SOUND RADIATION FROM RANDOMLY VIBRATING BEAMS

OF FINITE CIRCULAR CROSS SECTION

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INTRODUCTION

Previous studies of the radiation of sound from vibrating cylindrical beams have been concerned with radiation from resonant modes of these finite beams, with specified end boundary conditions. These include studies by Yousri and Fahy (ref. 1) and Kuhn and Morfey (ref. 2). The radiation efficiency or radiation loss factor determined in this manner represents the contribution of a single mode to the radiation at a given frequency. More recently, Yousri and Fahy have presented (ref. 3) a more general derivation which shows that the radiation efficiency for a cylindrical beam is a summation of terms that represent contributions from various modes. This summing over modes is necessary whenever more than one mode is excited in the frequency band of interest.

The results of the present study are given in a form which depends only on the frequency of the beam vibrations and the physical characteristics of the beam and its surroundings. A statistical consideration of random beam vibrations allows this result to be independent of the boundary conditions at the ends of the beam. The acoustic power radiated by the beam can be determined from a knowledge of the frequency band vibration data without a knowledge of the individual modal vibration amplitudes.

A practical example of the usefulness of this technique is provided by the application of the theoretical calculations to the prediction of the octave band acoustic power output of the picking sticks of an automatic textile loom. Calculations are made of the expected octave band sound pressure levels based on measured acceleration data. These theoretical levels are subsequently compared with actual sound pressure level measurements of loom noise.

THEORY

A beam of finite length is modelled as a cylinder of infinite length situated on the z axis (see Fig. 1). The transverse velocity of the beam vibrations is assumed to be zero except on the segment of the cylinder which lies between the points $z=\ell/2$ and $z=-\ell/2$. On this vibrating segment the transverse velocity consists of x and y components, v_x and v_y, which are expanded in a Fourier series of arbitrary fundamental time period ω

$$\vec{v}(z,t) = \sum_{n=-\infty}^{\infty} \vec{v}_n(z)\, e^{in\omega_o t} \tag{1}$$

The individual components of the Fourier series are represented as the superposition of two traveling waves moving in opposite directions on the beam

$$\vec{v}_n(z) = \vec{v}_n\,(e^{ik_b z} + e^{-i(k_b z + \psi)}) \qquad -\frac{\ell}{2} \leq z \leq \frac{\ell}{2} \tag{2}$$

where k_b is the wave number of the beam vibrations, and ψ represents the relative phase between the two traveling waves. Furthermore, $v_n(z)$ can be written as

$$\vec{v}_n(z) = \int_{-\infty}^{\infty} \vec{v}(\alpha)\, e^{i\alpha z}\, d\alpha \tag{3}$$

and the radial or normal velocity is

$$v_r(\theta,z) = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \cdot \int_{-\infty}^{\infty} \vec{v}(\alpha)\, e^{i\alpha z}\, d\alpha \tag{4}$$

The acoustic pressure due to these transverse vibrations is found by applying the acoustic boundary condition at the surface of the cylinder to the solution of the linear acoustic wave equation in cylindrical coordinates. The partial differential equation and the appropriate boundary condition are

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} + \frac{\partial^2 P}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{5}$$

$$\left.\frac{\partial P}{\partial r}\right|_{r=r_o} = in\omega_o \rho\, v_r \tag{6}$$

The solution to Eq. 5 can be written as a general linear combination of the seperable solutions (see ref. 4). The solution for outgoing waves of fixed frequency $\omega_n = n\omega_c$ can be written

$$P(r,\theta,z) = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} [A_m(\alpha)\cos(m\theta) + B_m(\alpha)\sin(m\theta)] H_m^{(1)}(k_r r) e^{i\alpha z}\, d\alpha \tag{7}$$

where $k_r = (k^2 - \alpha^2)$, $k=\omega/c$ and $H_m^{(1)}$ is the Hankel function of the first kind and of order m . Applying the boundary condition of Eq. 6 to the above expression, one finds that

$$P(r,\theta,z) = (\cos\theta\, \hat{i} + \sin\theta\, \hat{j}) \cdot \int_{-\infty}^{\infty} Z(\alpha,r)\vec{v}(\alpha) e^{i\alpha z}\, d\alpha \tag{8}$$

where
$$Z(\alpha,r) = i\,\omega_n \rho \frac{H_1^{(1)}(k_r r)}{k_r H_1^{(1)\prime}(k_r r_o)} \tag{9}$$

