LINEAR REGULATOR DESIGN FOR STOCHASTIC SYSTEMS BY A MULTIPLE TIME SCALES METHOD*

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ABSTRACT

This paper develops a hierarchically-structured, suboptimal controller for a linear stochastic system composed of fast snd slow subsystems. It is proved that the controller is optimal in the limit as the separation of time scales of the subsystems becomes infinite. The methodology is illustrated by design of a controller to suppress the phugoid and short period modes of the longitudinal dynamics of the F-8 aircraft.

INTRODUCTION

A common occurrence in engineering systems is the presence of phenomena that naturally evolve in widely separated time scales. Some examples drawn from the fields of aerospace and power engineering will serve to illustrate this point.

It is well-known [1] that the longitudinal dynamics of an aircraft are comprised of two distinct oscillatory modes - the phugoid and short period - of periods on the order of 100 sec. and 1 sec. A terrestrial inertial navigator has Schuler oscillations of period 84 min. and earth rate oscillations with period 24 hr. [2]. A dual spin satellite in synchronous orbit will be subject to an orbital oscillation with a 24 hr. period and a nutation with period on the order of 1-10 sec. [3,4]. These effects are often used in an ad hoc way in the design of filters and controllers, usually by assuming that the slow modes are constant if the fast modes are of concern, or by ignoring the fast modes when the slow modes are of interest. See [3, 5-8, 28] for examples.

Additional examples can be found in the field of electric power systems. An electrical machine has an oscillatory mode involving stator fluxes that is invariably neglected in favor of the much slower electromechanical oscillations in stability studies [9]. A similar approximation is made in studies of a large number of interconnected machines, in which the intermachine electromechanical swings are ignored when the much slower average frequency behavior is of primary concern [10].

In addition to these concrete examples, note that proponents of hierarchical control often suggest that the task of controlling a large scale system should be partitioned into subtasks by time scale. Thus the higher levels of the control system are concerned with slower phenomena, and the lower levels with faster phenomena [11-14]. It is difficult to point to any specific examples, with the possible exception of the interaction between automatic generation control and economic dispatch on electric power systems [15].

The multiple time scale phenomena alluded to above are conveniently

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modelled via perturbation theory [16]. There are a number of possible approaches, but we will adopt the framework of singular perturbation theory. This theory has been applied to a variety of control problems by a number of authors [17-21, 24-27] but the previous work most relevant to this paper is that of Kokotovic et. al. [18-20] and Haddad [21]. This paper is in the spirit of [18-21], and moreover requires several of the detailed results of these papers concerning singular perturbations of Ricatti equations.

Specifically, the paper begins by analysis of singular perturbations for linear stochastic systems with two time scales. An approximating system is obtained with the property that the mean-square error between the states of the actual and approximating systems approaches zero as the separation of time scales becomes infinite. The usefulness of this result is demonstrated by application to the stochastic optimal linear regulator problem. With the machinery properly set up, it is straightforward to identify the asymptotically optimal controller, using known results on singular perturbations of Riccati equations. The controller has an interesting hierarchical structure, with the implication of reduced on line computations.

These new theoretical results are illustrated by application to an important control problem. An asymptotically optimal two time scale controller is developed for the longitudinal dynamics of a jet aircraft. The two time scale controller is compared to the optimal controller, and it is demonstrated that there is negligible degradation in performance.

An attempt is made throughout to relegate technical details to the Appendix, so that the paper will be accessible to engineers interested only in the two time scale design procedure.

MAIN RESULTS

Singular perturbation theory is concerned with systems of the form

$$\dot{x}(t;\ \varepsilon) = f(x,\ y,\ \varepsilon) \tag{2.1}$$

$$\varepsilon\dot{y}(t;\ \varepsilon) = g(x,\ y,\ \varepsilon) \tag{2.2}$$

and the corresponding degenerate system

$$\dot{x}(t;\ 0) = f(x,\ y,\ 0) \tag{2.3}$$

$$0 = g(x,\ y,\ 0) \tag{2.4}$$

The basic question is whether (2.3) - (2.4) is an approximation to (2.1) - (2.2) in the sense that

$$\lim_{\varepsilon\to 0}\ x(t;\ \varepsilon) = x(t;\ 0) \tag{2.5}$$

$$\lim_{\varepsilon\to 0}\ y(t;\ \varepsilon) = y(t;\ 0) \tag{2.6}$$

Various technical assumptions are required to obtain (2.5) and (2.6), but under these assumptions the degenerate system is a valid reduced order approximation to the original system in the sense that for ε sufficiently small the solutions of the two systems are close.

In the stochastic case, the situation is more complex. Consider the linear system

$$\frac{d}{dt}\begin{bmatrix} x_1(t;\ \varepsilon) \\ \varepsilon x_2(t;\ \varepsilon) \end{bmatrix} = \begin{bmatrix} A_{11}(\varepsilon) & A_{12}(\varepsilon) \\ A_{21}(\varepsilon) & A_{22}(\varepsilon) \end{bmatrix}\begin{bmatrix} x_1(t;\ \varepsilon) \\ x_2(t;\ \varepsilon) \end{bmatrix} + \begin{bmatrix} L_1(\varepsilon) \\ L_2(\varepsilon) \end{bmatrix}\xi(t) \tag{2.7}$$

where

$$E\begin{bmatrix} x_1(0;\ \varepsilon) \\ x_2(0;\ \varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.8}$$

$$E\begin{bmatrix} x_1(0;\ \varepsilon)x_1^T(0;\ \varepsilon) & x_1(0;\ \varepsilon)x_2^T(0;\ \varepsilon) \\ x_2(0;\ \varepsilon)x_1^T(0;\ \varepsilon) & x_2(0;\ \varepsilon)x_2^T(0;\ \varepsilon) \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{2.9}$$

$$E\{\xi(t)\} = 0 \tag{2.10}$$

$$E\{\xi(t)\xi^T(s)\} = \Xi\delta(t-s) \tag{2.11}$$

$x_1(0)$, $x_2(0)$ are independent of $\xi_1(t)$, $\xi_2(t)$, and all matrices are continuous in ε at $\varepsilon = 0$. Moreover, $A_{22}(0)$ is stable.

An approximation to (2.7) is desired that is valid for small ε and is simpler than (2.7). Note that setting $\varepsilon = 0$ in (2.7) is inadequate; since

$$x_2(t;\ 0) = -A_{22}^{-1}(0)A_{21}(0)x_1(t;\ 0) - A_{22}^{-1}(0)L_2(0)\xi(t) \tag{2.12}$$

has a white noise component and therefore has infinite variance. Consequently,

$$E\left\{\left(x_2(t;\ \varepsilon) - x_2(t;\ 0)\right)^T\left(x_2(t;\ \varepsilon) - x_2(t\ ;\ 0)\right)\right\} = +\infty \tag{2.13}$$

so that $x_2(t;\ 0)$ is not an approximation to $x_2(t;\ \varepsilon)$ (in the least squares sense).

