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STATISTICAL PROPERTIES OF FILTERED PSEUDO-RANDOM DIGITAL SEQUENCES

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SECTION I

INTRODUCTION

The purpose of this report is to:

(1) Give a tutorial presentation of pseudo-random digital sequences, their generation and properties; and

(2) Report the results of a study of filtered pseudo-random sequences, and their statistical properties.

This study resulted from a need for specific design information for a pseudo-random signal generator. The generator, to be used in a telemetry communications system test unit, must generate its pseudo-random signals by filtering a long digital sequence. Desired signal properties include:

(1) Approximately Gaussian amplitude probability density function; and

(2) Signal spectral envelope approximately that of the filter being used in the generator.

Filtered maximum-length sequences have been used for this, and similar applications in the past.[1,2,3] The results reported were good for low-pass filtered sequences when the ratio of digital clock frequency to filter cutoff frequency was between fifteen and twenty. However, for higher values of this ratio, a definite skewing of the amplitude density function has been observed.[4] This skewing effect has been confirmed experimentally (See Section VI) for several digital generator configurations.

R. P. Gilson has reported that the skewing effect can be averted by using a digital sequence composed of the Modulo-2 sum of the outputs of two maximum-length digital sequence generators. His paper included the results of experiments with nine and eleven stage generators. Near Gaussian shaped

probability density functions were obtained for clock to cutoff ratios of up to eighty.

Experiments supporting this study have confirmed Gilson's results; however, it was found experimentally and by computer simulation, that sequence generation by summing two maximum-length sequences did not always result in a non-skewed amplitude probability density function. To the contrary, in some cases a single maximum-length sequence gave better results in this respect than did the sum of two maximum-length sequences. Therefore, finding a relation between the digital sequence generator feedback configuration and the sequence statistical structure was important to meeting the objectives of the study.

This effort includes an investigation of the phase distribution of the spectral components of various binary sequences, and a direct analysis of the relation between third moment (skewing indicator) of subsequences of long binary sequences and the sequence characteristic equation. Each approach yields useful information when evaluating the quality of a pseudo-random sequence.

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SECTION II

SEQUENCE GENERATION

The type sequences to be considered in this section can be described as "linear shift-register generated sequences". The sequences are linear when the generator feedback operations are restricted to addition, and not multiplication. Such sequences obey the laws of linear mathematics such as the law of superposition. Figure II-1 is an example of a sequence generator. Each box represents a digital storage element, and feedforward and feedback are accomplished with the use of "exclusive-or", or "modulo-2 half adder" gates. The unit is clock driven, and the output sequence can be taken from any stage. The generator in the figure is called a "multiple-return generator" because of the presence of several feedforward and feedback paths. If feedforward is eliminated, the feedback is limited to the first stage of the register, then the unit is called a "simple shift register generator" (SSRG). Figure II-2 illustrates a generator of this type.

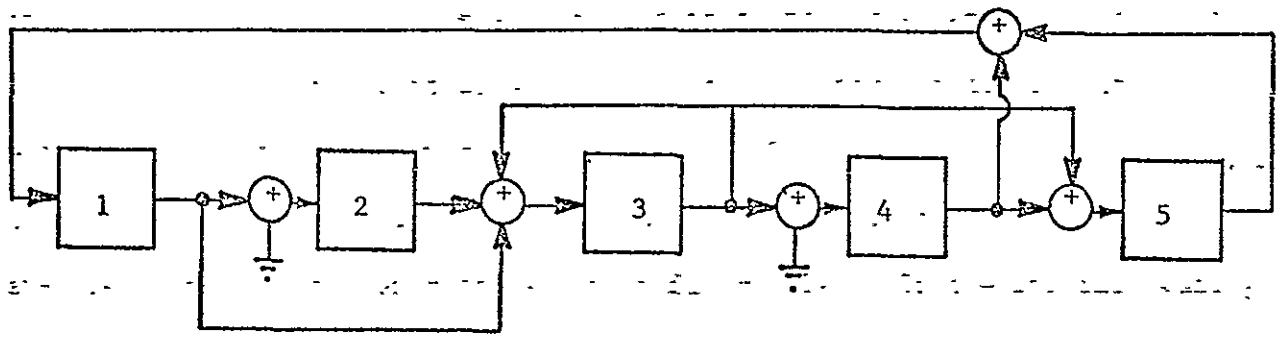

Figure II-1. Multiple-Return Generator.

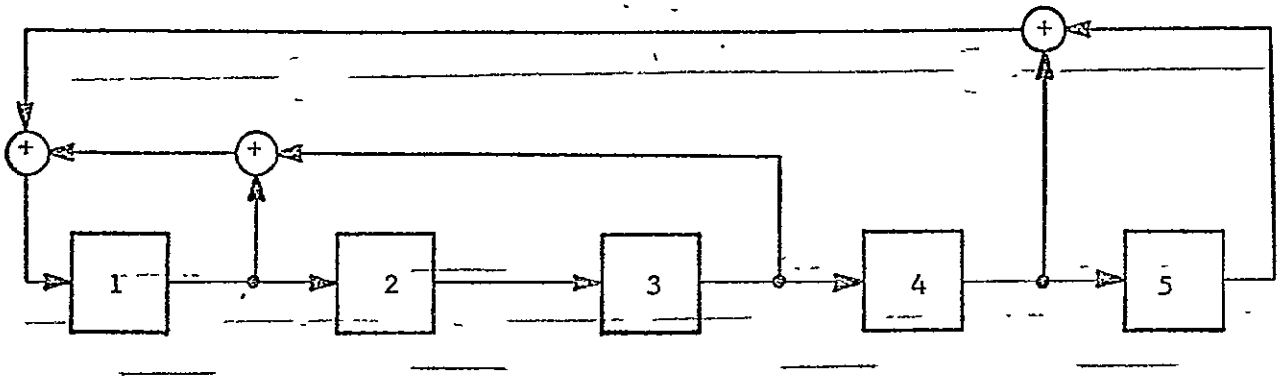

Figure II-2. Simple Shift Register Generator.

These digital generators are capable of producing repetitive sequences of length

$$L = 2^n - 1 \text{ bits,} \qquad \text{(II-1)}$$

where n is the number of stages in the register. Depending upon the particular feedback configuration, the sequence may cycle in less than L clock periods. If this is the case, the generator is called a "non-maximum length sequence generator".

The sequence output of a non-maximum length generator may depend upon the initial loading of the generator. In this case, the generator register may pass through a few initial stages, to which it never returns, before settling into a non-maximum repetitive cycle. The initial output (before getting into the repetitive cycle) is called a "transient output". For example, consider the generator shown in figure II-3. If the initial loading is 0111 for the four stages respectively, the output is 111001100110 ..... The 1110 is the transient portion of the output and 0110 is the cyclic portion. In this case, the cycle is of length four.

