NONLINEAR PERIODIC WAVES*

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ABSTRACT

A review of systematic perturbation procedures for the analysis of nonlinear problems is presented. The cases when the multiplicity of an eigenvalue is finite or infinite are treated for self-sustained and for forced oscillations. The possibility of the formation of shock waves is discussed. Applications to acoustic problems are presented.

INTRODUCTION

Most of the problems in acoustics can be treated successfully by the linearized theory since the nonlinear terms in the governing equations are in general of higher order. The linearized theory predicts the natural frequencies. When there is only one eigenfunction associated with a natural frequency, we say the eigenvalue is simple and the mode of the free oscillation is given by the eigenfunction times a constant, the amplitude a, which is undefined. For forced oscillations the amplitude becomes infinite when the forcing term is in resonance.

When the small nonlinear terms are included, the periodic solutions can be constructed by the perturbation theory. The linear problem will yield the first term in the perturbation expansion of the solution, and further terms will also be determined by linear problems. This theory is based upon the original discovery by Lindstedt and Poincare that, to avoid the occurrence of secular terms in applying perturbation theory to periodic motions in celestial mechanics, it is necessary to make a perturbation expansion of the period of motion. This theory is frequently used for problems involving nonlinear ordinary differential equations. It has been applied systematically to periodic free vibration problems involving nonlinear partial differential equations by Keller and Ting (ref. 1). The solution and the frequency are assumed to be regular functions of a small parameter ε, the order of magnitude of the nonlinear terms.

When the eigenvalue is simple in the linear problem, the orthogonality of the eigenfunction with the inhomogeneous terms in the governing equation for the next order solution, and those for the higher ones, removes the secular terms and also defines the amplitude of the oscillation as a function of the frequency.

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When the multiplicity of an eigenvalue in the linear problem is equal to k, there are k linearly independent eigenfunctions. The linear solution is a linear combination of those k eigenfunctions. The orthogonality of the inhomogeneous terms of the governing equations for the higher solutions with k eigenfunctions yields k equations to relate the k coefficients in the linear combinations to the frequency. The nonlinear analyses define not only the amplitude but also specify the mode of the linear oscillation to one special linear combination of the k eigenfunctions.

When the multiplicity of an eigenvalue is infinite, the linear solution can be represented as an infinite series of the eigenfunctions. The orthogonality conditions with the eigenfunctions yield infinite numbers of equations for the coefficients in the series. The solution of these equations becomes very difficult, in general. Of course, we can construct an approximate solution by truncating the infinite series to a finite number of terms and imposing only a finite number of orthogonality conditions. This approximate solution will be useful only when the infinite series happens to converge very fast.

For a slightly nonlinear wave equation without dispersion, the multiplicity of each eigenvalue is infinite. The general solution of the linear wave equation can be represented by forward and backward waves with unknown wave forms. The orthogonality conditions were shown to be equivalent to an integral condition by Keller and Ting (ref. 1) and by Hale (ref. 2). Using the integral condition Fink, Hall, and Khalili (ref. 3) showed that it leads to a functional differential equation for the wave form and obtained explicit solutions in terms of elliptic functions for three types of nonlinear forcing terms. Generalization of the analysis and the establishment of an integral solvability condition for n-dimensional space have been made by Ting (ref. 4).

The systematic perturbation theory was applied to several interesting nonlinear boundary value problems by Millman and Keller (ref. 5). They also presented a systematic procedure for the construction of solutions of forced oscillations. The amplitude of the forcing term and the energy of the system are assigned. The solutions and the frequency are again regular functions of the small parameter ε . The energy equation guarantees the boundedness of the solution. With a simple eigenvalue, we obtain the finite amplitude solution of free oscillation when the forcing function is in resonance while its amplitude vanishes. When the eigenvalue is not simple, the force function can be any linear combination of the eigenfunctions with assigned amplitudes, and the solution of the forced oscillation will remain finite due to the energy condition. However, in order to recover the finite amplitude solution of free oscillation, the forcing function has to be a linear combination of all the eigenfunctions with all the coefficients approaching zero simultaneously while their ratios, which remain constant, are specified by the energy equation so that the wave form of the forcing function will be related to that of the free oscillation.

