PROPAGATION OF SOUND IN TURBULENT MEDIA*

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SUMMARY

A review of some of the perturbation methods commonly used to study the propagation of acoustic waves in turbulent media is presented. Emphasis is on those techniques which are applicable to problems involving long-range propagation in the atmosphere and ocean. Characteristic features of the various methods are illustrated by applying them to particular problems. It is shown that conventional perturbation techniques, such as the Born approximation, yield solutions which contain secular terms, and which therefore have a relatively limited range of validity. In contrast, it is found that solutions obtained with the aid of the Rytov method or the smoothing method do not contain secular terms, and consequently have a much greater range of validity.

INTRODUCTION

In many real problems involving wave propagation in random media, such as those arising out of investigations of sound propagation in the atmosphere or ocean, the propagation medium may be regarded as weakly inhomogeneous in the sense that it deviates only slightly from a uniform state. This is convenient from a theoretical standpoint, since it allows such problems to be solved by perturbation methods. However, conventional perturbation methods, such as the Born method, suffer from the drawback that approximations obtained with them are generally limited in their range of validity. As a consequence, such methods are applicable only to problems involving relatively short-range propagation. For example, under conditions of moderately strong daytime turbulence, the Born approximation for acoustic propagation in the atmosphere may break down in as little as 100 meters.

The failure of the Born approximation in cases of long-range propagation arises from the fact that it is a finite-order approximation; i.e., it includes only a finite sum of terms of the complete perturbation expansion of the solution. Since such expansions usually involve secular terms (i.e., terms which increase indefinitely in magnitude with propagation distance), the Born approximation itself is secular, and hence can not generally be uniformly valid in the sense that the resulting error is bounded independently of propagation distance.

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It follows that any uniformly valid approximation must include at least the sum of an infinite subseries of the complete perturbation expansion. It is for the purpose of obtaining such approximations that infinite-order methods, such as the two-variable method, the Rytov method, the smoothing method, diagram methods, etc., have been applied to problems involving propagation in inhomogeneous and random media. Two of these methods, the Rytov method and the smoothing method, are discussed in this paper.

In section 1 the advantage of the Rytov method over the Born method in the case of long-range propagation is illustrated by applying both methods to a simple non-random problem which can be solved exactly. In section 2 the essential features of the smoothing method are brought out by first developing the method in a general context, and then applying it to a particular problem involving propagation of sound in a turbulent fluid.

1. COMPARISON OF THE BORN AND RYTOV METHODS

The precise nature of the failure of the Born method in the case of long-range propagation, as well as the improvement represented by the Rytov method, can best be illustrated by means of an example.

Consider the one-dimensional, non-random problem defined by the equation

$$u'' + k^2(1+\varepsilon)^2 u = 0 \ , \tag{1}$$

where the primes denote differentiation with respect to x. Here k and ε are real constants, with $k>0$ and ε a small parameter. We seek a solution of (1) representing rightward-propagating waves in the region $x>0$, subject to the boundary condition $u(0) = 1$. The exact solution of this problem can, of course, be written down immediately, and is

$$u(x;\varepsilon) = \exp\{ik(1+\varepsilon)x\} \ . \tag{2}$$

Now let us solve this problem by the Born method, with ε as the perturbation parameter. The procedure is as follows. We assume a solution of (1) of the form

$$u(x;\varepsilon) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots \ , \tag{3}$$

substitute into (1), expand in powers of ε, and equate the individual coefficients of the resulting series to zero. This yields a sequence of differential equations and boundary conditions for the functions u_0, u_1, u_2, etc., which can be solved successively. By inserting the result into (3) we obtain the expansion

$$u(x;\varepsilon) = (1 + i\varepsilon kx - \tfrac{1}{2}\varepsilon^2 k^2 x^2 + \cdots)\exp(ikx) \ , \tag{4}$$

which we recognize as being just the series expansion in powers of ε of the exact solution. Termination of this procedure after the calculation of $n+1$ terms of the series yields the nth Born approximation for this problem. Note that the result is a finite-order expansion; i.e., it consists of a finite sum of terms of the complete perturbation expansion given by (4).

It is clear that, for any fixed, bounded range of x, the nth Born approximation can be made to approximate u as closely as we please by choosing n sufficiently large. However, for any fixed n, no matter how large, the nth Born approximation is not uniformly valid for all x. This is due to the presence of secular terms (i.e., terms which involve x raised to some positive power) in the expansion given by (4), which causes the resulting approximate expression for u to increase indefinitely in magnitude as $x \to \infty$. In contrast, the exact solution is obviously bounded as $x \to \infty$.

