THERMAL STRESSES IN A SPHERICAL PRESSURE VESSEL HAVING TEMPERATURE-DEPENDENT, TRANSVERSELY ISOTROPIC, ELASTIC PROPERTIES

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SUMMARY

Rayleigh-Ritz and modified Rayleigh-Ritz procedures are used to construct approximate solutions for the response of a thick-walled sphere to uniform pressure loads and an arbitrary radial temperature distribution. The thermoelastic properties of the sphere are assumed to be transversely isotropic and nonhomogeneous; variations in the elastic stiffness and thermal expansion coefficients are taken to be an arbitrary function of the radial coordinate and temperature. Numerical examples are presented which illustrate the effect of the temperature-dependence upon the thermal stress field. A comparison of the approximate solutions with a finite element analysis indicates that Ritz methods offer a simple, efficient, and relatively accurate approach to the problem.

INTRODUCTION

Modern engineering structures are often subject to thermal environments in which the temperature causes significant variations in the thermal and mechanical properties of the material. Over certain temperature ranges the material may behave elastically, but have variable stiffness and thermal expansion characteristics. In addition, modern materials of construction (e.g. composites) often possess anisotropy and nonhomogeneity. While most classical thermoelastic solutions are not applicable to situations involving temperature-dependent anisotropic behavior, some progress has been made in this direction. For example, the problem of a hollow sphere with temperature-sensitive isotropic elastic properties has been studied by Nowinski (ref. 1) and Stanisic and McKinley (ref. 2). More recently Hata and Atsumi (ref. 3) investigated the response of a transversely isotropic sphere exposed to a sudden temperature rise on its internal surface.

In the present paper a transversely isotropic hollow sphere having temperature sensitivity and/or initial nonhomogeneity is considered. The variability of the thermoelastic properties may result from manufacturing processes, in which case the properties depend upon position but not temperature, or the nonhomogeneity may be a consequence of the materials' temperature sensitivity.

FORMULATION OF THE PROBLEM

Consider a hollow elastic sphere of inner radius r_1 and outer radius r_2, exposed to a temperature distribution T(r) in addition to internal and external pressures, p_I and p_{II}, respectively. Owing to the spherical symmetry of the problem, the nonvanishing strain components depend upon the radial displacement u according to the relations

$$e_{rr} = \frac{du}{dr} \qquad e_{\phi\phi} = e_{\theta\theta} = \frac{u}{r} \tag{1}$$

Assuming transverse isotropy, the thermal stresses are related to the strains and temperature rise by

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\phi\phi} \\ \sigma_{\theta\theta} \end{Bmatrix} = \begin{Bmatrix} A_{11} \\ A_{12} \\ A_{12} \end{Bmatrix} e_{rr} + \begin{Bmatrix} A_{12} \\ A_{22} \\ A_{23} \end{Bmatrix} e_{\phi\phi} + \begin{Bmatrix} A_{12} \\ A_{23} \\ A_{22} \end{Bmatrix} e_{\theta\theta} - \int_0^T \begin{Bmatrix} \beta_1(T,r) \\ \beta_2(T,r) \\ \beta_2(T,r) \end{Bmatrix} dT \tag{2}$$

in which $A_{ij}(T,r)$ denote the elastic stiffnesses and $\beta_i(T,r)$ are the stress-temperature coefficients. Alternatively, the strains may be expressed in terms of the stresses and temperature as

$$\begin{Bmatrix} e_{rr} \\ e_{\phi\phi} \\ e_{\theta\theta} \end{Bmatrix} = \begin{Bmatrix} a_{11} \\ a_{12} \\ a_{12} \end{Bmatrix} \sigma_{rr} + \begin{Bmatrix} a_{12} \\ a_{22} \\ a_{23} \end{Bmatrix} \sigma_{\phi\phi} + \begin{Bmatrix} a_{12} \\ a_{23} \\ a_{22} \end{Bmatrix} \sigma_{\theta\theta} + \int_0^T \begin{Bmatrix} \alpha_1(T,r) \\ \alpha_2(T,r) \\ \alpha_2(T,r) \end{Bmatrix} dT \tag{3}$$

where $a_{ij}(T,r)$ and $\alpha_i(T,r)$ are the compliances and the coefficients of thermal expansion, respectively.

