SOUND PROPAGATION THROUGH NONUNIFORM DUCTS*

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SUMMARY

A critical review is presented of the state of the art regarding methods of determining the transmission and attenuation of sound propagating in nonuniform ducts with and without mean flows. The approaches reviewed include purely numerical techniques, quasi-one-dimensional approximations, solutions for slowly varying cross sections, solutions for weak wall undulations, approximation of the duct by a series of stepped uniform cross sections, variational methods, and solutions for the mode envelopes.

INTRODUCTION

The prediction of sound propagation in nonuniform ducts is a problem whose solution has application to the design of numerous facilities, such as central airconditioning and heating installations, loud speakers, high-speed wind tunnels, aircraft engine-duct systems, and rocket nozzles.

The mathematical statement of sound propagation in a nonuniform duct that carries compressible mean flows can be obtained as follows. Each flow quantity $q(\vec{r},t)$ can be expressed as the sum of a mean flow quantity $q_0(\vec{r})$ and an acoustic quantity $q_1(\vec{r},t)$, where $\vec{r}$ is a dimensionless position vector and t is a dimensionless time. In nonuniform ducts, $q_0(\vec{r})$ is a function of the axial dimensionless coordinate z as well as the transverse dimensionless coordinates x and y. Substituting these representations into the equations of state and conservation of mass, momentum, and energy and subtracting the mean quantities, we obtain

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \vec{v}_1 + \rho_1 \vec{v}_0) = NL \tag{1}$$

$$\rho_0\left(\frac{\partial \vec{v}_1}{\partial t} + \vec{v}_0 \cdot \nabla\vec{v}_1 + \vec{v}_1 \cdot \nabla\vec{v}_0\right) + \rho_1\vec{v}_0 \cdot \nabla\vec{v}_0 + \nabla p_1 = \frac{1}{Re}\nabla \cdot \underline{\underline{\tau}}_1 + NL \tag{2}$$

$$\rho_0\left(\frac{\partial T_1}{\partial t} + \vec{v}_0 \cdot \nabla T_1 + \vec{v}_1 \cdot \nabla T_0\right) + \rho_1\vec{v}_0 \cdot \nabla T_0 - (\gamma-1)\left(\frac{\partial p_1}{\partial t} + \vec{v}_0 \cdot \nabla p_1 + \vec{v}_1 \cdot \nabla \rho_0\right) = \frac{1}{Re}\left[\frac{1}{Pr}\nabla \cdot (\kappa_0 \nabla T_1 + \kappa_1 \nabla T_0) + (\gamma-1)\Phi_1\right] + NL \tag{3}$$

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$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} \tag{4}$$

where $\underline{\underline{\tau}}_1$ and Φ_1 are the linearized viscous stress tensor and dissipation function and NL stands for the nonlinear terms in the acoustic quantities. These equations are supplemented by initial and boundary conditions.

No solutions to eqs. (1)-(4) subject to general initial and boundary conditions are available yet. To determine solutions for the propagation and attenuation of sound in ducts, researchers have used simplifying assumptions. In the absence of shock waves, the viscous acoustic terms produce an effective admittance at the wall that leads to small dispersion and attenuation (ref. 1). For lined ducts, this admittance produced by the acoustic boundary layer may be neglected, but it cannot be neglected for hard-walled ducts as demonstrated analytically and experimentally by Pestorius and Blackstock (ref. 2).

Most of the existing studies neglect the nonlinear acoustic terms in eqs. (1)-(4) and the boundary conditions. However, the assumption of linearization is not valid for high sound pressure levels. The effects of the nonlinear acoustic properties of the lining material become significant when the sound pressure level exceeds about 130 dB (re 0.0002 dyne/cm^2), while the gas nonlinearity becomes significant when the sound pressure level exceeds about 160 dB. In particular, the nonlinearity of the gas must be included when the mean flow is transonic.

