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# THEORY OF THIN SCREEN SCINTILLATIONS FOR A SPHERICAL WAVE

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**THEORY OF THIN SCREEN  
SCINTILLATIONS FOR A  
SPHERICAL WAVE**

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# THEORY OF THIN SCREEN SCINTILLATIONS FOR A SPHERICAL WAVE

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## ABSTRACT

A thin screen scintillation theory for a spherical wave is presented under the "quasi-optical" approximation. We calculate the "scattering angle", the "observed angle", the intensity correlation function and the temporal pulse broadening for the random wave. It is found that as the wave propagates outward away from the phase screen, the correlation scale of the intensity fluctuation increases linearly while the "observed angle" decreases linearly. The calculations are carried out for both Gaussian and power-law spectra of the turbulent medium.

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# THEORY OF THIN SCREEN SCINTILLATIONS FOR A SPHERICAL WAVE

## I. INTRODUCTION

Phenomena such as the twinkling of starlight, the ionospheric, interplanetary and interstellar radio wave scintillations involve the scattering of an electromagnetic wave by a layer of random medium. When the thickness of the irregularity layer is much smaller than the distance between an observer and the irregularity layer, "thin phase-changing screen approximation" (Salpeter 1967, Lee and Jokipii 1975 b, c) can be used to analyze the observed scintillation data. Applications of the "thin screen scintillation" theory have been successful in interplanetary scintillations (e.g. Cohen et al. 1967), in interstellar scintillations (e.g. Salpeter 1969, Lee and Jokipii 1976), and in scintillations of radio waves during occultation by comet's tail (Lee 1976).

The existing theory of "thin screen scintillation" assumes an initial plane wave impinging on the irregularity layer (Mercier 1962, Salpeter 1967, Lovelace 1970, Lee and Jokipii 1975 b, c). However, most stars or radio sources are compact and emit spherical waves. The "plane wave" assumption applies only to cases where  $(r_1/r_2) \simeq 1$ . Here  $r_1$  is the distance between the source and the irregularity layer, and  $r_2$  is the distance between the source and an observer. For interplanetary and comet's tail scintillations,  $r_1/r_2 \simeq 1$  is satisfied and the plane-wave thin screen scintillation theory can be applied. For interstellar scintillations,  $r_1/r_2 \simeq 0.5$  and the plane-wave theory can still be applied without much risk. However, recent observations of angular broadening and temporal broadening of the Crab Nebula pulsar's radiation indicate the existence of a scattering region of plasma turbulence within the Crab Nebula (Rankin and Counselman 1973, Lyne and Thorne 1975, Vandenberg 1976). In this case,  $r_1/r_2 \ll 1$  and the plane wave approximation can not be applied. In order to analyze the observed data correctly, a theory of thin screen scintillation for a spherical wave is needed.

It is the purpose of the present paper to present a thin screen scintillation theory for a spherical wave. In Section II, we formulate the scintillation problem for a spherical wave. In Section III, we derive the scattering angle  $\theta_s$  and the observed apparent angle  $\theta_o$  of the radiowave caused by the plasma irregularity layer. In Section IV, we consider the intensity fluctuations for both weak and strong scintillations. In Section V, we develop a theory for the temporal broadening of pulses. In Section VI, we summarize the results and compare them with the plane wave case. The theory developed here will be applied to analyze the temporal and angular broadening data of the Crab Nebula pulsar in a future paper.

## II. GENERAL CONSIDERATION FOR A SPHERICAL WAVE

We consider a spherical wave, of wavenumber  $k = 2\pi/\lambda$  and frequency  $(\omega/2\pi)$ , propagating outward from a radio source situated at  $r = 0$ ,

$$E_\omega(\underline{r}, t) = F(\underline{r}, k) e^{-i\omega t} = u(\underline{r}, k) e^{-i(kr - \omega t)/r} \quad (1)$$

Let  $\underline{r} = (r, \theta, \phi)$  be the spherical coordinates in space. As shown in Figure (1), the wave propagates freely until it hits the "thin phase-changing screen" situated at  $r = r_1$ . After passing the screen, the phase of the wave is randomized and is characterized by the function  $\Phi_k(\theta, \phi)$ . Then we have

$$u(r = r_1, \theta, \phi, k) = e^{i\Phi_k(\theta, \phi)} \quad (2)$$

where we have normalized  $u(\underline{r}, k)$  such that

$$u(r < r_1 - D, k) = 1.$$

We will consider in this paper the propagation of a wave with  $\omega \gg \omega_p$ , the plasma frequency of the plasma medium, through the plasma layer. Then the phase function  $\Phi_k(\theta, \phi)$  is related to the electron density fluctuation  $\delta N_e(\underline{r})$  by

$$\Phi_k(\theta, \phi) = (-4\pi r_e/k) \int_{r_1-D}^{r_1} \delta N_e(r', \theta, \phi) dr' \quad (3)$$

where  $r_e$  is the classical electron radius and  $D$  is the thickness of the plasma layer. In deriving Equation (3), we have neglected the diffraction effect inside the plasma layer between  $r_1 - D$  and  $r_1$  (Lee and Jokipii 1975b).

