STABILITY OF NEUTRAL EQUATIONS

WITH CONSTANT TIME DELAYS

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SUMMARY

A method has been developed for determining the stability of a scalar neutral equation with constant coefficients and constant time delays. A neutral equation is basically a differential equation in which the highest derivative appears both with and without a time delay. Time delays may appear also in the lower derivatives or the independent variable itself. The method is easily implemented and an illustrative example is presented.

INTRODUCTION

Ordinary differential equations with time delays are called differential-difference equations (ref. 1). Two basic types of differential-difference equations are retarded and neutral equations. The stability of the solutions of these equations is related to the roots of a characteristic equation. Generally this characteristic equation is transcendental and thus has an infinite number of roots.

A convenient method is developed in reference 2 for examining the stability of retarded equations with many time delays (not necessarily distinct) and a scalar neutral equation with one delay. The purpose of the present paper is to develop the basic method of reference 2 for neutral equations with many time delays.

SYMBOLS

a_j, b_j, c, d	real constants
$H_K(s)$	function of s in equation (11)
i	imaginary unit, $\sqrt{-1}$
$J_K(s)$	function of s in equation (12)
j	integer

K	refers to τ_K
$L(s) = 0$	characteristic equation
$L_0(s)$	resulting polynomial with zero delays in $L(s)$
N	highest derivative in neutral equation
$N(\tau_K, -\hat{\mu})$	number of roots of $L(s)$ with $\sigma > -\hat{\mu}$ at τ_K for fixed τ_j, $j \neq K$
$P(s)$	function of s in equation (20)
p	integer
$Q(s)$	function of s in equation (21)
s	complex variable, $\sigma + i\omega$
$\lvert s \rvert_m$	an upper bound on magnitude of s which satisfies $L(s) = 0$, where $s = -\hat{\mu} + i\omega$
t	time
$W_K(\sigma, \omega)$	testing function defined in equation (17)
$x(t)$	scalar function of time
α_1, α_2	real numbers
ϵ	small positive number
μ	positive real number
$\hat{\mu}$	specified value of μ
ξ	real gain constant
σ	real part of s
σ_∞	asymptote of real part of large modulus roots
τ, τ_j, τ_K	constant real time delays
$\bar{\tau}_K$	final desired value of τ_K
$\psi(t)$	yaw angle, radians
ω	imaginary part of s
ω_m	an upper bound on ω in $L(s) = 0$, where $s = -\hat{\mu} + i\omega$

Mathematical notations:

$\lvert\ \rvert$	absolute value or magnitude
arg	argument
0^+	arbitrarily small positive values

Dots over a symbol denote derivatives with respect to time.

ANALYSIS

A method is developed herein for determining the stability of the neutral equation

$$\sum_{j=0}^{N} [a_j\, x^{(j)}(t) + b_j\, x^{(j)}(t - \tau_j)] = 0 \tag{1}$$

where $a_N \neq 0$, $b_N \neq 0$, $0 < \tau_j \leqq \tau_N$, for $j = 0, 1, \ldots, N - 1$, and $x^{(j)}(t)$ denotes the jth derivative of $x(t)$.

The characteristic equation associated with equation (1) is

$$L(s) = \sum_{j=0}^{N} (a_j + b_j\, e^{-\tau_j s})s^j = 0 \tag{2}$$

It has been shown (ref. 3) that if all the roots $s = \sigma + i\omega$ of equation (2) satisfy the property

$$\sigma \leqq -\mu < 0 \tag{3}$$

where μ is a positive constant, then the solution of equation (1) is of exponential order as $t \rightarrow \infty$; that is

$$|x(t)| < d\ e^{-c\mu t} \tag{4}$$

where $d > 0$ is a constant real number and c is arbitrary on the interval (0, 1). Hence, if all the characteristic roots have negative real parts and are not asymptotic to the imaginary axis, then $x(t) \to 0$ as $t \to \infty$ (asymptotically stable).

If there is a root of $L(s)$ with positive real part, then equation (1) has a divergent mode and is said to be unstable.

Relative Stability

If it can be determined that there are no roots of the characteristic equation with real parts greater than a specified negative real number, then the solution to the neutral equation is asymptotically stable.