The procedures for the construction of the perturbation solutions and the statements regarding the solutions will be demonstrated in the following two sections for slightly nonlinear one dimensional wave equations with or without a finite dispersion term. When there is a finite dispersion term, the eigenvalue of the linear equation is simple. Without the dispersion term, the multiplicity of the eigenvalue is infinite. For each case, we will treat the problem of the free oscillations first and then that of forced oscillations.

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SYMBOLS

A	amplitude of the forcing function
a	amplitude of the oscillation
b_j	coefficients of linear combinations in eq. (31)
C_{jk}	coefficients of in eq. (15)
E	energy of the system
f	nonlinear term
g, G	the wave form of the solution and the forcing term respectively
j, k, m, n	positive integers
t	time
u	the solution of the nonlinear equation
x	space variable
α	dispersion coefficient
β	defined in eq. (28)
γ	coefficient in the nonlinear term when $f(u) = \gamma u^3$
ε	small dimensionless parameter denoting the order of magnitude of the nonlinear terms
λ	eigenvalue or ω^2
ξ	the phase variable
τ	integration variable
ω	frequency
$(\cdot)$	differentiation with respect to ε at $\varepsilon = 0$

Subscripts:

o	the leading term, i.e., when $\varepsilon = 0$
x,t	partial derivatives with respect to x or t

NONLINEAR WAVE EQUATION FOR A DISPERSIVE SYSTEM

Analysis of Free Oscillations

Let us construct the periodic solution of the nonlinear equation with a finite dispersion coefficient,

$$u_{tt} - u_{xx} + \alpha u = \varepsilon f(u) \quad 0 < x < \pi \tag{1}$$

subjected to the periodicity and boundary conditions,

$$u(x,\ t + 2\pi/\omega) = u(x,t) \tag{2}$$

$$u(0,t) = u(\pi,t) = 0 \tag{3}$$

ε is a prescribed small parameter and $\varepsilon\, f(u)$ represents the nonlinear force. ω is an undetermined angular frequency. Let us introduce $t' = \omega t$ and $u'(x,t') = u(x,t)$ in eqs. (1),(2),(3) so that the period in the new time variable is 2π. We will drop the primes and eqs. (1),(2), (3) become

$$\omega^2 u_{tt} - u_{xx} + \alpha u = \varepsilon f(u) \tag{4}$$

$$u(x,t + 2\pi) = u(x,t) \tag{5}$$

$$u(0,t) = u(\pi,t) = 0 \tag{6}$$

We shall seek a solution $u(x,t,\varepsilon)$ and a corresponsing angular frequency $\omega(\varepsilon)$ which are representable by finite Taylor series in ε, i.e.

$$u(x,t,\varepsilon) = u_o(x,t) + \varepsilon\dot{u}(x,t) + \tfrac{1}{2}\varepsilon^2\ddot{u} + \ldots \tag{7}$$

and $$\lambda(\varepsilon) = \omega^2(\varepsilon) = \omega^2_o + \varepsilon\dot{\lambda} + \tfrac{1}{2}\varepsilon^2\ddot{\lambda} + \ldots . \tag{8}$$

where $u_o = u(x,t,o)$, $\omega_o = \omega(0)$ and $(\dot{\ })$ denotes differentiation with respect to ε at $\varepsilon = 0$.

By setting $\varepsilon = 0$, eq. (4) becomes the linear equation for the zero-order solution u_o, i.e.,

$$\omega^2_o u_{ott} - u_{oxx} + \alpha u_o = 0 \tag{9}$$

and the same periodicity and boundary conditions eqs. (5), (6) hold for u_o. By choosing the origin of t appropriately we can require $u_t = 0$ at $t = 0$ and write these solutions as

$$u_o = a \sin nx \cos t \tag{10}$$

$$\lambda_o = \omega^2_o = \alpha + n^2, \quad n = 1, 2, \ldots . \tag{11}$$

For each integer n, ω_o or the eigenvalue λ_o is defined. When α is an irrational number we can show that there is only one eigenfunction, namely sin nx cos t. The amplitude a is undetermined so far.