The time average acoustic power radiated by the beam at a given frequency is found by taking the time average of the integral over the surface of the cylinder of the product of the acoustic pressure at the surface of the cylinder $P(r_o,\theta,z)$ and the normal velocity $v_r(\theta,z)$

$$W = \frac{1}{2} \mathrm{Re} \left\{ \int_{-\infty}^{\infty} \int_{0}^{2\pi} P(r_o,\theta,z)\; v_r^*(\theta,z)\; r_o d\theta\, dz \right\} \tag{10}$$

where Re denotes the real part, and * indicates the complex conjugate. The ensemble average acoustic power radiated in a given frequency band is found by taking the ensemble average of a sum over the frequencies within the band of the result of Eq. 10. The ensemble average is performed assuming all relative phases to be equally probable.

The result of the θ integration in Eq. 10 is π , and the z integration produces a dirac delta function $2\pi\delta(\alpha-\alpha')$. This allows one of the α integrations to be performed by inspection, yielding

$$W = \pi^2 r_o \int_{-\infty}^{\infty} \mathrm{Re}\,\{Z_o(\alpha)\}\; (|v_x(\alpha)|^2 + |v_y(\alpha)|^2)\; d\alpha \tag{11}$$

where
$$Z_o(\alpha) = Z(\alpha,r_o) = i\omega_n \rho \frac{H_1^{(1)}(k_r r_o)}{k_r H_1^{(1)\prime}(k_r r_o)} \tag{12}$$

This expression for the acoustic power radiated by a finite cylinder can be simplified by applying some of the properties of Hankel functions. For

$k^2 < \alpha^2$ the argument for the Hankel functions is imaginary and $Re\{Z_o(\alpha)\}$ is zero. For $k^2 > \alpha^2$ the argument is real and the Wronskian relation for the Hankel functions gives (ref. 5)

$$Re\{Z_o(\alpha)\} = \frac{2\omega\rho}{\pi r_o} \frac{1}{k_r^2 |H_1^{(1)\prime}(k_r r_o)|^2} \tag{13}$$

Referring back to Eqs. 2 and 3, one can write $\vec{v}(\alpha)$ as the inverse transform of $\vec{v}(z)$, which when evaluated gives

$$\vec{v}(\alpha) = \frac{\vec{v}_n}{\pi} \left[\frac{\sin(k_b + \alpha)\frac{\ell}{2}}{(k_b + \alpha)} e^{-i\psi} + \frac{\sin(k_b - \alpha)\frac{\ell}{2}}{(k_b - \alpha)} \right] \tag{14}$$

Therefore

$$|v_x(\alpha)|^2 + |v_y(\alpha)|^2 = \frac{|v_{nx}|^2 + |v_{ny}|^2}{\pi^2} \left[\frac{\sin^2(k_b + \alpha)\frac{\ell}{2}}{(k_b + \alpha)^2} + \right.$$

$$\left. \frac{\sin^2(k_b - \alpha)\frac{\ell}{2}}{(k_b - \alpha)^2} + 2\cos\psi \frac{\sin(k_b + \alpha)\frac{\ell}{2} \sin(k_b - \alpha)\frac{\ell}{2}}{(k_b^2 - \alpha^2)} \right] \tag{15}$$

The $\cos\psi$ term goes to zero in taking the average assuming all ψ to be equally probable. It can be shown, by taking the time average of $\vec{v}\cdot\vec{v}$, that the mean square velocity in a given frequency band is equal to a sum over the frequencies in the band of the magnitude squared of $\vec{v}_n$

$$\langle v_x^2 \rangle = \sum |v_{nx}|^2 \quad \text{and} \quad \langle v_y^2 \rangle = \sum |v_{ny}|^2 \tag{16}$$

assuming x and y vibrations to be uncorrelated.