For convenience in later operations the following dimensionless quantities are introduced:

$$\left.\begin{array}{lll} \rho = r/r_2 & \rho_1 = r_1/r_2 & v = u/r_2 \\ \Theta = T/T_o & q_I = p_I/\beta_o T_o & q_{II} = p_{II}/\beta_o T_o \\ t_{\rho\rho} = \sigma_{rr}/\beta_o T_o & t_{\phi\phi} = \sigma_{\phi\phi}/\beta_o T_o & t_{\theta\theta} = \sigma_{\theta\theta}/\beta_o T_o \\ B_{ij}(\Theta,\rho) = A_{ij}(T,r)/\beta_o T_o & & b_{ij}(\Theta,\rho) = a_{ij}(T,r)\beta_o T_o \end{array}\right\} \tag{4}$$

$$\gamma_i(\Theta,\rho) = \beta_i(T,r)/\beta_o \qquad \epsilon_i(\Theta,\rho) = \alpha_i(T,r)T_o$$

in which β_o denotes an arbitrary reference stress-temperature coefficient and T_o represents some reference temperature.

In formulating the problem through the use of energy principles, we require specification of the total potential energy of the sphere. For the case of quasi-static loading, the *total potential energy* Π consists of the strain energy U plus the potential V_E of the external forces. General expressions for the strain energy in anisotropic, temperature-sensitive, elastic bodies are given in reference 4. Based upon these expressions the total potential energy for a transversely isotropic sphere with strains given by equation (1) is

$$\Pi = 4\pi\beta_o T_o r_2^3 \left\{ \int_{\rho_1}^{1} \left[\tfrac{1}{2}B_{11}\left(\frac{dv}{d\rho}\right)^2 + 2B_{12}\left(\frac{dv}{d\rho}\right)\left(\frac{v}{\rho}\right) + \left(B_{22}+B_{23}\right)\left(\frac{v}{\rho}\right)^2 \right.\right.$$
$$- \frac{dv}{d\rho}\int_0^{\Theta}\gamma_1(\Theta,\rho)d\Theta - 2\frac{v}{\rho}\int_0^{\Theta}\gamma_2(\Theta,\rho)d\Theta + \tfrac{1}{2}\int_0^{\Theta}\epsilon_1(\Theta,\rho)d\Theta\int_0^{\Theta}\gamma_1(\Theta,\rho)d\Theta \tag{5}$$
$$\left.\left. + \int_0^{\Theta}\epsilon_2(\Theta,\rho)d\Theta\int_0^{\Theta}\gamma_2(\Theta,\rho)d\Theta \right]\rho^2 d\rho - \rho_1^2\; q_I v(\rho_1) + q_{II}v(1)\right\}$$

in which the integral expressions constitute the strain energy, and the terms involving q_I and q_{II} represent the potential of the pressure loads.

A complementary variational approach to the problem, in which stresses rather than displacements represent the varied quantities, involves the *total complementary energy*. When tractions are specified over the entire boundary of the body, the total complementary energy Π^* is equal to the complementary strain energy U^*. From the general results given in reference 4 it can be shown that for the sphere

$$\Pi^* = 4\pi\beta_o T_o r_2^3 \int_{\rho_1}^{1}\left[\tfrac{1}{2}b_{11}t_{\rho\rho}^2 + b_{12}t_{\rho\rho}(t_{\phi\phi}+t_{\theta\theta}) + \tfrac{1}{2}b_{22}(t_{\phi\phi}^2+t_{\theta\theta}^2)\right.$$
$$\left. + b_{23}t_{\phi\phi}t_{\theta\theta} + t_{\rho\rho}\int_0^{\Theta}\epsilon_1(\Theta,\rho)d\Theta + (t_{\phi\phi}+t_{\theta\theta})\int_0^{\Theta}\epsilon_2(\Theta,\rho)d\Theta\right]\rho^2 d\rho \tag{6}$$

Before developing approximate solutions to the problem, it is noted that the governing differential equation and natural boundary conditions can be derived through direct application of the *principle of minimum potential energy*. Requiring that the first variation of the total potential energy be equal to zero ($\delta\Pi = 0$), and performing integration by parts, one obtains the displacement equation of equilibrium

$$B_{11}\left(\frac{d^2v}{d\rho^2}+\frac{2}{\rho}\frac{dv}{d\rho}\right)-2(B_{22}+B_{23}-B_{12})\frac{v}{\rho^2}+\frac{dB_{11}}{d\rho}\frac{dv}{d\rho}+2\frac{dB_{12}}{d\rho}\frac{v}{\rho}$$
$$=\frac{d}{d\rho}\int_0^{\Theta}\gamma_1(\Theta,\rho)d\Theta+\frac{2}{\rho}\int_0^{\Theta}[\gamma_1(\Theta,\rho)-\gamma_2(\Theta,\rho)]d\Theta \tag{7}$$

and the natural boundary conditions

$$\left.\begin{aligned} B_{11}(\rho_1)\frac{dv(\rho_1)}{d\rho}+2B_{12}(\rho_1)\frac{v(\rho_1)}{\rho_1}-\int_0^{\Theta(\rho_1)}\gamma_1(\Theta,\rho_1)d\Theta=-q_I \\ B_{11}(1)\frac{dv(1)}{d\rho}+2B_{12}(1)\,v(1)-\int_0^{\Theta(1)}\gamma_1(\Theta,1)d\Theta=-q_{II}\end{aligned}\right\} \tag{8}$$