In order to proceed further, we specify the two-point correlation function

$$P_N(\underline{r}) = \langle \delta N_e(\underline{x}) \delta N_e(\underline{x} + \underline{r}) \rangle$$

of the fluctuating electron density  $\delta N_e(\underline{r})$ , from which the statistical properties of the phase function  $\Phi_k(\theta, \phi)$  can be obtained through Equation (3). Assume that the plasma medium is statistically homogeneous and isotropic. Then the correlation function  $P_N(\underline{r})$  depends only on  $r = |\underline{r}|$ . It is convenient here to work with the spatial power spectrum  $\hat{P}_N(q)$  of the fluctuating electron density,

which is related to the correlation function through

$$\hat{P}_N(q) = \int d^3r \tilde{P}_N(\tilde{r}) \exp(i \tilde{q} \cdot \tilde{r}). \quad (4)$$

We consider two types of power spectrum for  $\hat{P}_N(q)$ : a Gaussian spectrum and a power-law spectrum. For a Gaussian spectrum, we write

$$\hat{P}_N(q) = B_G \exp(-q^2 a^2 / 2) \quad (\text{Gaussian}) \quad (5a)$$

where  $a$  is the correlation scale of the fluctuations in electron density. For a power-law spectrum,  $\hat{P}_N(q)$  has the following form:

$$\hat{P}_N(q) = B_P (1 + q^2 L^2)^{-\alpha/2} \exp(-q^2 \ell^2 / 2) \quad (\text{Power-law}) \quad (5b)$$

where  $L$  is the correlation scale of electron density fluctuations and  $\ell$  is the cutoff (or inner) scale. Usually  $\ell \ll L$ , and Equation (5b) for  $\alpha = 11/3$  is the Kolmogorov turbulence spectrum. Note that in each case  $B$  may be related to the mean square electron density fluctuations. We have

$$B_G = (2\pi)^{3/2} a^3 \langle \delta N_e^2 \rangle \quad (6a)$$

$$B_P = 8\pi^{3/2} L^3 \langle \delta N_e^2 \rangle \Gamma(\alpha/2) / \Gamma(\alpha/2 - 3/2) \quad (6b)$$

where  $\langle \delta N_e^2 \rangle$  is the rms fluctuation electron density.

Define  $\tilde{\xi} = (\theta, \phi)$ ,  $\xi = |\tilde{\xi}| = (\theta^2 + \phi^2)^{1/2}$ . From Equation (3), we obtain the correlation of phase fluctuation,

$$\begin{aligned} \Phi_0^2 P_\phi(\theta, \phi) &= \langle \Phi_k(\theta_1, \phi_1) \Phi_k(\theta_1 + \theta, \phi_1 + \phi) \rangle \\ &= (4\pi r_e / k)^2 D \cdot A_N(\theta, \phi), \end{aligned} \quad (7)$$

where



$$A_N(\theta, \phi) = \frac{1}{2} \int_{-\infty}^{\infty} \langle \delta N_e(r_1, \theta_1, \phi_1) \delta N_e(r_1 + r, \theta_1 + \theta, \phi_1 + \phi) \rangle dr \quad (8)$$

Here  $\phi_0^2 = (4\pi r_e/k)^2 A_N(0)$  is the rms phase fluctuation and  $P_\phi$  is the normalized phase function, i.e.  $P_\phi(0) = 1$ . In obtaining Equation (7), we have assumed that  $D$  is much greater than the correlation scale  $a$  or  $L$ . Assuming  $D \ll r_1$ , we then have from Equations (5a), (6a) and (8),

$$A_N(\xi) = \frac{B_G}{4\pi a^2} \exp(-r_1^2 \xi^2 / 2a^2) \quad (9)$$

for the Gaussian spectrum. Now consider the power-law spectrum in Equation (5b). We have for  $r_1 \xi \gg \ell$ , by neglecting the effect of the cutoff at  $q > \ell^{-1}$ ,

$$A_N(\xi) = \frac{B_p (r_1 \xi / L)^\mu K_\mu(r_1 \xi / L)}{2^{2+\mu} \pi L^2 \Gamma(\mu + 1)} \quad (10)$$

where  $\mu = \frac{\alpha}{2} - 1$ ,  $2\mu + \frac{3}{2} > 0$ , and  $K_\mu$  denotes a modified Bessel function of the second kind. One can further show that for  $L \gg r_1 \xi \gg \ell$ ,

$$A_N(\xi) \simeq \frac{B_p}{4\pi (\alpha - 2) L^2} \left[ 1 - \frac{\Gamma\left(2 - \frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \left(\frac{r_1 \xi}{2L}\right)^{\alpha-2} \right]. \quad (11)$$

However, for  $r_1 \xi \leq \ell$ ,

$$A_N(\xi) = A_N(0) + \frac{A_N''(0)}{2} \xi^2 + \frac{A_N^{(4)}(0)}{4!} \xi^4 + \dots, \quad (12)$$

where

$$A_N(0) = \frac{3 B_P}{20\pi L^2},$$

$$A_N''(0) = \left. \frac{d^2 A_N(\xi)}{d\xi^2} \right|_{\xi=0} = \frac{-B_P}{16\pi} L^{-11/3} (\ell^2/2)^{-1/6} \Gamma\left(\frac{1}{6}\right) r_1^2,$$

and

$$A_N^{(4)}(0) = \left. \frac{d^4 A_N(\xi)}{d\xi^4} \right|_{\xi=0} = \frac{3 B_P}{64\pi} L^{-11/3} (\ell^2/2)^{-7/6} \Gamma\left(\frac{7}{6}\right) r_1^4,$$

for  $\alpha = 11/3$ .