This secular behavior, which is characteristic of finite-order approximations and which limits their range of validity, constitutes the main drawback of this type of approach. This is a practical, as well as a theoretical, problem, since, for example, investigations of sound propagation in the atmosphere and ocean often involve propagation ranges that are greater than the range of validity of the Born approximation.

The analysis given above, in addition to delineating the difficulty arising from the presence of secular terms in the perturbation expansion, also furnishes a clue as to how this difficulty may be overcome. Comparison of equation (2) with equation (4) shows that the sum of an infinite series of secular terms may be non-secular. This suggests the general idea of avoiding secular behavior by summing infinite series of secular terms. Of course, when dealing with more complicated problems involving propagation in inhomogeneous or random media, we cannot expect, in general, to be able to sum the entire perturbation series, as we did in the simple example treated above, since that would be tantamount to writing down the exact solution. It may, however, be possible to sum an infinite sub-series of the complete perturbation series, thereby obtaining a non-secular approximation. This idea; i.e., the idea of summing an infinite sub-series of the complete perturbation series, is central to methods such as the two-variable method, the Rytov method, the smoothing method, diagram methods, etc., which we call infinite-order methods.

With these thoughts in mind we turn now to a discussion of the Rytov method. To apply this method to the problem considered above, we first write the solution of (1) in the form

$$u = \exp(i\psi) \quad , \tag{5}$$

where ψ, the new unknown function, is assumed to have an expansion of the form

$$\psi(x;\varepsilon) = \psi_0(x) + \varepsilon\psi_1(x) + \varepsilon^2\psi_2(x) + \cdots \quad . \tag{6}$$

The functions ψ_0,ψ_1,ψ_2,etc., can be determined by substituting (5) into (1), after which the resulting equation for ψ is transformed and then solved by a perturbation technique similar to that described above. The details of this procedure are given in reference 1 for the general case of propagation in a multi-dimensional random medium. (Note that Tatarski refers to the Rytov method as the method of smooth perturbations.) An alternate approach, which makes use of the corresponding Born series, has been suggested by Sancer and Varvatsis (ref. 2). In this approach equation (6) is substituted into equation (5), the right-hand side of which is then expanded in a power series in ε. Since the resulting series must be identical to that given by equation (3), we can equate coefficients to obtain

$$u_0 = \exp(i\psi_0) \ , \quad u_1 = i\psi_1 u_0 \ , \quad u_2 = (i\psi_2 - \tfrac{1}{2}\psi_1^2)u_0 \ ,$$

etc., from which it follows that

$$\psi_0 = -i\log u_0 \ , \quad \psi_1 = -i\frac{u_1}{u_0} \ , \quad \psi_2 = -i\left[\frac{u_2}{u_0} - \frac{1}{2}\left(\frac{u_1}{u_0}\right)^2\right] \ , \tag{7}$$

etc. The nth Rytov approximation is obtained by terminating this process after the calculation of $n+1$ terms in the expansion of ψ and substituting the resulting truncated series into (5).

The essential feature of the resulting nth Rytov approximation is that, for $n>0$, it is equivalent to the summation of an infinite sub-series of the complete perturbation expansion of u. For example, the first Rytov approximation,

$$u_1^{(R)} = \exp\{i(\psi_0+\varepsilon\psi_1)\} \ , \tag{8}$$

is obviously equivalent to the summation

$$u_1^{(R)} = (1+i\varepsilon\psi_1 - \tfrac{1}{2}\varepsilon^2\psi_1^2 + \cdots)\exp(i\psi_0) \ ,$$

which, from (7), is the same as

$$u_1^{(R)} = u_0 + \varepsilon u_1 + \tfrac{1}{2}\varepsilon^2 \frac{u_1^2}{u_0} + \cdots \tag{9}$$

It is for this reason that the range of validity of the Rytov approximation is, in general, much greater than that of the Born approximation. As an example, the first Rytov approximation for the problem treated above is, from (8), (7), and (4),

$$u_1^{(R)} = \exp\{ik(1+\varepsilon)x\} \quad , \tag{10}$$

which is the same as the exact solution. Thus, in this case, the first Rytov approximation is equivalent to the summation of the entire perturbation series for u, and consequently has an infinite range of validity.