Instead, define the stochastic degenerate system associated with (2.7) to be the system

$$\dot{x}_{1d}(t;\ \varepsilon) = A_{11d}(\varepsilon)x_{1d}(t;\ \varepsilon) + L_{1d}(\varepsilon)\xi(t),\ x_{1d}(0;\ \varepsilon) = x_1(0) \tag{2.14}$$

$$\dot{x}_{2d}(t;\ \varepsilon) = A_{21d}(\varepsilon)x_{1d}(t;\ \varepsilon) + A_{22d}(\varepsilon)x_{2d}(t;\ \varepsilon) + L_{2d}(\varepsilon)\xi(t),$$
$$x_{2d}(0;\ \varepsilon) = x_2(0) \tag{2.15}$$

where

$$A_{11d}(\varepsilon) = A_{11}(\varepsilon) - A_{12}(\varepsilon)A_{22}^{-1}(\varepsilon)A_{21}(\varepsilon) \tag{2.16}$$

$$L_{1d}(\varepsilon) = L_1(\varepsilon) - A_{12}(\varepsilon)A_{22}^{-1}(\varepsilon)L_2(\varepsilon) \tag{2.17}$$

$$A_{21d}(\varepsilon) = A_{21}(\varepsilon) \tag{2.18}$$

$$L_{2d}(\varepsilon) = L_2(\varepsilon) \tag{2.19}$$

Notice that the stochastic degenerate system is of the same order as the original system, unlike the situation for deterministic singular perturbations.

Theorem 1

Consider the linear stochastic system (2.7) - (2.11) and a corresponding stochastic degenerate system (2.14) - (2.15). Assume that all matrices in the two systems are continuous in ε at $\varepsilon = 0$, and that $A_{22}(0)$ and $A_{11}(0) - A_{12}(0)A_{22}^{-1}(0)A_{21}(0)$ are stable[1] (i.e., have eigenvalues in the open left half complex plane). Then the stochastic degenerate system is an approximation to the original system in the sense that

$$\lim_{\varepsilon\to 0} E\{(x_1(t;\ \varepsilon) - x_{1d}(t;\ \varepsilon))(x_1(t;\ \varepsilon) - x_{1d}(t;\ \varepsilon))^T\} = 0 \tag{2.22}$$

$$\lim_{\varepsilon\to 0} E\{(x_2(t;\ \varepsilon) - x_{2d}(t;\ \varepsilon))(x_2(t;\ \varepsilon) - x_{2d}(t;\ \varepsilon))^T\} = 0 \tag{2.23}$$

uniformly for $0 \leq t < \infty$.

Proof

The proof of the theorem is quite involved. Differential equations for

$$\Sigma_{11}(t;\varepsilon) \triangleq E\{\tilde{x}_1(t;\varepsilon)\tilde{x}_1^T(t;\varepsilon)\} \triangleq E\{(x_1(t;\varepsilon)-x_{1d}(t;\varepsilon))(x_1(t;\varepsilon)-x_{1d}(t;\varepsilon))^T\} \tag{2.24}$$

$$\Sigma_{22}(t;\varepsilon) \triangleq E\{\tilde{x}_2(t;\varepsilon)\tilde{x}_2^T(t;\varepsilon)\} \triangleq E\{(x_2(t;\varepsilon)-x_{2d}(t;\varepsilon))(x_2(t;\varepsilon)-x_{2d}(t;\varepsilon))^T\} \tag{2.25}$$

are obtained, and the limits are evaluated by (non-stochastic) singular perturbation theory to establish (2.22) and (2.23). See the Appendix for details.

Remarks

1. Note that the stochastic degenerate system is the same order as the original system, so that (2.23) is valid for $t = 0$.

2. Clearly, the assumption $A_{11}(0) - A_{12}(0)A_{22}^{-1}(0)A_{21}(0)$ stable is only necessary to insure uniform convergence in (2.22), (2.23) on the infinite interval. Without this assumption, a theorem of Tihonov [16] can be invoked which insures uniform convergence in (2.22), (2.23) for sets of the form $[0,T]$.

3. Proof is easily generalized to cover uniformly asymptotically stable time-varying systems at the expense of some additional notation.

Consider now the system

$$\dot{x}(t;\ \varepsilon) = A(\varepsilon)x(t;\ \varepsilon) + B(\varepsilon)u(t) + L(\varepsilon)\xi(t) \tag{2.26}$$

with observations

$$y(t;\ \varepsilon) = Cx(t;\ \varepsilon) + \theta(t) \tag{2.27}$$

and cost

$$J = \lim_{T\to\infty} \frac{1}{T} [E\{\int_0^T x^TQx + u^TRu\ dt\}] \tag{2.28}$$

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[1] Note that this assumption implies that there exists an $\varepsilon_0 > 0$ such that the system matrix in (2.7) is stable for all $0 \leq \varepsilon < \varepsilon_0$ [22].

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \frac{B_2}{\varepsilon} \end{bmatrix}$$

$$L = \begin{bmatrix} L_1 \\ \frac{L_2}{\varepsilon} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{21} \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0, \quad R > 0$$

$$E\{\xi(t)\} = 0, \qquad E\{\theta(t)\} = 0$$

$$E\{\xi(t)\xi^T(s)\} = \Xi\delta(t-s), \quad E\{\theta(t)\theta^T(s)\} = \Theta\delta(t-s)$$

$$\Xi > 0, \qquad \Theta > 0$$

and $\xi(t)$, $\theta(t)$ are independent and Gaussian. The assumptions

$$[A(\varepsilon),\ B(\varepsilon)], \quad [A(\varepsilon),\ L(\varepsilon)] \quad \text{controllable} \tag{2.29}$$

$$[A(\varepsilon),\ C], \quad [A(\varepsilon),\ \sqrt{Q}] \quad \text{observable} \tag{2.30}$$

are made, $0 < \varepsilon \leq \varepsilon_0$.

As is well known, the optimal control law is

$$u(t;\ \varepsilon) = -\ G(\varepsilon)\hat{x}(t;\ \varepsilon) \tag{2.31}$$

where

$$G(\varepsilon) = R^{-1}B^T K(\varepsilon) \tag{2.32}$$

and $K(\varepsilon)$ satisfies

$$0 = -\ K(\varepsilon)A(\varepsilon) - A^T(\varepsilon)K(\varepsilon) - Q(\varepsilon) + K(\varepsilon)B(\varepsilon)R^{-1}B^T(\varepsilon)K(\varepsilon) \tag{2.33}$$

The estimate satisfies the equation

$$\dot{\hat{x}}(t;\ \varepsilon) = A(\varepsilon)\hat{x}(t;\ \varepsilon) + H(\varepsilon)(y(t;\ \varepsilon) - C\hat{x}(t;\ \varepsilon)) + B(\varepsilon)u(t;\ \varepsilon) \tag{2.34}$$

where

$$H(\varepsilon) = \Sigma(\varepsilon)C'\Theta^{-1} \tag{2.35}$$

and $\Sigma(\varepsilon)$ satisfies

$$0 = \Sigma(\varepsilon)A^T(\varepsilon) + A(\varepsilon)\Sigma(\varepsilon) + L(\varepsilon)\,\Xi\, L^T(\varepsilon) - \Sigma(\varepsilon)C^T R^{-1} C\Sigma(\varepsilon) \tag{2.36}$$