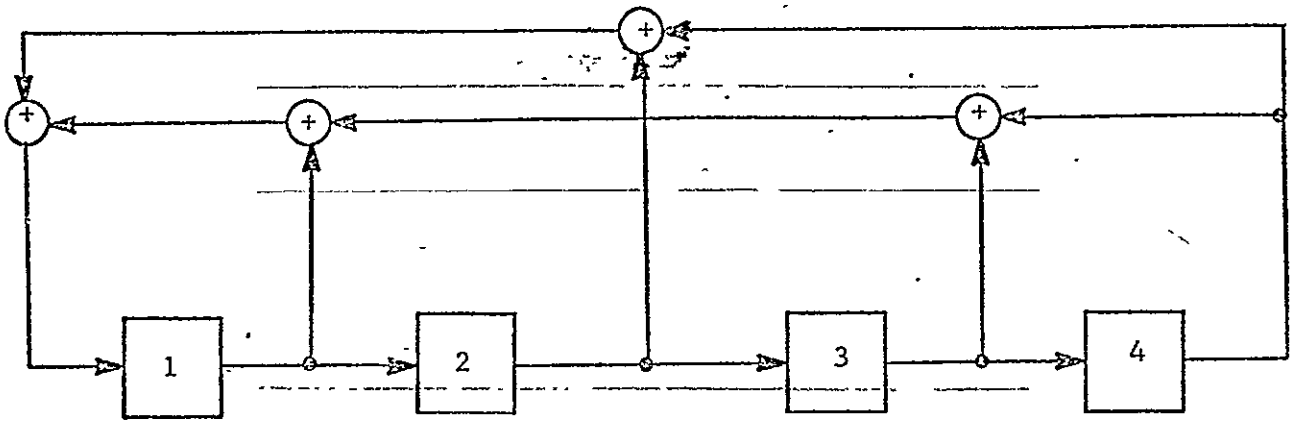

Figure II-3. Non-Maximum Generator with Transients.

Figure II-4 is an example of a non-maximum generator with no transients; however, the cycle's sequence depends on initial loading.

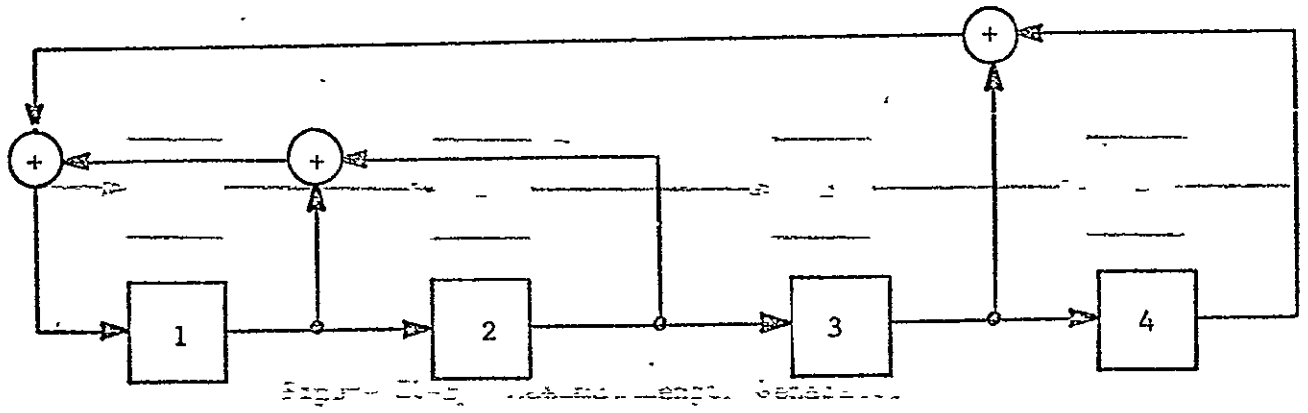

Figure II-4. Non-Maximum Generator with no Transients.

Table II-1 lists initial loading and cyclic sequence output for the generator in figure II-4.

Figure II-5 is a "maximum length generator". The output sequence (except for phase reference) does not depend upon initial loading. If the reference (initial loading) were 1111, the output sequence would be 111101011001000.

This report deals primarily with simple shift register generators. It

Table II-1. Initial Loading and Cyclic Sequence Output.

INITIAL LOADING	CYCLIC SEQUENCE
1111	11110
1000	00011
1010	01010

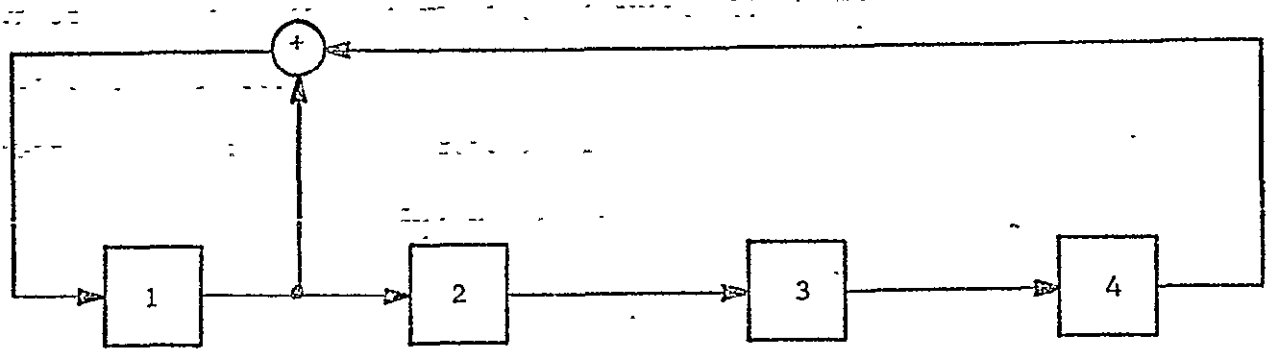

Figure II-5. Maximum Length Generator.

can be shown that every multiple-return generator (that has no transients) possesses an equivalent simple generator. That is, a simple generator exists that generates exactly the same sequence.

Before describing a formal analysis procedure for simple shift register generators, several characteristics of these generators and their sequences will be given.

No simple shift-register generator which utilizes an odd number of feed-

back taps can produce a maximum length sequence. Any shift register generator producing a maximum length sequence would necessarily include the all 1 state of the shift register. If the modulo-2 feedback included an odd number of taps, the feedback would be 1, and the next state of the register would be the all 1 state. In fact, the generator with an odd number of feedback taps would "lock-up" in the all 1 condition, and no maximum length sequence could be produced.

If an N-stage simple shift register generator has feedback on stages n, k, m, ..., the reverse generator will have taps on stages n, n-k, n-m, ... For example, the generator in figure II-5 has taps on stages 1 and 4 and generates the sequence 111101011001000. The reverse generator has taps on stages 3 and 4 and generates the sequence 111100010011010. The justification for this is as follows:

Represent the state of the register as

$$x_1, x_2, \ldots x_n$$

After one clock period, the state of the register becomes

$$b_1, x_1, \ldots x_{n-1}$$

where

$$b_1 + x_i + x_j + x_k + \ldots x_n = 0, \qquad \text{(II-1)}$$

and i, j, k, etc. are the stages included in the feedback. Now consider the state

$$x_n, x_{n-1}, \ldots x_1$$

in a generator whose feedback is from stages n-i, n-j, n-k, etc. The previous state of the register was

$$x_{n-1}, x_{n-2}, \ldots x_1, c_1$$

where

X(j): content vector during jth clock period

where

$$c_1 + x_i + x_j + x_k + x_n = 0 \quad . \tag{II-2}$$

Comparing (II-1) and (II-2)

$$b_1 = c_1, \tag{II-3}$$

notice that

$$x_{n-1}, x_{n-2}, \cdots x_1, b_1$$

is the reverse of

$$b_1, x_1, \cdots x_{n-1}$$

and the second generator produces a sequence that is the reverse of that produced by the first generator.