Finding an exact solution to these equations does not appear possible for a sphere of general nonhomogeneity.

RAYLEIGH-RITZ METHOD

In the Rayleigh-Ritz method a kinematically admissible displacement field is assumed, and the principle of minimum potential energy is used to determine unknown coefficients in the assumed solution. Here we shall represent the radial displacement $v(\rho)$ by the power series

$$v=\sum_{i=-m}^{n}a_i\rho^i=a_{-m}\rho^{-m}+\ldots+a_o+\ldots+a_n\rho^n \tag{9}$$

in which the number of nonzero coefficients a_i is arbitrary. Although it is only necessary to satisfy displacement boundary conditions when applying the Rayleigh-Ritz method, generally it is desirable to satisfy traction conditions as well. Relations (8) will be satisfied identically by the displacement field (9) if the coefficients a_i satisfy

$$\left.\begin{aligned} f_1(a_i)=B_{11}(\rho_1)\sum_i i\,a_i\,\rho_1^{i-1}+2B_{12}(\rho_1)\sum_i a_i\rho^{i-1}-\int_0^{\Theta(\rho_1)}\gamma_1(\Theta,\rho_1)d\Theta+q_I=0 \\ f_2(a_i)=B_{11}(1)\sum_i i\,a_i+2B_{12}(1)\sum_i a_i-\int_0^{\Theta(1)}\gamma_1(\Theta,1)d\Theta+q_{II}=0\end{aligned}\right\} \tag{10}$$

These equations can be used in order to eliminate two of the coefficients a_i from the assumed solution. Alternatively, equation (9) can be retained in its original form and conditions (10) satisfied by the method of Lagrange multipliers, as described in reference 4. In this case the restrictions (10) are written in terms of Lagrange multipliers λ_1 and λ_2 as

$$\lambda_1 f_1(a_i) = 0, \quad \lambda_2 f_2(a_i) = 0 \tag{11}$$

Necessary conditions for a minimum value of the total potential energy Π, subject to the subsidiary conditions (11), then are given by

$$\frac{\partial \tilde{\Pi}}{\partial a_j} = 0 \ (j = -m, \ldots, n), \ \frac{\partial \tilde{\Pi}}{\partial \lambda_s} = 0 \ (s = 1, 2) \tag{12}$$

where

$$\tilde{\Pi} = \Pi + \lambda_1 f_1 + \lambda_2 f_2 \tag{13}$$

Substituting the assumed solution (9) into the potential energy expression (5), and differentiating $\tilde{\Pi}$ with respect to a_j as indicated in equation (12), gives

$$\sum_{i=-m}^{n} G_{ji}\, a_i + \sum_{s=1}^{2} g_{js}\, \lambda_s = H_j \ (j = -m, \ldots, n) \tag{14}$$

in which

$$\left.\begin{aligned} G_{ji} &= \int_{\rho_1}^{1} \left[ijB_{11} + 2(i+j)B_{12} + 2(B_{22} + B_{23}) \right] \rho^{i+j} d\rho \\ H_j &= \int_{\rho_1}^{1} \left[\int_0^{\Theta} [j\gamma_1(\Theta,\rho) + 2\gamma_2(\Theta,\rho)]d\Theta \right] \rho^{j+1} d\rho + \rho_1^{j+2} q_I - q_{II} \\ g_{j1} &= B_{11}(\rho_1)\ j\ \rho_1^{j-1} + 2B_{12}(\rho_1)\rho_1^{j-1}, \ g_{j2} = B_{11}(1)\ j + 2B_{12}(1) \end{aligned}\right\} \tag{15}$$

The Ritz coefficients a_i are then found by solving the algebraic equations (14) together with the constraint equations (10).

MODIFIED RAYLEIGH-RITZ METHOD

The modified Rayleigh-Ritz method consists of assuming a state of stress which satisfies equilibrium and traction boundary conditions, and then determining unknown coefficients in the assumed solution by applying the *principle of minimum complementary energy*.