We shall be interested in the statistical properties of the fluctuating wave at  $r = r_2$ . After passing the screen, the radio wave propagates in free space and the wave function  $F(\underline{r}, k)$  satisfies the following equation,

$$(\nabla^2 + k^2) F(\underline{r}, k) = 0. \quad (13)$$

Equation (13) can also be written in terms of  $u$  as

$$2ik \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0. \quad (14)$$

We are interested in situations where fluctuation scales of  $u$  in the radial direction are much larger than wavelength of the radiation  $\lambda$ , so that  $\partial^2 u / \partial r^2$  in Equation (14) can be neglected. The omission of the term  $\partial^2 u / \partial r^2$  is called the "quasi-optical" or "parabolic" approximation (Lee and Jokipii 1975a). Assuming the scattering angle caused by the random phase screen is small, Equation (14) can be simplified as

$$2ik \frac{\partial u}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} \right) = 0 \quad (15)$$

for waves propagating in the direction  $\theta \simeq \pi/2$ ,  $\phi \simeq 0$ . Let the observer be situated near  $(r = r_2, \theta = \pi/2, \phi = 0)$ . We then have from Equation (15),

$$u(r_2, \theta, \phi) = \frac{-i k s}{2\pi} \iint d\theta_1 d\phi_1 e^{i\Phi_k(\theta_1, \phi_1) + \frac{i}{2} k s [(\theta - \theta_1)^2 + (\phi - \phi_1)^2]} \quad (16)$$

where  $s = r_1 r_2 / (r_2 - r_1)$ . Equation (16) relates the wave function  $u$  at  $r_2$  to the random phase function  $\Phi_k(\theta, \phi)$  at  $r_1$ . We will use Equation (16) to calculate the scattering angle, the intensity correlation, and the pulse broadening in the following sections.

### III. ANGULAR SCATTERING

Consider the Fourier transform of the random wave function  $u(r_2, \vec{\xi})$  observed at  $r_2$ ,

$$\hat{u}(r_2, \vec{\Theta}) = \frac{(k r_2)^2}{4\pi^2} \iint d\vec{\xi}_2 u(r_2, \vec{\xi}_2) \exp(-i k r_2 \vec{\Theta} \cdot \vec{\xi}_2). \quad (17)$$

It follows that

$$u(r_2, \vec{\xi}_2) = \iint d\vec{\Theta} \hat{u}(r_2, \vec{\Theta}) \exp(i k r_2 \vec{\Theta} \cdot \vec{\xi}_2). \quad (18)$$

For small scattering angles, it can be shown from Equations (11) and (18) that  $\hat{u}(r_2, \vec{\Theta})$  is the Fourier component of the random wave  $u$ , propagating in the direction with angle  $\vec{\Theta}$  relative to the radial direction  $\hat{r}$ . We call  $\hat{u}(r_2, \vec{\Theta})$  the "angular spectrum" for a spherical random wave. The plane-wave case has been treated by Booker, Ratcliffe and Shinn (1950).

Let the two-point correlation of the random wave  $u(r_2, \vec{\xi}_2)$  be

$$I_2(r_2, \vec{\xi}_2) = \langle u(r_2, \vec{\xi}_2) u^*(r_2, \vec{\xi}_2 + \vec{\xi}) \rangle \quad (19)$$

where  $*$  denotes complex conjugate. It may be shown that for the random wave

$u$ , the probability of distribution of intensity over angle  $\Theta$ ,  $\psi(r_2, \Theta)$ , is related to  $\Gamma_2$  by the relationship

$$\psi(r_2, \Theta) = \frac{k^2 r_2^2}{4\pi^2} \iint d\xi \Gamma_2(r_2, \xi) \exp(-ikr_2 \xi \cdot \Theta) \quad (20)$$

(Booker, Ratcliffe and Shinn 1950, Lee and Jokipii 1975a).  $\psi(r_2, \Theta)$  is the "angular power spectrum" of the random wave  $u$ . Equation (20) states that "the correlation function of a random wave is proportional to the Fourier transform of its angular power spectrum," which is called the Wiener's theorem (Wiener 1930).