All these results are incorporated into Eq. 11 with the result that the acoustic power radiated in a given frequency band is

$$W = \frac{2\omega\rho}{\pi} (\langle v_x^2 \rangle + \langle v_y^2 \rangle) \int_{-K}^{K} \frac{1}{k_r^2 |H_1^{(1)\prime}(k_r r_o)|^2} \left[\frac{\sin^2(k_b+\alpha)\frac{\ell}{2}}{(k_b+\alpha)^2} + \frac{\sin^2(k_b-\alpha)\frac{\ell}{2}}{(k_b-\alpha)^2} \right] d\alpha \tag{17}$$

where all frequency terms are evaluated at the center frequency of the band.

This result can be written in the form of a radiation efficiency for the beam. For a cylinder the radiation efficiency is

$$\sigma = \frac{W}{\rho c \pi r_o \ell (\langle v_x^2 \rangle + \langle v_y^2 \rangle)} \tag{18}$$

With the change of variables $\beta = \alpha/k$ and the above expression, one can write the radiation efficiency for a cylindrical beam

$$\sigma = \frac{4}{\pi^2 k^2 r_o \ell} \int_{-1}^{1} \frac{1}{(1-\beta^2)|H_1^{(1)\prime}(\sqrt{1-\beta^2} k r_o)|^2} \frac{\sin^2(\varepsilon+\beta)\frac{k\ell}{2}}{(\varepsilon+\beta)^2} d\beta \tag{19}$$

where $\varepsilon = k_b/k$ and a symmetry of the integral was used to simplify the expression.

RESULTS

In the low frequency or small radius limit, kr_o is small. An approximation may be used in place of the Hankel function in order to simplify the integral in Eq. 19. The first term in the series expansion for the derivative of the Hankel function gives

$$|H_1^{(1)\prime}(z)|^2 = \frac{4}{\pi^2} \frac{1}{z^4} \tag{20}$$

With this approximation and the change of variables $u = (\varepsilon+\beta)k\ell/2$, one gets

$$\sigma = \frac{(kr_o)^3}{2} \int_{(\varepsilon-1)\frac{k\ell}{2}}^{(\varepsilon+1)\frac{k\ell}{2}} [1 - (\frac{2u}{k\ell} - \varepsilon)^2] \frac{\sin^2 u}{u^2} du \tag{21}$$

Well below the coincidence frequency $(k_b=k)$, u^2 in the denominator can be approximated as $u^2 = (k\ell/2)^2$. The result of this approximation is

$$\sigma = \frac{2}{3} (kr_o)^2 (\frac{r_o}{\ell}) (\frac{k}{k_b})^2 [1 - \frac{3\cos k_b\ell}{(k\ell)^2} \{\frac{\sin k\ell}{(k\ell)} - \cos k\ell\}] \tag{22}$$

This expression is directly comparable to the results of Kuhn and Morfey (ref. 2) in their low frequency approximation of the Yousri and Fahy (ref. 1) expression for the radiation efficiency of a simply supported beam. The radiation efficiency for the simply supported case gives a result which is exactly twice that given by Eq. 22 when evaluated at the same resonance frequency. The reason for this difference lies in the fact that the simply supported boundary condition implies a specific phase ψ in Eq. 2. This phase happens to be one which maximizes the radiation efficiency. Other possible phase relationships result in lower values for the radiation efficiency. It will be shown later that these differences disappear at higher frequencies.