An important characteristic of maximum length sequences that is used extensively in the study of their statistical structure is the "shift-and-add" property. A statement of the shift-and-add property is as follows:
If a ML sequence is modulo-2 added with a shifted version of itself, the result is a shifted version of the original sequence.

Another similar characteristic of maximum length sequences is the sampling property:
If the sequence is sampled every k bits where $k=2^g$, g an integer, then the resulting sequence is a shifted version of the original sequence. These two properties will later be verified using the generator characteristic equation.

The contents of a shift register has been represented as a vector,

$$X = x_1, x_2, \cdots x_n \quad . \tag{II-4}$$

The operation of the generator can be represented as a matrix operating on the content vector. The operating matrix, called the "A" matrix, operates on the content vector such that

$$AX(j) = X(j+1), \tag{II-5}$$

where

$X(j)$: content vector during j_{th} clock period

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$X(j+1)$: content vector during $j+1_{th}$ clock period

Then, from (II-5)

$$x_i(j+1) = \sum_{k=1}^{n} a_{i,k}\, x_k(j) \qquad \text{(II-6)}$$

where

$x_i(j+1)$: i_{th} element of the content vector during the $(j+1)_{th}$ clock period,

$x_k(j)$: k_{th} element of the content vector during the j_{th} clock period.

As an example, consider the generator shown in figure II-6.

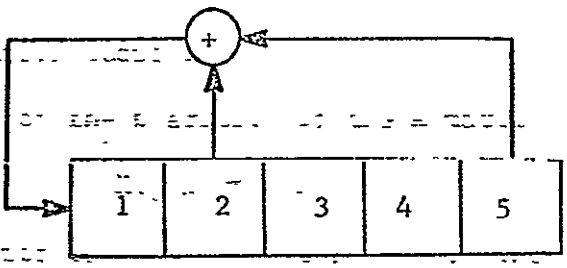

Figure II-6. Five Stage Shift Register Generator.

The A matrix for this generator is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{(II-7)}$$

This can be verified by noting

$$x_1(j+1) = x_2(j) + x_5(j)$$
$$x_2(j+1) = x_1(j)$$
$$x_3(j+1) = x_2(j)$$
$$x_4(j+1) = x_3(j)$$
$$x_5(j+1) = x_4(j) \qquad \text{(II-8)}$$

The A matirx can be interpreted as representing the feed from the j_{th} stage to the i_{th} stage. If

$$a_{ij} = 1$$

then the j_{th} stage is part of the feed to the i_{th} stage, otherwise

$$a_{ij} = 0.$$

Notice that the matrix representation can be used to represent both maximum and non-maximum length sequences.

An extension of the operation of the A matrix yields the relation

$$X(j+k) = A^k X(j). \qquad \text{(II-9)}$$

Now, if the generator is a maximum length type, then

$$X(j+L) = A^L X(j) = X(j) \qquad \text{(II-10)}$$

where

$$L = 2^n - 1. \qquad \text{(II-11)}$$

Then

$$A^L = I; \qquad \text{(II-12)}$$

where I is the identity matrix. Likewise, if the sequence were non-maximum and of length P where $P<L$, then

$$A^P = I. \qquad \text{(II-13)}$$

The characteristic polynomial of the A matrix is

$$|A - \xi I| = \xi^n + c_{n-1}\xi^{n-1} + \cdots + c_0 = 0$$

which is the determinate value of $A - \xi I$.

The characteristic equation of the A matrix is

$$\left|A - \xi I\right| = 0. \qquad \text{(II-14)}$$

A formal relationship exists between the characteristic equation of a simple shift register generator and its feedback taps. If stages n, k, m, etc. are part of the feedback, then the characteristic equation is

$$\xi^n + \xi^{n-k} + \xi^{n-m} + \ldots + 1 = 0. \pmod 2 \qquad \text{(II-15)}$$

A short notation for the characteristic polynomial is

$$(n, n-k, n-m, \ldots 0). \qquad \text{(II-16)}$$

As an example of a characteristic polynomial, consider the SSRG of figure II-6. The characteristic polynomial is

$$\xi^5 + \xi^3 + 1. \pmod 2 \qquad \text{(II-17)}$$

The short form variation of this polynomial is

$$(5,3,0).$$

A short notation for describing the feedback arrangement for a SSRG will now be illustrated. If the feedback is from stages n, k, m, etc., the short notation for the feedback is

$$[n, k, m, \ldots 0]. \qquad \text{(II-18)}$$

The generator illustrated in figure II-6 has feedback

$$[5,2,0]$$

using the above notation.

The sequence generator characteristic equation can be related to the statistical properties of the sequence, and therefore is important to this study. The relation between the characteristic equation and the feedback taps is derived as follows:

From Algebra, with every polynomial of the form

$$A^n + C_1 A^{n-1} + C_2 A^{n-2} + \ldots + C_n = 0 \pmod 2 \qquad \text{(II-19)}$$

the output sequence could have been produced by a generator with content vectors

there exist a matrix (called the Companion Matrix) of the form

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n & C_{n-1} & C_{n-2} & C_{n-3} & \cdot & \cdot & \cdot & C_1 \end{bmatrix} \qquad \text{(II-20)}$$

If $|A - \xi I| = 0$ is evaluated, then the resulting polynomial has a C matrix given by

$$C = A^R, \qquad \text{(II-21)}$$

or the companion matirx is the rotate of the A matrix. Since the A matrix is related to the feedback taps, the companion matrix is also related to the feedback taps. Specifically, if tap k is part of the feedback, then

$$a_{1,k} = 1.$$

Since

$$C = A^R,$$

$$a_{1,k} = c_{n,n+1-k} \qquad \text{(II-22)}$$

and

$$c_{n;n+1-k} = C_k = 1. \qquad \text{(II-23)}$$

The characteristic equation will then have a term of the form

$$C_k \xi^{n-k} = a_{1,k} \xi^{n-k} = \xi^{n-k}, \qquad \text{(II-24)}$$

From the Caley-Hamilton Theorem,[5] a matrix satisfies its own characteristic equation. The characteristic equation can therefore be written

$$A^n + A^{n-k} + A^{n-m} + \ldots + I = 0 \pmod 2 \qquad \text{(II-25)}$$

Using the characteristic equation, the sampling principle stated earlier can be justified. If x is an initial content vector the succeeding content vectors are Ax, A^2x, A^3x, A^4x, ... If every second output bit is sampled, the output sequence could have been produced by a generator with content vectors

$$x,\ A^2x,\ A^4x,\ A^6x,\ \ldots,$$

and this is equivalent to having an A matrix of form "A^2". Now the characteristic equation is

$$A^n + A^{n-k} + A^{n-m} + \ldots + I = 0 \pmod 2 \qquad \text{(II-26)}$$

Squaring this equation,

$$A^{2n} + A^{2(n-k)} + A^{2(n-m)} + \ldots + I = 0 \pmod 2 \qquad \text{(II-27)}$$

where all cross-terms drop out because of the properties of modulo-2 addition. Notice (II-27) implies that A^2 satisfies the sequence characteristic equation. Therefore, two sequence generators with A matrix of form "A", and "A^2" respectively, produce the same sequence since they each satisfy the same characteristic equation. Because sampling every second bit can be represented as operating with A matrix of form A^2, then the sampled sequence must be equivalent to the original sequence.