Another popular assumption is that the mean flow is incompressible. Theories based on this assumption will not be applicable to evaluating the promising approach to the reduction of inlet noise by using a high subsonic inlet, or partially choked inlet, in conjunction with an acoustic duct liner. Numerous experimental investigations (refs. 3-20) of various choked-inlet configurations have been reported. Most, but not all, of these investigations have noted significant reductions of the noise levels when the inlet is choked. Further, most of the potential noise reduction is achieved by operation in the partially choked state (mean Mach number in the throat of 0.8 - 0.9). Some investigators (e.g. ref. 9) report the possibility of substantial "leakage" through the wall boundary layers, whereas others (e.g. ref. 12) report that leakage is minor. To evaluate these effects, one cannot neglect the viscous terms in the mean flow and perhaps in the acoustic equations. Since the mean flow is transonic at the throat, one has to include the nonlinear terms also because the linear acoustic solution is singular for sonic mean flows.

A fourth assumption being employed in analyzing sound propagation in ducts is the characterization of the effects of the linear by an admittance that is deterministic and homogeneous. On inspection of any liner, one can easily see that this is not the case. The analysis of the effects of stochastic admittances is in its infancy (ref. 21).

A fifth assumption which is usually employed is that of parallel mean flow in which the boundary layer is fully developed and the duct walls are parallel to the mean flow (ref. 22). Further, in some analyses, the fully developed mean flow is replaced by a plug flow, thereby neglecting the refractive effects of

the mean boundary layer which become increasingly more significant as the sound frequency increases. Certainly, theories based on the parallel flow assumption will not be capable of determining the attenuation and propagation characteristics in nonuniform ducts (ducts whose cross-sectional area changes along their axes). Recently, a number of approaches have been developed to treat sound propagation in nonuniform ducts. Each approach has unique characteristics and advantages as well as obvious limitations, either of a numerical or a physical nature. Some of these approaches were reviewed in reference 22. The purpose of the present paper is to present an updated critical review of these approaches.

DIRECT NUMERICAL TECHNIQUES

Direct numerical methods based on finite differences have been proposed (refs. 23-25). However, these methods have been restricted to simple cases of no-mean flow or one-dimensional mean flow and/or plane acoustic waves and promise to become unwieldy for more general cases. Methods were also based on finite elements (refs. 26 and 27). These purely numerical techniques would be impractical because of the excessive amount of computation time and the large round-off errors. The latter is a result of the necessity of using very small axial and transverse steps or very small finite elements to represent the axial oscillations and the rapidly varying shapes of each mode. In fact, a computational difficulty exists even in calculating the higher-order Bessel functions that represent the mode shapes in a uniform duct carrying uniform mean flow unless asymptotic expansions are used. Moreover, the axial step or finite element must be much smaller than the wavelength of the lowest mode in order to be able to determine the axial variation. These small steps and finite elements would cause the error in the numerical solution to increase very rapidly with axial distance and sound frequency.

To simplify the computation of the axial variation of the lowest mode in a two-dimensional duct with constant cross-sectional area but varying admittance, Baumeister (ref. 28) expressed the pressure as

$$p(x,y,t) = P(x,y)\exp[i(kx - \omega t)]$$

where k is the propagation constant corresponding to a hard-walled duct. Then, he used finite differences to solve for the "so-called" envelope P(x,y). This approach is suited for the lowest mode.

QUASI-ONE-DIMENSIONAL APPROXIMATIONS

The earliest studies of sound propagation in ducts with varying cross sections stemmed from the need to design efficient horn loudspeakers. Such horns are essentially acoustic transformers of plane waves and their efficiency depends on the throat and mouth area, the flare angle (wall slope), and the frequency of the sound. The walls of the horns are perfectly rigid and they do not flare so rapidly to keep the sound guided by the horn and prevent its spread-

ing out as spherical waves in free space.