At high frequencies the major contribution to the integral in Eq. 19 comes from the vicinity of $\beta = -\varepsilon$. Expanding the Hankel function term in a Taylor series about that point and keeping only the first term one gets

$$\sigma = \frac{2}{\pi k r_o} \frac{1}{(1-\varepsilon^2)\left|H_1^{(1)\prime}(\sqrt{1-\varepsilon^2}\, k r_o)\right|^2} \tag{23}$$

This is the radiation efficiency for a cylinder of infinite length. If $(1-\varepsilon^2)^{1/2} k r_o \gg 1$, then the derivative of the Hankel function may be expressed in terms of its asymptotic limit and

$$\sigma = \frac{\sqrt{1-\varepsilon^2}\,(k r_o)^2}{(1-\varepsilon^2)(k r_o)^2+1} \tag{24}$$

For purposes of a numerical evaluation of Eq. 19, the radiation efficiency is considered as a function of three independent variables only one of which is frequency dependent. These are kr_o, ℓ/r_o and

$$\varepsilon (k r_o)^{1/2} = \left(\frac{m}{B}\right)^{1/4} (c r_o)^{1/2} \tag{25}$$

where m is the mass per unit length of the beam and B is the bending stiffness. This last parameter is chosen so as to eliminate the frequency dependence of $\varepsilon = k_b/k$. Figs. 2 and 3 show the results of a numerical calculation of the radiation efficiency. Each graph shows σ as a function of kr_o for several values of $\varepsilon(kr_o)^{1/2}$ and for one value of ℓ/r_o. A valid comparison with the modal approach can be made by leaving the phase angle in Eq. 15 and continuing the derivation of the radiation efficiency. The result is

$$\sigma = \frac{4}{\pi^2 k^2 r_o \ell} \int_{-1}^{1} \frac{1}{(1-\beta^2)\left|H_1^{(1)\prime}(k r_o\sqrt{1-\beta^2})\right|^2} \left[\frac{\sin^2(\varepsilon+\beta)\frac{k\ell}{2}}{(\varepsilon+\beta)^2} + \cos\psi \, \frac{\sin(\varepsilon+\beta)\frac{k\ell}{2}\,\sin(\varepsilon-\beta)\frac{k\ell}{2}}{(\varepsilon^2-\beta^2)} \right] d\beta \tag{26}$$

Fig. 4 gives this comparison for the two extreme cases, where the $\cos\psi$ term is either always positive or always negative at the modal resonance frequencies. It is clear from the figure that the contribution from the second term to the integrand in Eq. 26 diminishes at higher frequencies, as the two extreme cases converge towards the average.

APPLICATION TO TEXTILE LOOM PICKING STICKS

In order to apply these results to the picking sticks of an automatic textile loom, it is necessary to generalize them to allow for different radiation efficiencies and perhaps different effective radii for the x and y vibrations. This is necessary to account for the fact that the picking stick is more nearly rectangular in cross section than circular. With this in mind one can write the acoustic power as

$$W = \rho c \pi \ell \quad [r_{ox} \langle v_x^2 \rangle \sigma_x + r_{oy} \langle v_y^2 \rangle \sigma_y] \qquad (27)$$

Values for the parameters were chosen to correspond to the characteristics of the picking sticks. The numerical technique used to evaluate σ and measured vibration data were used to determine the octave band acoustic power output for each of the two picking sticks on a loom. Fig. 5 shows a graph of the resulting power levels.

From these power levels, octave band sound pressure levels were calculated at a reference point one meter from the front of the loom, assuming symmetric cylindrical spreading,

$$SPL = 10 \log_{10} \frac{\langle P^2 \rangle}{P_{ref}^2} \qquad \langle P^2 \rangle = \frac{\rho c}{2\pi R \ell} [W_{left} + W_{right}] \qquad (28)$$

where R is the distance to the reference point and $P_{ref} = 2 \times 10^{-5}$ N/m^2 is the reference pressure. A comparison of this predicted sound pressure level with sound pressure levels actually measured at this position is shown in Fig. 6. This graph shows good agreement between theoretical and experimental results for frequencies above 125 Hz. The agreement is particularly close in the range of frequencies which have the highest sound pressure levels. Both curves in the figure represent an overall A-weighted level of 94 dBA. It is clear however that the low frequency predicted result falls short of the measured values. It is thought that there may be other noise sources on the loom which contribute to the higher levels at these low frequencies.