At this point, we are ready to apply Theorem 1 to approximate the controller (2.31), (2.34) by a two time scale controller. Because of Theorem 1, any system that has a stochastic degenerate system in common with the optimal closed-loop system will be asympotically optimal. The optimal closed-loop system can be written

$$\begin{bmatrix} \dot{x}_1(t;\varepsilon) \\ \dot{\hat{x}}_1(t;\varepsilon) \\ \varepsilon\dot{x}_2(t;\varepsilon) \\ \varepsilon\dot{\hat{x}}_2(t;\varepsilon) \end{bmatrix} =$$

$$\begin{bmatrix} A_{11} & -B_1G_1(\varepsilon) & A_{12} & -B_1G_2(\varepsilon) \\ H_1(\varepsilon)C_1 & A_{11}-B_1G_1(\varepsilon)-H_1(\varepsilon)C_1 & H_1(\varepsilon)C_2 & A_{12}-B_1G_2(\varepsilon)-H_1(\varepsilon)C_2 \\ A_{21} & -B_2G_1(\varepsilon) & A_{22} & -B_2G_2(\varepsilon) \\ \varepsilon H_2(\varepsilon)C_1 & A_{21}-B_2G_1(\varepsilon)-\varepsilon H_2(\varepsilon)C_1 & \varepsilon H_2(\varepsilon)C_2 & A_{22}-B_2G_2(\varepsilon)-\varepsilon H_2(\varepsilon) \end{bmatrix}$$

$$\begin{bmatrix} x_1(t;\varepsilon) \\ \hat{x}_1(t;\varepsilon) \\ \hat{x}_2(t;\varepsilon) \\ \hat{x}_2(t;\varepsilon) \end{bmatrix} + \begin{bmatrix} L_1 & 0 \\ 0 & H_1(\varepsilon) \\ L_2 & 0 \\ 0 & \varepsilon H_2(\varepsilon) \end{bmatrix} \begin{bmatrix} \xi(t) \\ \theta(t) \end{bmatrix} \tag{2.37}$$

Note that the stochastic degenerate system can be obtained by eliminating x_2, $\hat{x}_2$ from the equations for x_1, $\hat{x}_1$ using the algebraic relations that result when ε is set equal to zero in the left hand side of (2.37). Of course, the resulting system cannot be implemented since the value of ε in the x_1 and x_2 equations is not a design parameter.

An implementable system that has a stochastic degenerate system in common with (2.37) is obtained as follows. Assume that $(A_{22}-B_2G_2(\varepsilon)-\tilde{H}_2(\varepsilon)C_2)^{-1}$ exists. Set $\varepsilon = 0$ in the left hand side of only the $\hat{x}_2$ equations of (2.37) to obtain:

$$\hat{x}_2 = -(A_{22} - B_2G_2(\varepsilon) - \tilde{H}_2(\varepsilon)C_2)^{-1}[(A_{21}-B_2G_1(\varepsilon) - \tilde{H}_2(\varepsilon)C_1)\hat{x}_1 + \\ + \tilde{H}_2(\varepsilon)C_1x_1 + \tilde{H}_2(\varepsilon)C_2x_2 + \tilde{H}_2(\varepsilon)\theta] \tag{2.38}$$

Substitute into the $\hat{x}_1$ equation to obtain

$$\dot{\hat{x}}_{1d}(t;\ \varepsilon) = A_{11D}\hat{x}_{1d}(t;\ \varepsilon) + H_{1D}y(t;\ \varepsilon) \tag{2.39}$$

where

$$A_{11D}(\varepsilon) = A_{11}-B_1G_1(\varepsilon) - H_1(\varepsilon)C_1 - (A_{12}-B_1G_2(\varepsilon)-H_1(\varepsilon)C_2) \times \\ (A_{22}-B_2G_2(\varepsilon)-\tilde{H}_2(\varepsilon)C_2)^{-1}(A_{21} - B_2G_1(\varepsilon)- \tilde{H}_2(\varepsilon)C_1) \tag{2.40}$$

$$H_{1D}(\varepsilon) = H_1(\varepsilon)-(A_{12}-B_1G_2(\varepsilon)-H_1(\varepsilon)C_2)(A_{22}-B_2G_2(\varepsilon)-\tilde{H}_2(\varepsilon)C_2)^{-1}\tilde{H}_2(\varepsilon) \tag{2.41}$$

$$\tilde{H}_2(\varepsilon) = \varepsilon H_2(\varepsilon) \tag{2.42}$$

Based on this analysis, the following suboptimal closed-loop system is obtained.

$$\begin{bmatrix} \dot{x}_{1D}(t;\varepsilon) \\ \dot{\hat{x}}_{1D}(t;\varepsilon) \\ \varepsilon\dot{x}_{2D}(t;\varepsilon) \\ \varepsilon\dot{\hat{x}}_{2D}(t;\varepsilon) \end{bmatrix} =$$

$$
\begin{bmatrix}
A_{11} & -B_1G_1(\varepsilon) & A_{12} & -B_1G_2(\varepsilon) \\
H_{1D}(\varepsilon)C_1 & A_{11D}(\varepsilon) & H_{1D}(\varepsilon)C_2 & 0 \\
A_{21} & -B_2G_1(\varepsilon) & A_{22} & -B_2G_2(\varepsilon) \\
\tilde{H}_2(\varepsilon)C_1 & A_{21}-B_2G_1(\varepsilon)-\tilde{H}_2(\varepsilon)C_1 & \tilde{H}_2(\varepsilon)C_2 & A_{22}-B_2G_2(\varepsilon)-\tilde{H}_2(\varepsilon)C_2
\end{bmatrix}
$$

$$
\begin{bmatrix} x_{1D}(t;\varepsilon) \\ \hat{x}_{1D}(t;\varepsilon) \\ x_{2D}(t;\varepsilon) \\ \hat{x}_{2D}(t;\varepsilon) \end{bmatrix}
+
\begin{bmatrix} L_1 & 0 \\ 0 & H_{1D}(\varepsilon) \\ L_2 & 0 \\ 0 & \tilde{H}_2(\varepsilon) \end{bmatrix}
\begin{bmatrix} \xi(t) \\ \theta(t) \end{bmatrix}
\qquad (2.43)
$$

Clearly, (2.43) is not the stochastic degenerate system of (2.37), but has been constructed to have a stochastic degenerate system in common with (2.37)

Theorem 2.

