SPACE SCIENCES LABORATORY

NASA Contract NAS 8-31346

Feasibility of Coherent and Incoherent Backscatter Experiments From the AMPS Laboratory

FINAL REPORT

Technical Section

Principal Investigator

Prof. Forrest S. Mozer

April, 1976

Series 17 Issue 35

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TABLE OF CONTENTS

PART I

Introduction

In order to determine the feasibility of the space shuttle radar it was necessary to implement a computer simulation program at the University of California, Berkeley. This program simulates the spectrum which results when a radar signal is transmitted into the ionosphere for a finite time and received for an equal finite interval. The spectrum derived from this signal is statistical in nature because the signal is scattered from the ionosphere, which is statistical in nature. Thus, any estimator which uses the backscattered spectrum has a statistical character. That is, many estimates of any property of the ionosphere can be made. Their average value will approach the average property of the ionosphere which is being measured. However, due to the statistical nature of the spectrum itself, the estimators will vary about this average. The square root of the variance about this average is called the standard deviation, an estimate of the error which exists in any particular radar measurement. In order to determine the feasibility of the space shuttle radar, the magnitude of these errors for measurements of physical interest must be understood.

A complete reading of this report makes it possible to implement the computer simulation described herein. If such an implementation is not desired, reading of Part II will give a good understanding of the physical principles involved in implementation of the program and a reasonable overview of the mathematical tools used.

PART II

Theoretical Considerations and Computer Operation

The computer program simulates a spectrum having statistical properties which are estimated in frequency space. Before explaining exactly how the computer performs this simulation, it is necessary to discuss the physical ionosphere that is being simulated. This ionosphere is time stationary. This means that at any particular point in the ionosphere being simulated, a long time average will yield a well defined number for any physical property being measured. It also means that an ensemble average of many ionospheres will yield the same results as a long time average of any particular ionosphere being studied. The ionosphere is assumed to be uniform throughout the volume scattering signals back to the radar. This does not mean that there are no statistical fluctuations within the ionosphere, but, rather, that the point properties measured anywhere in the ionosphere possess the same long time average. Thus, although at any particular time there may be density fluctuations in the ionosphere, on the average the density at any point in the ionosphere is identical with the density of any other point from which scattered signal is being returned to the radar. The real ionosphere is not uniform. It is often true that over the time period during which data from the radar are being averaged or over the spatial region that one pulse of the radar covers, the ionosphere varies in its properties. This type of non-uniformity is not considered in this study, and the uniformity assumption should be

considered in extrapolating the data from the simulation to real situations in the ionosphere.

In addition to the uniformity of the scattering medium, the computer simulation is idealized in another respect. It is assumed throughout this study that the radar itself is ideal. For example, no asymmetries of the received filters are simulated. Nor is it assumed the receiver gain varies with time due to saturation by the transmitted signal, although both of these problems exist in real radars.

Consider a time stationary ionosphere in which density fluctuations can exist, and compare it with an ionosphere in which no density fluctuations exist. Assume that a signal with wavelength, λ, long by comparison to the Debye length and frequency, ω, high by comparison to the plasma frequency is transmitted into a fluctuationless ionosphere. There will be no return backscattered signal into the radar receiver. This can be shown as follows. Consider two planes in the ionosphere perpendicular to the vector of the incident radar wave. For convenience call the two planes P1 and P2 and allow the first of these to be at a distance d from the radar and the second at a distance $d + \lambda/4$. During the trip from radar to scattering plane and back, the scattered signal from the first plane travels a distance, 2d, while the scattered signal from plane P2 travels a distance $2d + \lambda/2$. The signals from these two planes are out of phase with one another and will cancel at the radar receiver. Now consider any plane, P3, in the medium from which scattered signal is

expected. It is possible to pick another plane, P4, which is related to P3 in the same way that P1 and P2 were related. In this way it can be seen that no scattered signal can return to the radar if the ionosphere is fluctuationless.

Now consider the time stationary ionosphere in which fluctuations exist. Consider a plane P1 and another plane P2 as before, but which have larger densities than the surrounding medium. These planes are now chosen a distance $\lambda/2$ apart. From the same arguments as in the previous paragraph, the two signals are in phase when scattered back to the radar and add, yielding a net return signal. It is now not possible to choose other identical planes in the medium that have the same density to cancel the signal already returned to the radar from the planes P1 and P2. Therefore signals are scattered from a medium in which density fluctuations exist in contrast to a fluctuationless medium. As a specific example, air contains density fluctuations. The sky is blue because light is scattered from the density fluctuations in the atmosphere. It is not blue when transmitted through water, because density fluctuations in this medium are many orders of magnitude smaller than the density fluctuations in air.

Care should be taken when considering arguments like the preceeding one. Although the argument is a useful heuristic tool, it is not a rigorous mathematical proof. One difficulty presents itself immediately since planes of atoms do not scatter signals since they are of negligible thickness. In the above analysis, one must use a volume in the medium which has a

large extent perpendicular to the incident wave and is thin relative to the wavelength. Then all the atoms in this particular volume of interest scatter waves back to the radar receiver in a relatively coherent fashion analogous to the thin planes considered earlier. The two planes a distance $\lambda/2$ apart would then be considered the boundaries of a scattering volume. This is, in fact, the smallest volume which can backscatter a radar wave of length, λ. The density fluctuations within a scattering volume or the density fluctuations which have scale sizes $\lambda/2$ are those which scatter radar waves. This statement is not rigorously true if the thin planes being considered are thin relative to the Debye length. However, in the simulated case, the radar wavelength is long by comparison to the Debye length and a thin plane relative to the radar wavelength does not have to be small by comparison to the Debye length so that it is possible to select planes in the medium which scatter the radar waves coherently.

It can be shown that the amplitude of a wave scattered from one particular scattering volume in the ionosphere is a Gaussian variable with zero mean. To prove this, it is necessary to consider the density fluctuations within the scattering volume, which generally contains a large number of ions. In the realm of the computer simulation, a typical number of particles within one scattering volume might be on the order of 10^8. The density fluctuations in a scattering volume can be considered to be due to the random motions of 10^8 particles. The central limit theorem of statistics implies that these fluctuations are Gaussian about the mean density unless they are of such enormous magnitude that they

approach the actual average density of the medium. In this case, it would be impossible for the distribution to have Gaussian tails since a negative density is impossible. Given these considerations,

$$A = k \nabla n \tag{1}$$

where A is the amplitude of the signal returning from a scattering volume under consideration and ∇n is the density fluctuation within that volume. Since A is a scattering wave amplitude and not a power, density waves within the scattering volume scatter incident radar waves coherently as can be seen from the arguments considering the planes P1 and P2. Since the volume scattering the wave has a particular size and position, the scattered wave considered in equation 1 has a specific wavelength and phase. This phase depends on the distance of the volume from the radar and upon the phase reference used -- a cosine wave or a sine wave. To understand this scattering further one can consider the phase of waves returning to the radar from scattering volumes. Since in the time stationary ionosphere no particular distance from the radar is preferred for scattering, the phase angle of waves scattered at a particular frequency is a uniformly distributed random variable. If the return signal s(t) is written

$$s(t) = A \cos \omega t + B \sin \omega t \tag{2}$$

for any particular frequency ω, the amplitudes A and B are Gaussian with zero mean since the density fluctuation in the scattering volume is

Gaussian with zero mean. Also, from the argument of uniform phase, A and B must have the same mean square. In other words, the power in the cosine component and the power in the sine component are equal on the average if no preferred phase exists in the return signal spectrum. It can also be stated that the A's and B's for each frequency component of the scattered wave are independent of each other in a statistical sense. These statements about the ionosphere may be summarized as follows:

$$\langle A^2 \rangle = \langle B^2 \rangle$$

$$\langle A\,B \rangle = 0 \qquad (3)$$

Consider again the limitations to equations 1, 2 and 3. One such limitation exists if the density fluctuation is on the order of the actual density in the scattering volume. It was stated that this is not an important restriction since a typical experimental radar does not receive signals from just one elementary scattering volume. Instead the radar averages over a relatively large portion of space which might contain on the order of 10^6 to 10^8 elementary scattering volumes. Even if each one of these volumes was itself coherent, i.e., a hard target, this would simply affect the magnitude of the signal scattered from the volume of the ionosphere which the radar was examining. The hard targets would fill the volume that the radar was observing in much the same manner that the small particles fill the much smaller scattering volume. Thus, for most kinds of coherent scatter, as well as incoherent scatter radars, the simulation being used in the program described here is

applicable. If the coherent target is as large as the volume of space being observed by the radar, the Gaussianess of the amplitudes and randomness of phases assumed in the simulation would no longer be valid. Thus far, no assumptions on the distribution of particle velocities within this ionosphere have been made. In the actual computer program it is assumed that the ionosphere is composed of drifting Maxwellians in which the electron and ion temperatures are allowed to differ. This does not affect the statistical distributions used in the computer but the fact that the ionosphere is of a Maxwellian character does affect the time average values of parameters of the spectrum. These time average values can be computed from theory once the particle distribution in the ionosphere is known. Given the known average values for the ionosphere parameters, the computer can simulate the fluctuations which must occur to produce these average values if the ionosphere is incoherent within a radar observation volume.

Now one can consider in more detail how the computer program models a signal scattered from a time-stationary uniform ionosphere. Figure 1a describes a particular input spectrum that is fed into the computer. This input spectrum is calculated from theory for the ionosphere described above for the following input parameters: electron temperature, T_e; ion temperature, T_i; bulk velocity, v_{bulk}; and density, n_e. These parameters determine the shape, width, and center of gravity of the spectrum shown in Figure 1a. The spectrum is computed under the assumption that a radar pulse is transmitted for inifinite time into the ionosphere

and received for an infinite time by a radar receiver. Since this does not correspond to physical reality, the spectrum must be modified in the computer to look like that of Figure 1b, which describes a spectrum that is spread because the radar pulse is finite in length. This pulse is not monochromatic and the spectrum which results when the pulse is scattered from the ionosphere is consequently broader than that which would result in the ideal case when only one frequency is present. In Figure 1c two spectra are shown. The first is an average spectrum of signal plus noise power. The second is an average noise spectrum. This average noise spectrum is computed from

$$\text{average noise power} = \text{average signal power/snr} \tag{4}$$

where snr = signal/noise ratio. Equation 4 refers to the total power in the signal and the total power in the noise. In order to create the noise spectrum of Figure 1c it is only necessary to assume that the total power is distributed equally among the frequency windows in the noise spectrum. This produces a flat spectrum which is consistent with the assumption of an ideal radar system. The signal plus noise spectrum is created by raising the signal on a pedestal so that the total power entering the receiver in any gate where signal is expected is the power of the signal plus the power of the noise.