For the case of no-mean flow, one writes the quasi-one-dimensional equivalent of eqs. (1)-(4). Combining these equations, he obtains Webster's equation (ref. 29).

$$\frac{1}{S}\frac{\partial}{\partial x}\left(S\,\frac{\partial p_1}{\partial x}\right) = \frac{\partial^2 p_1}{\partial t^2} \tag{5}$$

where S is the cross-sectional area of the duct. This equation can be derived alternatively as the first term in an expansion of the three-dimensional acoustic equations in powers of the dimensionless frequency (ref. 30). It can also be derived by integrating the acoustic equations across the duct. Solutions of equation (5) have been obtained and verified by many researchers (ref. 22). Using the method of multiple scales (ref. 31), Nayfeh (ref. 32) obtained an expansion for equation (5) with the nonlinear terms retained; the solution shows the variation of the position of the shock with the cross-sectional area.

In the case of mean flow, one writes the quasi-one-dimensional equivalent of equations (1)-(4). For linear waves and sinusoidal time variations, the resulting equations describing the axial variations were solved for a special duct geometry for which the equations have constant coefficients (ref. 33), for the case of short waves by using the WKB approximation (ref. 34), and for general duct geometry by using numerical techniques (refs. 35 and 36). The nonlinear case was treated by Whitham (ref. 37), Rudinger (ref. 38), Powell (refs. 39 and 40), and Hawkings (41).

In this quasi-one-dimensional approach, one can determine only the lowest mode in ducts with slowly varying cross sections and cannot account for transverse mean-flow gradients or large wall admittances.

SOLUTIONS FOR SLOWLY VARYING CROSS-SECTIONS

For slowly varying cross sections, the mean flow quantities are slowly varying functions of the axial distance; that is, $q_0 = q_0(z_1,x,y)$, where $z_1 = \varepsilon z$ with ε being a small dimensionless parameter that characterizes the slow axial variations of the cross-sectional area. For linear waves and sinusoidal time variations, the method of multiple scales (ref. 31) is used to express the acoustic quantities which are expressed in the form

$$q_1(x,y,z,t) = \sum_{n=0}^{N} \varepsilon^n Q_n(x,y,z_1,z_2,z_3,\ldots,z_N)\exp(i\phi) + O(\varepsilon^{N+1}) \tag{6}$$

where $z_n = \varepsilon^n z$ and

$$\frac{\partial\phi}{\partial t} = -\,\omega, \quad \frac{\partial\phi}{\partial z} = k_0(z_1) \tag{7}$$

Expressing each acoustic quantity as in equation (6), substituting these expressions into equations (1)-(4) and the boundary conditions, and equating co-

efficients of equal powers of ε yield equations to determine successively the Q_n. The zeroth-order problem is the same as the problem for a duct that is locally parallel with z_1 appearing as a parameter. The solution for the acoustic pressure can be expressed as

$$Q_0(x,y,z_1,z_2,\ldots,z_N) = A(x,y,z_1,z_2,\ldots,z_N)\psi(x,y,z_1) \tag{8}$$

where $\psi(x,y,z_1)$ is the quasi-parallel mode shape corresponding to the propagation constant $k_0(z_1)$. The function A is still undetermined to this level of approximation; it is determined by imposing the so-called evaluable conditions at the higher levels of approximation. To first order, one obtains the following equation for A:

$$f(z_1)\frac{dA}{dz_1} + g(z_1)A = 0 \tag{9}$$

where $f(z_1)$ and $g(z_1)$ are obtained numerically from integrals across the duct of ψ, q_0, k_0, and their derivatives.

Equation (9) has the solution

$$A(z_1) = A_0\exp[i\varepsilon\int k_1(z_1)dz] \tag{10}$$

where $k_1 = ig(z_1)/f(z_1)$. To first order, A_0 is a constant to be determined from the initial conditions. Then, to the first approximation,

$$p_1 = A_0\psi(x,y;z_1)\exp\left\{i\int[k_0(z_1) + \varepsilon k_1(z_1)]dz - i\omega t\right\} + O(\varepsilon) \tag{11}$$

According to this approach, one can determine the transmission and attenuation for all modes for hard-walled and soft-walled ducts with no-mean flow (ref. 42), two-dimensional ducts carrying incompressible and compressible flows (refs. 43 and 44), and annular ducts (ref. 45). Thus, in this approach one can include transverse and axial gradients, slow variations in the wall admittances, and boundary-layer growths, but the technique is limited to slow variations and the expansion needs to be carried out to second order in order to determine reflections of the acoustic signal.