Relative stability for a specified value $\hat{\mu}$ of μ in equation (3) is indicated herein by the number of roots of the characteristic equation with $\sigma > -\hat{\mu}$. For example, the neutral system is said to be relatively more stable when all the roots satisfy $\sigma < -\hat{\mu} < 0$, than when there is a root with $-\hat{\mu} < \sigma < 0$. Relative stability boundaries in the plane of two system parameters are boundaries corresponding to a root with $\sigma = -\hat{\mu}$.

The stability method to be presented is based on determining the number of roots of the characteristic equation with real parts greater than a specified negative real number $-\hat{\mu}$. The method is convenient for determining the number of roots of the characteristic equation with real parts located between specifie negative real numbers. The approach consists of separately examining the arbitrarily large modulus roots and the finite roots. The large modulus roots are examined by using a simple expression for their asymptote; whereas, the finite roots are examined by computing the magnitude of a complex-valued functic on a finite interval.

Large Modulus Roots

All roots of equation (2) must satisfy the inequality

$$\left| |a_N| - |b_N| e^{-\tau_N \sigma} \right| |s|^N \leq \sum_{j=0}^{N-1} \left(|a_j| + |b_j| e^{-\tau_j \sigma} \right) |s|^j \tag{5}$$

obtained from equation (2). It can be shown that since $a_N \neq 0$ and $b_N \neq 0$, the roots have bounded σ. Hence, in order for the large modulus roots ($|s| \to \infty$) to satisfy equation (5)

$$\lim_{|s| \to \infty} \left(|a_N| - |b_N| e^{-\tau_N \sigma} \right) = 0 \tag{6}$$

From equation (6), σ becomes arbitrarily close to

$$\sigma_{\infty} = -\frac{1}{\tau_N} \ln \left| \frac{a_N}{b_N} \right| \tag{7}$$

This relation represents the asymptote of the large modulus roots and is shown graphically in figure 1.

For $\left| \frac{a_N}{b_N} \right| < 1$ in figure 1, $\sigma_{\infty} > 0$; and equation (3) with $\sigma = \sigma_{\infty}$ is not satisfied. Now, consider $\left| \frac{a_N}{b_N} \right| > 1$ and let $\sigma_{\infty} = -\hat{\mu}$ correspond to $\tau_N = \hat{\tau}_N$ in figure 1. Then, $\sigma = \sigma_{\infty}$ satisfies equation (3) with $\mu = \hat{\mu}$, whenever

$$\tau_N < \hat{\tau}_N = \frac{1}{\hat{\mu}} \ln \left| \frac{a_N}{b_N} \right| \tag{8}$$

There are then no infinitely large modulus roots with $\sigma > -\hat{\mu}$ in the neutral system. It remains to examine the number of finite roots with $\sigma > -\hat{\mu}$.

Finite Roots

For $\tau_j \to 0^+$, $L(s)$ has N roots arbitrarily close to the N roots of the polynomial equation

$$L_0(s) = \sum_{j=0}^{N} (a_j + b_j)s^j = 0 \tag{9}$$

and the remaining roots have arbitrarily large moduli (ref. 2). For $\tau_N \to 0^+$ and $\left| \frac{a_N}{b_N} \right| > 1$ in equation (7), $\sigma_{\infty} \to -\infty < -\hat{\mu}$. Therefore, $L(s)$ and $L_0(s)$ have the same number of roots with $\sigma > -\hat{\mu}$ (initial relative stability). Since the complex roots occur in complex conjugate pairs, only roots with nonnegative imaginary parts ($\omega \geqq 0$) are considered.

As one of the time delays, say τ_K, is increased in a continuous manner with the remaining delays held fixed, the finite roots of $L(s)$ move in some continuous manner (ref. 2), generating root locus curves in the complex root plane (s-plane).

Intersection Points $s = -\hat{\mu} + i\omega$ and Corresponding Delays

A root locus curve must intersect the $-\hat{\mu}$-line (dashed line) in figure 2 in order for the number of roots of $L(s)$ with $\sigma > -\hat{\mu}$ to change. These intersection points $(-\hat{\mu}, \omega)$ and the corresponding values of the delay τ_K which result in these intersection points are discussed in this section. The change in the relative stability as a root locus curve crosses an intersection point is presented in the next section.