This principle can be extended to sampling every k bits, where $k=2^g$, g an integer. The sampling principle is not only useful in the study of sequence structure, but can also serve as the basis for the design of high data rate error checking systems.

The "shift-and-add" principle can also be justified with the characteristic equation. Consider figure II-7 which depicts two identical sequence generators. Each generator has the same A matrix. Assuming that generator A was started g bits earlier than generator B, there is a g bit phase shift between the two content vectors. The generator A content vector is

$$x,\ Ax,\ A^2x,\ A^3x,\ \ldots\ldots$$

while the generator B has content vector

$$A^gx,\ A^{g+1}x,\ A^{g+2}x,\ A^{g+3}x,\ \ldots\ldots$$

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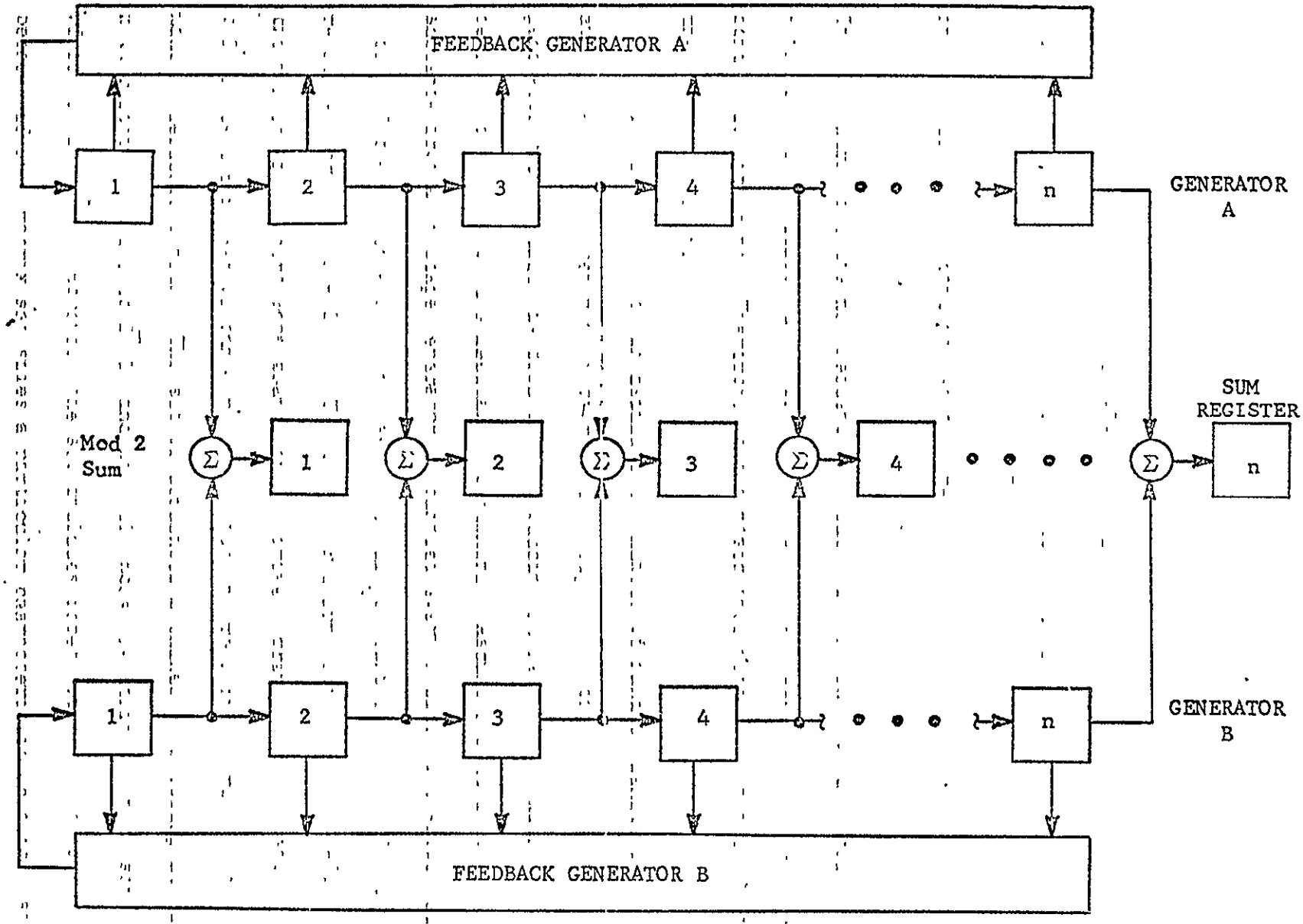

Figure II-7. Partial Correlation of Two Identical Sequences.

where x is the reference content vector. The content vectors of the sum register are

$$(A + A^{g+1})x,\ (A^2 + A^{g+2})x,\ (A^3 + A^{g+3})x,\ \ldots\ldots,$$

or rewriting, and using the properties of modulo-2 addition,

$$(A + A^{g+1})x,\ A(A + A^{g+1})x,\ A^2(A + A^{g+1})x,\ \ldots\ldots\ .$$

This is equivalent to operating the generator with an initial loading

$$x' = (A + A^{g+1})x, \qquad \text{(II-28)}$$

and the sequence of content vectors are

$$x',\ Ax',\ A^2x',\ A^3x',\ \ldots\ .$$

The "A" matrix for the new sequence generator is the same as the original A matrix that satisfied the original characteristic equation, only the initial content is different. Therefore, the sum sequence must be a shifted version of the original sequence.

It is of interest to investigate the number of maximum length sequences available from a given number of register stages. It can be shown that every generator with n stages is capable of producing at least one maximum length sequence depending on the feedback programming. Assuming a maximum length sequence from an n stage generator is sampled every k bits (one sample each k bits of the sequence), the new sequence will be maximum length if k is relatively prime to $L = 2^n-1$. The Euler's Phi function, $\phi(K)$, is defined as the number of integers less than K and relatively prime to K, therefore, $\phi(2^n-1)$ is the number of sampling rates that are relatively prime to the maximum sequence length. The set of sequences obtained by sampling at rates k (k relatively prime to 2^n-1) includes all possible maximum length sequences of length $L = 2^n-1$. However, the set includes repetitions of the same sequence because if an A matrix satisfies a particular characteristic equation, so does A^2, A^4, A^8, etc. If the sampling rate k_1 is related to the sampling rate k_2 by

$$k_2 = k_1^{2g},\ g \text{ an integer}$$

and both are relatively prime to L (k_1 relatively prime to L implies k_2 relatively prime to L), then both produce the same maximum length sequence. The number of times the same sequence appears in the set with population $\phi(2^n-1)$ must be determined in order to find the number of different maximum length sequences available from a given generator. Reference 5 gives a qualitative proof that the sampling rates fall into groups of size n, where n is the number of stages in the generator. The number of maximum length sequences available then is

$$N = \frac{\phi(2^n-1)}{n} . \qquad \text{(II-29)}$$

As an example, consider the case where n = 5. $2^n-1 = 31$ and the set of sampling rates, 1, 2, 3, ... 30 are all relatively prime to 2^n-1. Those sampling rates related by the square relation (Mod 2^n-1) are listed in sets below:

1, 2, 4, 8, 16

3, 6, 12, 24, 17

5, 10, 20, 9, 18

7, 14, 28, 25, 19

11, 22, 13, 26, 21

15, 30, 29, 27, 23

notice the size of each set of equivalent sampling rates is n=5. The number of different maximum length sequences available from a five stage generator is

$$\frac{\phi(2^5-1)}{5} = \frac{30}{5} = 6,$$

including reverse sequences.

