APPLICATION OF FINITE ELEMENT APPROACH TO TRANSONIC FLOW PROBLEMS*

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SUMMARY

A variational finite element model for transonic small disturbance calculations is described. Different strategy is adopted in subsonic and supersonic regions, and blending elements are introduced between different regions. In the supersonic region, no upstream effect is allowed. If rectangular elements with linear shape functions are used, the model is similar to Murman's finite difference operators. Higher order shape functions, nonrectangular elements, and discontinuous approximation of shock waves are also discussed.

INTRODUCTION

The plane, steady, inviscid flow past a smooth configuration near sonic speed can be described by a perturbation velocity potential ϕ satisfying the transonic small disturbance equation (TSDE)

$$(K-\phi_x)\phi_{xx} + \phi_{yy} = 0 \tag{1}$$

where K is a similarity parameter. This equation is nonlinear and of mixed hyperbolic-elliptic type. Its weak solution admits discontinuity in the pressure,

$$\langle K-\phi_x \rangle = -\left(\frac{dx^D}{dy}\right)^2 \tag{2}$$

and

$$\frac{dx^D}{dy} = -\frac{[\![\phi_y]\!]}{[\![\phi_x]\!]} \equiv [\![\phi]\!] = 0 \qquad (3)\ \&\ (3')$$

where $\langle\ \rangle$ and $[\![\ ]\!]$ signify the average and the jump across the shock $x^D(y)$. The flow field solution is required to determine the pressure distribution on the airfoil (unlike the methods of singularities, or Kernel methods, used for incompressible flow calculations). Recently, finite difference solutions have been obtained with marked success (refs. 1 - 4).

In this paper, the feasibility of applying a finite element approach to transonic flow problems will be studied. A finite element method should handle

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the same problems that finite differences did, namely, the change of the type of the equation in the domain of interest with a discontinuous solution satisfying prescribed jump conditions. In passing, the potential solution is completely reversible (no entropy changes), and an expansion shock must be excluded (using an artificial viscosity or a shock fitting procedure). Hopefully, complicated boundary conditions will be handled easily in the physical space, and the use of higher order shape functions will be efficient.

FINITE ELEMENTS - BACKGROUND

Elliptic Problems

Consider the classical boundary value problem,

$$L_e(\phi) = K\phi_{xx} + \phi_{yy} = f \text{ on a rectangular } \Omega \tag{4}$$

where ϕ is known on $\partial\Omega$; $K > 0$. The associated functional is

$$I(\phi) = \iint_{\Omega} K\phi_x^{\,2} + \phi_y^{\,2} + 2f\phi \; dxdy \tag{5}$$

The first variation is set equal to zero

$$\partial I(\phi) = \int_0^1\int_0^1 (K\phi_{xx} + \phi_{yy} - f) \;\; \delta\phi \, dxdy = 0 \tag{6}$$

and the second variation is positive definite.

If linear shape functions on triangular elements are used, the algebraic equations for the nodal values are identical to those obtained by applying a centered difference scheme.

A gradient method for solving this problem is

$$\delta\phi = \phi^{n+1} - \phi^{n} = - \rho\,\delta I(\phi^{n}) \tag{7}$$

where n indicates the iteration and the optimum ρ may be obtained in terms of the Residual and the Hessian.

Many nonlinear elliptic problems are solved iteratively by casting them in Poisson's form, where nonlinearity acts as a driving force (incompressible sources) =

$$\delta\phi_{xx} + \delta\phi_{yy} = - \omega R(\phi^{n}) \tag{8}$$

and where R is the Residual and ω is a relaxation parameter.

Argyris (ref. 5) calculated compressible subsonic flows by the Galerkin method and obtained impressive results within a few iterations. Similar applications were reported by Gelder (ref. 6), Norrie and DeVries (ref. 7), Periaux (ref. 8), and Chan and Brashears (ref. 9).

Hyperbolic Problems

Finite element methods were also developed for approximate solutions of initial value problems. Both variational and weighted Residual methods were used (see refs. 10 - 16). Most of these investigators used either finite element in space with finite difference in time, a quasi-variational principle, or a convolution bilinear form. A variational formulation for initial value problems is not possible in the classical context of the calculus of variations. Consider the simple linear wave equation

$$L_h(\phi) = K\phi_{xx} + \phi_{yy} = f, \; K < 0 \tag{9}$$

where $\phi(x = 0)$ and $\phi_x(x = 0)$ are given as initial conditions and x is the time-like coordinate. Application of Hamilton's principle requires knowledge of the conditions at the beginning and end of a time interval and does not apply here. This is difficult because we persist in employing boundary value techniques to solve an initial value problem.