It is easily verified that equilibrium is satisfied if the dimensionless stress components are expressed in terms of a stress function ψ as

$$t_{\rho\rho} = \frac{\psi}{\rho}, \quad t_{\phi\phi} = t_{\theta\theta} = \tfrac{1}{2}\left(\frac{\psi}{\rho} + \frac{d\psi}{d\rho}\right) \tag{16}$$

In this case the total complementary energy becomes

$$\Pi^* = 4\pi\beta_o T_o r_2^3 \int_{\rho_1}^{1} \Bigg[\tfrac{1}{2} b_{11}\left(\frac{\psi}{\rho}\right)^2 + b_{12}\left(\frac{\psi}{\rho}\right)\left(\frac{\psi}{\rho}+\frac{d\psi}{d\rho}\right) + \tfrac{1}{4}(b_{22}+b_{23})\left(\frac{\psi}{\rho}+\frac{d\psi}{d\rho}\right)^2 + \frac{\psi}{\rho}\int_0^{\Theta} \varepsilon_1(\Theta,\rho)\,d\Theta + \left(\frac{\psi}{\rho}+\frac{d\psi}{d\rho}\right)\int_0^{\Theta}\varepsilon_2(\Theta,\rho)\,d\Theta\Bigg]\rho^2 d\rho \tag{17}$$

We choose to represent the stress function ψ by the power series

$$\psi = \sum_{i=-m}^{n} a_i^* \rho^i = a_{-m}^* \rho^{-m} + \ldots + a_o^* + \ldots + a_n^* \rho^n \tag{18}$$

in which the number of nonzero coefficients a_i^* is arbitrary. In order that expression (18) yields stresses which satisfy the traction boundary conditions (8), the coefficients a_i^* must satisfy the relations

$$\begin{aligned} f_1^*(a_i^*) &= \sum_i a_i^* \rho_1^{i-1} + q_I = 0 \\ f_2^*(a_i^*) &= \sum_i a_i^* + q_{II} = 0 \end{aligned} \tag{19}$$

Proceeding as in the standard Rayleigh-Ritz technique outlined earlier, conditions (19) are next written in terms of the Lagrange multipliers λ_1^* and λ_2^*. Application of the principle of minimum complementary energy then leads to the set of equations

$$\sum_{i=-m}^{n} G_{ji}^* a_i^* + \sum_{s=1}^{2} g_{js}^* \lambda_s^* = H_j^* \tag{20}$$

where

$$\left.\begin{aligned} G_{ji}^* &= \int_{\rho_1}^{1}\left[b_{11} + (2+i+j)b_{12} + \tfrac{1}{2}(1+i)(1+j)(b_{22}+b_{23})\right]\rho^{i+j}d\rho \\ H_j^* &= -\int_{\rho_1}^{1}\left[\int_0^{\Theta}[\varepsilon_1(\Theta,\rho) + (1+j)\varepsilon_2(\Theta,\rho)]d\Theta\right]\rho^{j+1}d\rho \end{aligned}\right\} \tag{21}$$

$$g^*_{j1} = \rho_1^{\,j-1} \quad , \quad g^*_{j2} = 1$$

The coefficients a^*_i and the Lagrange multipliers λ^*_s then are found by solving equations (19) and (20).

FINITE ELEMENT TECHNIQUE

The energy formulation developed earlier also provides a convenient basis for constructing a finite element solution to the problem. In this case the sphere is idealized as a series of N hollow spherical subregions. A typical element j has an inner radius ρ_i and an outer radius ρ_j; the corresponding radial displacement components are denoted by v_i and v_j, and the radial stresses are taken to be $(t_{\rho\rho})_i$ and $(t_{\rho\rho})_j$, respectively.

It is assumed that the displacement varies linearly with ρ within each element, so that

$$v(\rho) = \left(\frac{\rho_j-\rho}{\rho_j-\rho_i}\right) v_i + \left(\frac{-\rho_i+\rho}{\rho_j-\rho_i}\right) v_j \tag{22}$$