The spectra that have been considered thus far are smooth, averaged spectra. An actual example of a single spectrum (Figure 1d) is selected at random in the computer by imposing statistical fluctuations on the

averaged spectrum of Figure 1c, to account for the previously discussed fluctuations of the ionosphere. In Figure 1e, the program takes account of the fact that the signal is received for a finite time. In the case of the particular simulation being considered here, the time of reception is equal to the time of transmission. This reception for a finite time causes spreading of the same nature as the spreading in Figure 1b that is caused by finite transmission time. In Figure 1f, the noise spectrum which entered the radar system from a portion of the ionosphere where no signal was expected, is subtracted from the signal plus noise spectrum which entered the radar system from a part of the ionosphere which is to be analyzed. What remains is a signal spectrum that represents one possible result of the physical experiment of transmitting a signal into the ionosphere for finite time and receiving the signal backscattered from the ionosphere for an equal finite time. This possible resulting spectrum can then be analyzed to determine the apparent parameters of the ionosphere which existed when the signal was scattered from it.

It should be noted that the diagrams of Figure 1 are schematic in nature and they do not explain two important points. First, these diagrams do not indicate in what way spreading of the signal is accomplished in the computer. Second, they do not explain how statistical principles are applied to produce the fluctuations in the spectrum of Figure 1d.

To clarify these points, one must digress to consider the autocorrelation function of the signal voltage $s(t)$, which is defined as

$$r(\tau) = \frac{1}{T}\int_0^{T-\tau} s(t)\, s(t+\tau)\, dt \tag{5}$$

where T is the sampling interval and τ is the lag time. To illustrate the properties of an autocorrelation function further, consider the specific function,

$$s(t) = A \cos(\omega t + \varphi) \tag{6}$$

For this s(t),

$$r(\tau) = \frac{a^2}{2T}\int_0^{T-\tau} \cos(2\omega t + \omega\tau + 2\varphi)\, dt + \frac{a^2}{2}\left(1 - \frac{\tau}{T}\right)\cos\omega\tau \tag{7}$$

As $T \to \infty$, equation 7 becomes

$$r_\infty(\tau) = \frac{a^2}{a} \cos\omega\tau \tag{8}$$

In the finite limit when it is assumed that the phase of s(t) is uniform and that therefore the phase term in equation 7 averages to zero one obtains

$$r(\tau) = \frac{a^2}{2}\left(1 - \frac{\tau}{T}\right)\cos\omega\tau \tag{9}$$

Equations 8 and 9 yield

$$r(\tau) = \left(1 - \frac{\tau}{T}\right) r_\infty(\tau) \tag{10}$$

This equation states that if the autocorrelation function is known in the limit where s(t) exists forever, it may be obtained for a signal which exists over a finite sample interval by multiplying by a factor $(1 - \tau/T)$. This multiplication factor looks like a ramp when graphed and the process

of multiplying by it is often called triangulation for this reason.

Triangulation is useful because the autocorrelation function and a signal spectrum are Fourier transforms of each other. To investigate this relationship, consider a signal of the form

$$s(t) = \sum_{n-1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \tag{11}$$

where

$$\omega_n = 2\pi n/T \tag{12}$$

and T is the length of the sampling interval. Consider a lag product of the form

$$s(t_1)s(t_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big[a_m a_n \cos \omega_m t_1 \cos \omega_n t_2 + a_m b_n \cos \omega_m t_1 \sin \omega_n t_2 + b_m a_n \sin \omega_m t_1 \cos \omega_n t_2 + b_m b_n \sin \omega_m t_1 \sin \omega_n t_2\Big] \tag{13}$$

$$\langle a_m b_n \rangle = 0 \text{ for all m and n} \tag{14}$$

$$\langle a_m a_n \rangle = \langle b_m b_n \rangle = 0 \text{ for } m \neq n \tag{15}$$

$$\langle a_n^2 \rangle = \langle b_n^2 \rangle \tag{16}$$

Then

$$\langle s(t_1)\, s(t_2)\rangle = \sum_{n=0}^{\infty} \langle a_n^2 \rangle \cos \omega_n \tau = r(\tau) \tag{17}$$

where

$$\tau = t_2 - t_1 \tag{18}$$

Equation 17 is the Fourier series equivalent for a real symmetric function of the statement that the autocorrelation function is the Fourier transform of the power spectrum. Notice that both the spectrum and the autocorrelation function are averaged. That is, the average value of the autocorrelation function is the Fourier transform of the average value of the spectrum. This is realized because of equations 14, 15, and 16, which are the generalization of equation 3. If equations 14, 15, and 16 do not hold, then the autocorrelation function of equation 17 is not the Fourier transform of the power spectrum, and furthermore, equation 17 has an explicit time dependence. This can be seen if the algebra leading to equation 17 is performed without assuming equations 14, 15, and 16. If the autocorrelation function is an explicit function of time, that is depends on something other than the time difference $t_2 - t_1$, then the process being considered cannot be a time stationary process. By its nature a time stationary process must be independent of the time at which one begins to measure physical phenomena associated with it. Another way of stating the same result is that if the autocorrelation function is a function of time, then the parameters of the ionosphere must be changing during the time that experiments are being performed. Such a situation is

contrary to the assumptions made earlier about the characteristics of the ionosphere.

To summarize, if equations 14, 15, and 16 are valid, the autocorrelation function is not a function of time and is the Fourier transform of the power spectrum. If on the other hand equations 14, 15, and 16 are invalid, the autocorrelation function is a function of time, is not necessarily the Fourier transform of the spectrum, and in general the process being described cannot be considered time stationary. Thus, it has been proved that for a time stationary process equations 14, 15, and 16 must hold and the autocorrelation function is a useful function to describe the spectral Fourier transform.

From the above analyses, it becomes possible to spread a signal in the following manner. If one has the spectrum of a signal which exists for an infinite time and which should be spread because it will be sampled for only a finite time, one first derives the autocorrelation function of the signal by Fourier transformation. Next, the Fourier transform is triangulated. Then the triangulated autocorrelation function is retransformed into frequency space to produce a spectrum equivalent to that which would have been derived from a signal existing for a finite time. The details of this equivalence are considered in the appendix and an example of a spectrum before and after spreading in this manner is given in Figure 2.

Now that the methods used for spectral spreading in the computer have been discussed, it is necessary to understand exactly how the

computer produces spectra with the proper statistics to simulate the signal scattered from the ionosphere. Such a spectrum, estimated from the N samples obtained of the waveform s(t), will contain N/2 points because the Fourier transform of these N points produces N/2 amplitudes of cosine terms and N/2 amplitudes of sineterms. The power at any frequency is the sum of the squares of the amplitudes of the cosine and sine terms at that frequency. Hence, the spectral power in the ith frequency window is proportional to $\langle A_i^2 + B_i^2 \rangle$ and is independent of the power in any other frequency window since the A's and B's are independent of each other for different frequencies.

Statistics are generated in the computer by use of

$$u_n = \sum_{i=1}^{n} a^2 , \tag{19}$$

where the a_i's are Gaussian variables with zero mean and are independent of each other. The quantity u_n of equation 19 is a chi square variable with n degrees of freedom and is used as follows. In any spectrum, such as that of Figure 1c, each spectral point, whether signal plus noise or noise, has a sine and a cosine component. These components are squared, added and averaged to form the power indicated. Since the A's and B's are independent Gaussian variables, each frequency point is the average of a chi-square variable with 2 degrees of freedom. Therefore, to select a particular possible spectrum from the ensemble of spectra generated from the scattering of signals from the ionosphere, it is only necessary to select each frequency point in Figure 1c from a distribution having the

correct average value to yield the power in that point and the statistics associated with a chi-square variable with 2 degrees of freedom. Such a selection is consistent with the physical case in which one pulse is transmitted into the ionosphere for a finite time and received for an infinite time. If one wished to generate statistics consistent with a case in which many pulses were sent into the ionosphere, received, and the data from these pulses were averaged, one could assign more degrees of freedom to the distribution for each spectral point. In particular, if P pulses are transmitted into the ionosphere and their return spectra averaged, then each independent frequency point would have 2P degrees of freedom. This is because each pulse yields data which is independent of the other pulses and, therefore, the number of independent variables being summed with P pulses are averaged is equal to 2P.

In summary, then, the average spectra in Figure 1c are used to generate spectra with the correct statistical behavior in the following way. First, the number of pulses transmitted into the ionosphere must be known. Second, each real frequency point is considered to be independent of every other real frequency point. Third, for each real frequency point a random number is selected from a chi-square distribution with 2P degrees of freedom and possessing an average value equal to the average power in that frequency window. This then generates the spectrum which has a statistical behavior consistent with that of signals being scattered from the uniform, time-stationary ionosphere already described.

PART III

Data Presentation

Two estimators are evaluated below. The first is an estimator of the ionospheric density and the second is an estimator of the bulk velocity of the ionosphere, which is equivalent to an estimation of the Doppler shift of the backscattered spectrum. These quantities are analyzed since they are examples of one relatively easy and one relatively difficult parameter to measure with reasonable accuracy using a Space Shuttle radar. Before presenting data from the computer simulation, it is necessary to understand how an actual error bar is computed from generation of many spectra such as that described in the previous section. Each such spectrum is analyzed by use of the estimator being studied with each estimate made by the estimator of interest being stored in the computer. The result of 1000 such estimates for a particular set of input parameters is presented in the histogram of Figure 4. In the simulation which produced this curve, a spectrum with the parameters and the signal-to-noise ratio listed in the figure was given to the computer as input data. The computer then produced the given histogram having an apparently Gaussian shape and a maximum at a mean value of almost 200 meters/second. The actual mean value estimated by the computer is 196 meters/second and the standard deviation is 32 meters/second.