We differentiate eqs. (4), (5), (6) with respect to ε and set $\varepsilon = 0$ to obtain the equations for the next order solution $\dot{u}$. They are:

$$\omega_o^2 \dot{u}_{tt} - \dot{u}_{xx} + \alpha \dot{u} = -\dot{\lambda} u_{o_{tt}} + f(u_o) \tag{12}$$

and eqs. (5) and (6) with u replaced by $\dot{u}$. They will have a solution (ref. 1) if the inhomogeneous part is orthogonal to the eigenfunction, i.e.

$$\frac{2}{\pi^2} \int_0^{2\pi} dt \int_0^{\pi} dx \quad [-\dot{\lambda} u_{o_{tt}} + f(u_o)] \quad \sin nx \cos t = 0. \tag{13}$$

We will give a physical meaning to this condition. The inhomogeneous term in the linear equation for $\dot{u}$ can be considered as the forcing term. The solution will be finite only when the forcing term is not in resonance with the normal mode of the homogeneous system. In other words, the coefficient of the Fourier component, sin nx cos t, of the forcing term should vanish as expressed by condition (13).

The solution is

$$\dot{u} = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty}{}' C_{jk} \sin j x \cos kt / [j^2 + \alpha - \omega_o^2 k^2] \tag{14}$$

The prime over the summation sign means $j \neq n$ when $k = 1$. The coefficients C_{jk} are defined by:

$$f(u_o) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} C_{jk} \cos kt \sin jx \qquad 0<x<\pi \tag{15}$$

The denominator in eq. (14) can be rewritten as $j^2 - n^2k^2 + \alpha(1-k^2)$ which will not vanish since α is an irrational number and $j \neq n$ when $k = 1$. Eq. (13) in turn yields the first order frequency-amplitude relationship,

$$\omega^2 - \omega_o^2 = \varepsilon\dot{\lambda} + 0(\varepsilon^2) = - C_{n1}/a + 0(\varepsilon^2) \tag{16}$$

For the special case of $f(u) = - u^3$, we have

$$C_n = -(9/16)a^3$$

and $\qquad \omega^2 = \alpha + n^2 + \varepsilon(9/16)a^2 + 0(\varepsilon^2) \qquad (17)$

Figure 1 shows the amplitude-frequency curves for $\alpha = \sqrt{2}$, $\varepsilon = 0.1$ in the neighborhood of the first two natural frequencies, i.e., n = 1,2,

Analysis of Forced Oscillations

We will modify the preceding problem by adding the distributed forcing function A sin j x cos ω t where j is a positive integer and A is the amplitude of the forcing function. The differential equation, eq. (4), must be changed to:

$$\omega^2 u_{tt} - u_{xx} + \alpha u - \varepsilon f(u) = A \sin jx \cos t \tag{18}$$

while eqs. (5) and (6) remain unchanged.

We will now establish the energy equation. We multiply eq. (18) by u_t, integrate with respect to x from 0 to π and carry out integration by parts for the second term with the aid of eqs. (5) and (6). We obtain

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^{\pi} \tfrac{1}{2}\{\omega^2 u_t^2 + u_x^2 + u^2\alpha - 2\varepsilon \int_0^{u} f(u)du - A\int_0^{t} u(k,\tau) \sin jx \cos \tau \, d\tau\}dx = 0$$

Therefore the energy E is a constant and in particular at t = 0, we have

$$\tfrac{1}{2}\int_0^{\pi} [\omega^2 u_t^2 + u_x^2 + u^2\alpha - 2\varepsilon \int_0^{u} f(u)\, du]dx\Big|_{t=0} = E \tag{19}$$