From Equation (16), we obtain

$$\begin{aligned} \Gamma_2(r_2, \xi) = & \frac{k^2 s^2}{4\pi^2} \iint d\xi_1 d\xi_2 \exp \left\{ -\phi_0^2 \left[ 1 - P_\phi(\xi_1 - \xi_2) \right] \right\} \\ & \cdot \exp \left\{ \frac{iks}{2} \left[ (\xi_1^1 - \xi_1)^2 - (\xi_1^1 + \xi_1 - \xi_2)^2 \right] \right\} \end{aligned} \quad (21)$$

where  $s = r_1 r_2 / (r_2 - r_1)$ . In deriving Equation (21), we have assumed the Gaussian statistics of the random phase function  $\Phi$ , or equivalently, of the electron density fluctuation  $\delta N_e$ . It can be shown from Equation (21) that

$$\Gamma_2(r_2, \xi) = \exp \left\{ -\phi_0^2 \left[ 1 - P_\phi(\xi) \right] \right\} = \Gamma_2(r_1, \xi). \quad (22)$$

Let the characteristic scale of  $\Gamma_2$  be  $\xi_c$ . For  $\phi_0 \leq 1$ , we have from Equation (22),

$$\begin{aligned} \xi_c &= a/r_1, \text{ for Gaussian spectrum} \\ &= L/r_1, \text{ for power-law spectrum.} \end{aligned} \quad (23a)$$

For  $\phi_0 > 1$ , we define  $\xi_c$  such that

$$\phi_0^2 [1 - P_\phi(\xi_c)] = 1.$$

Then we have for a Gaussian spectrum

$$\xi_c = \frac{\sqrt{2} a}{r_1 \phi_0}, \quad (23b)$$

and for a power-law spectrum

$$\xi_c = \left[ \frac{2L}{r_1} \right] \left[ \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(2 - \frac{\alpha}{2}\right) \phi_0^2} \right]^{\frac{1}{\alpha-2}}, \quad \text{if } r_1 \xi_c > \ell, \quad (23c)$$

$$\xi_c = \left[ \frac{8 L^{\alpha-2} (\ell^2/2)^{2-\frac{\alpha}{2}}}{(\alpha-2) \Gamma\left(2 - \frac{\alpha}{2}\right) \phi_0^2 r_1^2} \right]^{\frac{1}{2}}, \quad \text{if } r_1 \xi_c < \ell. \quad (23d)$$

The angular power spectrum  $\psi(r_1, \theta)$  at  $r_1$  is also related to the correlation  $\tilde{f}_2(r_1, \xi)$  by an equation similar to Equation (20). From Equation (22), we have

$$\psi(r_2, \theta) = \left( \frac{r_2}{r_1} \right)^2 \psi\left(r_1, \frac{\theta r_2}{r_1}\right). \quad (24)$$

Let the characteristic angles at  $r_1$  and  $r_2$  be  $\theta_{1c}$  and  $\theta_{2c}$  respectively. Then we have

$$\Theta_0 = \Theta_{2c} = \frac{r_1}{r_2} \Theta_{1c} \quad \frac{r_1}{r_2} \Theta_s = \frac{1}{k r_2 \sin \epsilon} \quad (25)$$

In Equation (25), we identify  $\Theta_{2c}$  as the "observed angle"  $\Theta_0$  at  $r_2$  and  $\Theta_{1c}$  as the "scattering angle"  $\Theta_s$  due to the plasma irregularity layer situated at  $r_1$ . Note that the observed angle  $\Theta_0$  is smaller than the scattering angle by a factor of  $(r_1/r_2)$ . One would obtain the relationship between  $\Theta_0$  and  $\Theta_s$  in Equation (25) from the geometry as shown in Figure (1) under the assumption of the validity of geometric optics. Note that for a plane-wave,  $\Theta_0$  and  $\Theta_s$  are of the same value.

One can also calculate the mean square angles which are defined as

$$\langle \Theta_i^2 \rangle = \langle \Theta_{1i}^2 \rangle + \langle \Theta_{2i}^2 \rangle = \int \int d\Omega_i (\Theta_{1i}^2 + \Theta_{2i}^2) \Psi(r_i, \Omega_i) \quad (26)$$

for  $i = 1, 2$ . Note that  $\Omega_i = (\Theta_{1i}, \Theta_{2i})$ . We then have

$$r_1^2 \langle \Theta_1^2 \rangle + r_2^2 \langle \Theta_2^2 \rangle = -r_0^2 k^{-2} \left. \frac{\partial^2 P_s(\frac{r}{r_0})}{\partial r^2} \right|_{r=0} \quad (27)$$

For a Gaussian spectrum, we have

$$\langle \Theta_{1i}^2 \rangle = \langle \Theta_{2i}^2 \rangle = \frac{1}{2} \langle \Theta_i^2 \rangle = \left( \frac{r_0}{k a} \right)^{1/2} \quad (28a)$$

For a power-law spectrum with  $\alpha < 4$ ,  $L \gg l$ , we have

$$\langle \Theta_1^2 \rangle = \frac{(\alpha - 2) r_0^2}{2 k^2} L^{(2-\alpha)} (l^2 + 2)^{\frac{\alpha}{2} - 2} \left( 2 - \frac{\alpha}{2} \right) \quad (28b)$$

#### IV. THE FLUCTUATIONS IN INTENSITY

In order to calculate the intensity correlation function of the random wave  $u$  of wavenumber  $k$ , we consider the fourth moment  $I_4$  which is defined as

$$\Gamma_4(\mathbf{r}, \xi_a, \xi_\beta) = \left\langle u(\mathbf{r}, \xi) u^*(\mathbf{r}, \xi + \xi_a) u^*(\mathbf{r}, \xi + \xi_\beta) u(\mathbf{r}, \xi + \xi_a + \xi_\beta) \right\rangle. \quad (29)$$