Now that the process producing the standard deviation or error bar has been described, the simulation estimates of errors in density and line

of sight velocity will be discussed. It is possible to develop a theoretical equation for the fractional error of the density, $\Delta n/n$ in the following way.

$$\frac{\Delta n}{n} = \Delta \frac{p_t}{p_t} \tag{20}$$

where p_t is the total power, the fractional error in the density may be obtained from the fractional error in the return signal power. To find the fractional error in the total power consider one point in frequency space, whose power is p. If the point contains signal and noise power then

$$\langle p_{s+n} \rangle = c\,(1 + \frac{1}{snr}) \tag{21}$$

where snr is the signal to noise ratio and c is a constant that represents the average signal power. Hence

$$\langle p_n \rangle = \frac{c}{snr} \quad ; \qquad \langle p_s \rangle = c \tag{22}$$

Equations 21 and 22 describe every point of a real signal plus noise and noise spectrum if both spectra are considered to be flat. Both the signal plus noise point and the noise point are chi-square variables with 2 degrees of freedom. For this type of distribution it can be shown that

$$\Delta u^2 = \langle u^2 \rangle = \langle u \rangle^2 \tag{23}$$

where u is a chi-square variable with 2 degrees of freedom, and Δu^2 is the variance. Since the variances of independent quantities are additive

$$\Delta (u_1 - u_2)^2 = \Delta u_1^{\,2} + \Delta u_2^{\,2} \tag{24}$$

if u_1 and u_2 are independent variables.

In order to obtain a point in the final signal spectrum as in Figure 1f it is necessary to take a point in the signal plus noise spectrum and subtract from it a point in the noise spectrum. From equations 21 through 24

$$\Delta(p_{s+n} - p_n)^2 = c^2\left[\left(1 + \frac{1}{snr}\right)^2 + \left(\frac{1}{snr}\right)^2\right] \tag{25}$$

From equations (25) and (22)

$$\Delta(p_{s+n} - p_n/\langle p_s\rangle = \Delta p_f/\langle p_s\rangle = \left[\left(1 + \frac{1}{snr}\right)^2 + \left(\frac{1}{snr}\right)^2\right]^{\frac{1}{2}} \tag{26}$$

where p_f is the final power in the particular point being considered.

Equation 26 applies to the power in only one point. However, there are N/2 independent frequency points. Each one of these points contributes to the power whose estimate is therefore improved by a factor of $1/\sqrt{N/2}$. In addition, when P pulses are transmitted and the data from the different pulses are averaged to yield a spectrum, another improvement by a factor $1/\sqrt{P}$ is accomplished. Thus, the final form for the fractional error in signal power, or from equation (20), for the fractional error in the density estimate is

$$\frac{\Delta n}{n} = \left[\left(1 + \frac{1}{snr}\right)^2 + \left(\frac{1}{snr}\right)^2\right]^{\frac{1}{2}} / (PN/2)^{\frac{1}{2}} \tag{27}$$

Since this formula was derived under the assumption that the signal spectrum is flat and since signal spectra of interest are not flat, the actual errors are expected to be somewhat greater than those predicted by this

theoretical formula. This is because all of the points do not contribute equally to the variance of the power. To give an extreme example, if all of the power in a signal were concentrated in one point, then there would be no enhancement of the accuracy by a factor $1/\sqrt{N/2}$ as is assumed in equation 27. However, for most spectra, the deviation of the signal spectrum from flatness is expected to increase errors by a factor of only about 10%. Thus, the theoretical formula for fractional density error derived in equation 27 is expected to be accurate over wide ranges of signal to noise ratio and numbers of pulses.

To verify this statement, consider Figure 5 which is a plot of $\Delta n/n$ estimated for a range of ionospheric parameters by the computer simulation versus that given by equation 27. The dots of this figure represent a high temperature case having an electron temperature of 3500° and an ion temperature of 2500° while the crosses represent a low temperature case with equal electron and ion temperatures of 1000°. If a case where the signal to noise ratio and number of pulses is high were simulated in the computer, this produced a point lying near the origin corresponding to small values of $\Delta n/n$. The points representing small signal to noise ratios and small numbers of pulses lie in the upper right hand portion of the graph.

If the equation agreed perfectly with the simulation, every dot and every cross would lie on a line at an inclination of 45°. As can be seen from the figure, the dots do lie close to this line over the wide range of parameters used in the simulation. However, it should be noted that the

crosses lie consistently above the line, indicating that the theoreti... equation consistently underestimates the density error associated with them by a typical factor of about 10%. This result is explained by the narrowness of the low temperature spectra associated with the crosses.

In summary, the computer simulation for broad spectra show that equation 27 is an excellent predictor of the fractional error in the density estimate. For relatively narrow spectra with low electron and ion temperatures, it underestimates density errors by about 10%. Since the feasibility of a space shuttle radar does not depend on 10% corrections to such error estimates, equation 27 will be used to evaluate the performance of such a radar.

In Figure 6, contours of constant $\Delta n/n$ as determined by equation 27 are plotted as functions of the signal to noise ratio (ranging from 0.1 to 100) and number of radar pulses, P (ranging from 1 to 10^5). Again this graph is a log-log plot as in Figure 5. A general characteristic of the contours in this figure is their tendency to parallel the snr axis for signal-to-noise ratios greater than about 2. This break point at snr $\approx$ 2 implies that the accuracy of a density measurement is not greatly improved by making the signal-to-noise ratio much greater than 2.

The break points in the contours can be understood by considering that beyond some value, no further increase in the signal-to-noise ratio will produce a meaningful decrease in the number of pulses for the same value of $\Delta n/n$ because there is error associated with the signal power

even in the absence of noise. When a value of the signal-to-noise is reached where most of the received power is signal, most of the error associated with the measurement is due to the statistical fluctuations in the return signal and is unrelated to the noise. These statistical fluctuations exist because the signal is scattered from a statistically fluctuating medium.

It is thus concluded that there is little point in designing a radar system having a large signal-to-noise ratio. After an optimum signal-to-noise ratio is reached, one can decrease the error bars in a measurement only by increasing the number of independent samples taken. For purposes of discussion, an optimum signal/noise ratio is considered to be 2.

The analysis presented above yields some useful order of magnitude numbers. The density can be measured to an accuracy of about 10% with about 10 radar pulses if the signal-to-noise ratio is bigger than about 2. Density measurements to an accuracy of 1% require about 1000 pulses. If the radar operates at 100 pulses/second and the space shuttle moves through the ionosphere at a speed of 10 km/sec, a measurement requiring 10 pulses would occur over a distance of 1 km. A measurement requiring 1000 pulses would involve 100 km of spatial averaging, which might destroy the utility of some types of measurements.

The velocity estimator will be examined by developing an equation for the uncertainty in the velocity analogous to equation 27. Since, at least in some range, the velocity estimator might be linear in the density

estimator, an attempt was made to design an empirical equation which would fit the velocity errors as closely as possible, remains simple, and have a form relatively similar to that used for the density estimator. The equation which resulted from this process is

$$\Delta v\,(\mathrm{m/sec}) = \frac{1000}{(NP/2)^{\frac{1}{2}}}\left[1+\left(\frac{2}{P}\right)^{\frac{1}{2}}\right]\left[\left(1+\frac{1}{snr}\right)^{2}+\left(\frac{1}{snr}\right)^{2}\right]^{\frac{1}{2}} \qquad (28)$$

This equation was developed through analyses of data such as that presented in tables 9 and 10. Table 9 compares the signal-to-noise dependence of equation 28 with the actual dependence of the average data. The columns in this table are velocity errors estimated from the computer simulation for different pulse numbers and two different temperature cases. The rows in the table represent different signal-to-noise ratios. The last column is an average of all the numbers in a particular row, with values in excess of 475 m/sec dropped from the table. This was done because the computer program has a cutoff which will not allow it to estimate any velocity greater than 800 m/sec. This means that velocity errors which approach this value are progressively underestimated and these data were eliminated from the table. The table values are normalized to the error associated with a signal-to-noise ratio of 100. Comparison of the average signal-to-noise dependence with that predicted by equation 28 produces close agreement, even though there are uncorrected temperature dependences in the data of Table 9.

In Table 10 the pulse dependence in equation 28 is compared with that of the computer simulation. Here the error is plotted in units of $\Delta v_P/\Delta v_{22,500}$, where Δv_P is the velocity uncertainty for P pulses. Examination of this table indicates that the equation predicts the pulse dependence for large numbers of pulses and overestimates it for small number of pulses. By examination of the tables the statement that equation 28 predicts velocity errors to within 15% typically and in the worst case to 30% accuracy can be verified.

Figure 7 presents a graphical comparison of equation 28 and the velocity uncertainties estimated by the computer simulation. Note that the points do not lie as close to the 45 degree fit line as they did for the density errors of Figure 5. Further note that not all of the dots and crosses lie consistently on the line and that for example sometimes crosses lie below the line while other times they lie above the line. This shows that in the estimation of velocity there are temperature dependences which are not being considered here. Note, however, that the overall accuracy of the equation is typically 15 or 20%. Since 15 or 20% errors in the uncertainty of the velocity estimate are small enough to be neglected when obtaining rules of thumb for operation of the space shuttle radar, such errors will be neglected.

Contours of constant velocity uncertainty are presented in Figure 8 as functions of the signal-to-noise ratio and number of radar pulses. These contours show many of the features of the density equation contours since the equations which produced both sets of contours are relatively

similar. In both curves, the break point for improvement of accuracy versus signal-to-noise occurs at a value of about 2 for all contours. For signal-to-noise ratios in excess of this value, one can obtain an accuracy of 100 m/sec in the velocity estimate with approximately 15 pulses, which involve a spatial averaging of about 1.5 km. If one wished to obtain an accuracy of 10 m/sec, one would need to collect approximately 1500 pulses. This would indicate spatial averaging over an area of perhaps 150 km, which may be too large for many classes of experiment. Thus, for experiments in the auroral zone, where typical bulk velocities of the ionosphere are much larger than 100 m/sec, the space shuttle radar would be a useable tool for analysis. In the equatorial zone, however, where typical velocities are less than 100 m/sec, the error in the measurement would be comparable to the magnitude of the velocity being measured unless one averaged over a large volume of space. Therefore, space shuttle experiments in the equatorial zone designed to measure bulk velocities of the ionosphere are probably not feasible.

