RH}{\rho H - \rho_f R g_1\left(\frac{\zeta_f}{R}\right)} - \frac{3RH + \zeta_f^2}{\rho H + \rho_f R g_0\left(\frac{\zeta_f}{R}\right)} = 0 \tag{39}$$

Equation (39) can be solved numerically to obtain ζ_f. It is noted that ζ_f depends only on the shell parameters H and R and the density ratio ρ_f/ρ. Since ζ_f is the quantity that is known and not t_f, the equations (32) and (33) are now reformulated with ζ being the independent variable and $t(\zeta)$ and $V_o(\zeta)$ being the dependent variables. Thus,

$$\frac{dV_o}{d\zeta} = \frac{dt}{d\zeta}\,\phi(\zeta) \tag{40}$$

and

$$\frac{d}{d\zeta}\left(\frac{V_o}{\zeta}\right) = \frac{dt}{d\zeta}\,\psi(\zeta) \tag{41}$$

$$z_1 \le \zeta \le \zeta_f$$

The new initial conditions are

$$t(z_1) = t_e \tag{42}$$

and

$$V_o(z_1) = \frac{K_1 t_e}{\rho H} \tag{43}$$

Equations (40) and (41) are solved numerically. Finally, the maximum plastic deformation $U_o(t)$ can be obtained as,

$$U_o(t) = \frac{K_1 t^2}{2\rho H} \qquad 0 \le t \le t_e \tag{44}$$

and $$U_o(t) = U_o(t_e) + \int_{t_e}^{t} V_o(\bar{t})d\bar{t} \qquad 0 < t \le t_f$$

or $$U_o(t) = \frac{K_1 t_e^2}{2\rho H} + \int_{z_1}^{\zeta} V_o(\bar{\zeta}) \frac{dt}{d\bar{\zeta}} d\bar{\zeta} \qquad z_1 \le \zeta \le \zeta_f \tag{45}$$

The integral in equation (45) can be numerically evaluated after the numerical solutions $V_o(\zeta)$ and $t(\zeta)$ have been obtained.

For the special case in which z_1 coincides with ζ_f, the solution for $t > t_e$ discussed above is not valid. For this case, equations (32) and (33) with (35) yield,

$$\zeta = z_1 = \zeta_f \tag{46}$$

$$V_o(t) = (t - t_e)\phi(z_1) + \frac{K_1 t_e}{\rho H} \tag{47}$$

where $\phi(\zeta)$ is defined in equation (32). Note $\phi(z_1) < 0$. From equation (47) we see

$$t_f = t_e \left\{1 - \frac{K_1}{\rho H \phi(z_1)}\right\} \tag{48}$$

From equations (47) and (44) we can show that

$$U_o(t) = \frac{\phi(z_1)}{2}(t - t_e)^2 + \frac{K_1 t_e}{2\rho H}(2t - t_e) \tag{49}$$

Finally we have, for the maximum plastic deformation in this special case

$$U_o(t_f) = \frac{K_1 t_e^2}{2\rho H}\left(1 - \frac{K_1}{\rho H \phi(z_1)}\right) \tag{50}$$

NUMERICAL EXAMPLE

For a shell with $H/R = 1/36$, $L_e/R = 1/4$ surrounded by a fluid of $\rho_f/\rho = 1/10$, the complete numerical solution is determined for the admissible range of loads P_e. As is seen from figure 3, the range of P_e/P_o that will give rise to motion in the assumed phase is between: 1.33 and 2.97. The same range for a shell in vacuum is 1.33 to 2.19. Figure 4 shows the final maximum plastic deformation, $U_o(t_f)$, (non-dimensionalized as $\rho H U_o/P_o t_e^2$) as a function of P_e/P_o. For the sake of comparison the corresponding curve for a shell deforming in vacuum is also shown in the same figure.

In the numerical methods used, non-linear algebraic equations such as (27) and (39) were solved by Newton's iteration method and the system of differential equations (40), (41) by a method using automatic step change (ref. 5).

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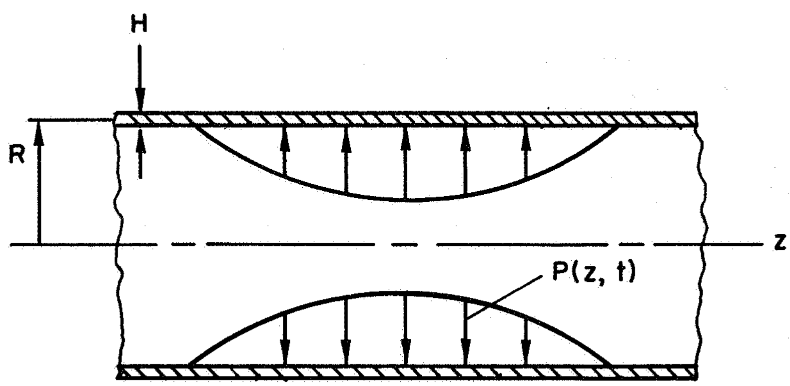

Figure 1.- Circular cylindrical shell immersed in fluid and loaded by internal pressure pulse.