More detailed discussions of the Rytov method can be found in references 1, 2, and 3.

2. THE SMOOTHING METHOD

One of the more useful perturbation techniques for treating problems involving wave propagation in random media is the smoothing method. It is, like the Rytov method, an infinite-order method, as we will show. However, the smoothing method is more convenient than the Rytov method for treating problems involving propagation in random media since it yields directly equations for the desired statistical properties of the wave field.

Our development of the method is quite general and follows closely that of Keller (ref. 4). We should emphasize here that the analysis which follows is entirely formal; except for some special cases, rigorous proofs of convergence of the series involved have not yet been given.

We begin our discussion of the smoothing method by considering the equation

$$(D+\varepsilon R)u = f \quad , \tag{11}$$

where D and R are linear operators on some vector space and ε is a small parameter. Here D is assumed to be deterministic with a known inverse, whereas R is assumed to be random with $\langle R \rangle = 0$ (the angular brackets denote an ensemble average). The source term f is assumed to be deterministic.

Since R is random, the solution u of (11) will also be random. We shall therefore be interested in solving the following type of problem: Given the operator D and the source term f, along with some appropriate statistical properties of the operator R, find some specified statistical properties of the solution u. In the analysis which follows we shall be concerned primarily with $\langle u \rangle$, the ensemble average of u.

We begin the analysis of $\langle u \rangle$ by multiplying equation (11) by D^{-1} and writing the resulting equation in the form

$$u = D^{-1}f - \varepsilon D^{-1}Ru \quad . \tag{12}$$

Solving (12) by iteration yields

$$u = D^{-1}f - \varepsilon D^{-1}RD^{-1}f + \varepsilon^2 D^{-1}RD^{-1}RD^{-1}f + \cdots \quad , \tag{13}$$

which is just the Neumann series for u. By averaging (13) and using the fact that $\langle R\rangle = 0$ we obtain

$$\langle u\rangle = D^{-1}f + \varepsilon^2 D^{-1}\langle RD^{-1}R\rangle D^{-1}f + \cdots \quad . \tag{14}$$

The series expansion for $\langle u\rangle$ given by equation (14) is analogous to the Born series (i.e., equation (4)) of section 1. It can be shown that, like the Born series, this series generally contains secular terms, and hence no finite sub-series of it can be expected to yield a uniformly valid approximation for $\langle u\rangle$.

In order to get a uniformly valid approximation for $\langle u\rangle$, we proceed as follows. First, we note that, from equation (14),

$$D^{-1}f = \langle u\rangle + O(\varepsilon^2) \quad .$$

It follows, by replacing the term $D^{-1}f$ in the second term on the right-hand side of (14) by $\langle u\rangle$, that

$$\langle u\rangle = D^{-1}f + \varepsilon^2 D^{-1}\langle RD^{-1}R\rangle\langle u\rangle + O(\varepsilon^3) \quad . \tag{15}$$

Now let w be a solution of the equation obtained by dropping the term of order ε^3 from (15); i.e., let w be a solution of

$$w = D^{-1}f + \varepsilon^2 D^{-1}\langle RD^{-1}R\rangle w \quad . \tag{16}$$

Then by writing w as a Neumann series; i.e., by writing

$$w = D^{-1}f + \varepsilon^2 D^{-1}\langle RD^{-1}R\rangle D^{-1}f + \varepsilon^4 D^{-1}\langle RD^{-1}R\rangle D^{-1}\langle RD^{-1}R\rangle D^{-1}f + \cdots \quad , \tag{17}$$

we see that w is the sum of an infinite series in ε , and also, by comparing (17) with (14), that $w - \langle u\rangle = O(\varepsilon^3)$. Thus, by solving equation (16) we obtain an approximation to $\langle u\rangle$ which is the sum of an infinite subseries of the complete perturbation expansion of $\langle u\rangle$, and which differs from $\langle u\rangle$ by terms of order ε^3 .

The procedure leading to equation (16) is called the smoothing method; the resulting equation is referred to as the first-order smoothing approximation for the mean field. The above analysis shows that the smoothing method is indeed an infinite-order method.

The approach described here can also be used to obtain higher-order statistics of u, such as the mean square, the correlation function, etc. These and other aspects of the smoothing method are discussed in more detail in reference 5.