We will prescribe the energy E and consider u and ω to be functions of E, A and ε. We will then represent u and ω as finite Taylor series in ε as in eqs. (7) and (8). Letting $\varepsilon = 0$ in eqs. (18), (19), (5) and (6), we obtain

$$\omega_o^2 u_{ott} - u_{oxx} + \alpha u_o = A \sin jx \cos t \tag{20}$$

and

$$\tfrac{1}{2}\int_0^{\pi}[\omega_0^2 u_{o_t}^2 + u_{ox}^2 + \alpha u_o^2]\Big|_{t=0} dx = E \tag{21}$$

while eqs. (5) and (6) hold for u_o. The solution of eqs. (20), (5) and (6) is

$$u_o = a \sin jx \cos t \tag{22}$$

where

$$a = A/[j^2 + \alpha - \omega_o^2] \tag{23}$$

ω_o is related to the energy E by eq. (21),

$$E = \pi a^2(j^2 + \alpha)/4$$
$$= (\pi/4)A^2(j^2 + \alpha)/(j^2 + \alpha - \omega_o^2)^2 \tag{24}$$

Since α is an irrational number, the forcing term is in resonance with the linear system only at the jth natural frequency. With $\omega_o^2 = j^2 + \alpha$, the amplitude a and the energy E in the linearized theory become infinite. For the nonlinear theory, we prescribe the amplitude A and the energy E and define $\omega_o^2(A,E,\varepsilon)$, from eq. (24). We will then proceed to determine $\omega^2(A,E,\varepsilon)$ from the next order analysis.

We differentiate eqs. (18) and (19) with respect to ε and the set $\varepsilon = 0$ to obtain

$$\omega^2_o \dot{u}_{tt} - \dot{u}_{xx} + \alpha \dot{u} = - \dot{\lambda} u_{o_{tt}} + f(u_o) \tag{25}$$

and

$$\int_0^\pi [u_o \dot{u}_\alpha + u_{ox} \dot{u}_x - \int_0^{u_o} f(u)du]_{t=0} \, dx = 0 \tag{26}$$

We make use of the fact that $u_{ot} = 0$ at $t = 0$. Again eqs. (5) and (6) hold also for $\dot{u}$. The solution of eqs. (25), (5), and (6) is

$$\dot{u} = \frac{\dot{\lambda} a + C_{j1}}{j^2 - \omega_o^2 + \alpha} \sin j x \cos t \tag{27}$$

$$+ \sum_{m=0}^{\infty} \sum_{k=1}^{\infty}{}' \frac{C_{km}}{k^2 + \alpha - \omega^2_o m^2} \sin k x \cos mt$$

The symbol ($'$) means that $k \neq j$ when $m = 1$. Here the coefficients C_{km} are defined by eq. (15). Insertion of eq. (27) into eq. (26) yields

$$\dot{\lambda} = \frac{-1}{a} \left\{ C_{j1} + (j^2 + \alpha - \omega_o^2) \left[\sum_{\substack{m=0 \\ m \neq 1}}^{\infty} \frac{C_{jm}}{j^2 + 1 - \omega_o^2 m^2} \right. \right. \tag{28}$$

$$\left. \left. - \frac{2}{\pi a (j^2 + \alpha)} \int_0^\pi dx \int_0^\beta f(u)du \right] \right\}$$

where $\beta = u_o(x,0) = a \sin jx$. Use of eq. (28) in eq. (8) yields ω^2 in terms of A and E to the first order in ε.

From eq. (24) it is clear that for a nontrivial solution, $a \neq 0$ and E is positive. We can solve for ω_o^2,

$$\omega^2_o = j^2 + \alpha \pm [\pi A^2 (j^2 + \alpha)/(4E)]^{\frac{1}{2}} \tag{29}$$

In the limit $A \to 0$, we obtain the results for free oscillations,

$$\omega_o^2 \to j^2 + \alpha$$

and

$$\dot{\lambda} \to - C_{j1}/a$$

since $f(u_o)$ is finite. These results are in agreement with the results of eqs. (11), (14) and (16) obtained in the preceding subsection by a different procedure.

