A CORRESPONDENCE PRINCIPLE FOR STEADY-STATE WAVE PROBLEMS*

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SUMMARY

A correspondence principle has been developed for treating the steady-state propagation of waves from sources moving along a plane surface or interface. This new principle allows one to obtain, in a unified manner, explicit solutions for any source velocity. To illustrate the correspondence principle in a particular case, the problem of a load moving at an arbitrary constant velocity along the surface of an elastic half-space is considered.

INTRODUCTION

Certain problems in the linear theory of wave propagation are of fundamental importance to a wide variety of fields. One of these is the response of a plane interface between two different materials to moving transient sources of disturbance. The reflection and refraction of plane transient waves at an interface (refs. 1-3) and the generation of waves from specified sources moving at a constant velocity along an interface (refs. 4-7) are two important examples of this type of problem. In such cases it is usually assumed that the surrounding media is in plane motion and that a steady-state wave pattern exists relative to an observer moving with the source of disturbance. The resulting two-dimensional steady-state boundary value problem can then be solved either by transform techniques or by the use of complex function theory and the method of characteristics (see refs. 8 and 9).

In many problems, however, both of these traditional methods are very inefficient. This is because it is necessary to pose and solve separately the special cases when the source velocity is less than or greater than each of the characteristic wavespeeds in the surrounding media. This paper demonstrates that it is possible to treat all such special cases in a simple, unified manner through the application of a newly developed correspondence principle. In addition, this correspondence principle leads to a new and direct representation for the general solution of steady-state interface problems.

PROBLEM STATEMENT

Consider two homogeneous, isotropic semi-infinite media (either fluid or solid) which are in contact along the plane $\bar{y} = 0$ and which contain disturbances traveling at a constant velocity U in the negative $\bar{x}$-direction. We assume that these disturbances are uniform in the $\bar{z}$-direction and that a steady-state motion exists in the semi-infinite media. Under these conditions, the governing equations of motion in the two media reduce to (ref. 8):

*This work was supported by the Engineering Research Institute, Iowa State University.

$$(1 - U^2/c_m^2)\partial^2\phi_m/\partial x^2 + \partial^2\phi_m/\partial y^2 = 0 \qquad (m = 1,\ldots j) \tag{1}$$

in a set of moving coordinates (x,y,z) defined by $x = \bar{x} + Ut$, $y = \bar{y}$, $z = \bar{z}$. In equation (1), $\phi_m = \phi_m(x,y)$ and c_m are the displacement potentials and their corresponding wavespeeds, respectively, for the two media.

Along the surface $y = 0$, the ϕ_m must satisfy a certain set of boundary or continuity conditions which, in general, can be written in terms of the second order partial derivatives of the ϕ_m as

$$E_n(\partial^2\phi_m/\partial x^2,\ \partial^2\phi_m/\partial x\partial y,\ \partial^2\phi_m/\partial y^2) = P_n(x) \qquad (n = 1,\ldots j) \tag{2}$$

where the E_n also depend on the source speed U and the material properties of the two media and are linear functions of their arguments. The vector $\underline{P} = \{P_n(x)\}$ is determined by the values of the source disturbances along the plane $y = 0$ and is assumed to be given.

Since the linear operators which appear in equation (1) are hyperbolic if $U > c_m$ and elliptic if $U < c_m$, the steady-state solutions of these equations will depend on the relative size of U and the wavespeeds c_m. Consider first the "totally supersonic" (TSS) case (i.e. where $U > c_m$ is satisfied for all m in equation (1)).

Totally Supersonic Case

In the TSS case, the general solutions to the equations of motion (1) can be written as

$$\phi_m = F_m(x - \beta_m|y|) \tag{3}$$

where $\beta_m = (U^2/c_m^2 - 1)^{1/2}$ and the bars denote "absolute value of". In equation (3), solutions of the type $F_m(x + \beta_m|y|)$ have been rejected since they represent disturbances traveling in the positive x-direction and, hence, would violate the "radiation conditions" (ref. 8).