We present now results obtained by applying the smoothing method to a problem involving propagation of acoustic waves in a turbulent fluid. The starting point of the analysis is equation (60) of reference 6 which is written here in the form

$$(c^{-2}D_t^2-\nabla^2)p = 0 \quad . \tag{18}$$

This is a convected wave equation governing the propagation of high-frequency acoustic disturbances in a moving, inhomogeneous fluid medium. Here p is the acoustic pressure, c is the sound speed of the medium, and $D_t = \partial_t + \underline{u}\cdot\nabla$, where $\underline{u}$ $[=(u_1,u_2,u_3)]$ is the fluid velocity. Also $\nabla = (\partial_1,\partial_2,\partial_3)$, where $\partial_t \equiv \frac{\partial}{\partial t}$, $\partial_i \equiv \frac{\partial}{\partial x_i}$; t is time and $\underline{x}$ $[=(x_1,x_2,x_3)]$ is the position vector. Since the basic flow is assumed here to be turbulent, both c and $\underline{u}$ are to be regarded as random functions of $\underline{x}$ and t.

We assume that the basic flow represents a small perturbation of a uniform fluid at rest. Accordingly we write

$$c = c_0(1+\tfrac{1}{2}\varepsilon\mu) \quad , \tag{19}$$

$$\underline{u} = \varepsilon c_0\hat{\underline{u}} \quad , \tag{20}$$

where μ and $\hat{\underline{u}}$ are dimensionless random functions with zero mean, c_0 is the average sound speed of the medium, and ε is a small parameter measuring the deviation of the medium from a uniform motionless state. By inserting (19) and (20) into (18), expanding in powers of ε, and (in accordance with the assumption of high-frequency waves) dropping derivatives of flow quantities, we obtain

$$[L_0+\varepsilon L_1+\varepsilon^2 L_2+O(\varepsilon^3)]p = 0 \quad , \tag{21}$$

where the operators L_0, L_1, and L_2 are given by

$$L_0 = c_0^{-2}\,\partial_t^2 - \nabla^2 \quad , \quad L_1 = 2c_0^{-1}(\hat{\underline{u}}\cdot\nabla)\partial_t - c_0^{-2}\mu\partial_t^2 \quad ,$$

$$L_2 = \hat{u}_i\hat{u}_j\partial_i\partial_j - 2c_0^{-1}\mu(\underline{\hat{u}}\cdot\nabla)\partial_t + \frac{3}{4}c_0^{-2}\mu^2\partial_t^2 \quad .$$

Equation (21) can be written in the same form as equation (11), provided we define

$$D = L_0 + \varepsilon\langle L_1\rangle + \varepsilon^2\langle L_2\rangle + O(\varepsilon^3) \ ,$$

$$R = L_1 - \langle L_1\rangle + \varepsilon(L_2 - \langle L_2\rangle) + O(\varepsilon^2) \quad .$$

It follows that the smoothing method, as described above, is applicable to this problem.

A detailed analysis of this problem based on the smoothing method is described in reference 6 . The main result of that analysis is an approximate expression for the quantity $\langle p\rangle$ (the coherent wave) which, for the case of a plane, time-harmonic wave propagating in the x_1 direction through a statistically homogeneous and isotropic medium which is slowly varying in time, can be written in the form

$$\langle p(x_1,t)\rangle = A\exp\{i(kx_1-\omega t)\} \ , \tag{22}$$

where A is an arbitrary amplitude factor, ω is the frequency, and

$$k = k_0 + \tfrac{1}{2}\varepsilon^2 k_0\,[4v^2(1+\tfrac{1}{2}im_0k_0\ell)+\langle\mu^2\rangle(1+\tfrac{1}{2}im_0'k_0\ell)] \ . \tag{23}$$

Here $k_0 = \omega/c_0$, $v^2 = \langle\hat{u}_1^2\rangle = \langle\hat{u}_2^2\rangle = \langle\hat{u}_3^2\rangle$, m_0 and m_0' are positive constants of order one, and ℓ is the correlation length of the turbulence.

Equation (23) shows that $\mathrm{Im}k > 0$ and also that $\mathrm{Re}k > k_0$. Thus, the turbulence causes an attenuation of the coherent wave as well as a reduction in its phase speed. The aspect of the solution given by equations (22) and (23) which is of most interest to us, however, in view of the preceding development, is that it is non-secular in the propagation distance x_1 . Note also that this solution can be written as the sum of an infinite sub-series of the complete perturbation series for $\langle p\rangle$, as can be seen by substituting (23) into (22) and expanding in powers of ε .

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