For a specific time delay τ_K, equation (2) can be written as

$$L(s) = H_K(s) - J_K(s)\, e^{-\tau_K s} = 0 \tag{10}$$

where

$$H_K(s) = \sum_{j=0}^{N} a_j s^j + \sum_{\substack{j=0 \\ j\neq K}}^{N} b_j s^j e^{-\tau_j s} \tag{11}$$

and

$$J_K(s) = -b_K s^K \tag{12}$$

At an intersection point $s = -\hat{\mu} + i\omega$, equation (10) is equivalent to

$$|H_K(-\hat{\mu}, \omega)| = |J_K(-\hat{\mu}, \omega)|\, e^{\hat{\mu}\tau_K} \tag{13}$$

and

$$\tau_K = \frac{1}{\omega} \arg\left[\frac{J_K(-\hat{\mu}, \omega)}{H_K(-\hat{\mu}, \omega)} + 2p\pi\right] \tag{14}$$

where $H_K(-\hat{\mu}, \omega) = H_K(-\hat{\mu} + i\omega)$, $J_K(-\hat{\mu}, \omega) = J_K(-\hat{\mu} + i\omega)$, and

$$-\pi < \arg \frac{J_K(-\hat{\mu}, \omega)}{H_K(-\hat{\mu}, \omega)} \leqq \pi \tag{15}$$

It is assumed that $\omega \neq 0$ and $H_K(-\hat{\mu}, \omega) \neq 0$. To handle these special cases, the approach used in reference 2 may be followed. Only non-negative values of the integer p in equation (14) are of interest because $\tau_K \geqq 0$ and $\omega > 0$.

Equation (13) gives the points $(-\hat{\mu}, \omega)$ where the root locus curves intersect the $-\hat{\mu}$-line in figure 2, and equation (14) gives the corresponding values of τ_K which result in these intersection points. In general, the values of ω at an intersection point must be found by an iteration process. The values of ω which may satisfy equation (14) are restricted to some finite interval $(0, \omega_m]$, where ω_m is an upper bound on ω determined from equation (5). Also, a useful bound on the integer p in equation (14) is obtained as

$$|p| \leqq \frac{1}{2} + \frac{\omega_m \bar{\tau}_K}{2\pi} \tag{16}$$

where $\tau_K \leqq \bar{\tau}_K$ and $\omega \leqq \omega_m$.

Change in Number of Roots With $\sigma > -\hat{\mu}$

Let $N(\tau_K, -\hat{\mu})$ denote the number of roots of $L(s)$ with $\sigma > -\hat{\mu}$ at τ_K for fixed τ_j, $j \neq K$; and define the testing function

$$W_K(\sigma, \omega) = \frac{J_K(\sigma, \omega)}{H_K(\sigma, \omega)} e^{-\sigma\tau_K} \tag{17}$$

Then, the following theorem can be used to determine the change in the number of roots of $L(s)$ with $\sigma > -\hat{\mu}$ as τ_K varies.

Theorem: Let $(-\hat{\mu}, \omega)$ be an intersection point with corresponding delay τ_K. Let $\alpha_1 < \omega$ and $\alpha_2 > \omega$ be real numbers for which $W_K(-\hat{\mu}, \alpha_1)$ and $W_K(-\hat{\mu}, \alpha_2)$ are defined, and such that there are no other intersection points with imaginary parts which lie on the interval $[\alpha_1, \alpha_2]$. Then, for ϵ an arbitrarily small positive number

(1) $N(\tau_K + \epsilon, -\hat{\mu}) = N(\tau_K, -\hat{\mu}) + 1$

if $|W_K(-\hat{\mu}, \alpha_1)| > 1$ and $|W_K(-\hat{\mu}, \alpha_2)| < 1$;

(2) $N(\tau_K + \epsilon, -\hat{\mu}) = N(\tau_K, -\hat{\mu}) - 1$

if $|W_K(-\hat{\mu}, \alpha_1)| < 1$ and $|W_K(-\hat{\mu}, \alpha_2)| > 1$; and

(3) $N(\tau_K + \epsilon, -\hat{\mu}) = N(\tau_K, -\hat{\mu})$ if both

$|W_K(-\hat{\mu}, \alpha_1)|$ and $|W_K(-\hat{\mu}, \alpha_2|$ are greater than 1 or both less than 1.