The thermoelastic properties are taken to be constant over each element, in which case the following average values will be used

$$\left.\begin{aligned} \overline{B}_{kl} &= \tfrac{1}{2}\left[B_{kl}(\rho_j) + B_{kl}(\rho_i)\right] \; , \; \overline{\Gamma}_k = \tfrac{1}{2}\left[\int_0^{\Theta(\rho_j)} \gamma_k(\Theta,\rho_j)\,d\Theta + \int_0^{\Theta(\rho_i)} \gamma_k(\Theta,\rho_i)\,d\Theta\right] \\ \overline{E}_k &= \tfrac{1}{2}\left[\int_0^{\Theta(\rho_j)} \varepsilon_k(\Theta,\rho_j)\,d\Theta + \int_0^{\Theta(\rho_i)} \varepsilon_k(\Theta,\rho_i)\,d\Theta\right] \end{aligned}\right\} \tag{23}$$

By analogy with equation (5), the total potential energy for element j is

$$\begin{aligned} \Pi^{(j)} = 4\pi\beta_o T_o r_2^{\,3}\Bigg\{\int_{\rho_i}^{\rho_j}\Bigg[&\tfrac{1}{2}\overline{B}_{11}\left(\frac{dv}{d\rho}\right)^2 + 2\overline{B}_{12}\left(\frac{dv}{d\rho}\right)\left(\frac{v}{\rho}\right) + \left(\overline{B}_{22}+\overline{B}_{23}\right)\left(\frac{v}{\rho}\right)^2 \\ &- \overline{\Gamma}_1\frac{dv}{d\rho} - 2\overline{\Gamma}_2\frac{v}{\rho} + \tfrac{1}{2}\left(\overline{E}_1\overline{\Gamma}_1 + \overline{E}_2\overline{\Gamma}_2\right)\Bigg]\rho^2 d\rho + \rho_i^{\,2}(t_{\rho\rho})_i v_i - \rho_j^{\,2}(t_{\rho\rho})_j v_j\Bigg\} \end{aligned} \tag{24}$$

Substituting equation (22) into (24), and minimizing $\Pi^{(j)}$ with respect to v_i and v_j gives

$$-\rho_i^{\,2}(t_{\rho\rho})_i - \frac{(\rho_j^{\,3}-\rho_i^{\,3})}{3(\rho_j-\rho_i)}\overline{\Gamma}_1 + \frac{(\rho_j^{\,3}-3\rho_j\rho_i^{\,2}+2\rho_i^{\,3})}{3(\rho_j-\rho_i)}\overline{\Gamma}_2 = k_{11}v_i + k_{12}v_j \tag{25a}$$

$$\rho_j^2 (t_{\rho\rho})_j + \frac{(\rho_j^3 - \rho_i^3)}{3(\rho_j - \rho_i)} \bar{\Gamma}_1 + \frac{(\rho_i^3 - 3\rho_i\rho_j^2 + 2\rho_j^3)}{3(\rho_j - \rho_i)} \bar{\Gamma}_2 = k_{12} v_i + k_{22} v_j \qquad (25b)$$

where the element stiffness coefficients k_{ij} are

$$\left.\begin{aligned}
k_{11} &= \frac{(\rho_j^3 - \rho_i^3)}{3(\rho_j - \rho_i)^2} \bar{B}_{11} - \frac{2(\rho_j^3 - 3\rho_j\rho_i^2 + 2\rho_i^3)}{3(\rho_j - \rho_i)^2} \bar{B}_{12} + \frac{2(\rho_j - \rho_i)}{3} (\bar{B}_{22} + \bar{B}_{23}) \\
k_{22} &= \frac{(\rho_j^3 - \rho_i^3)}{3(\rho_j - \rho_i)^2} \bar{B}_{11} + \frac{2(\rho_i^3 - 3\rho_i\rho_j^2 + 2\rho_j^3)}{3(\rho_j - \rho_i)^2} \bar{B}_{12} + \frac{2(\rho_j - \rho_i)}{3} (\bar{B}_{22} + \bar{B}_{23}) \\
k_{12} &= -\frac{(\rho_j^3 - \rho_i^3)}{3(\rho_j - \rho_i)^2} \bar{B}_{11} + \frac{(\rho_j - \rho_i)}{3} (\bar{B}_{22} + \bar{B}_{23} - \bar{B}_{12})
\end{aligned}\right\} \qquad (26)$$

Application of equations (25) to each of the N elements provides a system of 2N linear equations for the N + 1 displacement components and the N-1 interface stresses. The interface stresses can be eleminated, resulting in a set of N + 1 equations for the unknown displacement components.