Since the amplitude a of the solution is equal to $\{4E/[\pi(j^2+\alpha)]\}^{\frac{1}{2}}$ we will plot a vs ω^2 for each constant forcing amplitude A instead of E vs ω^2 Fig. 2 shows the curves for A = 0, 0.2, 1 with $\alpha = \sqrt{2}$, $\varepsilon = 0.1$, $j = 1$, $f(u) = -u^3$. Of course, the curve for A = 0 agrees with the curve in Fig. 1 near the first natural frequency.

We will now explain why these two apparently different methods for the construction of finite amplitude free oscillations are equivalent.

In the first method, the underlying principle is that the solution u or u_0, $\dot{u}$ etc., have to be finite. The orthogonality condition eq. (13) guarantees that u is finite since the eigenvalue is simple and the solution $\dot{u}$ in eq.(14) is therefore not in resonance with the natural mode which is proportional to u_0. In the second method, we note that u_0 is finite for finite energy E. Hence $f(u_0)$ is finite, and the first order energy equation (26) says that the part of $\dot{u}_x$ which is orthogonal to u_{0x} has to be finite. u_0 is an eigen-solution. Since the eigenvalue is simple, eq. (26) is sufficient to guarantee that $\dot{u}$ is finite. When the eigenvalue is not simple, and the forcing function contains only one of the eigenfunctions, eq. (26) is not sufficient to insure that $\dot{u}$ is finite although eq. (26) will produce a relationship between ω^2 and E for A = 0. Of course, if we continue to the second order energy equation, the appearance the $\dot{u}_x^2$ term would then require $\dot{u}$ to be finite. The procedures for handling the problems with nonsimple eigenvalues are described in the next section.

NONLINEAR WAVE EQUATION FOR A NONDISPERSIVE SYSTEM

Analysis of Free Oscillation

The governing equations (4), (5), (6) in the preceding section remain applicable when we set $\alpha = 0$. The perturbation expansions eqs. (7) and (9) are the same. The governing equations for the zero order solution are

$$\omega^2_0 \, u_{ott} - u_{oxx} = 0 \tag{30}$$

and eqs. (5) and (6) with u replaced by u_0. Again we choose the origin of t such that $u_t = 0$ at t = 0, the solution for a given integer n can be written as

$$u_0 = a \, [\sum_{j=1}^{\infty} b_j \sin nj\,x \cos nj t] \tag{31}$$

with

$$\lambda_0 = \omega^2_0 = n^2 \tag{32}$$

u_0 is expressed in terms of a linear combination of all the eigenfunctions associated with λ_0. Since we factor out the amplitude a, we can impose a condition on b_j's, say

$$\sum_1^{\infty} b_j^2 \, j^2 = 1 \tag{33}$$

The governing equations for the next order solution $\dot{u}$ are eqs. (12), (5), and (6) with $\alpha = 0$. The solution $\dot{u}$ is bounded when the inhomogeneous part of eq. (12) is orthogonal to all the eigenfunctions $\sin njx \cos njt$ for all j. The results are

$$\dot{\lambda} a b_j + \frac{2}{\pi^2} \int_0^{2\pi} dt \int_0^{\pi} dx\, f(u_0) \sin njx \cos njt = 0 \tag{34}$$

for j = 1, 2..... Eqs. (34) and (33) are the equations for all the b_j's and a in terms of $\dot{\lambda}$. They are difficult to solve.

It was observed in ref. 4 that at least for the special case of $f(u) = \gamma u^3$ where γ is a constant, the approximate solution which was obtained by keeping only two terms in eq. (31) and applying eq. (34) for j = 1 and 2 differs from the exact solution by less than 0.1%. The construction of the approximate solution is the same as if the multiplicity of the eigenvalue were finite (say equal to 2). For a general nonlinear problem, it would be advisable to construct the exact solution by the following procedure.