The initial value of  $\Gamma_4$  at  $\mathbf{r}_1$  can be obtained from Equation (2) as

$$\begin{aligned} \Gamma_4(\mathbf{r}_1, \xi_a, \xi_\beta) = \exp \left\{ -\phi_0^2 \left[ 2P_\phi(0) - 2P_\phi(\xi_a) - 2P_\phi(\xi_\beta) \right. \right. \\ \left. \left. + P_\phi(\xi_a + \xi_\beta) + P_\phi(\xi_a - \xi_\beta) \right] \right\}. \end{aligned} \quad (30)$$

The intensity correlation  $P_I$  is related to  $\Gamma_4$  by

$$P_I(\mathbf{r}, \xi) = \Gamma_4(\mathbf{r}, \xi_a = \xi, \xi_\beta = 0) - 1. \quad (31)$$

The scintillation index  $M_r$  is defined as

$$M_r^2 = P_I(\mathbf{r}, 0). \quad (32)$$

Assuming that the plasma irregularity layer is statistically homogeneous, we have from Equation (16)

$$\frac{\partial \Gamma_4(\mathbf{r}, \xi_a, \xi_\beta)}{\partial \mathbf{r}} = \frac{i}{k r^2} \nabla_{\xi_a} \cdot \nabla_{\xi_\beta} \Gamma_4, \quad (33)$$

for  $\mathbf{r} > \mathbf{r}_1$ . Define

$$\eta = -\frac{1}{r}, \quad \eta_1 = -\frac{1}{r_1}, \quad \eta_2 = -\frac{1}{r_2}.$$

Equation (33) can also be written as

$$\frac{\partial \Gamma_4(\eta, \xi_a, \xi_\beta)}{\partial \eta} = \frac{i}{k} \nabla_{\xi_a} \cdot \nabla_{\xi_\beta} \Gamma_4, \quad (33')$$

which is Equation (65) of Lee and Jokipii (1975c) for the plane-wave case if  $\eta$  is replaced by  $z$ .

From Equation (33) or (33'), we have

$$\Gamma_4(\eta_2, \tilde{\xi}_\alpha, \tilde{\xi}_\beta) = \left[ \frac{k}{2\pi(\eta_2 - \eta_1)} \right]^2 \iint \Gamma_4(\eta_1, \tilde{\xi}_\alpha, \tilde{\xi}_\beta) \cdot \exp \left[ - \frac{ik}{(\eta_2 - \eta_1)} (\tilde{\xi}_\alpha - \tilde{\xi}_\alpha') \cdot (\tilde{\xi}_\beta - \tilde{\xi}_\beta') \right] d\tilde{\xi}_\alpha d\tilde{\xi}_\beta. \quad (34)$$

Let  $\hat{P}_I(r, q)$  and  $\hat{P}_\phi(r, q)$  be respectively the Fourier transforms of  $P_I(r, \tilde{\xi})$  and  $P_\phi(\tilde{\xi})$  in the transverse coordinate  $\tilde{\xi}$ . We have

$$\hat{P}_I(r, q) = (2\pi)^{-2} \iint P_I(r, \tilde{\xi}) e^{-i\tilde{\xi} \cdot q} d\tilde{\xi}$$

$$\hat{P}_\phi(r, q) = (2\pi)^{-2} \iint P_\phi(\tilde{\xi}) e^{-i\tilde{\xi} \cdot q} d\tilde{\xi}.$$

It can be shown from Equation (34) that

$$\hat{P}_I(r_2, q) = 4\phi_0^2 \hat{P}_\phi(q) \sin^2 \left[ \frac{(r_2 - r_1) q^2}{2k r_1 r_2} \right] \quad (35)$$

for weak scintillations where  $M_r^2 \ll 1$  (Salpeter 1967, Jokipii 1970).

For small scattering angle, the local transverse Cartesian coordinate  $\rho$  is related to  $\tilde{\xi}$  approximately as:  $\rho = r \tilde{\xi}$ . Let the Fourier transforms in  $\tilde{\rho}$  of  $P_I$  and  $P_\phi$  be respectively,



$$Q_I(r_2, \kappa) = \left(\frac{1}{2\pi}\right)^2 \iint P_I\left(r_2, \xi = \frac{\rho}{r_2}\right) e^{-i\rho \cdot \kappa} d\rho$$

and

$$Q_\phi(\kappa) = \left(\frac{1}{2\pi}\right)^2 \iint P_\phi\left(\xi = \frac{\rho}{r_1}\right) e^{-i\rho \cdot \kappa} d\rho.$$

Then Equation (35) can be written as

$$Q_I(r_2, \kappa) = 4\phi_0^2 \sin^2 \left[ \frac{r_1(r_2 - r_1)(\kappa r_2 / r_1)^2}{2k r_2} \right] Q_\phi(r_2 \kappa / r_1) \quad (35')$$

where  $\kappa = |\kappa|$ . From Equation (35'), we see that the scale of Fresnel zone for a spherical wave is

$$\rho_F = \left[ \lambda r_1(r_2 - r_1) / r_2 \right]^{1/2} \quad (36)$$

(c.f. Born and Wolf 1959, page 372). Consider two limiting cases for  $\rho_F$ :

(i) For  $r_1 / r_2 \approx 1$ , we have

$$\rho_F \approx \left[ \lambda (r_2 - r_1) \right]^{1/2}. \quad (37a)$$

Here  $(r_2 - r_1)$  is the distance between the phase-changing screen and the observer. The plane-wave approximation is applicable in this case.