Contrary to the conventional shooting method (an initial value technique), which employs a marching (step-by-step) scheme to solve a boundary value problem, here we will solve the initial value problem by a formal application of Hamilton's principle. The success of the shooting methods depends on the assumption that a variation in the initial slope has a one-to-one correspondence with a variation in the end position; hence, the problem can be solved iteratively. At each iteration, only an initial value problem is solved. For linear problems, iterations may not be needed. The reverse of this process is valid if the same assumption holds, namely, initial value problems can be solved iteratively, with each iteration consisting of a boundary value problem. Again, iterations may not be needed for linear problems.

So, if we assume that the end value $\phi(x = X)$ is known instead of the initial slope $\phi_x(x = 0)$, the associated functional (potential and kinetic energy) would be

$$I(\phi) = \int_0^X \int_0^1 K\phi_x^{\,2} = \phi_y^{\,2} + 2f\phi \; dxdy \tag{10}$$

which can be discretized and expressed as a sum over finite elements. A basic requirement for application of Hamilton's principle is that we not vary the extreme positions of the physical system. The missing equation (the variation with respect to the end position) is replaced by an equation prescribing the variation with respect to the initial slope (see fig. 1).

Note that the second variation is not positive (stationary but not extremum), and there may be no advantage over weighted Residual methods with a sensible choice of suitable weighting functions. We note also that arrangement of the elements is not completely arbitrary, and sometimes the element size is restricted by stability requirements. For example, if we use linear shape functions on triangular elements, the algebraic equations for the nodal values are identical to those obtained by explicit centered difference schemes. These requirements arise because a hyperbolic system has characteristics (or preferred directions of propagation) and by just minimizing the energy, we have not taken these features into account. Implicit (unconditionally stable) schemes will be discussed below.

For many nonlinear hyperbolic equations, the following iterative procedure can be used:

$$- \alpha\delta\phi_{xx} + \delta\phi_{yy} = - \omega R(\phi^{n}) \tag{11}$$

where α is determined to guarantee convergence of iterations (the approximate domain of dependence contains the exact one).

TRANSONIC FLOWS

Consider the functional

$$I(\phi) = \iint \frac{1}{2} (K\phi_x^2 + \phi_y^2) - \frac{1}{6} \phi_x^3 dxdy - \int_{s_1} g\phi \, ds \tag{12}$$

Perturbing ϕ in any direction η(η is an admissible function)

$$I(\phi + \varepsilon\eta) = I(\phi) + \varepsilon \iint (K\phi_x - \frac{1}{2} \phi_x^2)\eta_x + \phi_y \eta_y dxdy - \int_{s_1} g\eta \, ds$$

$$+ \frac{\varepsilon^2}{2} \iint (K - \phi_x)\eta_x^2 + \eta_y^2 \, dxdy + \frac{\varepsilon^3}{6} \iint \eta_x^3 \, dxdy \tag{13}$$

Vanishing of the first variation gives

$$\iint (K\phi_x - \frac{1}{2} \phi_x^2)\eta_x + \phi_y \eta_y dxdy - \int_{s_1} g\eta \, ds = 0 \tag{14}$$

Applying Green's theorem, equation (3) becomes

$$\iint \left[(K\phi_x - \frac{1}{2} \phi_x^2)_x + (\phi_y)_y \right] \eta dxdy - \int_{s_1} (\phi_n - g)\eta \, ds = 0 \tag{15}$$

Note that the second variation is not always positive.

Iterative Procedures

For a nonlinear problem, we need a linearization procedure and a discretization technique. In general, they do not commute.

If we start by discretizing the integral expression, minimization will lead to a nonlinear system of algebraic equations to be solved iteratively (e.g., Newton's method). On the other hand, consider the sequence of functionals

$$I_n(\phi) = \iint (K-\phi_x^{n-1})\phi_x^2 + \phi_y^2 - \int_{s_1} 2g\phi ds \tag{16}$$

At each iteration, only a linear system of equations will be solved.

Discretization Procedures

The finite element method has been used to solve efficiently subsonic flow problems, with complex geometries employing nonrectangular elements, with a better approximation of the boundary conditions than finite differences. Although the matrix for the nodal values will not have the same regular structure as in finite differences, the number of unknowns is usually less (for higher order elements), and the matrix inversion procedure is different (banded Gaussian Elimination).