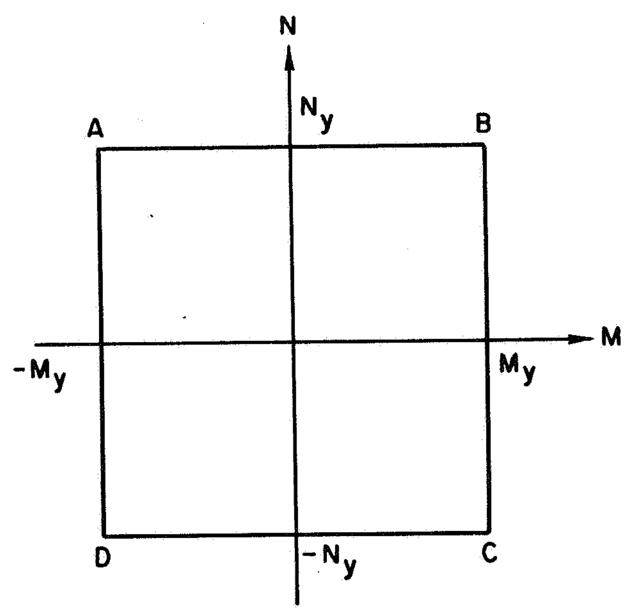

Figure 2.- Yield condition.

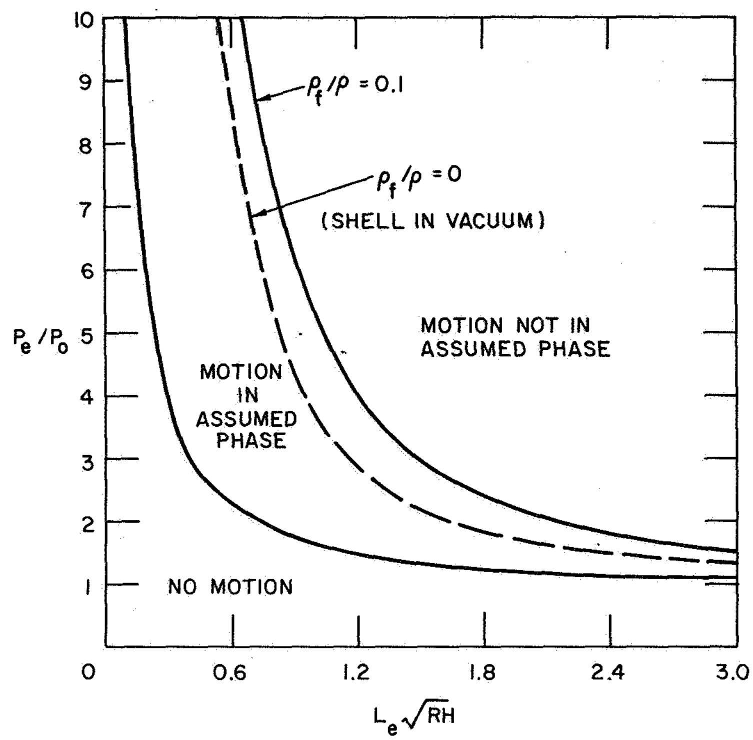

Figure 3.- Range of pulse intensity initiating motion in assumed phase.

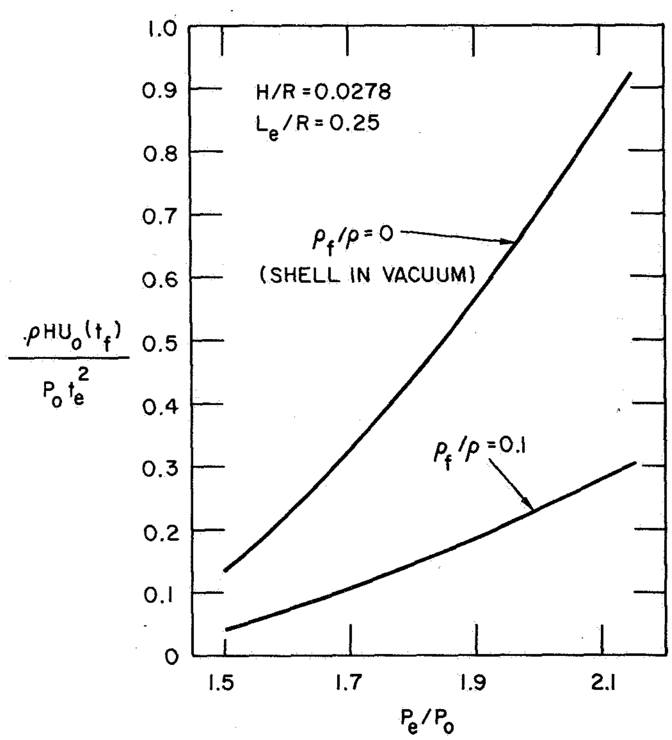

Figure 4.- Maximum plastic deformation as a function of pulse intensity.