Consider the LQG problem defined by (2.26) - (2.28). Assume that the controllability and observability assumptions (2.29), (2.30) hold and that $(A_{22} - B_2G_2(\varepsilon)-H_2(\varepsilon)C_2)^{-1}$ exists, $0< \varepsilon \leq \varepsilon_0$. Then the suboptimal closed-loop system (2.43) is asymptotically optimum in the following sense:

$$\lim_{\varepsilon\to 0} E\{(x_1(t;\varepsilon) - x_{1D}(t;\varepsilon))(x_1(t;\varepsilon)-x_{1D}(t;\varepsilon))^T\} = 0$$

$$\lim_{\varepsilon\to 0} E\{(\hat{x}_1(t;\varepsilon)-\hat{x}_{1D}(t;\varepsilon))(\hat{x}_1(t;\varepsilon)-\hat{x}_{1D}(t;\varepsilon))^T\} = 0$$

$$\lim_{\varepsilon\to 0} E\{(x_2(t;\varepsilon)-x_{2D}(t;\varepsilon))(x_2(t;\varepsilon)-x_{2D}(t;\varepsilon))^T\} = 0$$

$$\lim_{\varepsilon\to 0} E\{(\hat{x}_2(t;\varepsilon)-\hat{x}_{2D}(t;\varepsilon))(\hat{x}_2(t;\varepsilon)-\hat{x}_{2D}(t;\varepsilon))^T\} = 0$$

Proof

As noted above, (2.37) and (2.43) have a common stochastic degenerate system which governs their behavior as $\varepsilon \to 0$. Therefore, all that is required is the verification of the hypotheses of Theorem 1.

The required continuity properties follow from [18, 20]. Note that under the stated controllability and observability assumptions the degenerate and boundary layer system controllability and observability conditions required are satisfied [24]. The required stability properties are established as a consequence of the stability of the closed loop LQG design and a previously quoted result [22]. Full details can be found in [23].

Remarks

1. Note that the above results are not the most general possible, since the time-varying and finite horizon case could probably also be solved. However, the above results are of the greatest practical interst.

2. Extension to more than two time scales in straightforward [22].

3. Investigation of the rather ad hoc assumption $(A_{22}-B_2(\varepsilon)G_2(\varepsilon)-H_2(\varepsilon)C_2)^{-1}$ exists for $0 < \varepsilon \leq \varepsilon_0$ would be of theoretical interest since verification is difficult. As a practical matter, this issue is less important, since invertibility for the value of ε of interest is easily checked. The performance of the two time scale controller can then be directly assessed.

4. The proposed suboptimal controller is illustrated in Figure 1. Notice that there is a unidirectional interface between the slow and fast filters. Thus there is opportunity for considerable reduction in on-line computational effort since the two sets of filter equations can be numerically integrated in different time scales (i.e., with different step sizes).

5. The above development assumed, for simplicity, that the optimal gain matrices $H(\varepsilon)$, $G(\varepsilon)$ were computed. In fact, by a more elaborate analysis, it is possible to show that only asymptotic approximations to $H(\varepsilon)$, $G(\varepsilon)$ near $\varepsilon = 0$ need to be computed [22]. It is possible to compute these approximations by the methods extensively studied in [18-21], thus obtaining a potential reduction in off-line as well as on-line computation.

6. Notice that it is the two time scale nature of the closed loop system that is required. In the above analysis, as $\varepsilon \to 0$ the closed loop system automatically has fast and slow modes. In a physical system ε has a fixed, non-zero value. Consequently, it is possible that the open loop dynamics can have fast and slow modes, but in the closed loop these slow modes are eliminated.

TWO TIME SCALE CONTROL OF AIRCRAFT LONGITUDINAL DYNAMICS

The particular problem to be addressed in this section is the design of a feedback control system for the longitudinal dynamics of an F-8 aircraft. Specifically the controller must produce elevator commands to keep the aircraft in steady level flight in the face of wind disturbances. For simplicity, the wind disturbance is modelled as white.

The equations of motion of an airplane are a set of coupled nonlinear equations in the longitudinal and lateral state variables. If the equations are linearized about nominal state and control variables, then the resulting linear equations are found to approximately decouple into separate sets for the longitudinal and lateral dynamics. See [1] for an excellent discussion of the modelling issues.

The aircraft's longitudinal variables are

$$x = \begin{bmatrix} V \\ \gamma \\ \alpha \\ q \end{bmatrix}, \quad u = \delta_e$$

where

V = horizontal velocity deviation in ft/sec

γ = flight path angle in radians,

α = angle of attack in radians,

q = pitch rate in rad/sec.

δ_e = is the elevator deflection in radians

The interpretation of these variables is given in Figure 2. Table 1 gives the system matrices. It is assumed that velocity and pitch rate measurements, both corrupted by wideband noise, are available.

Figures 3-4 show the system response to an initial pitch $\theta(0) = 1^0$, $V(0) = 100$ ft/sec in the absence of the wind disturbance. The two time scale behavior is well illustrated here. Table 2 gives the system's eigenvalues and eigenvectors. Note that the variables V, γ dominate the slower phugoid mode, and the variables α, q dominate the faster short period mode. The physical nature of these oscillations is beautifully described on pages 320-328 of [1].

From the above discussion and inspection of the system A matrix, the equations for V and γ are the logical candidates for the slow dynamics and the equations for α and q are suggested as the fast dynamics. For determination of ε, the following procedure is suggested. Note that the state equations can be written

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \varepsilon x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon A_{21} & \varepsilon A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ \varepsilon B_2 \end{bmatrix} u + \begin{bmatrix} L_1 \\ \varepsilon L_2 \end{bmatrix}$$

where $x_1^T = [V\ \gamma]$, $x_2^T = [\alpha\ q]$. For $\varepsilon = .01$, the matrix

$$\varepsilon A_{22} \quad \begin{bmatrix} -.012 & .0100 \\ -.0901 & .0069 \end{bmatrix}$$

has eigenvalues comparable to those of A_{11}. Determination of a value of ε is actually not required for design of the two time scale controller, but is useful for judging the appropriateness of the approximation $\varepsilon = 0$.

Before proceeding with the regulator design, a few remarks are in order. First, the aircraft longitudinal variables more often include the pitch θ instead of the flight path angle γ. It was only after considerable difficulty that the above formulation, in which the choice of fast and slow variables is clear, was hit upon. Second, the extensive literature on singular perturbation theory contains almost no discussion of the choice of fast and slow variables, or the choice of ε. But determination of these quantities is the first problem one has to face up to in applications of the theory.

A LQG controller was designed for the Q and R matrices Q = diag [.01,0,3260,3260], R = [3260] using standard routines [29]. Design goals were to (i) achieve a damping ratio $\zeta > .707$ for both modes (ii) reduce the state variables response to the wind disturbance. Closed-loop eigenvalues and RMS state variable and estimation error standard deviations are given in Table 3. As noted in remark 6 of Theorem 2, it is critical that the open loop separation of modes be present also in the closed loop. From Table 2, the optimal design has this property. The closed-loop eigenvalues of the two time scale controller and corresponding RMS state variable and estimation error standard deviations are also listed in Table 3. Note the generally good correspondance between the optimal and suboptimal designs.