For transonic small disturbance theory, the streamlines are almost parallel to the x-axis, and the body boundary condition can be applied at $y = 0$. Moreover, in the supersonic bubble, x is the time-like coordinate, and the nodes may be located along x = constant lines. Finite differences suit the problem very well. The small disturbance simplifications eliminate the advantages of finite elements. The situation will be different, however, if the full potential equation is considered where the flow direction is unknown and if the exact boundary conditions are applied at the surface of the body.

Nevertheless, we will consider a simple example and use rectangular elements to study the feasibility of using a finite element approach to a mixed type equation. As a matter of fact, efficient finite difference schemes for elliptic and parabolic equations are constructed this way (see refs. 17 - 19).

Semi-Discretization

Let

$$\phi = \sum_{i=1}^{m} X_i(x)\ Y_i(y)$$

where m is the number of strips in the $y-$direction. The functional I_n becomes

$$I_n(\phi) = \sum_{i=1}^{m} \sum_{j=1}^{m} \left\{ K_{ij} \int_{x_1}^{x_2} X_i X_j \, dx + \int_{x_1}^{x_2} M_{ij} \frac{dX_i}{dx} \frac{dX_j}{dx} \, dx \right\}$$

where

$$K_{ij} = \int_{0\,y=y_F}^{y=y_F} \frac{dY_i}{dy} \frac{dY_j}{dy} \, dy$$

and

$$M_{ij} = \int_{y=0}^{y=y_F} (K-\phi_x^n) \; Y_i Y_j dy \equiv \int_{y=0}^{y=y_F} K_\ell Y_i Y_j dy \tag{17}$$

The kinetic and potential energies are

$$T = \frac{1}{2}\sum\sum M_{ij} \frac{dX_i}{dx} \frac{dX_j}{dx} \qquad V = \frac{1}{2}\sum\sum K_{ij} X_i X_j$$

The Euler-Lagrange equation reads

$$\delta \int (T-V)dx = 0 \;\left(\text{i.e., } - (M\dot{X})^{\cdot} + KX = 0\right) \tag{18}$$

where M and K are the mass and the stiffness matrices. Or, in the canonical form,

$$M\dot{X} = P \qquad \dot{P} = + KX \tag{18'}$$

where $X_i(x)$ must satisfy the essential boundary conditions. For local hyperbolic regions, the end value $X_i(x = x_2)$ will be replaced by an initial condition, $\frac{dX_i}{dx}$ $(x = x_1)$.

Full-Discretization

Instead of solving a system of ordinary differential equations along lines, we will consider different discretization procedures also in the x-direction.

<u>Finite Element in Space, Finite Difference in Time</u>. - If linear hat functions in y are used, M will be a triadiagonal matrix

$$-\frac{h}{6}K_\ell \begin{bmatrix} 1 & 4 & 1 \end{bmatrix} \quad \text{and} \quad K \text{ will read } \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}\frac{1}{h}$$

These two matrices will be modified by introduction of the boundary conditions.

In the x-direction, centered differences in the subsonic segment will give star A, as shown in figure 2, while backward differences in the supersonic segment will give star B. At the parabolic point P, K_ℓ is set equal to zero. At the shock point S the locally normal shock relation $\langle K-\phi_x \rangle = 0$ provides ϕ_x downstream of the shock and is used as a derivative boundary condition for the rest of the unknowns on the line.

Finite Element in Space and Time. - If linear hat functions in both y and x are used, both stars A and B will be the same as in figure 3.

Higher Order Shape Functions: Linear Hat Functions in y and Hermite Cubics in x. - The cubic polynomial on $0 \le x \le \Delta x$, which takes on the four prescribed values ϕ_o, ϕ_{x_o}, ϕ_1, and ϕ_{x_1}, is

$$\{X_i\} = H_{00}, H_{10}, H_{01}, H_{11} \begin{bmatrix} \{\phi_0\} \\ \{\phi_{x_0}\} \\ \{\phi_1\} \\ \{\phi_{x_1}\} \end{bmatrix}$$

with

$$\begin{aligned} H_{00} &= 2\theta^3 - 3\theta^2 + 1 \\ H_{01} &= -2\theta^3 + 3\theta^2 \\ H_{10} &= (\theta^3 - 2\theta^2 = \theta)\Delta x \\ H_{11} &= (\theta^3 - \theta^2)\Delta x \\ (\theta &= \frac{x}{\Delta x}) \end{aligned} \tag{19}$$