WEAK WALL UNDULATIONS

In this approach, one assumes that the cross section of the duct deviates slightly from a uniform one. For example, the dimensionless radius of a cylindrical duct can be expressed as

$$R(z) = 1 + \varepsilon R_1(z) \tag{12}$$

and the dimensionless positions of the walls of a two-dimensional duct can be expressed as

$$\begin{aligned} y &= 1 + \varepsilon d_1(z) \\ y &= 1 + \varepsilon d_2(z) \end{aligned} \tag{13}$$

where ε is a small dimensionless parameter and R_1, d_1, and d_2 need not be slowly varying functions of z.

Taking advantage of the small deviation of the duct cross-section from a uniform one, a number of researchers (refs. 46-49) sought straightfoward expansions (called Born approximations in the physics literature). For two-dimensional ducts and sinusoidal time variations, the expansions have the form

$$q_1(y,z,t) = \exp(i\omega t) \sum_{n=1}^{N} \varepsilon^n Q_n(y,z) + O(\varepsilon^N) \tag{14}$$

Substituting expressions like equation (14) for each flow quantity in equations (1)-(4) and the boundary conditions and expanding the results for small ε, one obtains equations and boundary conditions for the successive determination of the Q_n.

Isakovitch (ref. 46), Samuels (ref. 47), and Salant (ref. 48) obtained straightforward expansions for waves propagating in two-dimensional ducts when d_1 and d_2 vary sinusoidally with z. Under these conditions, first-order expansions are unbounded for certain frequencies called the resonant frequencies; hence, the straightforward expansion is invalid near these resonant frequencies. Nayfeh (ref. 50) used the method of multiple scales and obtained an expansion that is valid near these resonant frequencies. He pointed out that resonances occur whenever the wavenumber of the wall undulations is equal to the difference of the wavenumbers of two propagating modes. These results show that these two modes interact and neither of them exists in the duct without strongly exciting the other modes. These results were extended by Nayfeh (ref. 51) to the case of two-dimensional ducts carrying uniform mean flows in the absence of the wall undulations.

Tam (ref. 49) obtained a first-order expansion for waves incident in the upstream direction on a throat or a constriction in a cylindrical duct. His results show that substantial attenuation of wave energy is possible for an axial flow Mach number of about 0.6 and throats of reasonable area reduction. It should be noted that the straightforward expansion is not valid for long distances and it might break down near resonant frequencies. These deficiencies can be removed by using the method of multiple scales. Then, one can account for all effects except large axial variations.

APPROXIMATIONS BY STEPPED UNIFORM SECTIONS

In this approach, one analyzes the effects of the continuous variations in the wall admittance and/or the cross-sectional variations by approximating the duct by a series of sections, each with a uniform admittance (refs. 52 and 53) and a uniform cross-section (ref. 54). Then, one matches the pressure and the velocity at all interfaces of the different uniform sections. Hogge and Ritzi (ref. 55) approximated the duct by a series of cylindrical and conical sections and matched the pressure and velocity at the approximate interfaces between sections. Since the end surfaces of the conical sections are spherical rather

than planar, the interfaces between sections do not match exactly and some error is introduced.

This approach is most suited for cases in which the wall liner consists of a number of uniform segments (refs. 52,53,56-61) and/or cases in which the duct cross-section consists of uniform but different segments (ref. 62). In the latter case, determining the mean flow can be a formidable problem if viscosity is included. In approximating a duct with a continuously varying cross-sectional area by a series of stepped uniform ducts, a large number of uniform segments are needed to provide sufficient accuracy for the solution when the axial variations are large.