NUMERICAL EXAMPLES

To illustrate the influence of temperature-dependent material properties upon the thermoelastic response, and at the same time to demonstrate the applicability of Ritz methods in thermal stress problems, numerical results are presented for a sphere subject to various temperature and pressure conditions. The ratio of the sphere's inner and outer radii is taken to be $\rho_1 = 0.8$. It is assumed that the body is initially homogeneous, and that the thermal expansion coefficients vary linearly with temperature, while the elastic stiffnesses exhibit a quadratic variation. In particular we let

$$\varepsilon_i = \varepsilon_i^o (1 + b\Theta), \quad B_{ij} = B_{ij}^o (1 - c\Theta^2) \qquad (27)$$

in which b and c are constants. The initial (zero-temperature) thermoelastic coefficients are taken to be

$$\left.\begin{aligned}
&B_{11}^o = 3.0 \times 10^4 \quad B_{12}^o = 1.0 \times 10^4 \quad B_{22}^o + B_{23}^o = 31.0 \times 10^4 \\
&\gamma_1^o = 1.0 \quad \gamma_2^o = 1.5
\end{aligned}\right\} \qquad (28)$$

These values are representative of certain fiber reinforced composite materials, reinforced in the circumferential (ϕ and θ) directions.

As a first example let us consider a sphere subject to a uniform temperature rise $\Theta = 1$ and zero internal and external pressure. Values of the thermal displacements and stresses found using the Rayleigh-Ritz and the modified Rayleigh-Ritz methods are compared with the exact solution (ref. 5) for the limiting case of temperature-independent properties (b=c=0) in Table I. It is evident that the accuracy of the approximate solutions generally improves as additional terms are included in the assumed solution. When the Rayleigh-Ritz approximation contains 3 independent coefficients (i.e., a total of the 5 coefficients $a_{-2}, a_{-1}, a_0, a_1, a_2$ of which 2 may be eliminated using the boundary conditions), the value of the maximum stress amplitude $|t_{\phi\phi}(0.8)|$ exceeds the exact value by 0.9%. For 5 independent coefficients the error is reduced to 0.3%. On the other hand the maximum stresses predicted by the modified Rayleigh-Ritz procedure using 3 and 5 independent coefficients are 2.3% and 1.6% smaller than the exact value. When the powers of ρ in either the standard or modified Rayleigh-Ritz approximation are taken to be -5, 4, and 1, the computed values of the displacements and stresses are exact, since the assumed solution then has precisely the form of the exact solution.

Results of finite element analyses are compared with the exact solution to this same problem in Table II. Naturally the accuracy of the finite element solutions improves as the number of independent displacement components is increased. When the finite element solution is based upon 3 independent displacement components (2 elements), the maximum stress $|t_{\phi\phi}(0.8)|$ exceeds the exact value by 2.6%. The error is reduced to 0.7% when 13 displacement components (12 elements) are used. However for this problem it was found that the computations required to achieve a given level of accuracy were less time consuming when one of the Ritz methods was used than when the finite element technique was applied.

To demonstrate the influence of temperature-dependent behavior upon the circumferential stress in the sphere, Ritz solutions based upon 5 independent coefficients are plotted in figures 1-3. Each of the figures shows the stress distributions associated with various values of the temperature-dependent parameters b and c for temperature alone and for combined temperature plus internal pressure. Figure 1 shows the stresses induced by a uniform temperature rise Θ=1. Results for the linearly varying temperature distributions Θ=5-5ρ and Θ=-4+5ρ are given in figure 2 and 3, respectively. Each of the Ritz solutions plotted in the figures was compared with a finite element solution based upon a 12-element model. Agreement between the values of the maximum absolute stress predicted by the two methods varied between 0.1% and 1.6%, with one exception. In the case of Θ=-4+5ρ and zero internal pressure q_I=0 (fig. 3) the maximum stress was relatively small, and the discrepancy was nearly 5.0%.

As would be expected for the purely temperature loadings (q_I=q_{II}=0), the maximum stresses diminish with increasing values of c(i.e., with decreasing stiffness), whereas they become larger with increasing values of b (increasing thermal expansion). The influence of temperature sensitivity is less predictable in the case of combined temperature and pressure, since both the pressure-induced and temperature-induced stresses are affected by the nonhomogeneity.

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Table I. Ritz approximations for the thermal displacements and stresses caused by a uniform temperature rise $\Theta = 1$ when $b = c = 0$.