In the subsonic region, the contribution of the neighboring element will be included through the assembly of the elemental expression into the global system (see figure 4). In the supersonic region, the stationary value with respect to $\phi_x(0)$ and $\phi_x(\Delta x)$, assuming $\phi(0)$ and $\phi(\Delta x)$ are known, will give two algebraic equations that will be used to solve for $\phi(\Delta x)$ and $\phi_x(\Delta x)$ (according to the inverse shooting method described earlier), namely,

$$\int_0^{\Delta_x} \begin{Bmatrix} H_{10} \\ H_{11} \end{Bmatrix} \left(\frac{\partial}{\partial x} \left(M \frac{\partial}{\partial x} \right) + K \right) \{X\} \quad dx = 0 \tag{20}$$

or

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \{\phi_1\} \\ \{\phi x_1\} \end{bmatrix} = - \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \{\phi_0\} \\ \{\phi_{x_0}\} \end{bmatrix} + \begin{bmatrix} \{f_0\} \\ \{f_1\} \end{bmatrix} \tag{20'}$$

Note, no upstream effect is allowed in the supersonic region.

Nonrectangular Elements. - All the previous approximates were special cases of tensor products. To relax this restriction, consider

$$\phi = \sum_i \phi_i N_i(x,y) \tag{21}$$

where N_i are the global shape functions. (For example, the isoparametric element with four nodes, where ϕ, ϕ_x, and ϕ_y are given at each node, curved boundaries are allowed with the restriction that the nodes in the supersonic region lie on x = constant lines.)

Element Equations and Assembly Procedures

For simplicity, consider a bilinear element with four nodes:

$$\phi_e = a + bx + cy + dxy$$

The coefficients a, b, c, and d are given in terms of the four nodal values. (If the elements were rectangular, this case would reduce to the tensor product of linear hat functions in x and y.) If we consider the element equations rather than the nodal equations, the usual finite element assembly procedure in the supersonic region must be modified according to the inverse shooting method, as shown in figure 5.

The transition between the elliptic and hyperbolic parts of the flow is achieved by introducing blending elements between different regions. Two such elements are used: one for the sonic line; one for shock waves.

Sonic Elements

For sonic elements, the average of $(K-\phi_x^n)$ is set to zero. These elements act as a "buffer zone" between subsonic and supersonic elements. We can show that the system matrix will be positive definite if the above assembling strategy is adopted and if the sonic element is included.

Shock Elements

In transonic small disturbance calculations by finite differences, shocks are either captured (using artificial viscosity) or fitted (as a discontinuity). The artificial viscosity term required to smooth out the discontinuity is usually of the same order as the mesh size (because of large, but finite, gradients of the solution in the shock region, even if higher order schemes are used). The same comment holds for finite elements. On the other hand, the discontinuous finite element approximation of shock waves proved to be efficient in nonlinear elasticity (see ref. 20). Here we will describe a finite element analogue for the shock fitting procedure used by Hafez and Cheng (ref. 21).

Consider a shock element, as shown in figure 6. The Rankine-Hugoniot relations under the transonic small disturbance assumptions are given in equations (13) and (14).

The first relation can be derived actually from the weak solution admitted by TSDE, while the second is consistent with the irrotationality condition, which is equivalent to $[\![\phi]\!] = 0$. The equation for the nodal value at $i - 1$ will not be affected. The equation at i, however, will be different since only the contribution of segment II downstream of the shock will be considered. To the first order of accuracy, knowing ϕ_{i-1} and ϕ_{i-2}, we know the condition upstream of the shock. X_D can be determined according to relation (2) and in terms of ϕ_i . The righthand side $(dx/dy)^2$ may be evaluated from a previous iteration as the average of the slope of the shock in the adjacent elements. If this term is neglected, the scheme will reduce to the shock point operator, as discussed earlier. The compatibility relation (3') is satisfied by using linear shape functions in upstream and downstream segments. Thus, (in finite difference calculations) the introduction of shock relations will not make the system matrix singular or disturb the convergence of iterations.

As an alternative approach, instead of altering the nodal equation at the shock point to admit the jump in ϕ_x between i and i-1, according to equation (1), we may use the divergence theorem to obtain an integral relation as a conservation of mass over the element. The element equation will read

$$\iint \nabla\cdot\vec{g}\,dA = \oint \vec{g}\cdot n\,ds = 0 \tag{22}$$

$$\vec{g} = \begin{bmatrix} K\phi_x - 1/2\ \phi_x^2 \\ \phi_y \end{bmatrix}$$

Bilinear shape functions in I thru IV (fig. 6) may be used with a jump in ϕ_x across the shock. Similarly, the irrotationality condition (existence of potential) implies zero vorticity over each element and, by Stokes theorem, zero circulation, namely

$$\iint \nabla \times \vec{V}\cdot n\,dA = \oint \vec{V}\cdot d\vec{s} = 0 \tag{23}$$

where

$$\vec{V} = \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix}$$

So, as an alternative approach, relations (2) and (3) are replaced by relations (22) and (23).