VARIATIONAL METHODS

In the variational approach, one uses either the Rayleigh-Ritz procedure, which requires the knowledge of the Lagrangian describing the problem, or the method of weighted residuals (ref. 63). Since the Lagrangian is not known yet for the general problem, the Galerkin procedure (a special case of the method of weighted residuals) is the only applicable technique at this time. According to this approach, one chooses basis functions (usually the mode shapes of a quasi-parallel problem) and represents each flow quantity as

$$q_1(x,y,z,t) = \sum \psi_n(z)\phi_n(x,y)\exp(i\omega t) \tag{15}$$

where the ϕ_n are the basis functions, which, in general, do not satisfy the boundary conditions. On expressing each flow quantity as in equation (15), substituting the result into equations (1)-(4) and the boundary conditions, and using the Galerkin procedure, one obtains coupled ordinary-differential equations describing the ψ_n. These equations are then solved numerically.

Stevenson (ref. 64) applied this approach to the problem of waves propagating in hard-walled ducts with no-mean flow. Beckemeyer and Eversman (ref. 65) used the variational approach with the Lagrangian for waves propagating in hard-walled ducts with no-mean flow, Eversman, Cook, and Beckemeyer (ref. 66) applied the Galerkin approach to two-dimensional lined ducts with no-mean flow, and Eversman (ref. 67) applied it to ducts carrying mean flows.

Since the $\psi_n(z)$ vary rapidly even for a uniform duct, $\psi_n(z) \propto \exp(ik_n z)$ and k_n can be very large for the lower modes, very small axial steps must be used in the computations resulting in large computation time, which increases very rapidly with axial distance and sound frequency.

THE WAVE ENVELOPE TECHNIQUE

According to this approach, one uses the method of variation of parameters to change the dependent variables from the fast varying variables to others that vary slowly. Thus, each acoustic quantity q_1 is expressed as

$$q_1(x,y,z,t) = \sum_{n=1}^{N} A_n(z)\exp[i\int k_n(z)dz - i\omega t]Q_n(x,y,z)$$

$$+ \tilde{A}_n(z)\exp[-i\int k_n(z)dz - i\omega t]\tilde{Q}_n(x,y,z) \qquad (16)$$

where the $Q_n(x,y,z)$ are the quasi-parallel modes corresponding to the quasi-parallel propagation constants $k_n(z)$, the tilde refers to upstream propagation, N is the number of modes used, and $A_n(z)$ is a complex function whose modulus and argument represent, in some sense, the amplitude and the phase of the nth mode. Since k_n is complex, the exponential factor contains an estimate of the attenuation rate of the nth mode. Thus,

$$|A_n|\exp[-\int\alpha_n(z)dz]$$

is the envelope of the nth mode.

To use this method, one determines first the functions $\psi_n^{(1)}(x,y,z)$, $\psi_n^{(2)}(x,y,z)$, $\psi_n^{(3)}(x,y,z)$, $\psi_n^{(4)}(x,y,z)$, and $\psi_n^{(5)}(x,y,z)$ which are solutions of the adjoint quasi-parallel problem corresponding to the propagation constant k_n. Multiplying equations (1)-(4), respectively, by $\psi_n^{(1)}$, $\psi_n^{(2)}$, $\psi_n^{(3)}$, $\psi_n^{(4)}$, and $\psi_n^{(5)}$, adding the resulting equations, integrating the result by parts across the duct to transfer the transverse derivatives from the dependent variables to the ψ's, and using the boundary conditions, one obtains 2N integrability conditions (constraints), one corresponding to each k_n. Substituting the truncated expansion (eq. 16) into these integrability conditions, one obtains 2N first-order ordinary differential equations for the A_n. Then, these equations are solved numerically.

This technique has been applied by Kaiser and Nayfeh (ref. 68) to the propagation of multimodes in two-dimensional, nonuniform, lined ducts with no-mean flow. The results show that the present technique is superior to the variational approach especially for large sound frequencies and axial distances. This approach is being applied to the inlet problem by Nayfeh, Shaker, and Kaiser.

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