Equation (II-29) is exact, however, it becomes tedious to evaluate for large values of n, and an upper bound on (II-29) is useful. Such an upper bound is devised as follows:

$$\phi(2^n-1) = 2^n-2 \qquad \text{(II-30)}$$

if n^2-1 is prime, and this is the largest possible value of $\phi(2^n-1)$. Then

$$\frac{\phi(2^n-1)}{n} \leq \frac{2^n-2}{n} < \frac{2^n}{n} = \frac{2^n}{2^{\log_2 n}} = 2^{n-\log_2 n}. \qquad \text{(II-31)}$$

An upper bound is then

$$U = 2^{n-\log_2 n}. \qquad \text{(II-32)}$$

For example, if n = 16

$$U = 2^{16-4} = 2^{12} = 4096.$$

Most of the mathematics necessary for the analysis of the statistical properties of maximum length sequences has been presented. The determination of the connections necessary to produce a maximum length sequence with an n stage generator is not central to the purposes of this report; however, a few comments are in order. It can be proven that no sequence generator whose characteristic equation is a reducible polynomial can produce a maximum length sequence. Some generators whose characteristic equations are irreducible do not produce maximum length sequences; however, all generators that are of the maximum length type do have irreducible characteristic equations. It has already been proven that only those generators which utilize an even number of feedback taps (odd number of terms in the characteristic equation) produce maximum length sequences. Therefore the starting point in the search for maximum length sequence generators is the set of irreducible polynomials with an odd number of terms. This defines the class of polynomials to be related to the statistical properties of maximum length sequences of modulo-2 sums of maximum length sequences.

SECTION III

SPECTRAL PROPERTIES OF DIGITAL SEQUENCES

A characteristic of white noise is that the spectral components have a uniform phase distribution because all the signal spectral phases must be independently and uniformly distributed. It is of interest to study the phase distribution of the spectral components of a filtered pseudo random digital sequence. The pseudo random sequences considered here are the maximum length sequences produced by simple linear shift register generators, or the modulo-2 sum of several such sequences.

The phase distribution of the sequence spectral components can be investigated by calculating the Fourier transform of the entire sequence. As a first step, a time reference is established. A maximum length sequence has

$$L = 2^n - 1 \qquad \text{(III-1)}$$

bits in its sequence. If the true reference (t=0) is chosen as the center of the m_{th} bit where

$$m = 2^{n-1} - 1 \qquad \text{(III-2)}$$

then there are $m+\frac{1}{2}$ bits during the time interval $t<0$ and the same number in the time interval $t>0$. If the bit interval is T_b, the total sequence period is

$$T = T_b(2^n - 1) = T_b\,(2m+1). \qquad \text{(III-3)}$$

Figure III-1 gives a graphic presentation of the time frame and time reference. During the bit interval the time function is defined as follows:

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BIT VALUE	TIME FUNCTION
1	P(t)
-1	-P(t)

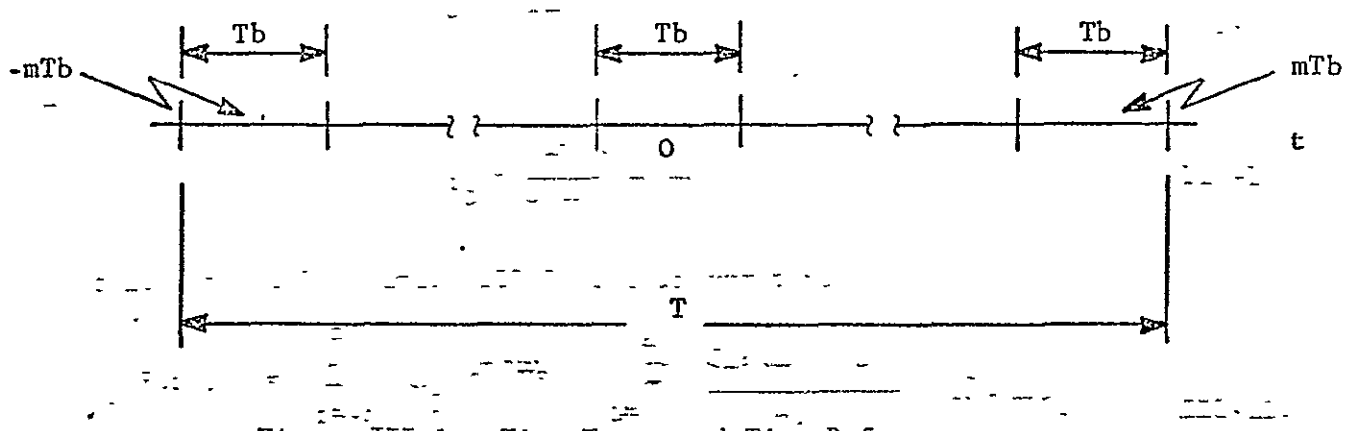

Figure III-1. Time Frame and Time Reference.

The time representation of the sequence is

$$x(t) = \sum_{g=-m}^{m} A_g P(t-gT_b) \qquad \text{(III-4)}$$

where A_g is the bit value during the g_{th} time interval.

Most storage devices available for use in shift register generators have time functions of form

$$P(t) = u(t+T_b/2) - u(t-T_b/2) \qquad \text{(III-5)}$$

where u(t) is the unit step function.

The Fourier transform of the sequence is

$$F(x(t)) = \int_{-\infty}^{\infty} \sum_{g=-m}^{m} A_g P(t-gT_b)\, \epsilon^{-j\omega t} dt$$

$$= \sum_{g=-m}^{m} A_g \int_{-\infty}^{\infty} P(t-gT_b)\, \epsilon^{-j\omega t} dt \qquad \text{(III-6)}$$

Now, changing the variable of integration,

$$F(x(t)) = \sum_{g=-m}^{m} A_g \int_{-\infty}^{\infty} P(t)\, \epsilon^{-j\omega(t+gTb)} dt$$

$$= \sum_{g=-m}^{m} A_g \epsilon^{-j\omega g T_b} \int_{-\infty}^{\infty} P(t)\, \epsilon^{-j\omega t} dt. \qquad \text{(III-7)}$$

A repetitive binary sequence (Period T) can be represented as

$$P(t) = \sum_{\alpha=-\infty}^{\infty} r_\alpha \epsilon^{j\alpha\omega_o t} \qquad \text{(III-8)}$$

where

$$\omega_o = \frac{2\pi}{T} \qquad \text{(III-9)}$$

and

$$r_\alpha = \frac{\mathrm{Sin}(\alpha\omega_o T_b/2)}{\alpha\omega_o T/2}. \qquad \text{(III-10)}$$