REMARKS AND COMMENTS

Mixed Variational Principles

Note that higher order shape functions, namely, Hermite cubics, lead to equations (20) and (20') for ϕ and ϕ_x at the nodes. The resulting algebraic equations can be considered as finite difference approximations of two differential equations: the first is the TSDE (1), and the second is the x-derivative of the TSDE. Instead, the problem can be formulated in terms of two unknown functions ϕ and u , where ϕ is governed by the TSDE and u is governed by a compatability relation $u = \phi_x$. A mixed variational principle (in terms of ϕ and u), together with a dual iterative procedure for TSDE, is studied in a separate paper where the merits and the efficiency of the new method is assessed.

Weighted Residual Methods

Chan and Brashears (ref. 9) used least squares to solve TSDE. Straightforward application of the method fails (the solution diverges), so results can be obtained by changing the system matrix. The element matrices are constructed in the usual manner. Before assembling the element matrices into the system matrix, the rows corresponding to the nodes along the upstream side of any element in the supersonic zone are zeroed out; hence, no upstream effect is allowed there. Applying a similar procedure using the Galerkin method and cubic elements in the x-direction gives

$$\int_0^{\Delta x} \begin{Bmatrix} H_{01} \\ H_{11} \end{Bmatrix} \left\{ \frac{\partial}{\partial x} \left(M \frac{\partial}{\partial x}\right) + K \right\} \left\{ X \right\} dx = 0 \qquad (24)$$

or

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \{ \phi_1 \} \\ \{ \phi_{x1} \} \end{bmatrix} = - \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \{ \phi_0 \} \\ \{ \phi_{x0} \} \end{bmatrix} + \begin{bmatrix} \{ f_0 \} \\ \{ f_1 \} \end{bmatrix} \qquad (24')$$

Note that equations (24) and (24') differ from equations (20) and (20') since different weighting functions are used.

Type-Insensitive Methods

In our method, a different strategy is adopted in subsonic and supersonic regions. A unified, type-insensitve method may be simpler, but not efficient, since different requirements in each region must be satisified simultaneously.

To obtain such a procedure, the steady problem is embedded in a higher dimensional space, where the problem is more amenable for analysis. The extra dimension may have a physical meaning, as in the unsteady (time-dependent) method or may be just a mathematical trick, like the use of complex characteristics or any parameter as in the method of parametric differentiation. Also, extra dependent variables may be used, as in the mixed variational principle. The usefulness of these imbedding techniques depends on how fast the limit solution will be obtained. As an example of a unified procedure, consider the TSDE in the form of a system of first order equations,

$$K_\ell u_x = v_y = f \qquad K_\ell = K-u$$

$$u_y - v_x = g$$

or

$$\begin{pmatrix} K_\ell & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} f \\ g \end{pmatrix}$$

For cases where K_ℓ was a linear function of y, Friedrichs (ref. 22) and Chu (ref. 23) found a transformation that put this system into a positive symmetric form. As shown by Lesaint (ref. 24) and reported by Levanthal and Aziz (ref. 25), the finite element method can be applied successfully using this transformation. In general, however, such a transformation may not exist. Nevertheless, if the problem is considered as the asymptotic limit of an unsteady problem, where the vector $\begin{pmatrix} f \\ g \end{pmatrix}$ is replaced by $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_t$, the situation is different. The modified system is symmetric and hyperbolic. Unlike the equilibrium equations, for symmetric hyperbolic equations positivity could always be attained by a simple transformation, as shown by Friedrichs (ref. 22). For such a modified system, no special treatment for subsonic and supersonic regions is needed.

However, based on the finite difference calculations of the Euler equations, where centered differences are used everywhere in space, this "iterative" procedure may be slow. On the other hand, it seems that efficient applications of finite element methods to the full potential equation may require such imbedding techniques (artificial time-dependent and viscosity terms).

CONCLUSIONS

Applications of a finite element approach to transonic flow problems have been discussed. Only small disturbance equations with streamlines almost parallel to the x-axis (hence, the nodes are located along x = constant lines in the supersonic region) have been considered. Currently, computations of a simple numerical example are underway. Extension of this approach to the full potential equation is possible as long as the direction of the flow in the supersonic region is almost known a priori.