This theorem is developed in reference 3 by extending the τ-decomposition method, as refined by Lee and Hsu (ref. 4).

The theorem is interpreted as follows: Let $(-\hat{\mu}, \omega)$ be an intersection point, where $\hat{\mu}$ is specified and ω is a root of equation (13). If this is the only value of ω on the interval $\alpha_1 \leqq \omega \leqq \alpha_2$, which satisfies equation (13), then the change in the relative stability at the intersection point is determined by computing $|W_K(-\hat{\mu}, \alpha_1)|$ and $|W_K(-\hat{\mu}, \alpha_2)|$. For example, from condition 1 of the theorem, if $|W_K(-\hat{\mu}, \alpha_1)| > 1$ and $|W_K(-\hat{\mu}, \alpha_2)| < 1$, then the system gains exactly one root with $\sigma > -\hat{\mu}$; that is, $N(\tau_K + \epsilon, -\hat{\mu}) = N(\tau_K, -\hat{\mu}) + 1$.

The values of τ_K at all the intersection points are ordered by increasing magnitude to obtain the change in the relative stability as τ_K increases to its final desired value $\bar{\tau}_K$. As each delay is varied, that delay becomes τ_K in the theorem.

Intersection points $(-\hat{\mu}, \omega)$ satisfy equation (13), or $|W_K(-\hat{\mu}, \omega)| = 1$. In choosing α_1 and α_2 in the theorem, it is expedient to note that $|W_K(-\hat{\mu}, \omega)|$ increases as p increases for each value of $\omega \epsilon (0, \omega_m]$.

APPLICATION

The relative stability of the neutral equation

$$.01024\, \ddot{\psi}(t) + .00704\, \dot{\psi}(t) + .250\, \psi(t) + .163\xi\ddot{\psi}(t - \tau_K) = 0 \tag{18}$$

where ξ is a system gain constant and $\tau_K > 0$ is a constant time delay may now be determined. This equation was used in reference 5 in examining a yaw damper control system for an airplane with rudder deflection made proportional to the yawing acceleration.

The characteristic equation associated with equation (18) can be written as

$$L(s) = P(s) - Q(s)\xi e^{-\tau_K s} = 0 \tag{19}$$

where

$$P(s) = .01024s^2 + .00704s + .250 \tag{20}$$

$$Q(s) = .163s^2 \tag{21}$$

With $s = -\hat{\mu} + i\omega$, equation (19) can be used to write

$$|\xi| = \left| \frac{P(-\hat{\mu}, \omega)}{Q(-\hat{\mu}, \omega)} \right| e^{-\hat{\mu}\tau_K} \tag{22}$$

and

$$\tau_K = \frac{1}{\omega} \left[\arg \frac{Q(-\hat{\mu}, \omega)}{P(-\hat{\mu}, \omega)} + 2p\pi \right] \tag{23}$$

Now, with $\hat{\mu}$ specified, equations (22) and (23) can be used to partition the plane of ξ and τ_K into different regions as $\omega > 0$ is allowed to vary. The solid lines in figure 3 were generated in this manner. Any point on a partitioning line or boundary corresponds to a root locus curve intersecting the $-\hat{\mu}$-line in figure 2.

To examine the stability condition (stable or unstable) or the number of roots with $\sigma > -\hat{\mu}$ in the regions of figure 3, it is useful to write equation (19) in the form

$$L(s) = H_K(s) - J_K(s)e^{-\tau_K s} = 0 \tag{24}$$

where

$$H_K(s) = P(s) \tag{25}$$

and

$$J_K(s) = \xi Q(s) \tag{26}$$

The initial stability of equation (24) along the ξ-axis ($\tau_K \to 0^+$) is evaluated by using equations (7) and (9), which become

$$\sigma_\infty = -\frac{1}{\tau_K} \ln \left| \frac{.01024}{.163\xi} \right| \tag{27}$$