Substituting (III-8) into (III-7) and integrating,

$$F(x(t)) = \sum_{g=-m}^{m} A_g \epsilon^{-j\omega g T_b} \sum_{\alpha=-\infty}^{\infty} \frac{\mathrm{Sin}(\omega_o T_b/2)}{\omega_o T/2} \delta(\omega - \alpha\omega_o). \qquad \text{(III-11)}$$

Reversing the order of summation,

$$F(x(t)) = \sum_{\alpha=-\infty}^{\infty} \frac{\mathrm{Sin}(\omega_o T_b/2)}{\omega_o T/2} \delta(\omega-\omega_o) \sum_{g=-m}^{m} A_g C^{-j\omega g T_b}, \qquad \text{(III-12)}$$

which is a set of unit impulse functions at $\omega=\alpha\omega_o$, $\alpha=0, \pm1, \pm2, \ldots$. Each unit impulse represents a harmonic of the time frunction, x(t), and the phase of each harmonic (referenced to t=0, the mid point of the m_{th} bit in the sequence) can be calculated from (III-12),

$$\theta(x) = \mathrm{Tan}^{-1} \frac{\sum_{g=1}^{m} \mathrm{Sin}\left(\frac{2\pi\alpha g}{2^n-1}\right)(A_g - A_{-g})}{A_o + \sum_{g=1}^{m} \cos\left(\frac{2\pi\alpha l}{2^n-1}\right)(A_g + A_{-g})}. \qquad \text{(III-13)}$$

The A_g array is, of course, the digital sequence.

A computer program was written to calculate the phase of each harmonic. The program generates the A_g array from the binary set $\{-1,1\}$ rather than from $\{1, 0\}$.

As an example, consider the generator shown in figure III-2. With initial loading [-1, -1] the sequence output is -1, -1, 1. From (III-13), the phase of the α_{th} harmonic is

$$\theta(\alpha) = \text{Tan}^{-1}\left\{\frac{\text{Sin}(\frac{2\pi\alpha}{3})\ (2)}{-1}\right\} . \qquad \text{(III-14)}$$

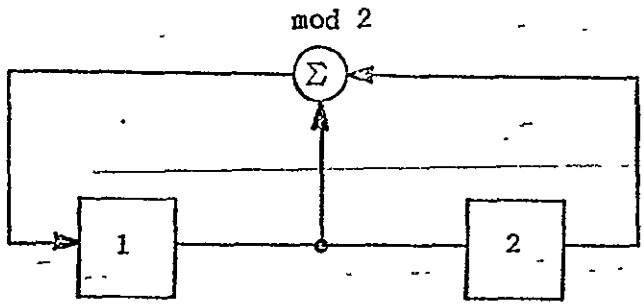

Figure III-2. Two Stage Sequence Generator.

Figure III-3 depicts the sequence spectrum.

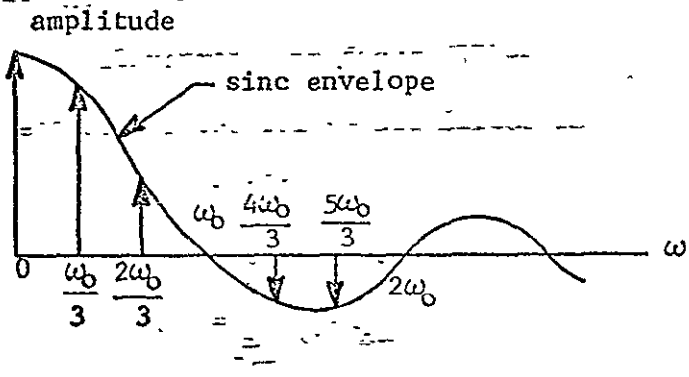

Figure III-3. Spectrum of Output Generator [2, 1, 0].

Notice, that in figure III-3 the third, sixth, etc. harmonic has zero amplitude. These are the harmonics that fall at multiplies of the clock frequency, ω_o. Table III-1 lists the phase of each harmonic. The phases listed for the

third, sixth, etc. harmonics are meaningless because their amplitudes are zero.

Table III-1. Phase of each harmonic from generator 2,1,0

HARMONIC	PHASE (RADIANS)
1	$4\pi/3$
2	$2\pi/3$
3	π
4	$4\pi/3 + \pi$
5	$2\pi/3 + \pi$
6	$\pi + \pi$
7	$4\pi/3$
8	$2\pi/3$

The additional π added to the fourth, fifth, and sixth harmonics is required because of the negative values of the sinc envelope for these harmonics. A maximum length sequence (sequence length L) has L harmonics before the zero crossing of the sinc envelope.

The computer program calculates the plots, the phase as a function of harmonic number, and also calculates and plots the statistical distribution of phase. The statistical distribution is calculated based on a truncation of the harmonics at α_{max}. This is equivalent to filtering with a perfect "brick-wall" filter as illustrated in figure III-4.

The impulse response of the brick-wall filter is

$$h(t) = F^{-1}(H(\omega)), \tag{III-15}$$

or

$$h(t) = \int_{-T_T}^{T_T} \epsilon^{j\omega t} d\omega, \tag{III-16}$$

where

$$T_T = \frac{\omega_o \alpha_{max}}{L}. \tag{III-17}$$

Evaluating (III-16)

$$h(t) = \frac{\mathrm{Sin}(tT_T)}{tT_T/2} T_T. \tag{III-18}$$

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Figure III-4. Sinc Envelope and Frequency Response of a "Brick-Wall" Filter.

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Figure III-5 is a block diagram of the generation of the time function r(t) where

$$r(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \qquad \text{(III-19)}$$

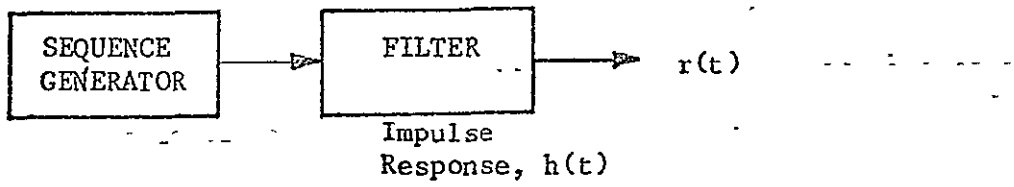

Figure III-5. Generation of r(t)

The pseudo-randomness qualities of r(t) can be investigated by a study of the phase distribution of its harmonic components. The function, r(t), is comprised of the harmonics of the sequence truncated at α_{max}, and therefore the plot of the distribution of phase provided by the computer program is the distribution of phase of r(t). Section VI gives plots of the phase and phase distribution for several maximum length generators, and maximum length sum generators. A maximum length sum generator is one in which the sequence is generated by modulo-2 summing two or more maximum length sequences.

SECTION IV

FILTERED SEQUENCES

In Section III an analysis of the spectral properties of low-pass filtered maximum length sequences was based upon a filter with frequency response as shown in figure IV-1 .