In summarzing the data the following conclusions can be made. Given reasonable pulse repetition rates (perhaps 100/sec) a sufficient signal-to-noise ratio (≥ 2), and a typical space shuttle speed (10 km/sec), one can make density measurements to the accuracy of 10% and velocity measurements to the accuracy of 100 m/sec, over small spatial intervals.

Certain cautions should be kept in mind when extrapolating from the data presented in part III. It must be remembered that the computer simulated a particular operating mode of the Chatanika radar. In this

mode the radar transmitted a pulse for 320 μsec and received a pulse for the same length of time. This corresponds to a range resolution of approximately 50 km. During the received pulse, 32 samples were taken, N = 32. This results in a bandwidth of 50 kHz containing 16 independent discrete frequency points. When one extrapolates from the data in part III these explicit dependences should be kept in mind if one wishes to do analyses for different range resolutions or different sampling rates.

APPENDIX

Throughout this appendix the following formalism will be followed. $s(t)$ will represent a continuous function. s_t will represent a discrete function existing only for specific values of t. It will also be assumed that the signal under consideration contains no d.c. nor Nyquist term, $a_0 = a_{N/2} = 0$. These terms are neglected because they complicate the mathematical formalism and add nothing to the understanding of discrete vs. continuous analysis.

I. Discrete Functions and Orthogonality

Beginning with continuous functions, one can study how the equations of Fourier analysis are modified as discrete sampling is introduced. If on an interval T

$$s(t) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \tag{1.1}$$

where $\omega_0 = 2\pi/T$, then for all n

$$a_n = \frac{2}{T} \int_0^T s(t) \cos n\omega_0 t \, dt \tag{1.2}$$

and

$$b_n = \frac{2}{T} \int_0^T s(t) \sin n\omega_0 t \, dt \tag{1.3}$$

These equations follow directly from equation 1 because of the orthogonality of sin and cos on the interval T. To form a discrete signal related to $s(t)$ one can divide the interval T into N intervals of length T/N, with

the first interval labeled as the 0 interval and the last interval labeled as N - 1. Imagine that at the beginning of each interval s(t) is sampled and its instantaneous value stored. This converts s(t) into a new function s_t with N discrete values on the sample interval. This new function has the value 0 except where

$$t = k \frac{T}{N} \tag{1.4}$$

$k = 0, 1, \ldots, N - 1$. s_t can be described mathematically as

$$s_t = s_{(kT/N)} = \int_0^T s(t)\, \Delta(t - kT/N)\, dt \tag{1.5}$$

or

$$s_{(k\,T/N)} = s(k\,T/N) \tag{1.6}$$

$k = 0, 1, 2, \ldots, N - 1$.

Once the interval length T and number of samples N are defined, equation 1.6 is a function of k alone. For this reason, the discrete signal is often called s_k.

At this point it is natural to inquire whether a signal of the type in equation 1.6 can be represented in a manner analogous to the representation of s(t) in equation 1.1. In fact, such a representation is possible but the proof of this point will be pursued indirectly. Before the proof can be understood, one must inquire into the nature of equations 1.2 and 1.3. These equations depend for their validity both on the representability of s(t) as in equation 1, and on the orthogonality relations for sin and cos terms in Fourier series.

The orthogonality relations in the continuous case are

$$\frac{1}{T}\int_0^T \cos m\omega_0 t \cos n\omega_0 t\, dt = \frac{1}{T}\int_0^T \sin m\omega_0 t \sin n\omega_0 t\, dt =$$

$$\tfrac{1}{2}\Delta_{m,n} ,$$

$$\frac{1}{T}\int_0^T \cos m\omega_0 t \sin n\omega_0 t\, dt = 0 \qquad (1.7)$$

for all m and n.

The orthogonality relations of equation 1.7 can be derived from the trigonometric identities and the fact that any sin or cos wave periodic on the interval T has integral 0 unless the term being considered is a constant, that is, unless the term being considered is a cos term with argument 0. As an illustration consider

$$\frac{1}{T}\int_0^T \sin m\omega_0 t \sin n\omega_0 t\, dt = \frac{1}{2T}\left[\int_0^T \cos (m-n)\omega_0 t\, dt - \int_0^T \cos (m+n)\omega_0 t\, dt\right] = \tfrac{1}{2}\Delta_{m,n} \qquad (1.8)$$

where m and n are positive integers.

If in the discrete case one were to discover that periodic functions summed over an interval behaved similarly to the periodic continuous functions one could prove the orthogonality relations at once.

If one considers an interval of length T divided into N parts then in accordance with equation 1.6 the orthogonality relations in equation 1.7 for discrete s_t become

$$\frac{1}{T}\sum_{k=0}^{N-1}\left[\cos m\omega_0 (k\,T/N)\cos n\omega_0 (k\,T/N)\right]\Delta t =$$

$$\frac{1}{T}\sum_{k=0}^{N-1}\left[\sin m\omega_0 (k\,T/N)\sin n\omega_0 (k\,T/N)\right]\Delta t =$$

$$\tfrac{1}{2}\Delta_{m-n,\,pN}$$

$$\frac{1}{T}\sum_{k=0}^{N-1}\left[\cos m\omega_0 (k\,T/N)\sin n\omega_0 (k\,T/N)\right]\Delta t = 0 \qquad (1.9)$$

Here, Δt equal T/N is the length of a sample interval and is analogous to the differential dt in the integrals of 1.7. From the definitions of Δt and ω_0 equation 1.9 can be written as

$$\frac{1}{N}\sum_{k=0}^{N-1}\cos(2\pi\, m\, k/N)\cos(2\pi\, n\, k/N) =$$

$$\frac{1}{N}\sum_{k=0}^{N-1}\sin(2\pi\, m\, k/N)\sin(2\pi n\, k/N) =$$

$$\tfrac{1}{2}\Delta_{m-n,\,pN}$$

$$\frac{1}{N}\sum_{k=0}^{N-1}\cos(2\pi\, m\, k/N)\sin(2\pi n k/N) = 0 \qquad (1.10)$$

where p is an integer.

To derive the orthogonality relations for discrete signals one needs only trigonometric identities and complete knowledge of f_q and g_q defined as

$$f_q = \frac{1}{N} \sum_{k=0}^{N-1} \cos 2\pi qk/N \ ,$$

$$g_q = \frac{1}{N} \sum_{k=0}^{N-1} \sin 2\pi \ qk/N \qquad (1.11)$$

To find f_q and g_q consider the relation

$$f_q + i_q = \frac{1}{N} \left(\sum_{k=0}^{N-1} \cos 2\pi qk/N + i \sin 2\pi qk/N \right) =$$

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{i2\pi qk/N} \qquad (1.12)$$

Rewriting of equation 1.12 yields

$$f_q + ig_q = \frac{1}{N} \sum_{k=0}^{N-1} \left(e^{i\,2\pi q/N} \right)^k \qquad (1.13)$$

By use of the algebraic identity

$$\sum_{k=0}^{N-1} x^k = \frac{1 - x^N}{1 - x} \qquad (1.14)$$

Equation 1.13 becomes

$$f_q + ig_q = \frac{1}{N} \ \frac{1 - \left(e^{i2\pi qk/N} \right)^N}{1 - e^{i2\pi q/N}} \qquad (1.15)$$

or

$$f_q + ig_q = \frac{1}{N} \ \frac{1 - e^{2\pi iq}}{1 - e^{2\pi iq/N}} \qquad (1.16)$$

This gives

$$f_q + ig_q = 0$$

$$q \neq pN \qquad p = 1, 2, \ldots \tag{1.17}$$

If q = pN, then both the numerator and denominator in equation 1.16 are 0, and one must use equation 1.12 with q = pN.

$$f_q + ig_q = \frac{1}{N} \sum_{k=0}^{N-1} \cos 2\pi pk + i \sin 2\pi pk = 1 \tag{1.18}$$

where q = pN and p and k are integers. Equating real and imaginary parts and combining the results of equations 1.17 and 1.18 yields

$$\frac{1}{N} \sum_{k=0}^{N-1} \cos 2\pi qk/N = 1$$

$$q = pN = 0 \text{ otherwise.}$$

$$\frac{1}{N} \sum_{k=0}^{N-1} \sin 2\pi qk/N = 0 \quad \text{for all integer } q. \tag{1.19}$$

These equations are analogous to the integral relations

$$\frac{1}{T} \int_0^T \cos m\omega_0 t \, dt = \frac{1}{T} \int_0^T \cos \frac{2\pi mt}{T} \, dt = 1$$

if m = 0 and 0 otherwise

$$\frac{1}{T} \int_0^T \sin m\omega_0 t \, dt = \frac{1}{T} \int_0^T \sin \frac{2\pi mt}{T} \, dt = 0 \quad \text{for all m} \tag{1.20}$$

Note here the periodicity in equation 1.19 which does not exist in equation 1.20. This leads to periodic behavior in the orthogonality relations equations 1.10 for discrete functions which does not exist in the case of

Fourier series intended to analyze continuous functions. This problem of periodicity will be discussed in the next section along with the problem of representability of a discrete function by a Fourier series.

II. Periodicity and Completeness in Frequency Space

This section contains explicit expressions for a_n and b_n for discrete functions analogous to equation 1.2 and 1.3. These expressions are derived using equation 1.10. However, because of the peculiar periodicity exhibited by equation 1.10, confusion about these results may occur unless efforts are made to understand in what way the relations for a_n and b_n are periodic.