		Rayleigh-Ritz		Modified Rayleigh-Ritz		Exact
Powers of ρ		-2,-1,0,1,2	-3,-2,-1, 0,1,2,3	-2,-1,0,1,2	-3,-2,-1, 0,1,2,3	-5,4,1
No. of indep. coefs.		3	5	3	5	3
Radial displ. x 10^7	$v(0.8)$	.058	.059	.066	.064	.060
	$v(0.9)$	.355	.355	.354	.354	.354
	$v(1.0)$	.636	.634	.633	.632	.634
Radial stress	$t_{\rho\rho}(0.8)$	0	0	0	0	0
	$t_{\rho\rho}(0.9)$	-.084	-.091	-.092	-.092	-.091
	$t_{\rho\rho}(1.0)$	0	0	0	0	0
Circumf. stress	$t_{\phi\phi}(0.8)$	-.948	-.943	-.918	-.925	-.940
	$t_{\phi\phi}(0.9)$	.001	-.001	-.006	-.003	-.003
	$t_{\phi\phi}(1.0)$	.762	.757	.755	.750	.758

Table II. Finite element solutions for the thermal displacements and stresses caused by a uniform temperature rise $\Theta = 1$ when $b = c = 0$.

		Finite Element				Exact
No. elements		2	4	6	12	-
No. indep. displ. comps.		3	5	7	13	-
Radial displ. x 10^7	$v(0.8)$	.062	.060	.060	.060	.060
	$v(0.9)$	.356	.355	.355	.355	.354
	$v(1.0)$	.636	.635	.635	.635	.634
Radial stress	$t_{\rho\rho}(0.8)$	-.105	-.064	-.046	-.025	0
	$t_{\rho\rho}(0.9)$	-.073	-.092	-.092	-.091	-.091
	$t_{\rho\rho}(1.0)$	-.033	-.022	-.016	-.008	0
Circumf. stress	$t_{\phi\phi}(0.8)$	-.965	-.959	-.954	-.947	-.940
	$t_{\phi\phi}(0.9)$	.012	.001	-.001	-.001	-.003
	$t_{\phi\phi}(1.0)$	.750	.752	.753	.756	.758

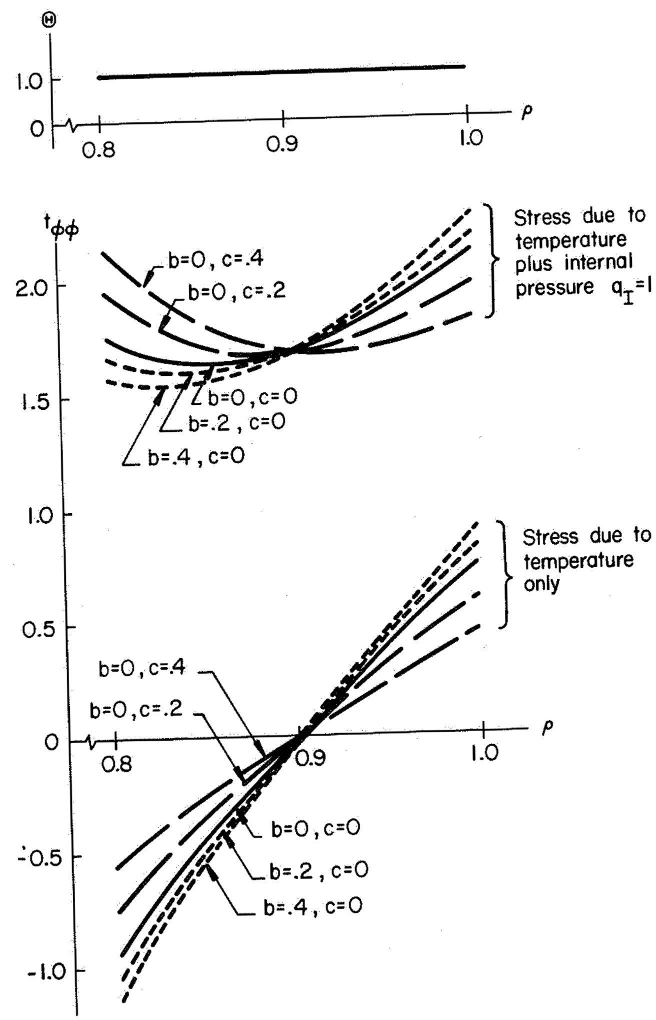

Figure 1.- Influence of temperature sensitivity coefficients b and c upon the circumferential stress distribution for the uniform