SUMMARY AND CONCLUSIONS

This paper has considered the reduction in on-line computational effort for an LQG design with fast and slow closed loop modes. Together with the results of [18-21], this paper demonstrates that the singular perturbation approach to the LQG problem offers the potential of near optimal performance with reduction in both on-and off-line computation.

The design procedure of this paper has been applied to control of the longitudinal dynamics of a jet aircraft. A two time scale design was obtained with performance extremely close to that of the optimal design. Note that recent proposals for adaptive flight control systems require multiple Kalman filters running in parallel [30, 31], so that reduction of on-line computation is of definite interest.

Several directions for future research are evident. First, procedures for systematically picking the fast and slow variables of a system with fast and slow modes, as well as for determining explicitly ϵ, would be highly useful in applications. Second, note that the design of Figure 1 has an interesting hierarchical structure. In fact, as pointed out in [14], if a system is composed of a number of fast subsystems with a slow interconnecting equation, a decentralized, hierarchical design is naturally obtained. Therefore, the results of this paper are of potential interest in hierarchical systems theory. Finally, note that singular perturbation theory is only one approach to the multiple time scale phenomenon. The results of Ramnath [32], for example, provide another approach that could be exploited in control theory.

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APPENDIX

Proof of Theorem 1:

From (2.7), (2.14), (2.15) the following equations are obtained

$$\begin{bmatrix} \dot{\tilde{x}}_1(t;\varepsilon) \\ \varepsilon\dot{\tilde{x}}_2(t;\varepsilon) \\ \dot{x}_1(t;\varepsilon) \\ \varepsilon\dot{x}_2(t;\varepsilon) \end{bmatrix} = \begin{bmatrix} A_{11d}(\varepsilon) & 0 & \tilde{A}_{13}(\varepsilon) & A_{12}(\varepsilon) \\ A_{21d}(\varepsilon) & A_{22d}(\varepsilon) & 0 & 0 \\ 0 & 0 & A_{11}(\varepsilon) & A_{12}(\varepsilon) \\ 0 & 0 & A_{21}(\varepsilon) & A_{22}(\varepsilon) \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t;\varepsilon) \\ \tilde{x}_2(t;\varepsilon) \\ x_1(t;\varepsilon) \\ x_2(t;\varepsilon) \end{bmatrix} +$$

$$\begin{bmatrix} \tilde{L}_1(\varepsilon) \\ 0 \\ L_1(\varepsilon) \\ L_2(\varepsilon) \end{bmatrix} \xi(t) \tag{A.1}$$

where

$$\tilde{A}_{13}(\varepsilon) \overset{\Delta}{=} A_{12}(\varepsilon)A_{22}^{-1}(\varepsilon)A_{21}(\varepsilon) \tag{A.2}$$

$$\tilde{L}_1(\varepsilon) \overset{\Delta}{=} A_{12}(\varepsilon)A_{22}^{-1}(\varepsilon)L_2(\varepsilon) \tag{A.3}$$

Define

$$\begin{bmatrix} \Sigma_{11}(t;\varepsilon) & \Sigma_{12}(t;\varepsilon) & \Sigma_{13}(t;\varepsilon) & \Sigma_{14}(t;\varepsilon) \\ \Sigma_{21}(t;\varepsilon) & \Sigma_{22}(t;\varepsilon) & \Sigma_{23}(t;\varepsilon) & \Sigma_{24}(t;\varepsilon) \\ \Sigma_{31}(t;\varepsilon) & \Sigma_{32}(t;\varepsilon) & \Sigma_{33}(t;\varepsilon) & \Sigma_{34}(t;\varepsilon) \\ \Sigma_{41}(t;\varepsilon) & \Sigma_{42}(t;\varepsilon) & \Sigma_{43}(t;\varepsilon) & \Sigma_{44}(t;\varepsilon) \end{bmatrix} =$$

$$= E\left\{ \begin{bmatrix} \tilde{x}_1(t;\varepsilon) \\ \tilde{x}_2(t;\varepsilon) \\ x_1(t;\varepsilon) \\ x_2(t;\varepsilon) \end{bmatrix} \begin{bmatrix} \tilde{x}_1^T(t;\varepsilon)\tilde{x}_2^T(t;\varepsilon)x_1^T(t;\varepsilon)x_2^T(t;\varepsilon) \end{bmatrix} \right\} \tag{A.4}$$

and let

$$\tilde{\Sigma}_{44}(t;\varepsilon) = \varepsilon\Sigma_{44}(t;\varepsilon) \tag{A.5}$$

Then the variance equations corresponding to (A.1) can be written

$$\dot{\Sigma}_{11} = A_{11d}(\varepsilon)\Sigma_{11} + \tilde{A}_{13}(\varepsilon)\Sigma_{31} + A_{12}(\varepsilon)\Sigma_{41} + \Sigma_{11}A_{11d}(\varepsilon) + \Sigma_{13}\tilde{A}_{13}^T(\varepsilon) + \Sigma_{14}A_{12}^T(\varepsilon) + \tilde{L}_1(\varepsilon)\Xi\tilde{L}_1^T(\varepsilon) \quad (A.6)$$

$$\dot{\Sigma}_{13} = A_{11d}(\varepsilon)\Sigma_{13} + \tilde{A}_{13}(\varepsilon)\Sigma_{33} + A_{12}(\varepsilon)\Sigma_{43} + \Sigma_{13}A_{11}^T(\varepsilon) + \Sigma_{14}A_{12}^T(\varepsilon) + \tilde{L}_1(\varepsilon)\Xi L_1^T(\varepsilon) \quad (A.7)$$

$$\dot{\Sigma}_{33} = A_{11}(\varepsilon)\Sigma_{33} + A_{12}(\varepsilon)\Sigma_{43} + \Sigma_{33}A_{11}^T(\varepsilon) + \Sigma_{34}A_{12}^T(\varepsilon) + L_1(\varepsilon)\Xi L_1^T(\varepsilon) \quad (A.8)$$

$$\varepsilon\dot{\Sigma}_{12} = \varepsilon A_{11d}(\varepsilon)\Sigma_{12} + \varepsilon\tilde{A}_{13}(\varepsilon)\Sigma_{32} + \varepsilon A_{12}(\varepsilon)\Sigma_{42} + \Sigma_{11}A_{21d}^T(\varepsilon) + \Sigma_{12}A_{22d}^T(\varepsilon) \quad (A.9)$$

$$\varepsilon\dot{\Sigma}_{14} = \varepsilon A_{11d}(\varepsilon)\Sigma_{14} + \varepsilon\tilde{A}_{13}(\varepsilon)\Sigma_{34} + A_{12}(\varepsilon)\tilde{\Sigma}_{44} + \Sigma_{13}A_{21}^T(\varepsilon) + \Sigma_{14}A_{22}^T(\varepsilon) + \tilde{L}_1(\varepsilon)\Xi L_2^T(\varepsilon) \quad (A.10)$$

$$\varepsilon\dot{\Sigma}_{22} = A_{21d}(\varepsilon)\Sigma_{12} + A_{22d}(\varepsilon)\Sigma_{22} + \Sigma_{21}A_{21d}^T(\varepsilon) + \Sigma_{22}A_{22d}^T(\varepsilon) \quad (A.11)$$