The general solution of eqs. (30), (5), and (6) with $\lambda_0 = n^2 = 1$, can be written as

$$u(x,t) = \tfrac{1}{2} a \; [g(t + x) - g(t - x)] \tag{35}$$

with

$$g(\xi + 2\pi) = g(\xi) \tag{36}$$

where g is the unknown wave form. We introduce an extra factor a so that a normalization condition on g can be introduced, say

$$\frac{1}{\pi} \int_0^{2\pi} [g'(\xi)]^2 d\xi = 1 \tag{37}$$

The series eq. (31) can be identified with g as

$$g(\xi) = \sum_1^{\infty} b_j \operatorname{Sin} j\xi. \tag{38}$$

The orthogonality condition of eq. (34) can now be replaced by an integral equation for the wave form g (ref. 1, 2)

$$-\frac{a\pi}{2}\dot{\lambda}\, g(t) + \int_0^{\pi} f[u_0(t-x,\, x)]\, dx = 0 \tag{39}$$

For the case of $f = \gamma u^3$, eq. (39) is reduced to a functional differential equation for g and the solution is an elliptic function (ref. 3). Consequently all the coefficients b_j in eqs. (38) or (31) are defined. Eq. (39) also provides a relationship between a and λ, i.e., the amplitude frequency relationship.

Analysis of Forced Oscillation

If we introduce a distributed forcing term proportional to an eigenfunction say the first one sin x cos t, eq. (30) becomes

$$\omega^2_o u_{ott} - u_{oxx} = A \sin x \cos t \tag{40}$$

The solution of eqs. (40), (5), and (6) is

$$u_o = a \sin x \cos t \tag{41}$$

where

$$a = A/[1 - \omega_o^2] \tag{42}$$

The energy equation is eq. (21) with $\alpha = 0$. Letting $\varepsilon = 0$, we relate ω_o to E

$$E = \pi a^2/4 = (\pi/4)\, A^2/(1 - \omega_o^2)^2 \tag{43}$$

or

$$\omega_o^2 = 1 \pm [\pi A^2/(4E)]^{\frac{1}{2}}$$

When the amplitude of the forcing function A and the energy E are nonzero, the amplitude a and the frequency ω_o are defined by eqs. (42) and (43). In particular $\omega_o^2 \neq 1$; therefore, a is finite.

The next order differential equation and energy equation for $\dot{u}$ and $\dot{\lambda}$ are given by eqs. (25) and (26) with $\alpha = 0$. Their solutions are given by eqs. (27) and (28) with $\alpha = 0$ and $j = 1$. In particular for $f(u) = -u^3$, we have

$$\dot{u} = \frac{\dot{\lambda} a - 9a^3/16}{1 - \omega_o^2} \sin x \cos t - \frac{3a^3}{16(1-9\omega_o^2)} \sin 3x \cos 3t + \frac{3a^3}{16(9-\omega_o^2)} \sin 3x \cos t + \frac{9a^3}{16(1-\omega_o^2)} \sin 3x \cos 3t \tag{44}$$

and

$$\dot{\lambda} = (9a^2/16)\,[1 + 3\omega_o^2 \quad (1 - \omega_o^2)/(1 - 9\omega_o^2)] \tag{45}$$

In the limit of $A \to 0$ and $\omega_o^2 \to 1$ such that the energy A and also the amplitude a remain constant, we obtain $\dot{\lambda} = 9a^2/16$ while $\dot{u}$ becomes unbounded because of the last term in eq. (44) unless E = a = 0.

In order to obtain nontrivial free oscillations as a limiting case of the forced oscillations, the forcing function should contain all the eigenfunctions in the form of an infinite series eq. (31) or be represented as general solution eq. (35) for $\lambda_o = 1$. Eq. (40) should now be

$$\omega^2_o u_{o_{tt}} - u_{oxx} = (A/2)[G(t + x) - G(t - x)] \tag{46}$$

The solution u_o is

$$u_o = (a/2)[g(t + x) - g(t - x)] \tag{47}$$

where

$$a = A/(1 - \omega_o^2) \tag{48}$$

$$g''(\xi) = -G(\xi) \quad (49)$$

with $g(\xi + 2\pi) = g(\xi)$ and $g(0) = 0$. Since we introduce an extra factor A we can normalize G or g by eq. (37). The energy is related to a or A by eq. (43).