REFERENCES

1. Yousri, S. N.; and Fahy, F. J.: Sound Radiation from Transversely Vibrating Unbaffled Beams. J. Sound Vib., vol. 26, 1973, pp. 437-439.

2. Kuhn, G. F.; and Morfey, C. L.: Radiation Efficiency of Simply Supported Slender Beams Below Coincidence. J. Sound Vib., vol. 33, 1974, pp. 241-245.

3. Yousri, S. N.,; and Fahy, F. J.: Acoustic Radiation by Unbaffled Cylindrical Beams in Multi-Modal Transverse Vibration. J. Sound Vib., vol. 40, 1975, pp. 299-306.

4. Morse, P. M.; and Ingard, K. U.: Theoretical Acoustics. McGraw Hill, New York, 1968, pp. 356-357.

5. Junger, M. C.: The Physical Interpretation of the Expression for an Outgoing Wave in Cylindrical Coordinates. J. Acoust. Soc. Am., vol. 25, 1953, pp. 40-47.

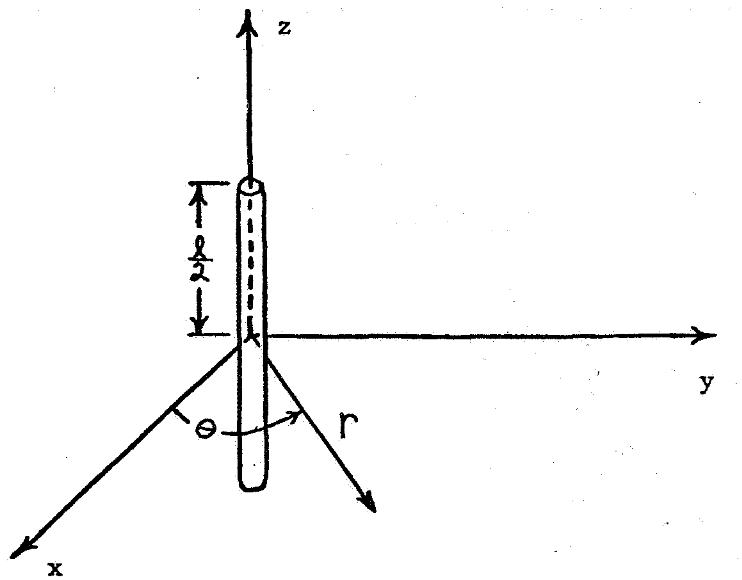

Figure 1.- Beam location and coordinate system.