FREQUENCY RESPONSE

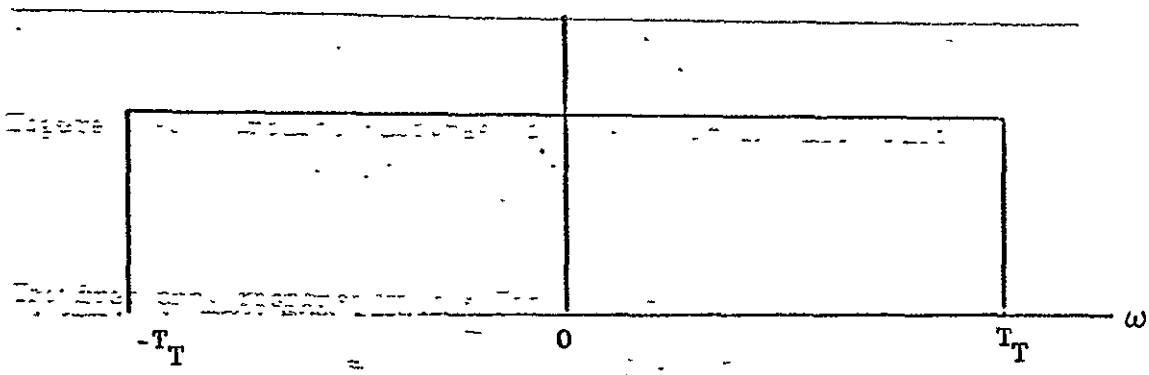

Figure IV-1. Frequency Response of Type 1 "Brick-Wall" Filter.

This filter has an impulse response given by equation (III-18),

$$h(t) = \frac{\text{Sin}\ (tT_T)}{(tT_T/2)}\ T_T.$$

In Section V an analysis of the statistical properties of a low-pass filtered maximum length sequences is based upon a filter with an impulse response of the "Brick-Wall" type as shown in figure IV-2.

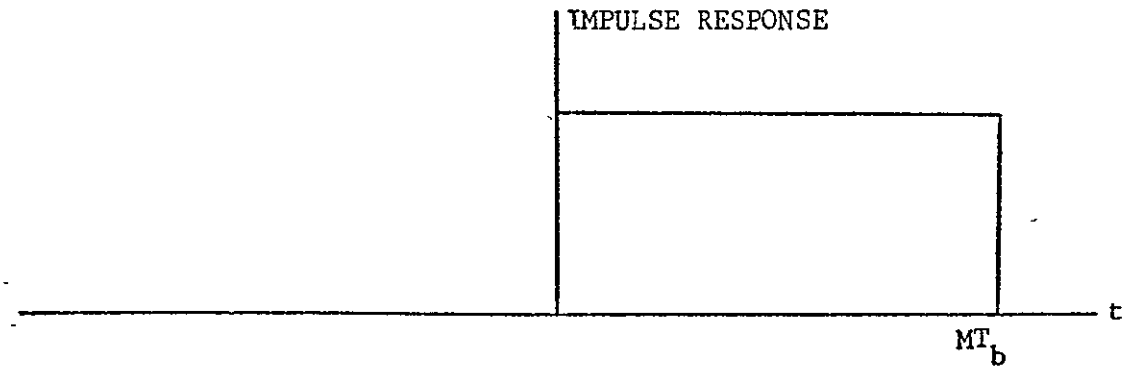

Figure IV-2. Impulse Response of Type 2 "Brick-Wall" Filter.

The frequency response of the Type 2 "Brick-Wall" filter is

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, \epsilon^{-j\omega t} dt = \frac{\mathrm{Sin}(\omega MT_b/2)}{\omega/2}\, \epsilon^{-j\omega MT_b/2} \qquad \text{(IV-1)}$$

Both of these filters are idealizations, no filter has a frequency response as shown by figure IV-1, and no filter has an impulse response as shown in figure IV-2. However, each filter serves as an approximation to a realistic filter. As an example, consider a first order filter with transfer function

$$H(j\omega) = \frac{1}{1+\frac{j\omega}{\omega_o}} \qquad \text{(IV-2)}$$

Figure IV-3 represents the frequency response of the first order filter.

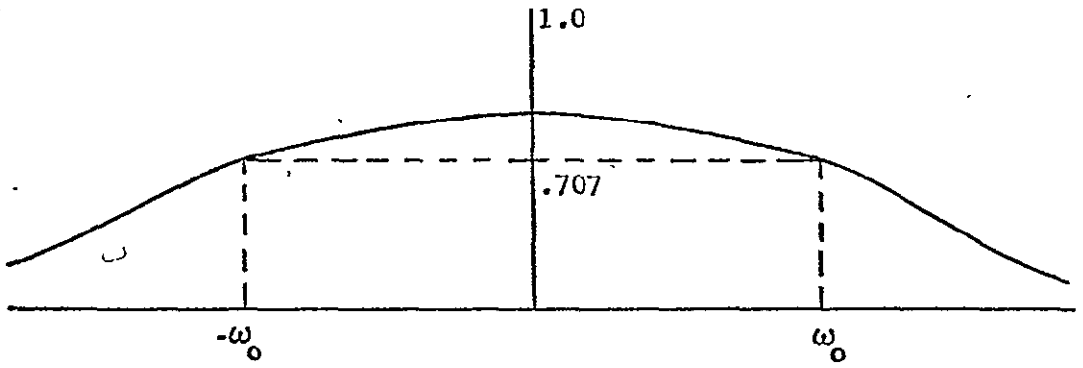

Figure IV-3. Frequency Response of First Order Filter.

The impulse response is

$$h(t) = F^{-1}\left[H(j\omega)\right] = \int_{-\infty}^{\infty} \frac{1}{1+j\frac{\omega}{\omega_o}} \epsilon^{j\omega t} d\omega$$

$$= 2\pi\omega_o \, \epsilon^{-\omega_o t} \quad 0 \leq t < \infty \quad . \tag{IV-3}$$

Figure IV-4 depicts the impulse response, h(t).

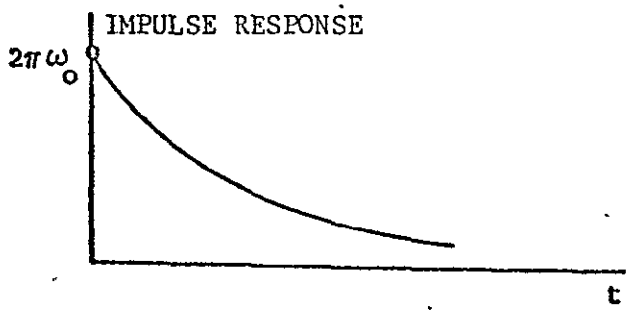

Figure IV-4. Impulse Response of the First Order Lowpass Filter.

The forms of the frequency response and impulse response, of course, depend on filter type and order. For example, as the order of the filter increases the quality of the "Brick-Wall" approximations usually increases. The primary justification for using the idealized filters in the analysis is the degree of correlation between the results of the analysis and experimental data obtained by filtering.