(ii) For  $r_1 \ll r_2$ , we have

$$\rho_F \approx (\lambda r_1)^{1/2}. \quad (37b)$$

Here  $r_1$  is the distance between the radio source and the screen and the plane-wave approximation is not applicable in this case.

Let  $\rho_I$  be the characteristic scale of  $P_I(r, \rho)$ . From Equation (35'), we have

(i) For Gaussian spectrum,

$$\rho_I = r_2 a / r_1; \quad (38a)$$

(ii) For power-law spectrum,

$$\rho_I = r_2 \rho_F / r_1, \text{ if } L > \rho_F > \ell \quad (38b)$$

$$= r_2 \ell / r_1, \text{ if } \ell > \rho_F. \quad (38c)$$

$$= r_2 L / r_1, \text{ if } \rho_F > L. \quad (38d)$$

Define  $r_{12}$  and the characteristic distance  $r_c$  respectively as

$$r_{12} = r_1(r_2 - r_1) / r_2 \quad (39)$$

and

$$r_c = k a^2 / \phi_0 \quad (\text{Gaussian spectrum}) \quad (40a)$$

$$= k L^2 \phi_0^{-4/(\alpha-2)} \quad (\text{power-law spectrum, } r_1 \xi_c > \ell) \quad (40b)$$

$$= k L^{\alpha-2} \ell^{4-\alpha} / \phi_0 \quad (\text{power-law, } r_1 \xi_c < \ell). \quad (40c)$$

It can be shown from Equation (35') that when

$$(i) \quad \phi_0 < 1 \quad (41a)$$

or

$$(ii) \quad \phi_0 \geq 1, \quad r_{12} \ll r_c, \quad (41b)$$

the scintillation is weak ( $M_r^2 \ll 1$ ) and Equation (35') gives us the power spectrum of intensity fluctuations.

For  $\phi_0 \geq 1$ ,  $r_{12} \gg r_c$ , one obtains from Equation (33) that

$$P_I(r_2, \xi) = \exp \left\{ -2\phi_0^2 \left[ 1 - P_\phi(\xi) \right] \right\} = \exp \left[ -2\phi_0^2 \right] \quad (42)$$

(Lee and Jokipii 1975c). In this case, the scintillation is strong ( $M_r^2 \simeq 1$ ). The scale of intensity fluctuations can be obtained from Equation (42)

$$\rho_I = 2^{-1/2} k^{-1} \Theta_0^{-1} = (a/\phi_0) \cdot (r_2/r_1) \quad (\text{Gaussian}) \quad (43a)$$

$$= 2^{-1/(a-2)} k^{-1} \Theta_0^{-1} \quad (\text{power-law, } L > r_1 \xi_c > l) \quad (43b)$$

$$= 2^{-1/2} k^{-1} \Theta_0^{-1} \quad (\text{power-law, } r_1 \xi_c < l) \quad (43c)$$

where  $\Theta_0$  and  $\xi_c$  are given by Equations (23) and (25). From Equations (38) and (43), we find that  $\rho_I \propto (r_2/r_1)$ . Thus the intensity correlation scale expands as the random wave propagates outward spherically. For the region where  $r_{12} \simeq r_c$ , the correlation scale  $\rho_I$  changes smoothly from the values given by Equations (38a, b, c) to those given by Equations (43a, b, c) (Lee 1974).

## V. TEMPORAL BROADENING OF PULSES

The temporal broadening of pulses has been considered by Lee and Jokipii (1975b) for the plane-wave case. We consider here the pulse broadening for a spherical wave. Let  $R(r, t)$  be the probability distribution of pulse intensity as a function of time  $t$  measured at  $r$ . Define the second moment of the random wave  $u$  between different wave-numbers as

$$\Gamma(r, \xi, \Delta k) = \left\langle u(r, \xi_1, k_1) u^*(r, \xi_2, k_2) \right\rangle \quad (44)$$

where  $\xi = \xi_1 - \xi_2$  and  $\Delta k = k_1 - k_2$ . It can be shown that

$$R(r, t) = \frac{c}{2\pi} \int d\Delta k \Gamma(r, 0, \Delta k) \exp(-i \Delta k c t) \quad (45)$$

(Lee and Jokipii 1975b). A transport equation for  $\Gamma(r, \xi, \Delta k)$  can be obtained from Equation (15),

$$\frac{\partial \Gamma(r, \xi, \Delta k)}{\partial r} + \frac{i \Delta k}{2 k^2 r^2} \nabla_{\xi}^2 \Gamma + O(k^3) = 0. \quad (46)$$