REFERENCES

1. Murman, E. M. and Cole, J. D.: AIAA J., vol. 9, 1971, p. 114.
2. Murman, E. M.: Proceedings of AIAA Computational Fluid Dynamics Conference, Palm Springs, July, 1973.
3. Martin, E. D. and Lomax, H.: AIAA Paper No. 74-11, 1974.
4. Jameson, A.: Proceedings of AIAA, Hartford, June, 1975.
5. Argyris, Y. H. and Mareczek, G.: Ingenieur-Archiv, Jan. 1973.
6. Gelder, D.: Int. J. Num. Methods Eng., vol. 3, 1971, p. 35.
7. Norrie, D. H. and DeVries, G.: The Finite Element Method, AP, 1973.
8. Periaux, J.: Int. J. Num. Methods Eng., vol. 9, 1975, p. 775.
9. Chan, S. T. K. and Brashears, M. R.: AFFDL-TR-74-11, Wright-Patterson Air Force Base, Ohio, March, 1974.
10. Zienkiewicz, O. C.: The Finite Element Method in Engineering Science, McGraw-Hill, 1971.
11. Oden, J. T.: Finite Elements of Nonlinear Continua, McGraw Hill, 1972.
12. Strang, G. and Fix, G. J.: An Analysis of the Finite Element Method, Prentice-Hall, 1973.
13. Noble, B.: Proceedings of the Brunel Univ. Conference of the Institute of Mathematics and Its Application, April, 1972.
14. Fried, J.: AIAA J., vol. 7, 1969, p. 1170.
15. Gurtin, M. E.: Archieve Ratl. Mech. Analy., vol. 16, 1964, p. 34.
16. Tonti, E.: Report, Instituts di Meccanica Razionale del Politechnico di Milano, Piazza da Vinci, Milano, Italy, 1971.
17. Swartz, B. and Wendroff, B.: Math. of Computations, vol. 23, no. 105, June, 1969.
18. Marchuk, G. I.: Methods of Numerical Mathematics, Springer-Verlag, 1975.
19. Greenspan, D. and Jain, P.: J. Math. Analysis Appl., vol. 18, 1967, p. 85.
20. Wellford, L. C. and Oden, J. T.: J. of Comp. Physics, vol. 19, no. 2, Oct. 1975, p. 179.

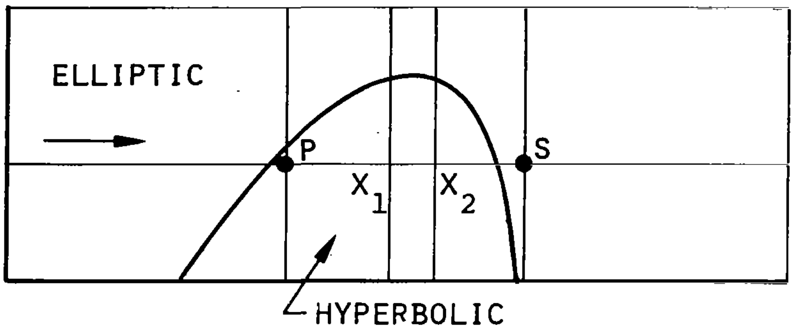
21. Hafez, M. M. and Cheng, H. K.: AIAA Paper No. 75-51, 1975.

22. Friedrichs, K. O.: Comm. Pure Appl. Math., vol. 11, 1958, p. 333.

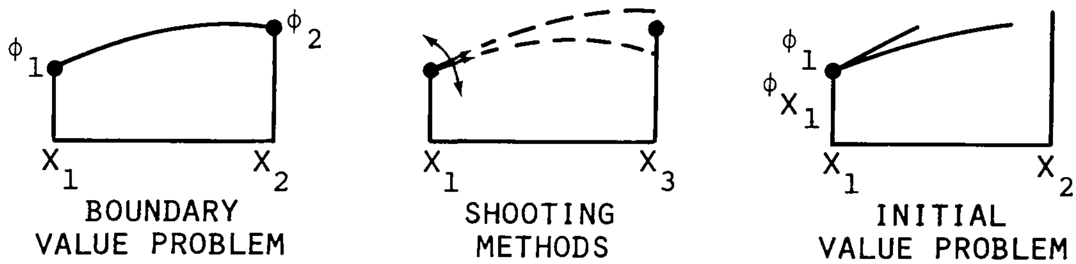
23. Chu, C. K.: Ph.D. Thesis, New York University, 1959.