If one writes expressions for discrete functions a_n and b_n naively, one obtains

$$a_n = \frac{2}{T} \sum_{k=0}^{N-1} s_{(kT/N)} \cos n\omega_0(kT/N) = \frac{2}{N} \sum_{k=0}^{N-1} s_k \cos 2\pi nk/N \qquad (2.1)$$

and

$$b_n = \frac{2}{N} \sum_{k=0}^{N-1} s_k \sin 2\pi nk/N \qquad (2.2)$$

Here

$$\omega_0 = 2\pi/T,$$
$$\Delta t = T/N \qquad (2.3)$$

where t is the length of the sampling interval, and N is the number of samples taken in the interval. Relations 2.1 and 2.2 give

$$a_n = a_{n+pN}$$

$$b_n = b_n + pN$$

$$p = 0, 1, 2.... \qquad (2.4)$$

and

$$a_n = a_{pN - n}$$

$$b_n = -b_{pN - n} \qquad (2.5)$$

Setting $p = 1$ gives

$$a_n = a_{N - n} \qquad b_n = -b_{N - n} \qquad (2.6)$$

It follows from equation 2.4 that if $a_0 \ldots a_{N-1}$ and $b_0 \ldots b_{N-1}$ are known, then a_n and b_n are known for all n. It follows from equation 2.6 that if a_n and b_n are known then a_{N-n} and b_{N-n} are known. Thus, one need know only $a_0 \ldots a_r$ and $b_0 \ldots b_r$ where r is the greatest integer less than or equal to $N/2$ in order to completely determine the frequency components a_n and b_n of s_k for all n.

Writing an expression analogous to equation 1.1 now yields

$$s_k = \sum_{n=0}^{\infty} a_n \cos n\omega_0 (kT/N) + b_n \sin n\omega_0 (kT/N) =$$

$$\sum_{n=0}^{\infty} a_n \cos \frac{2\pi nk}{N} + b_n \sin \frac{2\pi nk}{N} \qquad (2.7)$$

One may well ask at this point if it is necessary to extend the sum in equation 2.7 from 0 to ∞ in order to completely represent s_k. It is already known that the a's and b's repeat beyond the index r. If the functions in equation 2.7 multiplying the a_n's and b_n's repeat as do the coefficients a_n and b_n, then additional terms in the sum beyond r in equation 2.7 will contribute no new informabout about the functional form of s_k. To decide whether the sum in 2.7 is periodic, consider one term in that sum, the mth term, where m is between 0 and r. Call the contribution to the function s_k of this term, $s_{k,m}$. Then

$$s_{k,m+pN} = a_{m+pN} \cos \frac{2\pi(m+pN)k}{N} +$$

$$b_{m+pN} \sin \frac{2\pi(m+pN)k}{N} = a_m \cos \frac{2\pi mk}{N} +$$

$$b_m \sin \frac{2\pi mk}{N} = s_{k,m} \qquad (2.8)$$

Thus result follows from equation 2.4 and uses the same techniques in its derivation as were used in equation 2.4. Further,

$$s_{k,N-m} = a_{N-m} \cos \frac{2\pi(N-m)k}{N} + b_{N-m} \sin \frac{2\pi(N-m)k}{N}$$

$$= a_{N-m} \cos (2\pi - 2\pi mk/N) + b_{N-m} \sin (2\pi - 2\pi mk/N)$$

$$= a_{N-m} \cos \frac{2\pi mk}{N} - b_{N-m} \sin 2\pi mk/N$$

$$= a_m \cos \frac{2\pi mk}{N} + b_m \sin \frac{2\pi mk}{N}$$

$$= s_{k,m} \qquad (2.9)$$

Combining the results of equations 2.8 and 2.9 yields

$$s_{k,m} = s_{k,pN \pm m} \tag{2.10}$$

Equation 2.10 states that if $s_{k,m}$ is known for $0 < m < r$ then adding terms of the form on the right hand side of equation 2.10 to corresponding terms of the form on the left hand side of this equation for all values of m between 0 and r simply multiplies the sum in equation 2.7 by a constant but does not change the form of s_k. Thus, without loss of generality one can write

$$s_k = \sum_{n=0}^{r} s_{k,n} = \sum_{n=0}^{r} a_n \cos \frac{2\pi nk}{N} + b_n \sin \frac{2\pi nk}{N} \tag{2.11}$$

Generally, it follows from equation 2.10 that the limits on equation 2.11 can be written as

$$s_k = \sum_{n=pN}^{pN+r} a_n \cos \frac{2\pi nk}{N} + b_n \sin \frac{2\pi nk}{N} =$$

$$\sum_{n=r+1+pN}^{N-1+pN} a_n \cos \frac{2\pi nk}{N} + b_n \sin \frac{2\pi nk}{N} \tag{2.12}$$

where p is any integer. Equation 2.12 shows that, given a particular s_k, it is impossible to determine its frequency components. Specifically, for p = 0

$$s_k = \sum_{n=0}^{r} a_n \cos \frac{2\pi nk}{N} + b_n \sin \frac{2\pi nk}{N} =$$

$$\sum_{n=r+1}^{N-1} a_n \cos \frac{2\pi nk}{N} + b_n \sin \frac{2\pi nk}{N} \tag{2.13}$$

Since, $r \leq N/2$, equation 2.13 states that one cannot decide whether s_k has frequency components between 0 and $N\omega_0/2$ or between $N\omega_0/2$ and $N\omega_0$. Thus if one samples at a linear frequency $1/\Delta t = 1/(T/N)$ one can resolve components unambiguously only up to a frequency of $N\omega_0/2 \times 2\pi = N/2T = 1/2\Delta t$. Thus, the maximum frequency which can be unambiguously determined equals one half of the sampling frequency. This is a proof of Nyquist's theorem for discrete functions.

If on a fixed length sample interval $N \to \infty$ the number of a_n's and b_n's one must obtain before the coefficients repeat also approaches infinity. This is reasonable since in the limit $N \to \infty$ $s_k \to s(t)$, a continuous function. In other words as the structure of s_k becomes finer, rapidly oscillating sine and cosine terms must be added to the summation in equation 2.11 to represent the rapid fluctuations in the fine structure. A plausability argument for the fact that a finite number of terms is needed to represent a discrete function is that one would expect a function with a finite number of points to be representable with a finite number of functions which were independent on the interval of summation.

Now that it has been demonstrated that s_k need be represented with only a finite number of sine and cosine terms, one can show that every function s_k can be represented with a series of the form of equation 2.11, as follows. It has been noted that equations 1.19 and 1.20 are similar but that equation 1.20 possesses a periodic behavior. If one is limited to a series of the form of equation 2.11, the periodicity is removed from equation 1.20 and equation 1.19 and 1.20 imply identical orthogonality

relations for discrete and continuous functions. Thus, equation 1.10 becomes

$$\frac{1}{N}\sum_{k=0}^{N-1}\cos\frac{2\pi nk}{N}\cos\frac{2\pi mk}{N}=\frac{1}{N}\sum_{k=0}^{N-1}\sin\frac{2\pi mk}{N}\sin\frac{2\pi nk}{N}=\frac{1}{2}\Delta_{m,n} \tag{2.14}$$

$$\frac{1}{N}\sum_{k=0}^{N-1}\cos\frac{2\pi mk}{N}\sin\frac{2\pi nk}{N}=0 \tag{2.15}$$

$$\text{for } 0<m<r \text{ and } 0<n<r \tag{2.16}$$

In order to decide whether or not any function s_k can be represented in the form equation 2.11, it is only necessary to represent one particular point s_j of that function. The question of representability is equivalent to the question of completeness and can be stated in the following terms. If one computes the Fourier components of a function and then sums these Fourier components does one obtain the original function? If the answer is yes for all functions, then the set of Fourier components is said to be complete. In the case of the discrete function, if one can represent any one point in the set completely, one can represent any function by summing the representations for individual points by the principle of superposition. Let s_j be any point in the sample space. Define

$$s_k = c\Delta_{k,j} \tag{2.17}$$

which states that jth point on the sample interval has a value c.

$$a_{n,j} = \frac{2}{N} \sum_{k=0}^{N-1} c\,\Delta_{k,j} \cos \frac{2\pi nk}{N} \quad ,$$

$$b_{n,j} = \frac{2}{N} \sum_{k=0}^{N-1} c\,\Delta_{k,j} \sin \frac{2\pi nk}{N} \tag{2.18}$$

$$a_{n,j} = \frac{2c}{N} \cos \frac{2\pi nj}{N} \quad ,$$

$$b_{n,j} = \frac{2c}{N} \sin \frac{2\pi nj}{N} \tag{2.19}$$

Using equation 2.11

$$s_k = \sum_{n=0}^{r} \left[\frac{2c}{N} \cos \frac{2\pi nj}{N} \cos \frac{2\pi nk}{N} + \sin \frac{2\pi nj}{N} \sin \frac{2\pi nk}{N} \right] \tag{2.20}$$

Extending the interval of integration from 0 to N - 1 simply doubles s_k by equation 2.13. Therefore,

$$s_k = \frac{c}{N} \sum_{n=0}^{N-1} \left[\cos \frac{2\pi nj}{N} \cos \frac{2\pi nk}{N} + \sin \frac{2\pi nj}{N} \sin \frac{2\pi nk}{N} \right] \tag{2.21}$$

which gives

$$s_k = \frac{c}{N} \sum_{n=0}^{N-1} \cos \frac{2\pi n(j-k)}{N} \tag{2.22}$$

By equation 1.19

$$s_k = c\,\Delta_{k,j} \tag{2.23}$$

Equation 2.23 is the equivalent of the completeness relation for Fourier integrals. Its existence proves that s_k can be completely represented in the form equation 2.11.

Thus, a representation s_k in terms of a discrete Fourier series has been found. Further, this representation was found to be periodic in its coefficients. Lastly, it was proved that s_k can be completely represented by allowing the index in equation 2.11 to range from 0 to r only. If one inspects the number of terms thus included in the sum in equation 2.11, one discovers that the number of coefficients in the Fourier series is equal to the number of points to be represented. This is reasonable since all the coefficients multiply functions which are independent on the interval of summation, so that N independent points are represented by linear combinations of N independent functions.

III. An Analysis of Spreading by Use of Fourier Series

In sections I and II of the Appendix, the mathematical formalism used to derive amplitude spectra for discrete signals was developed. Next, certain applications of this formalism will be considered. The first of these is the uncertainty principle which expresses itself as a spreading of the signal width in frequency space when the signal is sampled for a finite time. It will be assumed throughout this section that the power in the signal spectrum is proportional to the square of the sum of the Fourier amplitudes for a given frequency just as is assumed for continuous representations using Fourier series.