$$\varepsilon\dot{\Sigma}_{23} = A_{21d}(\varepsilon)\Sigma_{13} + A_{22d}(\varepsilon)\Sigma_{23} + \varepsilon\Sigma_{23}A_{11}^T + \varepsilon\Sigma_{24}A_{12}^T \quad (A.12)$$

$$\varepsilon\dot{\Sigma}_{24} = A_{21d}(\varepsilon)\Sigma_{14} + A_{22d}(\varepsilon)\Sigma_{24} + \Sigma_{23}A_{24}^T(\varepsilon) + \Sigma_{24}A_{22}^T(\varepsilon) \quad (A.13)$$

$$\varepsilon\dot{\Sigma}_{34} = \varepsilon A_{11}(\varepsilon)\Sigma_{34} + A_{12}(\varepsilon)\tilde{\Sigma}_{44} + \Sigma_{33}A_{21}^T(\varepsilon) + \Sigma_{34}A_{22}^T(\varepsilon) + L_1(\varepsilon)\Xi L_2^T(\varepsilon) \quad (A.14)$$

$$\varepsilon\dot{\tilde{\Sigma}}_{44} = \varepsilon A_{21}(\varepsilon)\Sigma_{34} + A_{22}(\varepsilon)\tilde{\Sigma}_{44} + \varepsilon\Sigma_{43}A_{21}^T(\varepsilon) + \tilde{\Sigma}_{44}A_{22}^T(\varepsilon) + L_2(\varepsilon)\Xi L_2^T(\varepsilon) \quad (A.15)$$

To apply the theorem of Hoppensteadt to (A.6) - (A.15), conditions H_i - H_{viii} (The conditions are conveniently listed in [18]) must be verified. Conditions H_{ii}, H_{iv} and H_v are satisfied since (A.6) - (A.15) are linear time-invariant differential equations with coefficients continuous in ε.

Condition H_{iii} is verified by noting that (A.9) - (A.15) have a unique solution, for $\varepsilon = 0$ when Σ_{11}, Σ_{13}, and Σ_{33} are given, as follows. Since $A_{22}(0)$, $A_{22d}(0)$ are stable, $\tilde{\Sigma}_{44}$, Σ_{12}, and Σ_{23} are uniquely determined by

$$0 = A_{22}(0)\tilde{\Sigma}_{44} + \tilde{\Sigma}_{44}A_{22}^T(0) + L_2(0)\Xi L_2^T(0) \quad (A.16)$$

$$0 = \Sigma_{11}A_{21d}^T(0) + \Sigma_{12}A_{22d}^T(0) \quad (A.17)$$

$$0 = A_{21d}(0)\Sigma_{13} + A_{22d}(0)\Sigma_{23} \quad (A.18)$$

Therefore, Σ_{14}, Σ_{22}, and Σ_{34} are the unique solutions of

$$0 = A_{12}(0)\tilde{\Sigma}_{44} + \Sigma_{13}A_{21}^T(0) + \Sigma_{14}A_{22}^T(0) + \tilde{L}_1(0)\Xi L_2^T(0) \quad (A.19)$$

$$0 = A_{21d}(0)\Sigma_{12} + A_{22d}(0)\Sigma_{22} + \Sigma_{21}A_{21d}^T(0) + \Sigma_{22}A_{22d}^T(0) \quad (A.20)$$

$$0 = A_{12}(0)\tilde{\Sigma}_{44} + \Sigma_{33}A_{21}^T(0) + \Sigma_{34}A_{22}^T(0) + L_1(0)\Xi L_2^T(0) \quad (A.21)$$

Finally, Σ_{24} is uniquely determined by

$$0 = A_{21d}(0)\Sigma_{14} + A_{22d}(0)\Sigma_{24} + \Sigma_{23}A_{21}^T(0) + \Sigma_{24}A_{22}^T(0) \quad (A.22)$$

To verify H_{vi}, it must be shown that the solution defined by (A.16) - (A.22) is an asymptotically stable equilibrium of the following boundary layer system associated with (A.6) - (A.15).

$$\frac{d}{d\tau}\Sigma_{12} = \Sigma_{11}A_{21d}^T(0) + \Sigma_{12}A_{22d}^T(0) \quad (A.23)$$

$$\frac{d}{d\tau}\Sigma_{14} = A_{12}(0)\tilde{\Sigma}_{44} + \Sigma_{13}A_{21}^T(0) + \Sigma_{14}A_{22}^T(0) + \tilde{L}_1(0)\Xi L_2^T(0) \quad (A.24)$$

$$\frac{d}{d\tau}\Sigma_{22} = A_{21d}(0)\Sigma_{12} + A_{22d}(0)\Sigma_{22} + \Sigma_{21}A_{21d}^T(0) + \Sigma_{22}A_{22d}^T(0) \quad (A.25)$$

$$\frac{d}{d\tau}\Sigma_{23} = A_{21d}(0)\Sigma_{13} + A_{22d}(0)\Sigma_{23} \quad (A.26)$$

$$\frac{d}{d\tau}\Sigma_{24} = A_{21d}(0)\Sigma_{14} + A_{22d}(0)\Sigma_{24} + \Sigma_{23}A_{21}^T(0) + \Sigma_{24}A_{22}^T(0) \quad (A.27)$$

$$\frac{d}{d\tau}\Sigma_{34} = A_{12}(0)\tilde{\Sigma}_{44} + \Sigma_{33}A_{21}^T(0) + \Sigma_{34}A_{22}^T(0) + L_1(0)\Xi L_2^T(0) \quad (A.28)$$

$$\frac{d}{d\tau}\tilde{\Sigma}_{44} = A_{22}(0)\tilde{\Sigma}_{44} + \Sigma_{44}A_{22}^T(0) + L_2(0)\Xi L_2^T(0) \quad (A.29)$$

The stability of (A.23) - (A.29) is readily verified using the stability of $A_{22}(0)$, $A_{22d}(0)$ and working through the equations in the order indicated in (A.16)-(A.22).

The final conditions deal with the degenerate system of (A.6) - (A.15). Putting $\varepsilon = 0$ in (A.6)-(A.15), and after some algebra using (A.16)-(A.19), (A.21), the following equations are obtained:

$$\dot{\Sigma}_{33} = (A_{11}(0)-A_{12}(0)A_{22}^{-1}(0)A_{21}(0))\Sigma_{33}+\Sigma_{33}(A_{11}(0)-A_{12}(0)A_{22}^{-1}(0)A_{21}(0))^T + L_1(0)\Xi_1 L_1^T(0) \quad (A.30)$$

$$\dot{\Sigma}_{31} = (A_{11}(0)-A_{12}(0)A_{22}^{-1}(0)A_{21}(0))\Sigma_{31}+\Sigma_{31}(A_{11}(0)-A_{12}(0)A_{22}^{-1}(0)A_{21}(0))^T \quad (A.31)$$

$$\dot{\Sigma}_{11} = (A_{11}(0)-A_{12}(0)A_{22}^{-1}(0)A_{21}(0))\Sigma_{11}+\Sigma_{11}(A_{11}(0)-A_{12}(0)A_{22}^{-1}(0)A_{21}(0))^T \quad (A.32)$$

Since $A_{11}(0) - A_{12}(0)A_{22}^{-1}(0)A_{21}(0)$ is stable, H_i and H_{vi} are satisfied.