The second order partial derivatives of the ϕ_m then become

$$\left.\begin{aligned} \partial^2\phi_m/\partial x^2 &= F_m'' \\ \partial^2\phi_m/\partial x\partial y &= -\operatorname{sgn}(y)\beta_m F_m'' \\ \partial^2\phi_m/\partial y^2 &= \beta_m^2 F_m'' \end{aligned}\right\} \tag{4}$$

where F_m'' denote the second derivatives of F_m with respect to their arguments and sgn stands for "sign of". Placing equation (4) into equation (2) thus yields a set of linear equations in the F_m'' on $y = 0$, which using the summation convention can be written as

$$A_{nm}F_m''(x) = P_n(x) \qquad (n = 1,\dots j) \tag{5}$$

Here, the A is a real jxj matrix involving only the material properties of the given media and the disturbance speed U. Assuming that the matrix is non-singular, we can then solve for the F_m'', obtaining formally

$$F_m''(x) = A^{-1}_{mn}P_n(x) \tag{6}$$

and these results can be continued into the two media, using equation (3), to give:

$$\phi_m'' = A^{-1}_{mn}P_n(x - \beta_n|y|) \tag{7}$$

The stresses in each media can be obtained directly from equation (7) since they are simply linear functions of the ϕ_m''. The displacements or velocities, however, must be found from these results by a single integration. This TSS case is of fundamental importance for steady-state interface problems of the type we have been discussing because it contains implicitly, through equation (6), the solution for all other cases when this equation is interpreted in an operational sense. To prove this we now consider the general case.

General Case

Consider the general case when $U/c_m < 1$ for $m = 1,\dots k$, where $k \leq j$. Then the governing equations (1) are hyperbolic for $m > k$ and elliptic for $m \leq k$, and their general solutions are

$$\left.\begin{aligned} \phi_m &= \mathrm{Re}\{G_m(x + i\bar{\beta}_m|y|)\} \qquad & m \leq k \\ \phi_m &= F_m(x - \beta_m|y|) \qquad & k < m \leq j \end{aligned}\right\} \tag{8}$$

where $\bar{\beta}_m = (1 - U^2/c_m^2)^{1/2}$, Re denotes "real part of", and the G_m for $m \leq k$ are analytic functions of the complex variables $x + i\bar{\beta}_m|y|$. For $m > k$, the second order partial derivatives of the ϕ_m are again given by equation (4). For $m \leq k$, we now obtain instead

$$\left.\begin{aligned} \partial^2\phi_m/\partial x^2 &= \mathrm{Re}\{G_m''\} \\ \partial^2\phi_m/\partial x\partial y &= -\,\mathrm{sgn}(y)\,\bar{\beta}_m\,\mathrm{Im}\{G_m''\} \\ \partial^2\phi_m/\partial y^2 &= -\,\bar{\beta}_m^2\,\mathrm{Re}\{G_m''\} \end{aligned}\right\} \tag{9}$$

where Im denotes "imaginary part of". However, on the boundary y = 0 the real and imaginary parts of these G_m'' satisfy a pair of Hilbert transforms

$$\mathrm{Re}\{G_m''\} = H[\mathrm{Im}\{G_m''\}]$$

$$\mathrm{Im}\{G_m''\} = -\,H[\mathrm{Re}\{G_m''\}]$$

where the Hilbert transform is

$$H[f] = 1/\pi \int_{-\infty}^{\infty} f d\xi/(\xi - x)$$

and the integral is understood to be taken in the principal value sense. Hence, the partial derivatives on y = 0 can be written in terms of $Re\{G_m''\}$ only as

$$\left.\begin{aligned} \partial^2\phi_m/\partial x^2 &= Re\{G_m''\} \\ \partial^2\phi_m/\partial x\partial y &= sgn(0\pm)\bar{\beta}_m H[Re\{G_m''\}] \\ \partial^2\phi_m/\partial y^2 &= -\bar{\beta}_m^2 Re\{G_m''\} \end{aligned}\right\} \tag{10}$$

Note now that on the boundary y = 0, equations (4) and (10) will be identical if we make the following replacements in the TSS expressions (4) for $m \leq k$.

$$\beta_m \rightarrow -i\bar{\beta}_m \tag{11}$$

and identify the F_m'' in the TSS case with the G_m'' in the general case through the additional replacements for $m \leq k$ given by:

$$\left.\begin{aligned} F_m'' &\rightarrow Re\{G_m''\} \\ iF_m'' &\rightarrow H[Re\{G_m''\}] \end{aligned}\right\} \tag{12}$$

Thus, on the boundary y = 0 there is a one-to-one correspondence between the complex-valued TSS problem obtained by making the substitutions given by equation (11) and the general case problem if, as equation (12) shows, the appearance of the imaginary number i in the TSS problem is interpreted as representing the Hilbert transform operator in the general case. This correspondence also means that the complex-valued matrices $\underset{\sim}{A}$ and $\underset{\sim}{A}^{-1}$, which result in equations (5) and (6) from the substitutions given by equation (11), must be interpreted as representing matrix operators in the general case. In particular, breaking $\underset{\sim}{A}^{-1}$ into its real and imaginary parts, we have

$$A_{mn}^{-1} = a_{mn} + ib_{mn} \rightarrow a_{mn} + b_{mn}H[\cdot] \tag{13}$$

where a_{mn} and b_{mn} are both real. Using this result and equations (6) and (12), we see that on y = 0 the general case solution is given by

$$\left.\begin{aligned} F_m''(x) &= a_{mn}P_n(x) + b_{mn}H[P_n(x)] \qquad k < m \leq j \\ Re\{G_m''(x)\} &= a_{mn}P_n(x) + b_{mn}H[P_n(x)] \qquad m \leq k \end{aligned}\right\} \tag{14}$$