Consider a signal of the form

$$s_k = u \cos \frac{2\pi\mu k}{N} + v \sin \frac{2\pi\mu k}{N} \tag{3.1}$$

where the variable μ need not be an integer. Consistent with previous definitions, s_k is the discrete representation of a wave with linear frequency μ/T. An alternate representation is

$$s_k = a \cos \left(\frac{2\pi\mu k}{N} + \varphi \right) \tag{3.2}$$

where $\varphi = \tan^{-1}(-v/u)$ and $a = (u^2 + v^2)^{\frac{1}{2}}$. Since μ is not an integer, s_k in equation 3.1 is not periodic on the sampling interval. From equation 2.1 and 2.2

$$a_n = \frac{2}{N} \sum_{k=0}^{N-1} \cos \frac{2\pi nk}{N} \left(u \cos \frac{2\pi\mu k}{N} + v \sin \frac{2\pi\mu k}{N} \right) \tag{3.3}$$

$$b_n = \frac{2}{N} \sum_{k=0}^{N-1} \sin \frac{2\pi nk}{N} \left(u \cos \frac{2\pi\mu k}{N} + v \sin \frac{2\pi\mu k}{N} \right) \tag{3.4}$$

Define a new quantity $c_n = a_n + ib_n$. The spectral power per unit frequency is $p_\omega = \left(a_n^2 + b_n^2 \right)/2\omega_0 = |c_n|^2/2\omega_0$. Then

$$c_n = \frac{2}{N} \sum_{k=0}^{N-1} e^{i(2\pi nk/N)} \left(u \cos \frac{2\pi\mu k}{N} + v \sin \frac{2\pi\mu k}{N} \right) \tag{3.5}$$

$$c_n = \frac{1}{N} \sum_{k=0}^{N-1} e^{i(2\pi nk/N)} \left[u \left(e^{i2\pi\mu k/N} + e^{-i2\pi\mu k/N}\right) - iv \left(e^{i2\pi\mu k/N} - e^{-i2\pi\mu k/N}\right)\right] \tag{3.6}$$

$$\Delta = \frac{2\pi(n - \mu)}{N} \tag{3.7}$$

$$\gamma = \frac{2\pi(n + \mu)}{N} \tag{3.8}$$

Equation 3.6 becomes

$$c_n = \frac{1}{N} \sum_{k=0}^{N-1} u \left(e^{i\gamma k} + e^{i\Delta k}\right) - iv \left(e^{i\gamma k} - e^{i\Delta k}\right) \tag{3.9}$$

With the algebraic identity 1.14, equation 3.9 can be written

$$c_n = \frac{1}{N} \left[(u - iv) \frac{1 - e^{i\gamma N}}{1 - e^{i\gamma}} + (u + iv) \frac{1 - e^{i\Delta N}}{1 - e^{i\Delta}}\right] \tag{3.10}$$

Consistent with the assumptions of the physical model presented earlier, assume that u and v are Gaussian variables with

$$\begin{aligned} \langle u \rangle &= \langle v \rangle = 0 \\ \langle u^2 \rangle &= \langle v^2 \rangle \\ \langle uv \rangle &= 0 \end{aligned} \tag{3.11}$$

This is a mathematical statement of the randomness of the phase and amplitude of a signal s_k scattered from the ionosphere. Define

$$\alpha = \frac{1 - e^{i\Delta N}}{1 - e^{i\Delta}} \quad ,$$

$$\beta = \frac{1 - e^{i\gamma N}}{1 - e^{i\gamma}} \tag{3.12}$$

Then equation 3.10 can be written as

$$c_n = \frac{1}{N} \left[\alpha (u + iv) + \beta (u - iv)\right] \tag{3.13}$$

or

$$|c_n|^2 = \frac{1}{N^2}\left[|\alpha|^2 (u^2 + v^2) + \alpha\beta^* (u^2 - v^2) + \right.$$

$$\left. \alpha^*\beta (u^2 - v^2) + |\beta|^2\right] \tag{3.14}$$

where β^* is the complex conjugate of β. With the use of the physical conditions represented by equation 3.11

$$\langle |c_n|^2 \rangle = \frac{1}{N^2} \langle u^2 + v^2 \rangle \left(\frac{1 - \cos N\Delta}{1 - \cos\Delta} + \frac{1 - \cos N\gamma}{1 - \cos\gamma} \right) \tag{3.15}$$

For the sake of explicitness this equation can be written as

$$\langle |c_n|^2 \rangle = \frac{1}{N^2} \langle u^2 + v^2 \rangle \left[\frac{1 - \cos 2\pi(n - \mu)}{1 - \cos 2\pi(n - \mu)/N} + \right.$$

$$\left. \frac{1 - \cos 2\pi(n + \mu)}{1 - \cos 2\pi(n + \mu)/N} \right] \tag{3.16}$$

Equation 3.16 is proportional to the average power per unit frequency in the nth frequency window. Note that one term in the equation is symmetric about the frequency μ/T while the other term is not. This means that if a signal which is not commensurate with a sample period is sampled for a finite time, it will be spread and this spreading is consistent with the uncertainty principle. However, the signal will not be spread symmetrically about its original frequency.

Before 3.16 can be applied to a spectrum, one must know the conditions under which equation 3.16 applies to a spectrum rather than simply to a monochromatic signal. It is possible to imagine cases in which cross terms would occur between different frequencies in the average power spectrum of equation 3.16 making it invalid for continuous spectra. To pursue this point further, assume a signal of the form

$$s_k = \sum_{\mu} u_{\mu} \cos 2\pi\mu/N + v_{\mu} \sin 2\pi\mu/N \qquad (3.17)$$

The sum over μ is purely formal since μ is not necessarily an integer. If μ is allowed to become a continuous variable, then s_k would correspond to a continuous signal and the spectrum would likewise be continuous. However, the subscript notation is retained for clarity and to preserve continuity with prior results. Under the conditions consistent with a time stationary Gaussian process of the type discussed earlier,

$$\langle u_\mu u_\nu \rangle = 0 \qquad \mu \neq \nu$$

$$\langle u_\mu^2 \rangle = \langle v_\mu^2 \rangle$$

$$\langle u_\mu v_\mu \rangle = 0 \qquad \text{for all } \mu \text{ and } \nu \tag{3.18}$$

Then equation 3.17 applies to a spectrum and we can write

$$\langle |c_n|^2 \rangle = \frac{1}{N^2} \sum_\mu \langle u_\mu^2 + v_\mu^2 \rangle \left(\frac{1 - \cos 2\pi(n - \mu)}{1 - \cos 2\pi(n - \mu)/N} + \frac{1 - \cos 2\pi(n + \mu)}{1 - \cos 2\pi(n + \mu)/N} \right) \tag{3.19}$$

This equation remains valid because the conditions of equation 3.18 insure that cross-terms from different μ values will not contribute to $\langle |c_n|^2 \rangle$. Since it is now indicated that equation 3.16 applies to continuous spectra in the form 3.19 when the process being considered is Gaussian and time stationary, one can draw certain conclusions about the spectra associated with such processes. The most important of these conclusions involves the center of gravity estimator. This estimator attempts to determine by what amount the center of the spectrum is shifted from a position symmetrically placed about the center of the bandpass. There are indications from the forms of equation 3.16 and 3.19 that the spectrum after being sampled for a finite time might have a center of gravity shifted relative to the center of gravity when the spectrum existed in a continuous state. This kind of shift could lead to systematic errors in the center of gravity estimate. The expression 3.19 for spectral spreading is an

extremely interesting one and will be discussed at much greater length in the next section. However, before this discussion it will be shown that the expression 3.19 obtained by squaring of Fourier components can also be obtained by Fourier transformation of the triangulated autocorrelation function. This is strong indication that expression 3.19 is a general characteristic of finite discrete sampling rather than an apparent characteristic resulting from some particular method of obtaining the spectrum.

To show that triangulation, the process discussed for continuous functions in equation 9, achieves the same effect as does the squaring of the Fourier amplitudes of a finitely sampled discrete signal, consider a discretely sampled function of the form

$$s_k = \sum_{\mu} u_{\mu} \cos 2\pi \mu k/N + v_{\mu} \sin 2\pi\mu k/N \qquad (4.1)$$

Here, as in section 3 of the Appendix, s_k is not necessarily composed of functions periodic on the sampling interval and the formal sum is physically equivalent to an integral since μ is not an integer. Forming the discrete autocorrelation function in accordance with methods outlined in section 1,

$$r(kT/N) = \frac{1}{T} \sum_{j=0}^{N-k-1} s(jT/N)\, s([j+k]T/N)\, \Delta t$$

or

$$r_k = \frac{1}{N} \sum_{j=0}^{N-k-1} s_j s_{j+k} \tag{4.2}$$

Substituting equation 4.1 into equation 4.2 gives

$$r_k = \sum_{j=0}^{N-k-1} \frac{1}{N} \left\{ \sum_{\mu} \left[u_\mu \cos 2\pi\mu j/N + v_\mu \sin 2\pi\mu j/N \right. \right.$$

$$\left. \left. \sum_{\nu} u_\nu \cos 2\pi\nu(j+k)/N + v_\nu \sin 2\pi\nu(j+k)/N \right] \right\} \tag{4.3}$$

$$r_k = \sum_{j=0}^{N-k-1} \frac{1}{N} \sum_{\mu} \sum_{\nu} \left[u_\mu u_\nu \cos \frac{2\pi\mu j}{N} \cos \frac{2\pi\nu(j+k)}{N} \right.$$

$$+ u_\mu v_\nu \cos \frac{2\pi\mu j}{N} \sin \frac{2\pi\nu(j+k)}{N} + v_\mu u_\nu \sin \frac{2\pi\mu j}{N} \cos \frac{2\pi\nu(j+k)}{N}$$

$$\left. + v_\mu v_\nu \sin \frac{2\pi\nu j}{N} \sin \frac{2\pi\mu(j+k)}{N} \right] \tag{4.4}$$

If one imposes constraints on the u_μ's and v_μ's consistent with those of equations 14, 15 and 16

$$\langle r_k \rangle = \frac{1}{N} \sum_{j=0}^{N-k-1} \sum_{\nu} \left\langle \frac{u_\nu^2 + v_\nu^2}{2} \right\rangle \cos 2\pi\nu k/N \tag{4.5}$$

Noting that equation 4.5 is symmetric in k and contains no j dependence it can be written as

$$\langle r_k \rangle = \left(1 - \frac{|k|}{N} \right) \sum_{\nu} \left\langle \frac{u_\nu^2 + v_\nu^2}{2} \right\rangle \cos 2\pi\nu k/N \tag{4.6}$$

From equation 17 it is known that $r(\tau)$ is the Fourier cosine transform of the power spectrum. It must be remembered however, that the auto-correlation function and the power spectral density are cosine transforms of each other only when they are real and symmetric. Therefore, one must perform the cosine transform over both positive and negative values of its argument. Otherwise, a real symmetric function would not have a Fourier transform which was real and symmetric. The fact that the transform is two-sided, that is, performed for positive and negative values of the argument, will become important when the discrete case is considered.