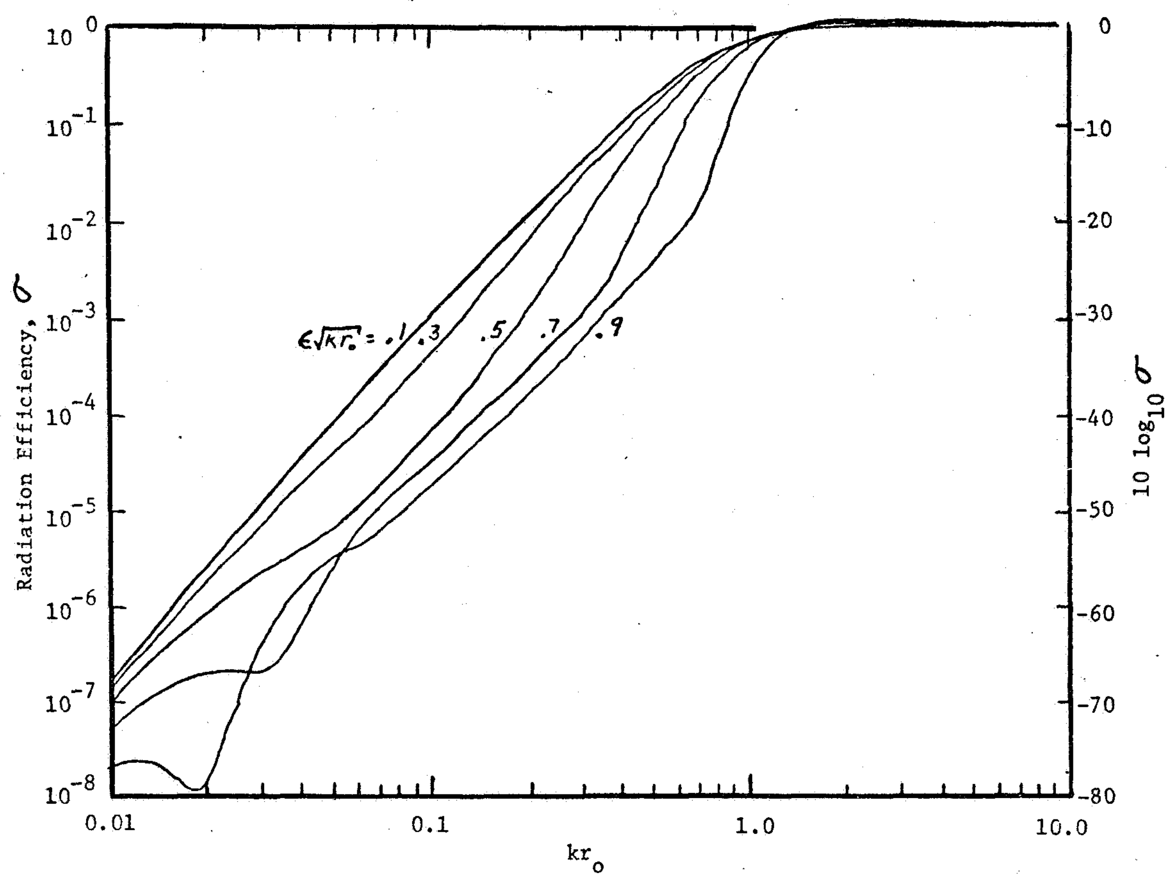

Figure 2.- Radiation efficiency of a beam for $\ell/r_o = 50$.

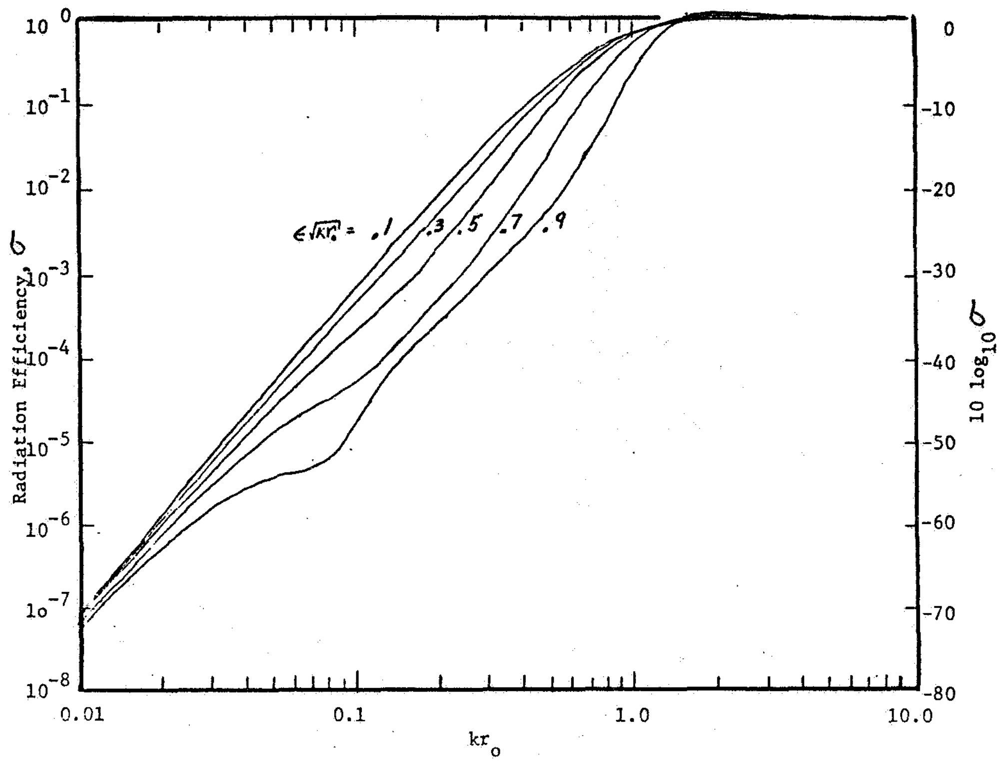

Figure 3.- Radiation efficiency of a beam for $\ell/r_o = 25$.