Since the general solutions for m > k are constant along the real characteristics $x - \beta_m|y|$ (see equation (8)), the F_m'' can be continued directly into the adjacent media and the general case solution written as

$$\phi_m'' = a_{mn}P_n(x - \beta_n|y|) + b_{mn}H[P_n(x - \beta_n|y|)] \qquad k < m \le j \tag{15}$$

For $m \le k$, however, our problem consists of finding the functions $G_m''(x + i\bar{\beta}_m|y|)$ which are analytic in the upper half of the complex-plane and whose real part is given on the real axis by equation (14). This is a standard problem in analytic function theory whose solution may be written as

$$G_m'' = 1/i\pi \int_{-\infty}^{\infty} A_{mn}^{-1}P_n(\xi)d\xi/(\xi - z_m) \qquad m \le k \tag{16}$$

provided the integral converges, where $z_m = x + i\bar{\beta}_m|y|$. As before, the stresses can be obtained directly from equations (15) and (16) although a further integration is necessary for displacements and velocities.

With the general solutions given by equations (15) and (16), it is now particularly easy to obtain the solution to steady-state interface problems for arbitrary source velocity. All that is needed is the inverse matrix $\underset{\sim}{A}^{-1}$ from the TSS case solution. In the general case this matrix becomes complex-valued when the substitution in equation (11) is made. A simple algebraic decomposition of $\underset{\sim}{A}^{-1}$ into its real and imaginary parts for each special case then gives the necessary matrices for the expressions in equations (15) and (16). To illustrate the use of this method we now consider a particular problem.

MOVING LOAD ON A HALF-SPACE

A number of authors (refs. 4-7) have previously considered the response of an elastic half-space to loads traveling at a constant velocity on the plane surface. Here, we will solve for the waves generated in the half-space $\bar{y} \ge 0$, $-\infty < \bar{x} < \infty$, $-\infty < \bar{z} < \infty$ by a moving distributed load of intensity P(x) in the moving coordinates $x = \bar{x} + Ut$, $y = \bar{y}$, $z = \bar{z}$ (figure 1). Then the normal stress, t_{yy}, and shearing stress, t_{xy}, on the surface are given by

$$\left.\begin{aligned} t_{yy} &= -P(x)\sin\theta \\ t_{xy} &= P(x)\cos\theta \end{aligned}\right\} \tag{17}$$

where θ is the angle between the direction of the applied load and the half-space surface. In this case there are only two displacement potentials ϕ_1 and ϕ_2, which correspond to dilatational and shear wave disturbances, respectively, and two corresponding wavespeeds c_1 and c_2. Application of the boundary conditions (17) yields the matrix $\underset{\sim}{A}$ and vector $\underline{P}$ given by (ref. 8):

$$\underset{\sim}{A} = \begin{bmatrix} (M_2^2 - 2) & -2\beta_2 \\ -2\beta_1 & -(M_2^2 - 2) \end{bmatrix} \qquad \underline{P} = \begin{bmatrix} -P\sin\theta/\mu \\ P\cos\theta/\mu \end{bmatrix} \tag{18}$$

where μ is the shear modulus and $M_2 = U/c_2$. Then the inverse matrix $\underset{\sim}{A}^{-1}$ is given by

$$\underset{\sim}{A}^{-1} = \begin{bmatrix} (M_2^2 - 2)/D & -2\beta_2/D \\ -2\beta_1/D & -(M_2^2 - 2)/D \end{bmatrix} \quad (19)$$

where $D = (M_2^2 - 2)^2 + 4\beta_1\beta_2$. Table 1 shows the breakdown of this inverse matrix into its real and imaginary parts. In that table $D_1 = (M_2^2 - 2)^4 + 16\bar{\beta}_1\beta_2$ and $D_2 = (M_2^2 - 2)^2 - 4\bar{\beta}_1\bar{\beta}_2$. When those results are placed back into equations (15) and (16), the problem is then formally complete. To illustrate the use of these expressions for a particular loading, consider the case of a moving concentrated line load, i.e. $P(x) = P\delta(x)$ where P is a constant and $\delta(x)$ is the Dirac delta function. Then we obtain:

Totally Supersonic Case ($U > c_1 > c_2$)

$$\phi_1'' = (-a_{11}\sin\theta + a_{12}\cos\theta)P\delta(x - \beta_1 y)/\mu$$

$$\phi_2'' = (-a_{21}\sin\theta + a_{22}\cos\theta)P\delta(x - \beta_2 y)/\mu$$

Transonic Case ($c_2 < U < c_1$)

$$G_1'' = (a_{11}\sin\theta - a_{12}\cos\theta)P/i\pi\mu Z_1 + (b_{11}\sin\theta - b_{12}\cos\theta)P/\pi\mu Z_1$$

$$\phi_2'' = (-a_{21}\sin\theta + a_{22}\cos\theta)P\delta(x - \beta_2 y)/\mu + (b_{21}\sin\theta - b_{22}\cos\theta)P/\pi\mu(x - \beta_2 y)$$

Subsonic Case ($U < c_2$)

$$G_1'' = (a_{11}\sin\theta - ib_{12}\cos\theta)P/i\pi\mu Z_1$$

$$G_2'' = (-a_{22}\cos\theta + ib_{21}\sin\theta)P/i\pi\mu Z_2$$

Similar results to these have been derived by the traditional complex variable and characteristics approach in the treatise by Eringen and Suhubi (ref. 9).

CONCLUDING REMARKS

The correspondence principle developed above has led to a new unified form of the solution for steady-state interface problems (equations 15 and 16) which can be efficiently used to treat a number of problems. In addition, this principle clearly demonstrates the close relationship that exists between the structure of the general solution and the TSS case. This relationship is currently being extended to steady-state problems in anisotropic media.

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TABLE I. - REAL AND IMAGINARY PARTS OF $\underset{\sim}{A}^{-1}$

CASES	MATRIX $\underset{\sim}{a}$		MATRIX $\underset{\sim}{b}$	
$U > C_1$	$\underset{\sim}{a} = \underset{\sim}{A}^{-1}$ (eq. (19))		$\underset{\sim}{b} = 0$	
$C_2 < U < C_1$	$(M_2^2 - 2)^3/D_1$	$-2\beta_2(M_2^2 - 2)^2/D_1$	$4\bar{\beta}_1\beta_2(M_2^2 - 2)/D_1$	$-8\bar{\beta}_1\beta_2^2/D_1$
	$-8\bar{\beta}_1^2\beta_2/D_1$	$-(M_2^2 - 2)^3/D_1$	$2\bar{\beta}_1(M_2^2 - 2)^2/D_1$	$-4\bar{\beta}_1\beta_2(M_2^2 - 2)/D_1$
$U < C_2$	$(M_2^2 - 2)/D_2$	0	0	$2\bar{\beta}_2/D_2$
	0	$-(M_2^2 - 2)/D_2$	$2\bar{\beta}_1/D_2$	0

$$D_1 = (M_2^2 - 2)^4 + 16\bar{\beta}_1\beta_2$$

$$D_2 = (M_2^2 - 2)^2 - 4\bar{\beta}_1\bar{\beta}_2$$

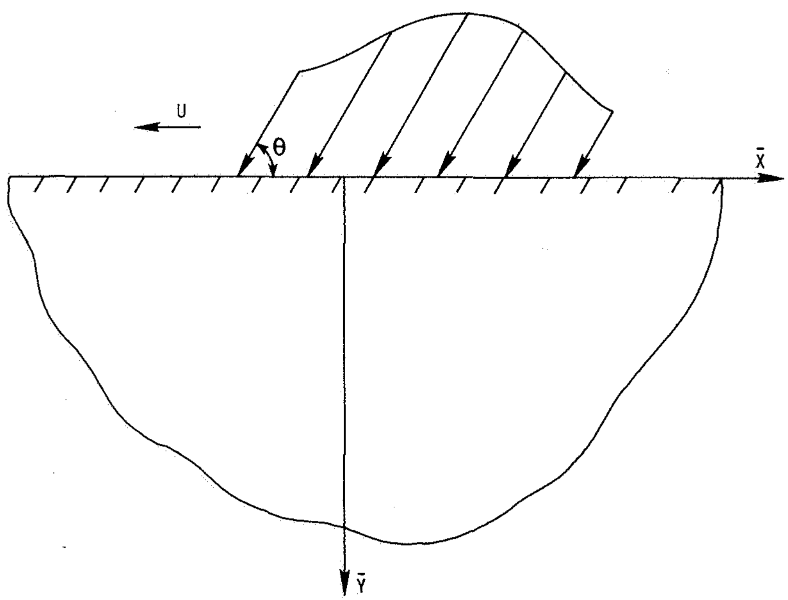

Figure 1.- Moving load on a half-space.