We assume  $\Delta k$  is small and  $|\Delta k| \ll k$  and neglect terms of order  $\Delta k^3$  in Equation (46). The initial value of  $\Gamma$  at  $r_1$  can be obtained from Equation (2) as

$$\Gamma(r_1, \xi, \Delta k) = \exp \left\{ -\frac{\Delta k^2}{2 k^2} \phi_0^2 \right\} \exp \left\{ -\phi_0^2 [1 - P_{\phi}(\xi)] \right\}. \quad (47)$$

By employing the same technique used by Lee and Jokipii (1975b), we obtain from Equations (45, 46, 47),

$$R(r_2, t) = R_1(t) \star R_2(r_2, t) \quad (48)$$

where  $\star$  denotes convolution of two functions in the variable  $t$ , and

$$R_1(t) = (c k / \phi_0) \pi^{-1/2} \exp(-c^2 k^2 t^2 / \phi_0^2) \quad (49a)$$

$$R_2(t) = \frac{2\pi c r_1}{r_2(r_2 - r_1)} \psi \left( r_2, \Theta = \left[ \frac{2 c r_1 t}{r_2(r_2 - r_1)} \right] \right), \quad t \geq 0, \quad (49b)$$

$$= 0, \quad t < 0. \quad (49b)$$

In Equation (49b),  $\psi(r_2, \Theta)$  is the probability distribution of wave intensity over angle as discussed in Section III.

$R_1(t)$  is the broadening due to the "pure refraction" effect because it gives the effect of the differing transit times of different rays due to the varying index of refraction. Note that  $R_1(t)$  is a symmetric Gaussian function with characteristic time

$$t_1 = \phi_0 / c k . \quad (50)$$

$R_2(r_2, t)$  gives the pulse profile due to the "pure diffraction" effect. From Equation (49b), we obtain the characteristic time scale  $t_2$  of  $R_2$  as

$$t_2 = \frac{r_2(r_2 - r_1)}{2 c r_1} \Theta_0^2 = \frac{r_1(r_2 - r_1)}{2 c r_2} \Theta_s^2 \quad (51)$$

where  $\Theta_0$  and  $\Theta_s$  are respectively the "observed angle" and "scattering angle" as given by Equation (25). The forms of  $R_2(t)$  for a Gaussian irregularity spectrum and a Kolmogorov power-law spectrum are exactly the same as for the plane-wave case and are given by Figure (6) of the Lee and Jokipii (1975b). For the Gaussian spectrum,  $R_2(t)$  is an exponential function. But for a power-law spectrum,  $R_2(t)$  does not have an exponential shape. The fact that  $R_2(t) = 0$  for  $t < 0$  indicates that all the scattered rays are delayed.

## VI. DISCUSSIONS

From the results obtained in the previous sections, we find that the plane-wave thin screen approximation is no longer applicable for cases where  $r_1/r_2 \ll 1$ . We summarize the thin screen scintillation results for a spherical wave as follows.

- (a) There are two characteristic angles, namely, the scattering angle  $\Theta_s$  and the observed angle  $\Theta_0$ . The scattering angle  $\Theta_s$  is determined by the amount of irregularity of the plasma layer and does not depend on  $r_1$  or  $r_2$ . However, the observed angle  $\Theta_0$  decreases as the wave propagates away from the scattering screen and is equal to  $(r_1 \Theta_s / r_2)$ . In the plane wave case,  $\Theta_0 = \Theta_s$ . The analysis of scattering data of the Crab Nebula Pulsar by Vandenberg (1976) is incorrect since the author uses  $\Theta_0$  as the angle related to the amount of irregularity in the plasma layer.
- (b) The scale of Fresnel size  $r_F$  is  $[r_1(r_2 - r_1) \wedge / r_2]^{1/2}$  instead of  $[(r_2 - r_1) \wedge]^{1/2}$  as in the plane-wave case.

- (c) The intensity correlation scale  $\rho_I$  is proportional to  $(r_2 / r_1)$  and increases as the wave propagates outward away from the screen. In the plane-wave case,  $\rho_I$  reaches a steady value for strong scintillations.
- (d) The conditions for strong scintillations are  $\phi_0 > 1$  and  $r_{12} = r_1(r_2 - r_1)/r_2 > r_c$ . For plane-wave case, the conditions are:  $\phi_0 > 1$ ,  $z = (r_2 - r_1) > r_c$ .
- (e) The characteristic time scale  $t_2$  of pulse broadening due to the diffraction effect is given by

$$t_2 = \frac{r_2(r_2 - r_1)}{2 c r_1} \theta_0^2 = \frac{r_1(r_2 - r_1)}{2 c r_2} \theta_s^2.$$

Finally we note that an observer usually measures the correlation time  $\tau_I$  of intensity fluctuations. Let the characteristic scale of  $P_I$  in Equation (42) be  $\xi_I$ . Then we have  $\xi_I = \rho_I / r_2$ . Assume the fluctuations in the plasma layer are frozen in and the plasma layer moves at an angular velocity  $\dot{\xi} = d\xi/dt$  transverse to the line of sight to the pulsar. Then

$$\tau_I = \xi_I / \dot{\xi} = \rho_I / (r_2 \dot{\xi}). \quad (52)$$

If  $\dot{\xi}$  is caused by the motion of the plasma layer with a transverse velocity  $v_1$  relative to the line of sight to the pulsar, then  $\dot{\xi} = v_1 / r_1$ . And we have

$$\tau_I = \frac{\rho_I r_1}{v_1 r_2}. \quad (53)$$

If  $\dot{\xi}$  is caused by the transverse motion of an observer with velocity  $v_2$ , then  $\dot{\xi} = v_2 / r_2$ . We have

$$\tau_I = \rho_I / v_2. \quad (54)$$

For example, if the temporal broadening of pulses from the Crab Nebula pulsar is due to Filament 159 as proposed by Vandenberg (1976), then  $r_1 / r_2 = 10^{-4}$ ,