and

$$L_0(s) = (.01024 + .163\xi)s^2 + .00704s + .250 = 0 \tag{28}$$

For $\tau_K \to 0^+$ and $\xi = .04$, there is one root with $\sigma = -.21$ and $\sigma_\infty \to -\infty$. As τ_K increases from 0^+ with $\xi = .04$ in figure 3, the relative stability boundary for $-\hat{\mu} = -.5$ is intersected. For this intersection point, it can be shown that α_1 and α_2 in the theorem can be chosen as $\alpha_1 = 3$ and $\alpha_2 = 4$. Then, since $|W_K(-.5, 3)| < 1$ and $|W_K(-.5, 4)| > 1$, condition 2 of the theorem applies. Thus, the neutral system loses one root with $\sigma > -.5$. (This is the root which originally had $\sigma = -.21$.) Inside the closed region for $-\hat{\mu} = -.5$, there are no roots with $\sigma > -.5$. This same procedure is used to determine which side of the curves in figure 3 should be hatched.

The hatching convention is as follows: Passing from the hatched (unhatched) side of a boundary line corresponding to a particular value of μ to the unhatched (hatched) side of the boundary results in the gain (loss) of exactly one root with $\sigma > -\hat{\mu}$.

At the point $(\xi, \tau_K) = (.04, .2)$, the system has no roots with $\sigma > -.7$. The value of σ_∞ in equation (27) at this point is $\sigma_\infty = -2.257$.

CONCLUDING REMARKS

A method has been developed for determining the stability and relative stability of scalar neutral equations, with constant coefficients and constant time delays. The approach was to determine the number of roots of the characteristic equation with real parts greater than specified negative real numbers. The method consists of separately examining the large modulus roots and finite roots. The large modulus roots are examined by using a simple expression for their asymptote; the finite roots are examined by computing the magnitude of a complex-valued function on a finite interval.

The stability method is convenient for determining the number of roots of the characteristic equation with real parts located between specified negative real numbers. An example which has occurred in practical application has been provided to illustrate the method.

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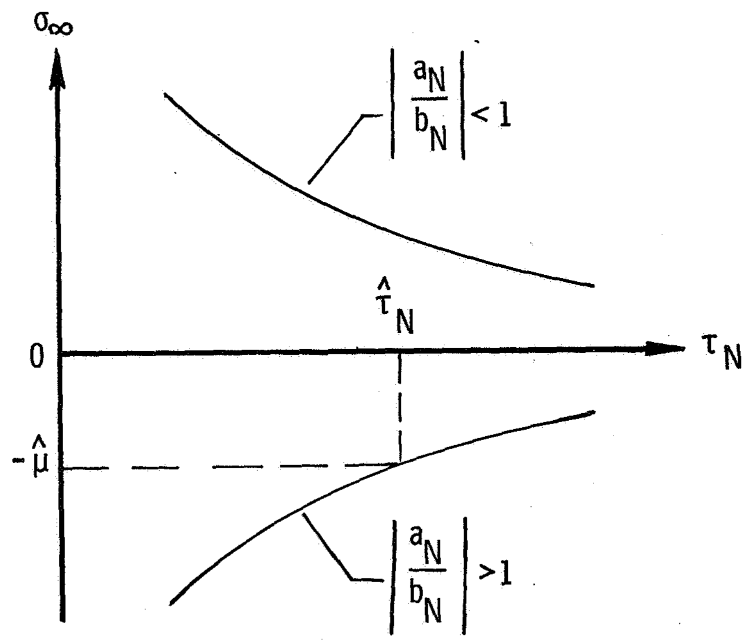

Figure 1.- Real part of large modulus roots.