The discrete relation corresponding to equation 17 is

$$\langle r(kT/N) \rangle = \sum_{n=0}^{r} \langle p_\omega \rangle \cos \frac{n\omega_0 kT}{N} \Delta\omega \tag{4.7}$$

where

$$p_\omega = p_{n\omega_0} = a_n^2 = \frac{a_n^2 + b_n^2}{2}$$

under the assumption that equal powers exist in the sine and cosine components. Equation 4.7 can be rewritten as

$$\langle r_k \rangle = \sum_{n=0}^{r} \langle \frac{a_n^2 + b_n^2}{2} \rangle \cos 2\pi nk/N \tag{4.8}$$

It should be noted that this function is symmetric in the variable k as it should be since it is the Fourier transform of a real symmetric spectrum. Because $\langle r_k \rangle$ is symmetric, it contains only cosine components and can be written as

$$\langle r_k \rangle = \sum_{n=0}^{r} A_n \cos \frac{2\pi nk}{N} \tag{4.9}$$

Equations 4.8 and 4.9 are identical in form if A_n is identified with the spectral power in a particular frequency window, $A_n = \left(a_n^2 + b_n^2\right)/2$. Thus, the spectral power can be obtained by determining the A_n. Substituting equation 4.6 into equation 4.9 leads to

$$A_n = \frac{2}{N} \sum_{k=0}^{N-1} (1 - |k|/N) \sum_{\nu} \langle \frac{u_\nu^2 + v_\nu^2}{2} \rangle \cos \frac{2\pi \nu k}{N} \cos \frac{2\pi nk}{N} \tag{4.10}$$

To simplify notation rewrite equation 4.10 as

$$A_n = \langle \frac{u^2 + v^2}{2} \rangle \frac{2}{N} \sum_{k=0}^{N-1} (1 - k/N) \cos \frac{2\pi \nu k}{N} \cos \frac{2\pi nk}{N} \tag{4.11}$$

where the index ν has been suppressed and notation is consistent with that used for the one component case.

Defining delta and gamma as in equation 3.7 and 3.8 with μ in place of ν gives

$$A_n = \langle \frac{u^2 + v^2}{2N} \rangle \sum_{k=0}^{N-1} (1 - k/N) (\cos \Delta k + \cos \gamma k) \tag{4.12}$$

Note that in equations 4.11 and 4.12 the absolute value sign has been dropped. This is because the summation indicated will be performed over only positive argument and then doubled. One must perform two sums in order to determine the value of equation 4.12 in closed form. They are

$$y = \sum_{n=0}^{N-1} \cos n\theta \tag{4.13}$$

$$z = \frac{1}{N} \sum_{n=0}^{N-1} n \cos n\theta \tag{4.14}$$

$$y = \mathrm{re} \sum_{n=0}^{N-1} e^{in\theta} \tag{4.15}$$

$$z = \frac{1}{N} \mathrm{im} \frac{d}{d\theta} \sum_{n=0}^{N-1} e^{in\theta} \tag{4.16}$$

Use of the algebraic identity 1.14 yields

$$y = \mathrm{re} \frac{1 - e^{iN\theta}}{1 - e^{i\theta}} \tag{4.17}$$

$$z = \frac{1}{N} \mathrm{im} \frac{d}{d\theta} \frac{1 - e^{iN\theta}}{1 - e^{i\theta}} \tag{4.18}$$

$$y = \frac{1 - \cos\theta - \cos N\theta + \cos (N-1)\theta}{2 - \cos\theta} \tag{4.19}$$

$$z = \frac{1}{N} \mathrm{im} \left[(1 - e^{iN\theta}) (1 - e^{i\theta})^{-2} ie^{i\theta} - (1 - e^{i\theta})^{-1} iNe^{iN\theta} \right] \tag{4.20}$$

$$z = \frac{1}{N} \mathrm{re} \left(\frac{1 - e^{iN\theta}}{(1 - e^{i\theta})^2} e^{i\theta} - \frac{Ne^{iN\theta}}{1 - e^{i\theta}} \right) \tag{4.21}$$

$$z = \frac{1}{N} \mathrm{re} \left(\frac{1 - e^{iN\theta}}{(1 - e^{i\theta})^2} e^{i\theta} - (1 - e^{-i\theta}) \frac{Ne^{iN\theta}}{2 - 2\cos\theta} \right) \tag{4.22}$$

$$z = \frac{1}{N} \mathrm{re} \left(\frac{1 - e^{iN\theta}}{(1 - e^{i\theta})^2} e^{i\theta} \right) - \frac{\cos N\theta - \cos (N-1)\theta}{2 - 2\cos\theta} \tag{4.23}$$

$$z = \frac{1}{N} re\left(\frac{1 - e^{iN\theta}}{(1 - e^{i\theta})^2} e^{i\theta}\right) + \frac{\cos (N - 1)\theta - \cos N\theta}{2 - 2 \cos \theta} \tag{4.24}$$

$$y - z = \frac{1}{2} - \frac{1}{N} re \frac{1 - e^{iN\theta}}{(1 - e^{i\theta})^2} e^{i\theta} \tag{4.25}$$

$$y - z = \frac{1}{2} - \frac{1}{N} re\left(\frac{1 - e^{iN\theta}}{e^{-i\theta} - 2 + e^{i\theta}}\right) \tag{4.26}$$

$$y - z = \frac{1}{2} + \frac{1}{N} \frac{1 - \cos N\theta}{2 - 2 \cos\theta} \tag{4.27}$$

Substitution of equation 4.27 into equation 4.12 gives

$$A_n = \langle \frac{u^2 + v^2}{2N} \rangle \left\{1 + \frac{1}{2N} \left(\frac{1 - \cos N\Delta}{1 - \cos\Delta} + \frac{1 - \cos N\gamma}{1 - \cos \gamma}\right)\right\} \tag{4.28}$$

Equation 4.12 for A_n includes terms ranging from k = 0 to k = N - 1. This is only half of the interval desired, since it was already shown that terms ranging from -(N - 1) to N - 1 should be included. If one were to double a result for equation 4.12 obtained in equation 4.28, one would make the mistake of including terms with k = 0 two times. As can be seen from equation 4.12, this would give 2 extra contributions, one from the term containing Δ and one from the term containing γ. Because cos 0 = 1 these two contributions have a total value of 2. Therefore, the desired coefficients are

$$A'_n = 2A_n - 2 \tag{4.29}$$

This yields in place of equation 4.28

$$A'_n = \left\langle \frac{u^2 + v^2}{2N^2} \right\rangle \left(\frac{1 - \cos N\Delta}{1 - \cos\Delta} + \frac{1 - \cos N\gamma}{1 - \cos\gamma} \right) \tag{4.30}$$

It should be noted that this equation is in exact agreement with equation 3.16 since $\langle |c_n|^2 \rangle$ has twice the power that is actually in each component of the frequency spectrum. In addition for equation 3.16 and 4.30 to be power spectra, they must be divided by ω_0, the width of a frequency window.

IV. Symmetry and Spacing

Since equation 4.30 was derived for a specific u_μ and v_μ, equation 4.30 can be written as

$$A'_{n,\mu} = \left\langle \frac{u_\mu^2 + v_\mu^2}{2N^2} \right\rangle \left(\frac{1 - \cos N\Delta}{1 - \cos\Delta} + \frac{1 - \cos N\gamma}{1 - \cos\gamma} \right) \tag{4.31}$$

or

$$A'_n = \sum_\mu A'_{n,\mu} = \sum_\mu \left\langle \frac{u_\mu^2 + v_\mu^2}{2} \right\rangle \left(\frac{1 - \cos N\Delta}{1 - \cos\Delta} + \frac{1 - \cos N\gamma}{1 - \cos\gamma} \right) \tag{4.32}$$

This equation is analogous to equation 3.19 in the same way that equation 4.30 is analogous to equation 3.16.

Equation 4.31 is a useful starting point for understanding the properties resulting from the discrete analysis of a signal which possesses a continuous spectrum. This is because of the ease of considering a monochromatic wave, and that results thereby obtained can be linearly superimposed in the limit of a Gaussian time stationary process according to equation 4.32. For convenience, rewrite equation 4.31 as

$$A'_{n,\mu} = \epsilon_\mu \left(\frac{1 - \cos 2\pi\ (n - \mu)}{1 - \cos 2\pi\ (n - \mu)/N} + \frac{1 - \cos 2\pi\ (n + \mu)}{1 - \cos 2\pi\ (n + \mu)/N} \right) \qquad (4.33)$$

where

$$\epsilon_\mu = \left\langle \frac{u_\mu^2 + v_\mu^2}{2N^2} \right\rangle$$

is proportional to the spectral power of the μth component in the continuous signal.