In the limiting case of $\omega^2 \to 1$ and $A \to 0$, while E and a remain finite, u_0 in eq. (47) is of the same form as eq. (35) for the free oscillation. The governing equation for $\mathring{u}$ is also the same as that for the free oscillation. On account of the next order energy equation, the part of $\mathring{u}$ which is infinite has to be orthogonal to all the eigenfunctions. Therefore, $\mathring{u}$ is finite and the wave form $g(\xi)$ will be the same as that for the finite free oscillation in ref. 3. The wave form for forcing term is in turn defined by eq. (49) as $A \to 0$.

CONCLUDING REMARKS

The perturbation theories for nonlinear free and forced oscillations are reviewed. In the case that the eigenvalue for the linear problem is simple, the solution for the forced oscillation yields that of free oscillation when the forcing term is in resonance while its amplitude approaches zero accordingly. This statement remains true when the multiplicity of the eigenvalue is finite or infinite provided that the forcing function is a linear combination of all the eigenfunctions. Their amplitudes vanish in such a menner that the wave form of the forcing function is defined by that for the free oscillation.

From eq. (24) we see that for a special combination of E and A, $k^2+\alpha-\omega_0^2m^2$ vanishes for a pair of (k,m) and the solution $\mathring{u}$ given by eq. (27) becomes infinite. We should then add a free oscillation mode b sin k x cos mt to u and the amplitude b is determined from the energy equation so that the secular term associated with the mode (k,m) in $\mathring{u}$ vanishes. Similar modifications should be made to avoid the appearance of a secular term of higher mode for the solution of a forced oscillation of a nondispersive system.

When the perturbation theory for the free oscillations yields only a trivial solution, we conclude that the small amplitude periodic solution which splits off from the state of rest does not exist. The longitudinal vibrations of a uniform bar with fixed or free ends are examples mentioned in ref. 1. This result is verified in ref. 1 by the method of characteristics. It yields a stronger result that, for certain initial boundary value problems, every nontrivial solution becomes singular in a finite time. Shock waves are formed afterwards. This result is the same as that of Lax but the proof is somewhat different (ref. 6). This result was also applied (ref. 1) to show that there are no nonsingular (shockless) finite amplitude sound waves in a closed tube. For long tubes with open ends, it can be extended to a problem periodic in x if the simple boundary condition of constant pressure is imposed. Consequently we can conclude again the nonexistence of nonsingular periodic solution. However, the realistic boundary condition for an open end of a long pipe should be imposed based on the analysis of Levine and Schwinger (ref.7). The reflection coefficient is not equal to -1. It depends on the wave number and furthermore its absolute value is less than unity due to the loss of energy propagated to infinity. Therefore an initial wave in an open tube will decay and a periodic free oscillation cannot be sustained. Detailed studies for this problem will be reported elsewhere.

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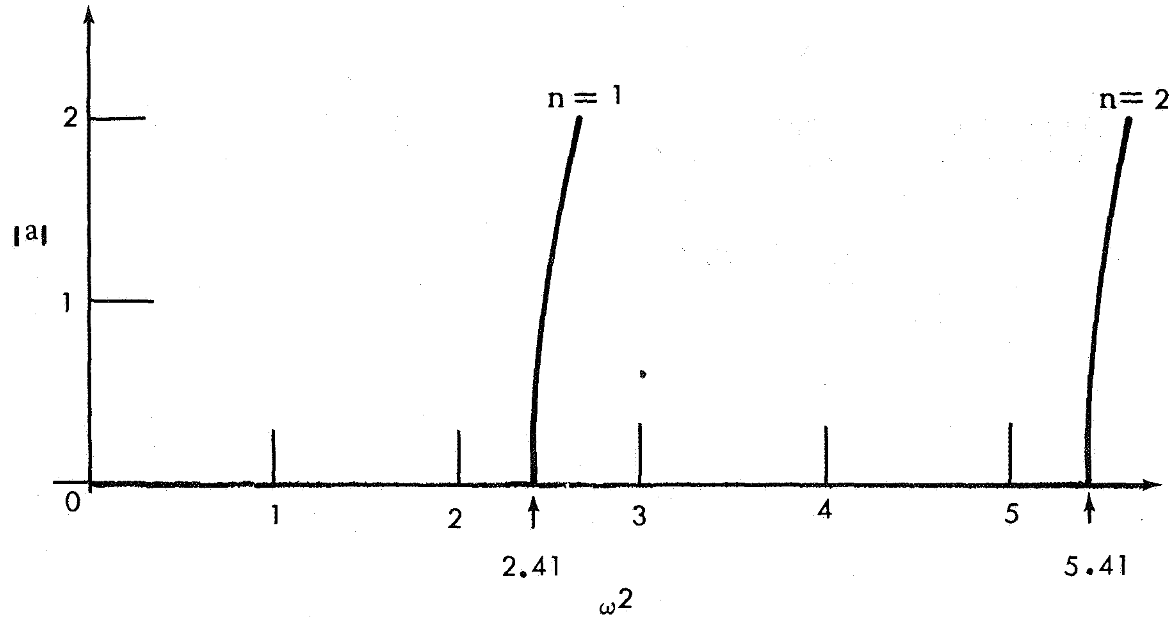