$v_1 = 30$  km/sec and  $\rho_1 = 2 \times 10^{10}$  cm. One obtains from Equation (53),  $\tau_1 = 0.67$  second, comparing to  $\tau_1 = \rho_1 / v_1 \simeq 2$  hours as would be obtained from the incorrect plane-wave theory.

In a later paper, we will apply the theory developed here to analyze the temporal and angular broadening data for the Crab Nebula pulsar.

## REFERENCES

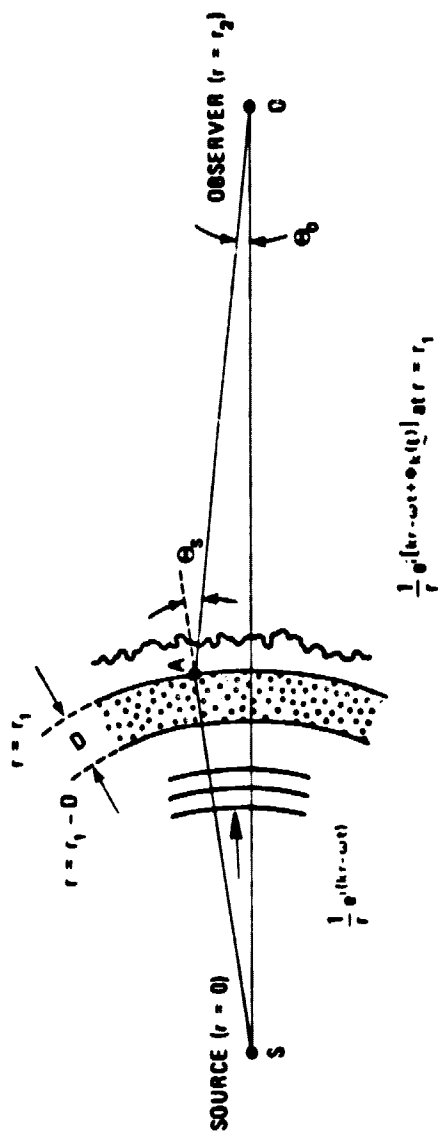
- Booker, H. G., Ratcliffe, J. A., and Shinn, D. H. 1950, *Phil. Trans. Roy. Phys. Soc.*, 242, A579.
- Born, M., and Wolf, E. 1959, *Principles of Optics*, Oxford, Pergamon Press, page 372.
- Cohen, M. H., Gundermann, E. J., Hardebeck, H. E., and Sharpe, L. E. 1967, *Ap. J.*, 147, 449.
- Jokipii, J. R. 1970, *Ap. J.*, 161, 1147.
- Lee, L. C. 1974, unpublished Ph.D. thesis, California Institute of Technology.
- Lee, L. C. To be published in the November 15, 1976 issue of *Ap. J.*
- Lee, L. C. and Jokipii, J. R. 1975a, *Ap. J.*, 196, 695.
- . 1975b, *Ap. J.*, 201, 532.
- . 1975c, *Ap. J.*, 202, 439.
- . 1976, *Ap. J.*, 206, 735.
- Lovelace, R. V. E. 1970, Ph.D. Thesis, Cornell Univ., Ithaca, N.Y.
- Lyne, A. G. and Thorne, D. J. 1975, *M.N.R.A.S.*, 172, 97.
- Mercier, R. P. 1962, *Proc. Roy. Soc.*, A58, 382.
- Salpeter, E. E. 1967, *Ap. J.*, 147, 433.
- Salpeter, E. E. 1969, *Nature*, 221, 31.
- Rankin, J. M. and Counselman, C. C. 1973, *Ap. J.*, 181, 875.
- Vandenberg, N. R. 1976, *Ap. J.*, 209, 578.
- Wiener, N. 1930, *Acta Math.* 55, 118.



# FIGURE CAPTION

Figure 1. A schematic sketch of the thin screen diffraction problem for a spherical wave. The random medium is confined to a thin layer of thickness  $D$  (from  $r = r_1 - D$  to  $r = r_1$ ). The spherical wave  $e^{i(kr - \omega t)}/r$  propagating from the source ( $r = 0$ ) hits the "thin screen". After passing the screen, the phase of the wave is randomized and is characterized by the function  $\Phi_k(\theta, \phi)$ . A ray path SAO is shown to demonstrate that  $\theta_0 = r_1 \theta_s / r_2$ .

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$$\frac{1}{r} e^{i(kr - \omega t + \phi_k(\hat{r}))} \text{ at } r = r_1$$

$$\frac{1}{r} e^{i(kr - \omega t)}$$

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