24. Lesaint, P.: Numer. Math., vol. 21, 1973, p. 244.

25. Leventhal, S. H. and Aziz, A. K.: Proceedings of Symposium on the Numerical Solution of Partial Differential Equations, Maryland, 1975.



(a) Character of equations.



(b) Solution methods.

Figure 1.- Mixed flow problems.

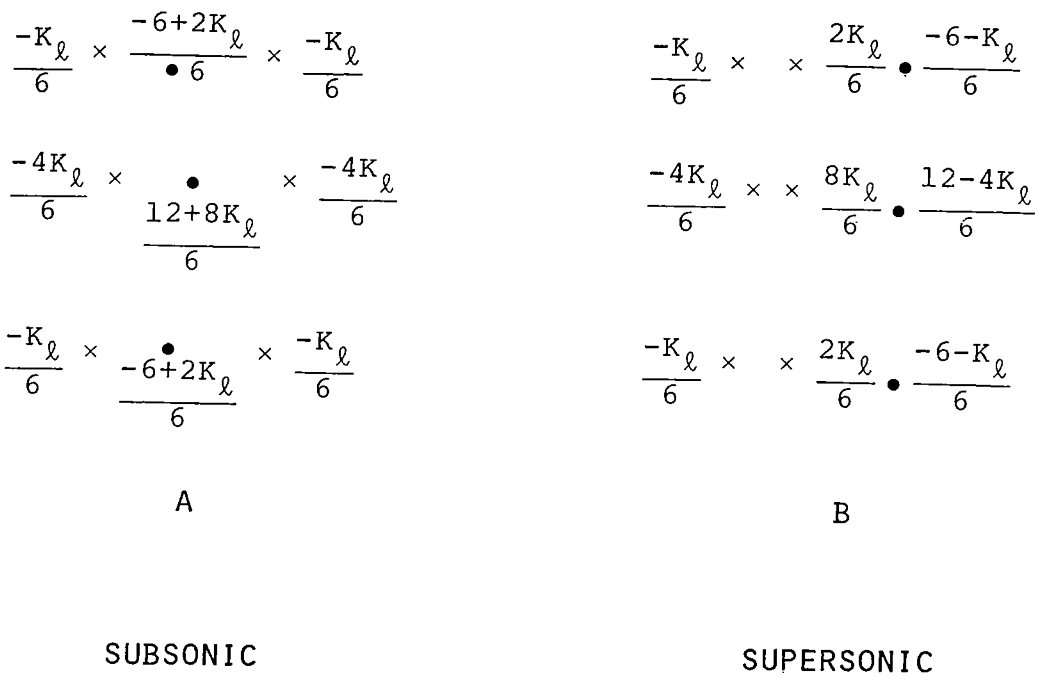


Figure 2.- Element for calculations using finite element in space, finite difference in time.

