RESPONSE OF LINEAR DYNAMIC SYSTEMS

WITH RANDOM COEFFICIENTS

John Dickerson
University of South Carolina

INTRODUCTION

Numerous models of physical systems contain parameters whose values are not known exactly. This paper attempts to address some of the physical and mathematical complexities arrising in the prediction of the statistical behavior of such systems. Although the discussions in the paper are far from providing a satisfactory solution to such problems, they perhaps, by utilization of simple examples, will create a greater awareness of the statistical effect of random parameters.

PROBLEM FORMULATION

Consider the problem of determining the statistical properties of the response of a finite dimensional linear dynamical system with random coefficients (constant with respect to time) and subjected to stochastic forces. Mathematically the problem is represented by the following equation:

$$\frac{dx(t)}{dt} = Ax(t) + f(t) \qquad 0 \leq t$$
$$x(o) = x_o \tag{1}$$

where $x(t)$, $f(t)$, and x_o are n-dimensional random vectors, A is an nxn random matrix. The problem is to determine statistical properties (mean value, variance, correlation function, spectral density, distribution, etc.) of $x(t)$ knowing the statistical properties of x_o, $f(t)$, and A.

EXISTENCE OF SOLUTION

If the derivative in equation (1) is interpreted in the almost sure sense then existence and uniqueness of a solution follows from appropriate results in R^n and if $f(t)$ is almost surely continuous then a solution in this sense would exist and be given by:

$$x(t) = e^{At}x_o + \int_o^t e^{A(t-\tau)} f(\tau)d\tau \tag{2}$$

However, since the discussions in this paper will be concerned with second moments of the solution, it would seem appropriate to require the derivative in (1) to be a mean square derivative and to consider the differential equation (1) over the Hilbert space Z^n where Z denotes the space of second order random variables. If A is a bounded operator over Z^n (probably equivalent to requiring A to be almost surely bounded) then the theory of ordinary differential equations would yield a unique solution given by (2), where the integral was a mean square integral, provided f(t) is mean square continuous. In general, however, A may not be bounded, i.e. the product of two second order random variables will not be a second order random variable and then the appropriate theory discussing existence of a solution to (1) would likely be a requirement that the solution be the action of a semigroup on the initial condition. For example, if there exists a real number λ_o such that:

$$||\lambda[\lambda+\lambda_o - A]^{-1}|| \leq C, \quad \text{for all complex } \lambda \text{ with Re } \lambda \geq 0, \ C_1 \text{ a real}$$

number and $|| \ ||$ denoting the norm over R^n, then there will be a solution in the mean square sense to (1) and further:

$$E[(e^{At}x_o)^T(e^{At}x_o)] \leq C_2 e^{2\lambda_o t} E[x_o^T x_o]$$

In particular if λ_o can be chosen to be negative then the solution will be asymptotically stable. This approach to the problem exhibits a solution with the only stipulations that $x_o \varepsilon Z^n$ and f(t) be mean square continuous. Another approach to finding a mean square solution to (1) would be to require conditions on x_o, A, f(t) such that (2) is a solution to (1). If x_o, A, and f(t) are mutually independent, then requiring e^{At} and Ae^{At} to have second moments would insure that (2) satisfies (1). The following elementary examples attempt to illustrate the above discussion.

EXAMPLES

Example 1

Consider the first order homogeneous equation (n = 1).

$$\frac{dx}{dt} = ax \qquad\qquad x(o) = x_o$$

with a uniformly distributed between α and β. Clearly a is a bounded operator over Z thus, for example, if x_o is independent of a it follows that:

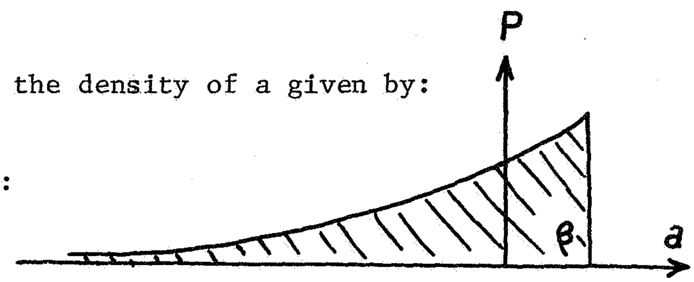
$$E[x(t)] = \frac{1}{(\beta-\alpha)t} (e^{\beta t} - e^{\alpha t}) E[x_o]$$

If $\beta>0$ then $E[x(t)]$ becomes arbitrarily large even if the mean of a is negative.

Example 2

Consider the above problem with the density of a given by:

$$P_a(a) = \alpha e^{\alpha(a-\beta)}$$ shown:



Although a is not a bounded operator over Z clearly $\left|\frac{\lambda}{\lambda+\lambda_o-a}\right| \leq C$ if $\lambda_o>\beta$ for all Re $\lambda>0$. Thus a solution exists. If x_o is independent of a then it can be shown that

$$E[x(t)] = \frac{\alpha e^{\beta t}}{t+\alpha} E[x_o].$$

Note again that if $\beta>0$ $E[x(t)]$ becomes arbitrarily large.

Example 3

Consider the above example with a Gaussian with mean μ and variance σ. Clearly a is not bounded and further no λ_o can be chosen to make $\left|\frac{\lambda}{\lambda+\lambda_o-a}\right| \leq C$.