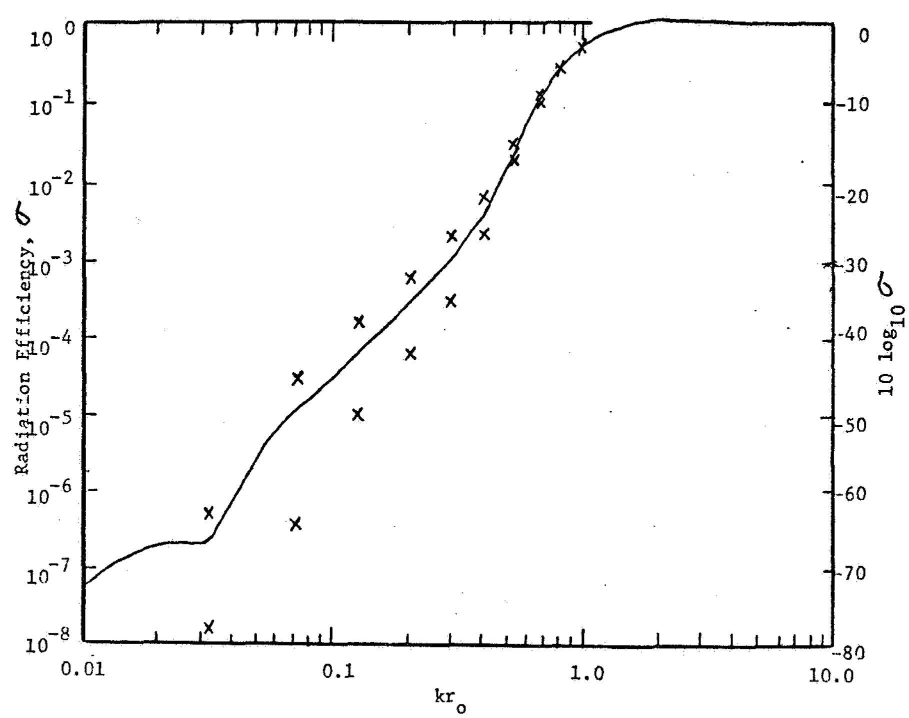

Figure 4.- A comparison of radiation efficiencies, — averaging technique, $\ell/r_o = 50$, $\varepsilon\sqrt{kr_o} = 0.7$, x modal approach.