Thus by Hoppensteadt's Theorem, $\lim_{\varepsilon\to 0} \Sigma_{11}(t;\varepsilon)$ and $\lim_{\varepsilon\to 0} \Sigma_{22}(t;\varepsilon)$ are equal to the solutions of (A.32) and (A.20). Since

$$\tilde{x}_1(0;\varepsilon) = x_1(0) - x_{1D}(0) = 0 \quad (A.33)$$

by choice of the initial conditions of the stochastic degenerate system, it follows that the initial conditions of (A.32) are

$$\Sigma_{11}(0;\ 0) = 0 \quad (A.34)$$

Thus

$$\lim_{\varepsilon\to 0} \Sigma_{11}(t;\varepsilon) \equiv 0 \quad (A.35)$$

From (A.35), (A.17) and (A.20),

$$\lim_{\varepsilon\to 0} \Sigma_{22}(t;\varepsilon) \equiv 0 \quad (A.36)$$

Note that (A.30) is valid at $t = 0$ by choice of initial conditions. Equations (A.34) and (A.36) are the desired result.

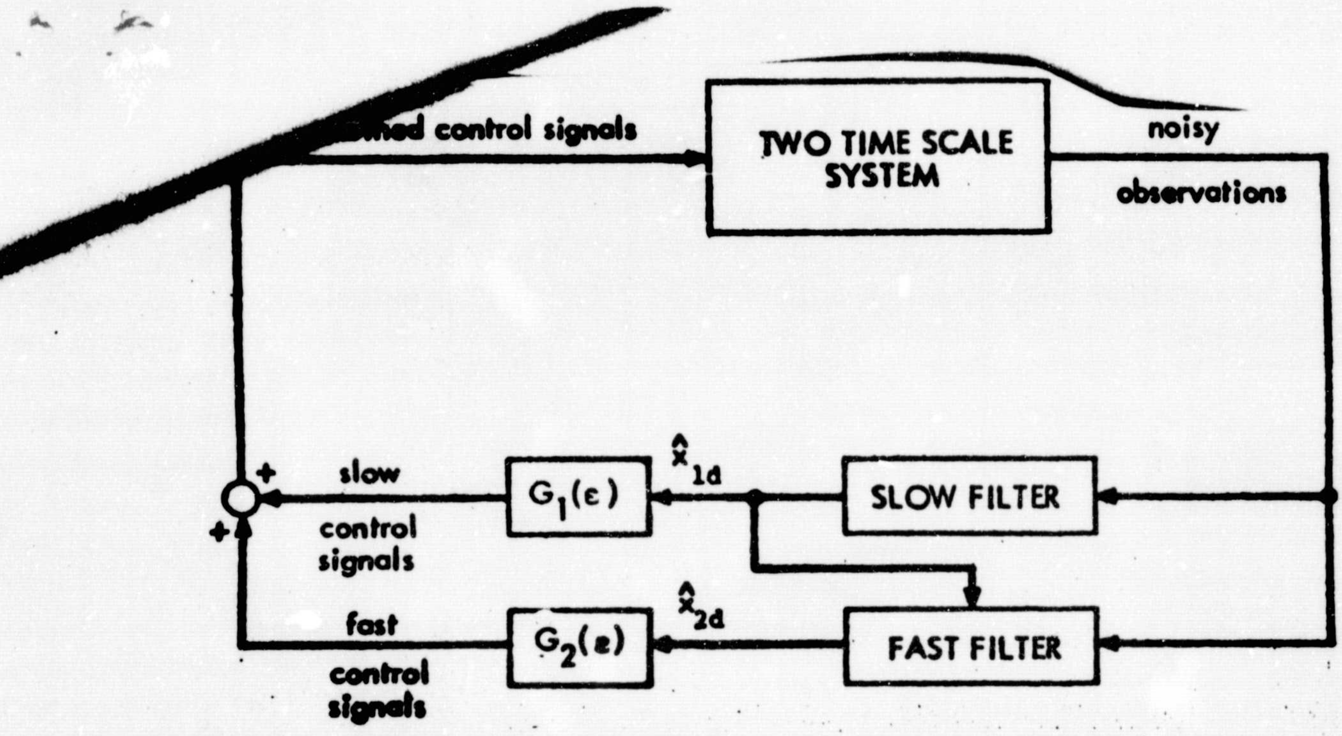

Fig. 1 Asymptotically Optimal Two Time Scale Controller

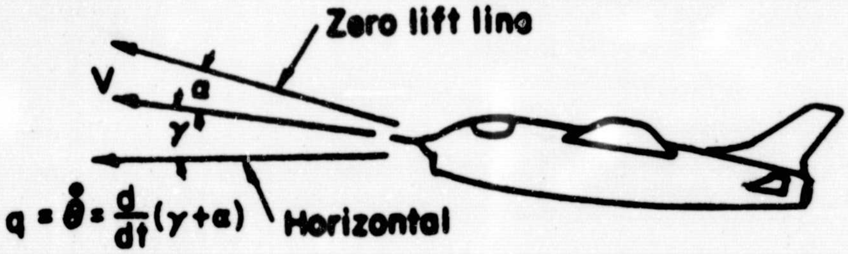

Figure 2 Aircraft Longitudinal Variables

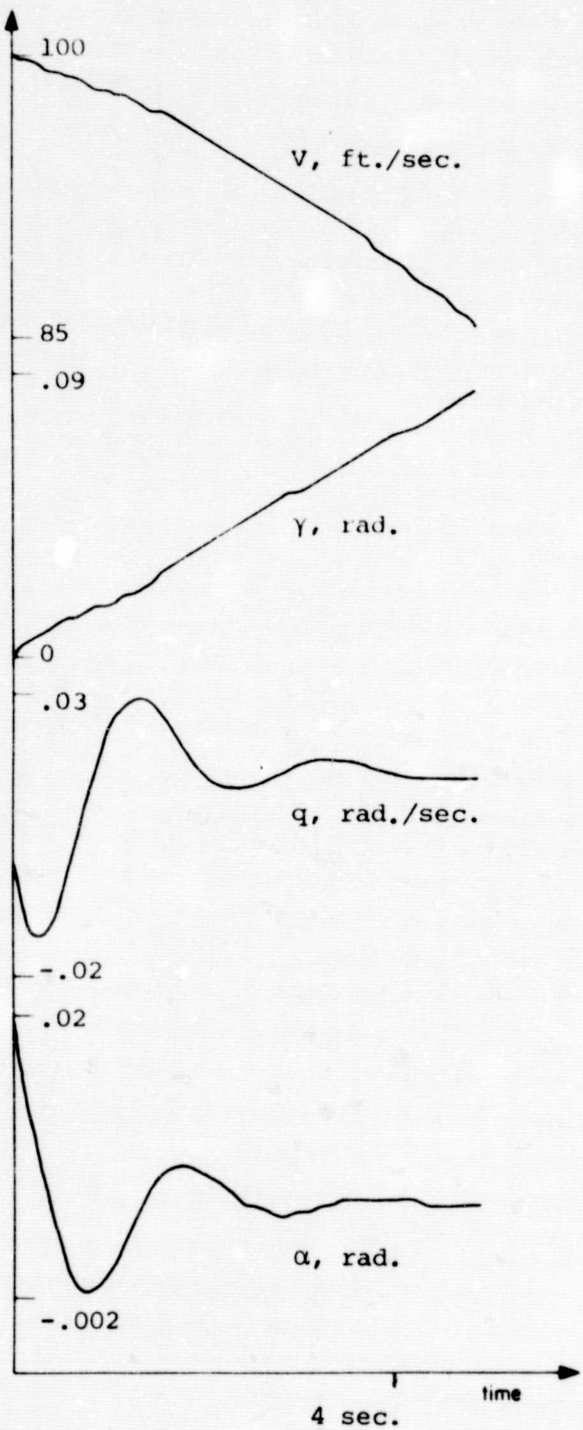

Figure 3.Response in Fast Time Scale

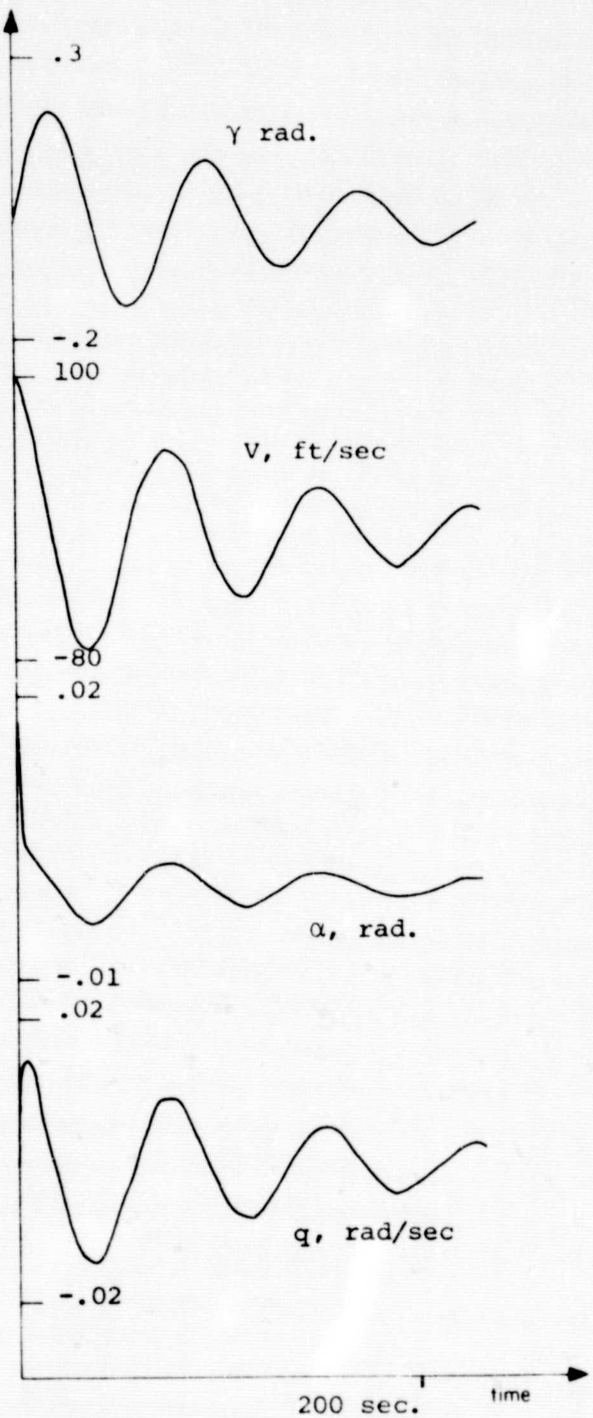

Figure 4.Response in Slow Time Scale

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$$A = \begin{bmatrix} -1.357\times10^{-2} & -3.220\times10^{1} & -4.630\times10^{1} & 0.000 \\ 1.200\times10^{-4} & 0.000 & 1.214 & 0.000 \\ -1.212\times10^{-4} & 0.000 & -1.214 & 1.000 \\ 5.700\times10^{-4} & 0.000 & -9.010 & -6.696\times10^{-1} \end{bmatrix}$$