V. Properties of Discretely Sampled Spectra

Consider the limit in which the μth component becomes periodic on the sampling interval T, that is the limit $\mu \rightarrow$ integer. This is represented by

$$\lim_{\mu \rightarrow \text{integer}} A'_{n,\mu} = \lim_{\mu \rightarrow \text{integer}} \left[\epsilon_\mu \frac{1 - \cos 2\pi(n - \mu)}{1 - \cos 2\pi(n - \mu/N} + \epsilon_\mu \frac{1 - \cos 2\pi(n - \mu)}{1 - \cos 2\pi(n + \mu)/N} \right] \qquad (4.34)$$

Since real radar systems possess filters which limit the continuous signal at the highest unambiguous sample frequency and because of arguments already made concerning the number of independent Fourier components needed for representation of a discretely sampled signal one may write

$$0 < \mu < r, \qquad 0 < n < r \qquad (4.35)$$

With the restrictions of equation 4.35

$$\lim_{\mu \rightarrow \text{integer}} A'_{n,\mu} = 0 \qquad (4.36)$$

if $\mu \neq n$. This is true because the numerators of both terms of equation 4.34 become 0 in the limit while the denominators do not. If, on the other hand, $\mu \to n$ then

$$\lim_{\mu \to n} A'_{n,\mu} = N^2 \epsilon_\mu \tag{4.37}$$

The limit is obtained by expanding the numerator and denominator in equation 4.34. Equation 4.36 and equation 4.37 can be combined if one denotes the limiting process simply by replacing μ by an integer symbol m. This gives

$$A'_{n,m} = \epsilon_m \Delta_{n,m} = \Delta_{n,m} \left\langle \frac{u_n^2 + v_n^2}{2} \right\rangle \tag{4.38}$$

Thus equation 4.34 states that a monochromatic signal which is periodic on a sampling interval develops no extra components when sampled. This is not surprising since if a signal were periodic on a sample interval its nature would be known for all times. Then, in accordance with the uncertainty principle this would mean that the frequency of the monochromatic periodic signal would be completely determined. Another statement of the same thing is that periodic signals cannot be spread.

Now examine equation 4.34 in the case where μ is not an integer. For a particular ϵ_μ, $A'_{n,\mu}$ has components for all values of n since neither numerator nor denominator of equation 4.34 are zero for any combination of n and μ. Under these conditions, however, the discretely analyzed continuous spectrum exhibits certain symmetry properties. It has already been stated that in general a particular component is not

spread symmetrically about its own frequency. In the special case in equation 4.38 the spreading is symmetric since there is no spreading at all. There is an interesting non-trivial case, however, in which spreading of components not commensurate with the sample period is symmetric.

This case can be understood as follows. In section 2 of the appendix it was shown that signals can be analyzed unambiguously from an angular frequency of 0 to an angular frequency corresponding to an integer $r = N/2$. The frequency corresponding to the integer r will be called the limit of the pass band. The center of the pass band is then

$$(\text{center of pass band}) = r\omega_0/2 = N\omega_0/4 \tag{4.39}$$

(Note: The numbers in the discrete case corresponding to these values are $r/2$ and $N/4$.)

Consider a case in which there are two components symmetrically placed above and below the center of the band pass with equal magnitudes and of the form

$$s_{k_1} = u \cos 2\pi (\nu + N/4)/N + v \sin 2\pi (\nu + N/4)/N$$

and

$$s_{k_2} = u \cos 2\pi (-\nu + N/4)/N + v \sin 2\pi (-\nu + N/4)/N \tag{4.40}$$

By equation 4.33 the components s_{k_1} and s_{k_2} possess spectra of the form

$$A'_{n,\nu+N/4} = \epsilon \frac{1 - \cos 2\pi (n - \nu - N/4)}{1 - \cos 2\pi (n - \nu - N/4)/N} +$$

$$\epsilon \frac{1 - \cos 2\pi (n + \nu + N/4)}{1 - \cos 2\pi (n + \nu + N/4)/N} \tag{4.41}$$

and

$$a'_{n,-\nu+N/4} = \epsilon \frac{1 - \cos 2\pi (n - \nu + N/4)}{1 - \cos 2\pi (n - \nu + N/4)/N} +$$

$$\epsilon \frac{1 - \cos 2\pi (n + \nu - N/4)}{1 - \cos 2\pi (n + \nu - N/4)/N} \tag{4.42}$$

To obtain the total spectrum of the signal $s_{k_{tot}} = s_{k_1} + s_{k_2}$ equations 4.41 and 4.42 must be added to obtain

$$a'_{n_{tot}} = \epsilon \left[\frac{1 - \cos 2\pi (n - \nu - N/4)}{1 - \cos 2\pi (n - \nu - N/4)/N} + \frac{1 - \cos 2\pi (n + \nu - N/4)}{1 - \cos 2\pi (n + \nu - N/4)/N} + \right.$$

$$\left. \frac{1 - \cos 2\pi (n + \nu + N/4)}{1 - \cos 2\pi (n + \nu + N/4)/N} + \frac{1 - \cos 2\pi (n - \nu + N/4)}{1 - \cos 2\pi (n - \nu + N/4)/N} \right] \tag{4.43}$$

To see whether $a'_{n,tot}$ is symmetric about the center of the bandpass one must compare $A'_{N/4+m,tot}$ with $A'_{N/4-m,tot}$. This comparison gives

$$A'_{N/4+m,tot} = \epsilon \left[\frac{1 - \cos 2\pi (m - \nu)}{1 - \cos 2\pi (m - \nu)/N} + \frac{1 - \cos 2\pi (m + \nu)}{1 - \cos 2\pi (m + \nu)/N} + \right.$$

$$\left. \frac{1 - \cos 2\pi (m + \nu) + N\pi}{1 + \cos 2\pi (m + \nu)/N} + \frac{1 - \cos 2\pi (m - \nu) + N\pi}{1 + \cos 2\pi (m - \nu)/N} \right] \tag{4.44}$$

and

$$A'_{N/4 - m, tot} = \epsilon\left[\frac{1 - \cos 2\pi(m + \nu)}{1 - \cos 2\pi(m + \nu)/N} + \frac{1 - \cos 2\pi(m - \nu)}{1 - \cos 2\pi(m - \nu)/N} + \right.$$

$$\left.\frac{1 - \cos 2\pi(m - \nu) + N\pi}{1 + \cos 2\pi(m - \nu)/N} + \frac{1 - \cos 2\pi(m + \nu) + N\pi}{1 + \cos 2\pi(m + \nu)/N}\right] \qquad (4.45)$$

Since equation 4.44 and equation 4.45 are identical and since superposition of spectral components holds in the time average limit it has been established that a continuous spectrum which is symmetric about the center of the pass band has this same symmetry property when resolved into discrete components. However, in general a continuous spectrum is not symmetric about the same frequency before and after discrete analysis. The general property of asymmetry induced by analyzing a spectrum for a finite time is called the principle of spectral asymmetry and leads as was earlier pointed out to mis-estimates of the spectral center of gravity.

VI. Spreading in the Computer

Equation 4.38 states that a signal periodic on the sample interval is not spread. Yet in the general discussion of signal and noise corruption in the main body of the report mention was made of corrupting only the frequency points which were obtained from a Fourier analysis of the backscattered signal. These points correspond exactly to functions which are periodic on the sample interval. Taking the autocorrelation function of these points triangulating it and re-transforming it to frequency space will cause no spreading in the frequency spectrum. One may wonder then how, in fact, a frequency spectrum is spread.

The spreading is accomplished by interpolation of points between those actually derived from a Fourier analysis of the incoming signal. If N samples are taken on a sample interval N autocorrelation functions and N/2 frequency points result. Before retransforming to autocorrelation space, N/2 frequency points are added. These points are interpolations between the N/2 "real" points already obtained. Now there are N points in frequency space. These N points are not all independent, since N/2 points are simply interpolated from the "real" points. The N points are symmetrized and the Fourier cosine transform is taken. That is, 2N points of a symmetric interpolated spectrum are transformed into a 2N point autocorrelation function. This function is symmetric and so N points are unnecessary since they contain no new information. Thus, only N points are used and these points are the new N point autocorrelation function, which is triangulated to produce spreading in frequency space. The interpolated points in frequency space are not periodic on the sample interval and thus triangulation causes these points to be spread.

To examine whether or not the spreading is of the proper magnitude, that is whether it has the same value as it would in a continuous spectrum, the following argument is employed. Imagine a continuous spectrum extending between two "real" (uninterpolated) points in frequency space. Assume that the interval under consideration contains one interpolated and one uninterpolated point. This interval is therefore representative of the way in which points were added in frequency space. For convenience allow the continuous spectrum to have an amplitude 1. Assume

the spectrum has height 0 everywhere except between the m and (m + 1)th real frequency points, with the (m + 1)th frequency point being excluded. By equation 4.32 the discretely analyzed spectrum has the form

$$A'_n = \int_m^{m+1} d\mu \left[\frac{1 - \cos 2\pi(n - \mu)}{1 - \cos 2\pi(n - \mu)/N} + \frac{1 - \cos (n + \mu)}{1 - \cos 2\pi(n + \mu)/N} \right] \tag{4.46}$$

The denominators are slowly changing functions by comparison to the numerators of equation 4.46. Thus, over the small interval m, m + 1 to a first approximation the denominators of the function of equation 4.46 can be considered constant and evaluated at $m + \frac{1}{2}$ while the numerators can be integrated. This yields

$$A'_{n,\, m+\frac{1}{2}} = \frac{1}{1 - \cos 2\pi(n - [m+1/2])/N} + \frac{1}{1 - \cos 2\pi(n+M+1/2)/N} \tag{4.47}$$

Since the cosine terms in the numerator cycle through one period between m and m + 1, hence integrate to 0. Now consider the discrete analog of equation 4.47. First in equation 4.47 the spectrum which was used had an amplitude of 1 over an interval of length 1. Thus the total power in the frequency interval between m and m + 1 was unity. To make a consistent analog in discrete space the total power must be the same. Thus, the powers in the points at m and m + 1/2 must be equal with amplitudes 1/2. It has already been shown that the point at m will not be spread. Thus, the spread power in the spectrum comes from the point at m + 1/2, and by equation 4.32

$$A'_n = \frac{A'_{n,\,m+1/2}}{2} = \frac{1}{2}\left(\frac{2}{1 - \cos 2\pi\,[n - (m + 1/2)]/N} + \frac{2}{1 - \cos 2\pi\,[n + (m + 1/2)/N]}\right) \tag{4.48}$$

Equations 4.47 and 4.48 are identical. Of course, they are only approximately so since equation 4.47 was obtained assuming that the denominators in equation 4.46 were constant. However, this is a good approximation except where these denominators are near 0. In such a case, however, components nearly periodic on the sample interval are being considered. As the denominator passes through 0 there is a positive and a negative contribution to the integral and these contributions tend to cancel to 0. This cancellation simply confirms the fact that nearly periodic functions tend to have little spreading. (Note: The same results for spreading are obtained if more than one point is interpolated for each real point. In fact, the approximate result obtained in equation 4.47 becomes exact in the infinite limit.)