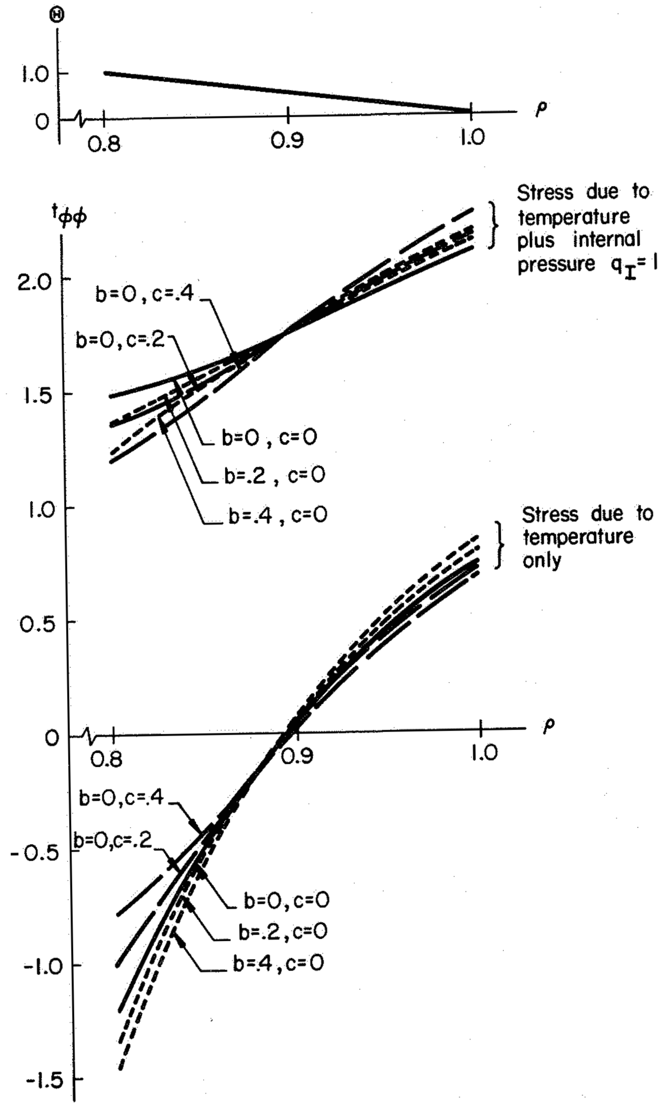

Figure 2.- Influence of temperature sensitivity coefficients b and c upon the circumferential stress distribution for the temperature

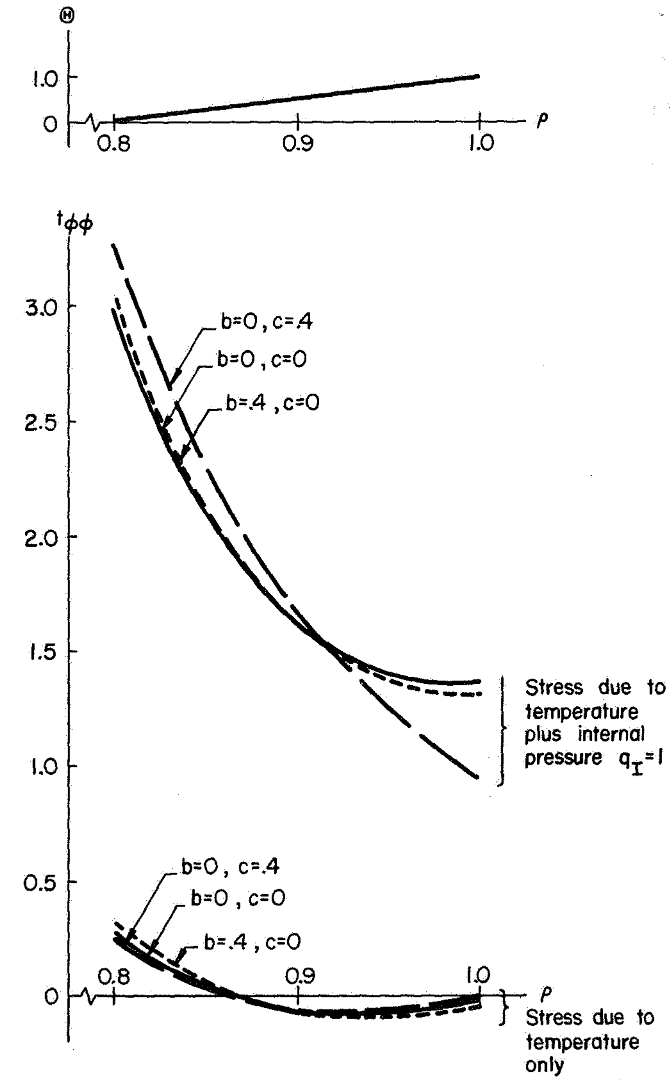

Figure 3.- Influence of temperature sensitivity coefficients b and c upon the circumferential stress distribution for the temperature field $\Theta = -4 + 5\rho$.