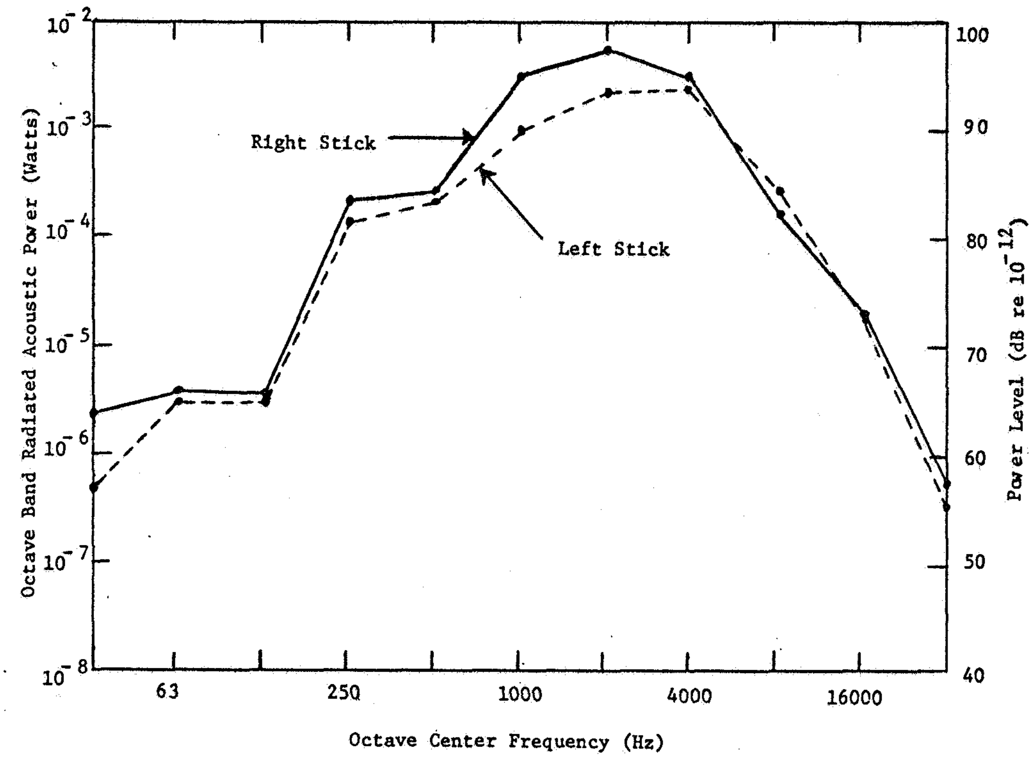

Figure 5.- Theoretical acoustic power output.

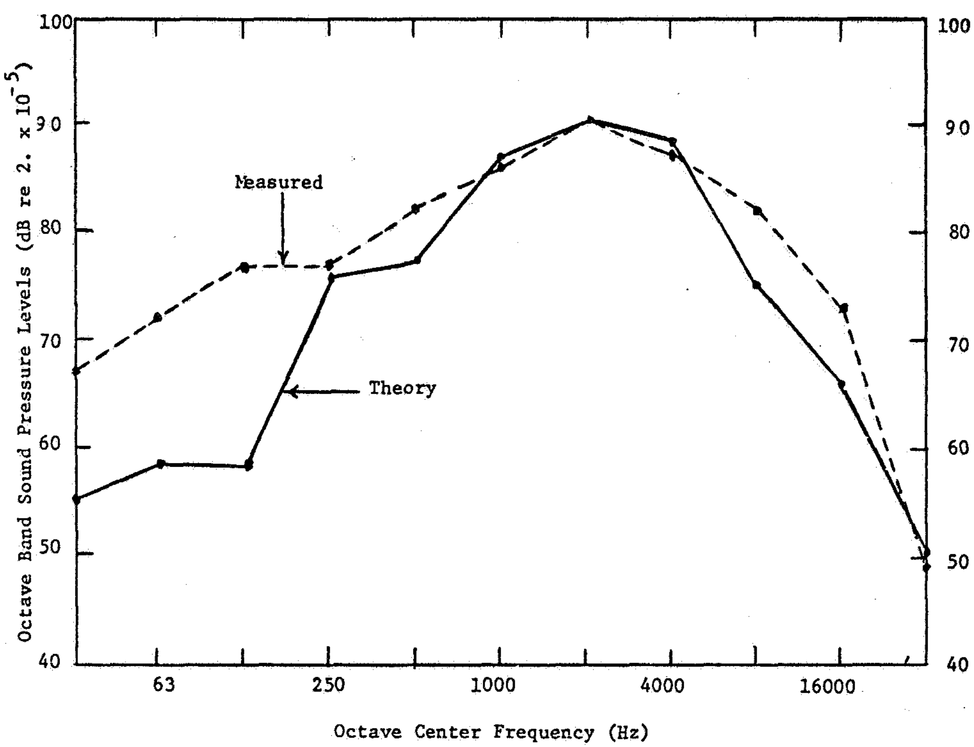

Figure 6.- Theoretical and experimental sound pressure levels.