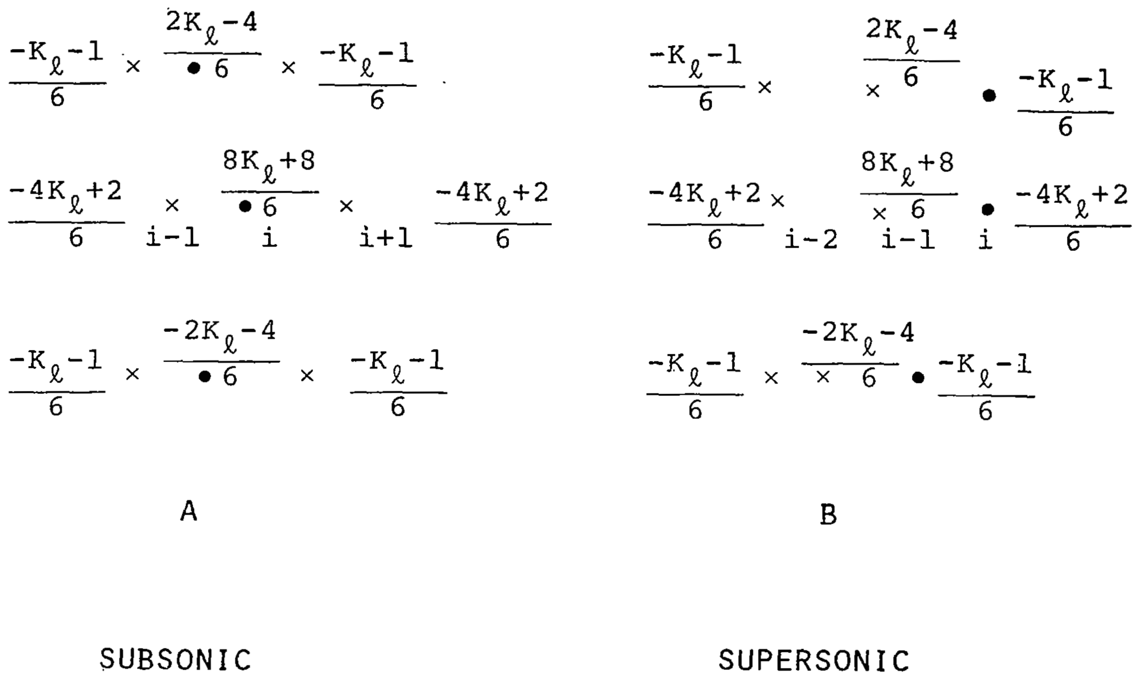


Figure 3.- Elements for calculations using finite element in space and time.

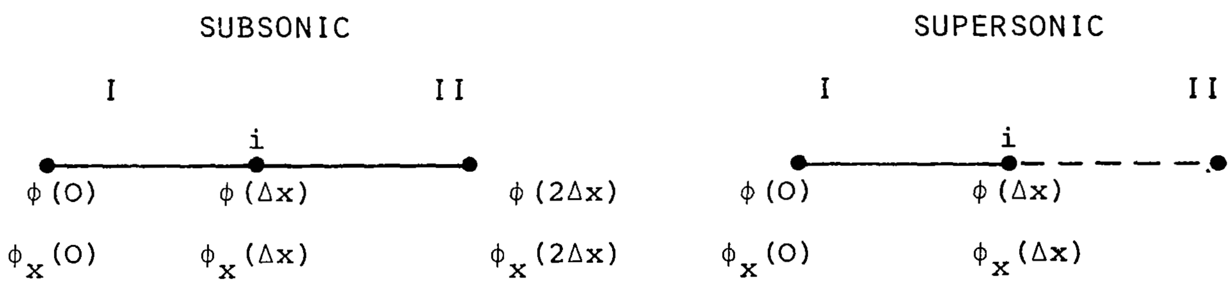


Figure 4.- Finite element calculations using higher order shape functions.

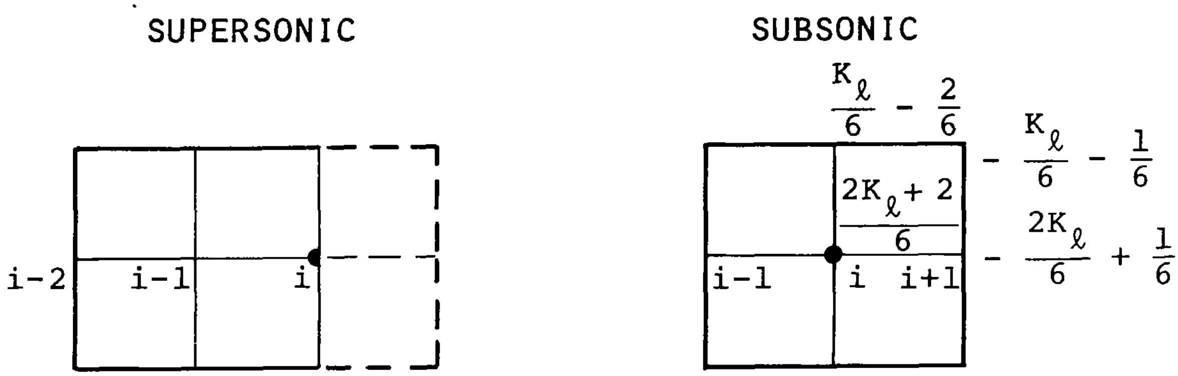


Figure 5.- Finite element assembly modified according to the inverse shooting method.

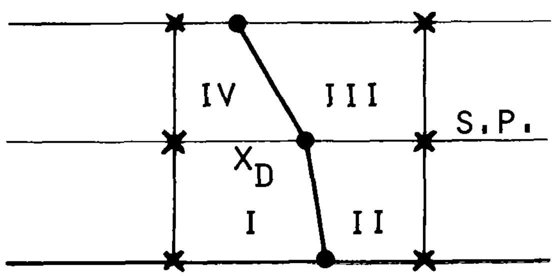


Figure 6.- Shock element in finite element scheme.