$$B = \begin{bmatrix} -4.330\times10^{-1} \\ 1.394\times10^{-1} \\ -1.394\times10^{-1} \\ -1.577\times10^{-1} \end{bmatrix} \quad L = \begin{bmatrix} -4.630\times10^{1} \\ 1.214 \\ -1.214 \\ -9.010 \end{bmatrix} \quad \Xi = [3.150\times10^{-4}]$$

$$C = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 1.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \quad \Theta = \begin{bmatrix} 6.859\times10^{-4} & 0.000 \\ 0.000 & 4.000\times10^{1} \end{bmatrix}$$

Table 1. System Matrices

eigenvalue		$-.94 \pm j.2.98$	$-.0075 \pm j.076$
eigenvector components	v	$1.3 \pm j.4.7$	$-4.3\times10^{2} \mp j1.1\times10^{2}$
	γ	$-5\times10^{-4} \mp j.13$	$-.15 \pm j1.0$
	α	$.33 \pm j.10$	$-.02 \mp j.005$
	q	$-.21 \pm j1.0$	$-.08 \mp j.02$

Table 2. Open Loop Eigenvalues and Eigenvectors.

Eigenvalues

open loop	$-.94 \pm j2.98$	$-.075 \pm j0.76$		
optimal	-2.9 ± 2.0	$-.20 \pm j.20$	$-3.8, -2.6$	$-.09 \pm j.10$
two time scale	$-3.1 \pm j2.3$	$-.18 \pm j.16$	$-3.8, -2.6$	$-.1 \pm j.11$

RMS Deviations

	$\sqrt{\overline{V^2}}$	$\sqrt{\overline{\gamma^2}}$	$\sqrt{\overline{\alpha^2}}$	$\sqrt{\overline{q^2}}$	$\sqrt{\overline{(V-\hat{V})^2}}$	$\sqrt{\overline{(\gamma-\hat{\gamma})^2}}$	$\sqrt{\overline{(\alpha-\hat{\alpha})^2}}$	$\sqrt{\overline{(q-\hat{q})^2}}$
open loop	8.8	.0024	.0031	.0082				
optimal	3.3	.0018	.0026	.0065	2.6	.0013	.0021	.0056
two time scale	3.4	.0018	.0026	.0065	2.6	.0015	.0021	.0056

Table 3. Performance Comparison