However if a is independent of x_o then ae^{at} and e^{at} do have second moments and it follows that:

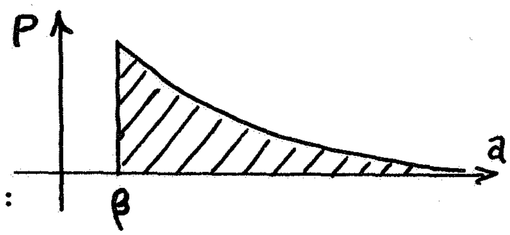
$$E[x(t)] = \exp \left\{\frac{t^2\sigma^4 + 2\mu t\sigma^2}{\sigma^2}\right\} E[x_o].$$

However, regardless of σ and μ, $E[x(t)]$ becomes arbitrarily large.

Example 4

Consider the above example with the density of a given by:

$$P_a(a) = \alpha e^{-\alpha(a-\beta)}$$ shown:



Again it is not possible to pick a λ_o such that:

$\left|\frac{\lambda}{\lambda+\lambda_o-a}\right| \leq C$ and even if x_o is independent of a it can be demonstrated that x(t) does not have a first moment for $t>\alpha$. Thus it makes no sense in this problem to attempt to calculate E[x(t)].

STATIONARY RESPONSE AND SPECTRAL DENSITY

Assume that the existence of the solution in the mean square sense to (1) is known and is expressible as:

$$x(t) = e^{At}x_o t \int_o^t e^{A(t-\tau)} f(\tau)d\tau$$

If A, x_o, and f(t) are mutually independent and further if f(t) is stationary with correlation matrix R_f then it follows that:

$$E[(x(t+\Delta) - E[x(t+\Delta)])\ (x(t) - E[x(t)])^T] = \int_o^{t+\Delta} \int_o^t E[e^{A(t+\Delta-\eta_1)} R_f(\eta_1-\eta_2)e^{A^T(t-\eta_2)}]\ d\eta_1\ d\eta_2$$

If it can further be shown that $||e^{At}|| \leq Ce^{\beta t}$ with $\beta<0$, then it follows in the usual way that as t goes to ∞ x(t) becomes stationary with:

$$R_x(\Delta) = \int_0^\infty \int_0^\infty E[e^{A\eta_1} R_f(\Delta-\eta_1 + \eta_2)e^{A^T\eta_2}]\ d\eta_1 d\eta_2$$

By taking the Fourier transform of $R_x(\Delta)$ it is easily shown that the spectral density of x(t) is given by:

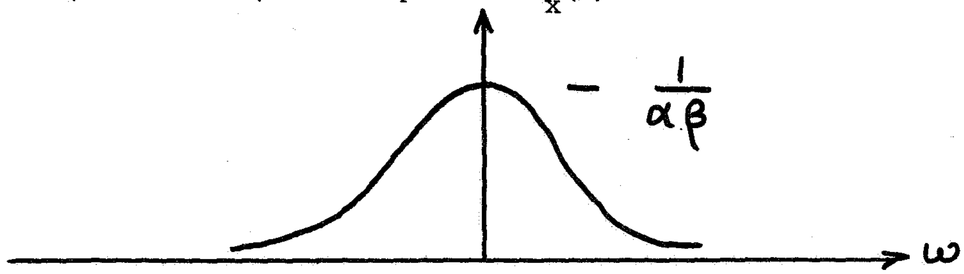
$$S_x(\omega) = E[\ [A+i\omega]^{-1}\ S_f(\omega)\ [A-i\omega]^{-1}\ ]$$

Example 5

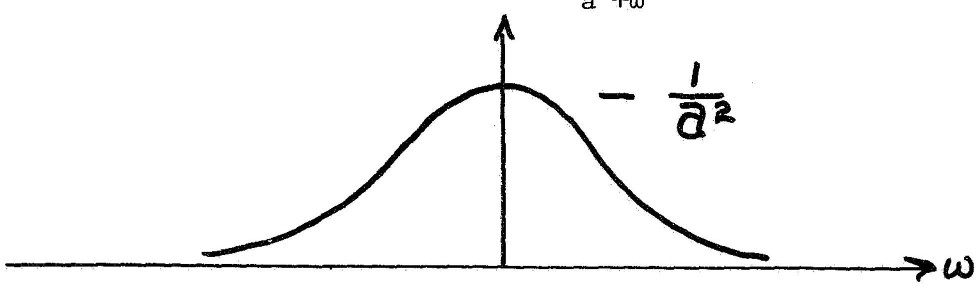
Of the previous examples only Example 1 with $\beta<0$ and Example 2 with $\beta<0$ eventually have stationary solutions. In example 1 (with $\beta<0$) it is easily seen that:

$$S_x(\omega) = \frac{1}{\omega} \left[\tan^{-1}\frac{\beta}{\omega} - \tan^{-1}\frac{\alpha}{\omega}\right] S_f(\omega)$$

If $S_f(\omega) = 1$ (white noise) then a plot of $S_x(\omega)$ follows:



If a was not a random variable then $S_x(\omega) = \frac{1}{a^2+\omega^2}$ and a plot of this follows:



SUMMARY

Those readers who have gotten to this point in the paper recognize it as a fraud. The paper (1) presents a physical problem, i.e.: how do you calculate the statistical properties of the response of dynamical systems which have random parameters, (2) presents possible mathematical models that pertain to the physical problem and (3) presents, via simple examples, where the problems are in trying to solve the problem. The result in example 3, where a is Gaussian, shows that regardless of how negative the mean value and how small the variance of a, the mean value of the solution goes to ∞ as time goes to ∞. In particular, it makes no sense to talk about the spectral density of the solution.

In the opinion of the author closed form solutions to problems beyond n=1 are not feasible and current work centers around the study of the accuracy of approximate methods that have been proposed in the literature.