Figure 1.- Amplitude-frequency curve for free oscillations.

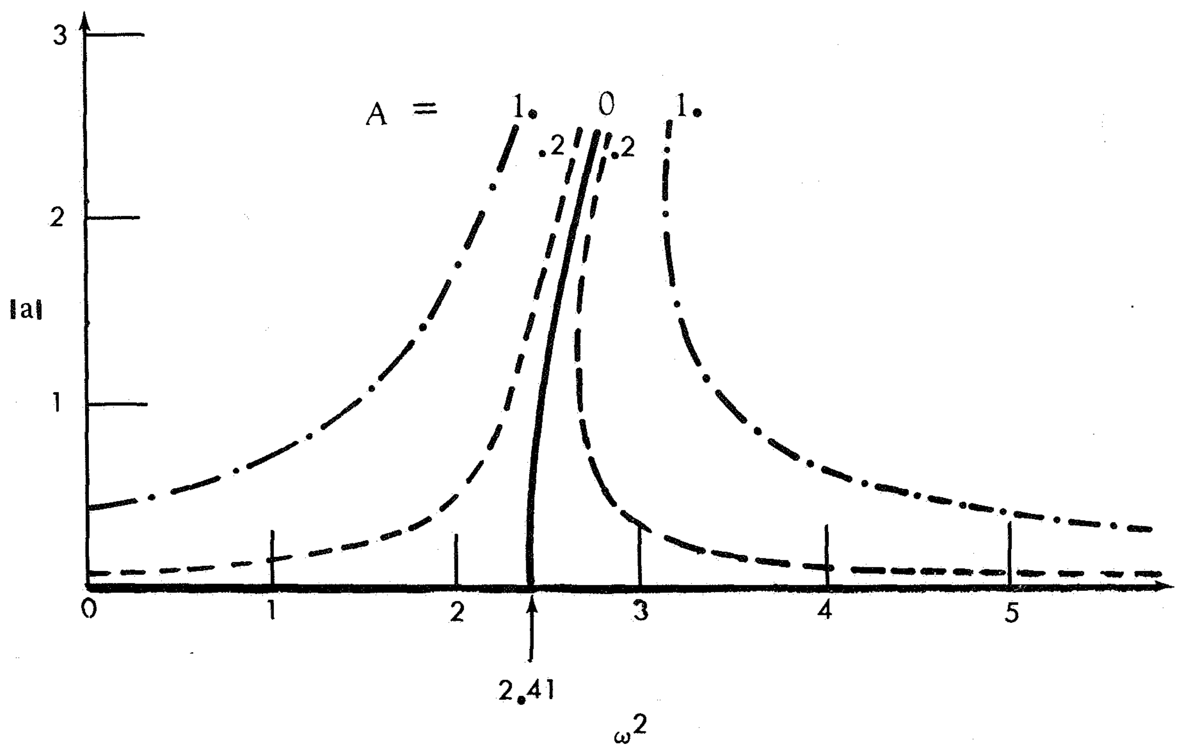

Figure 2.- Amplitude-frequency curve for forced oscillations.