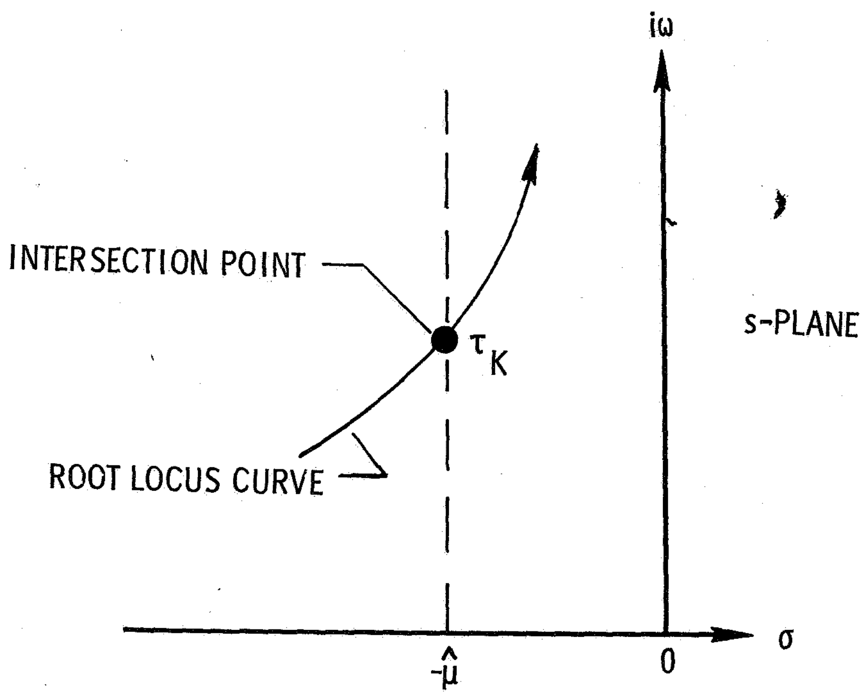

Figure 2.- Illustration of intersection point.

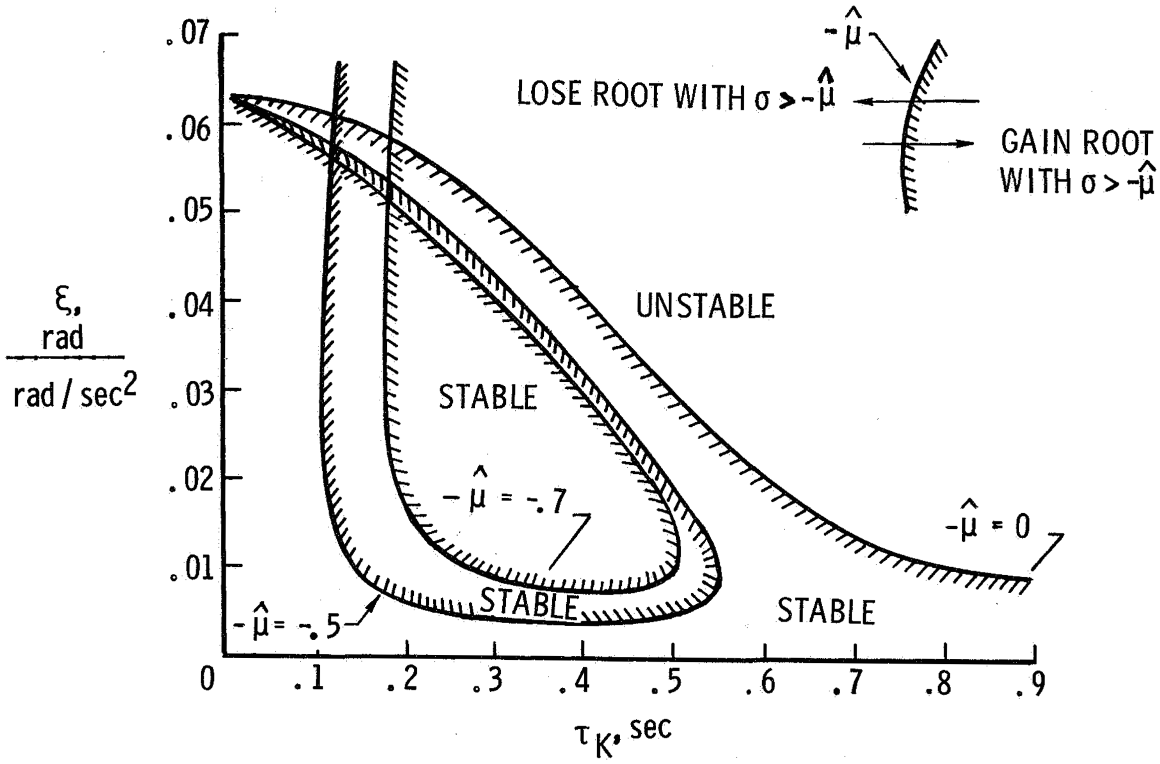

Figure 3.- Relative stability boundaries.