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[illegible]

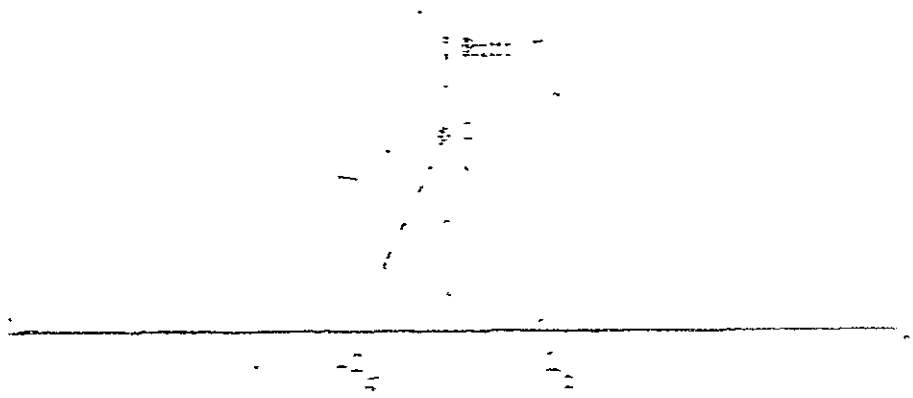

Figure [illegible] Autocorrelation of a Random Sequence [illegible]

[illegible]

[illegible]

where [illegible] and [illegible] [illegible].

Now assume the sequence [illegible] is generated by [illegible] [illegible] maximum length sequences [illegible]

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SECTION V

STATISTICAL PROPERTIES

This section is an analysis of some of the statistical properties of maximum length sequences. If x(t) is a random digital sequence, the autocorrelation function is

$$Rxx(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt, \qquad \text{(V-1)}$$

where the digital sequence is from the binary set $\{-1, 1\}$. If x(t) is random, then the autocorrelation function will have the form shown in figure V-1.

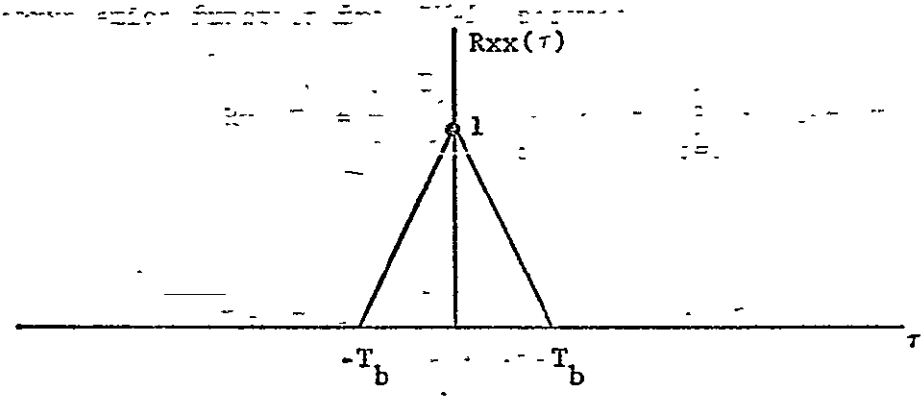

Figure V-1. Autocorrelation of a Random Sequence, X(t).

If x(t) is a maximum length sequence (sequence length = L) an appropriate autocorrelation function is

$$Rxx(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt \qquad \text{(V-2)}$$

where $T=L\ T_b$, and T_b is a bit period.

Now assume the sequence, x, is generated by summing (modulo-2) k maximum length sequences,

$$x(t) = \sum_{i=1}^{k} x_i(t) \qquad \text{Modulo 2} \qquad \text{(V-3)}$$

when $x(t)$ is from $\{1, 0\}$,

or

$$x(t) = \prod_{i=1}^{k} x_i(t) \qquad \text{(V-4)}$$

when $x(t)$ is from $\{-1, 1\}$.

The length of each constituent sequence is

$$L_i = 2^{n_i}-1$$

where n_i is the number of stages in the i_{th} generator. If none of the sequence lengths have common terms

$$L = \prod_{i=1}^{k} L_i. \qquad \text{(V-5)}$$

The autocorrelation function from (V-2) becomes

$$Rxx(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \prod_{i=1}^{k} x_i(t) \prod_{i=1}^{k} x_i(t+\tau)dt \qquad \text{(V-6)}$$

or

$$Rxx(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \prod_{i=1}^{k} x_i(t)x_i(t+\tau)dt. \qquad \text{(V-7)}$$

When using binary digits from the set $-1, 1$, the equivalent of the "shift-and-add" property becomes the "shift-and-multiply" property, and if

$$\tau \neq L_i \text{ for } i=1, 2, 3, \ldots k,$$

$$x_i(t)x_i(t+\tau) = x_i(t+\tau_i(\tau)). \qquad \text{(V-8)}$$

Substituting (V-8) into (V-7)

$$Rxx(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \prod_{i=1}^{k} x_i(t+\tau_i)dt \qquad \text{(V-9)}$$

where each τ_i is some function of τ. Since the sequence lengths have no common terms, all possible phase arrangements of the constituent sequences are cycled during the total sequence length T. This characteristic of the product

sequence can be used to change (V-9) to the form

$$Rxx(\tau\neq 0) = \frac{1}{L} \sum_{T_1=1}^{L_1} \sum_{T_2=1}^{L_2} \sum_{T_3=1}^{L_3} \cdots \sum_{T_k=1}^{L_k} \prod_{i=1}^{k} x_i(\tau_i)$$

$$= \frac{1}{L} \sum_{T_1=1}^{L_1} x_1(T_1) \sum_{T_2=1}^{L_2} x_2(T_2) \cdots \sum_{T_k=1}^{L_k} x_k(T_k). \qquad \text{(V-10)}$$

Now in each sequence there is one more -1 than +1 and

$$Rxx\ (\tau\neq 0) = \frac{(-1)^k}{L} \qquad \text{(V-11)}$$

If $\tau=0$, there is no shift and

$$Rxx\ (\tau=0) = 1. \qquad \text{(V-12)}$$

For the case where $\tau=L_j$ for $1\leq j\leq k$, then

$$Rxx(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \prod_{\substack{i=1 \\ i\neq j}}^{k} x_i(t)x_i(t+\tau). \qquad \text{(V-12a)}$$

From the shift and multiply property

$$Rxx(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} \prod_{\substack{i=1 \\ i\neq j}}^{k} x_i(t+\tau_i)dt$$

$$= \frac{1}{L} \sum_{T_1=1}^{L_1} \sum_{T_2=1}^{L_2} \cdots \sum_{T_k=1}^{T_k} \prod_{\substack{i=1 \\ i\neq j}}^{k} x_i(\tau_i)$$

$$= \frac{1}{L} \sum_{T_j=1}^{L_j} \left[\prod_{\substack{i=1 \\ j\neq j}}^{k} \sum_{T_i=1}^{L_i} x_i(T_i) \right] \qquad \text{(V-12b)}$$

or

$$Rxx(\tau=nL_j) = \frac{L_j}{L} (-1)^{k-1}. \qquad \text{(V-12c)}$$

Figure V-2 given an example of the autocorrelation function as a function of τ for even and odd values of k. The scale is exagerated for the purposes of

3/a

illustration.

The autocorrelation of a repetitive function is repetitive, as demonstrated in figure V-2. Notice the similarity of the autocorrelation function of a random sequence (Figure V-1) and the autocorrelation of a product maximum length sequence (Figure V-2).

In Section III the Pseudo randomness qualities of filtered maximum length sequences was studied from the approach of the phase distribution of the harmonics in the filtered sequence. Figure III-4 is the amplitude response of the assumed filter. In this section, a different type filter is assumed.

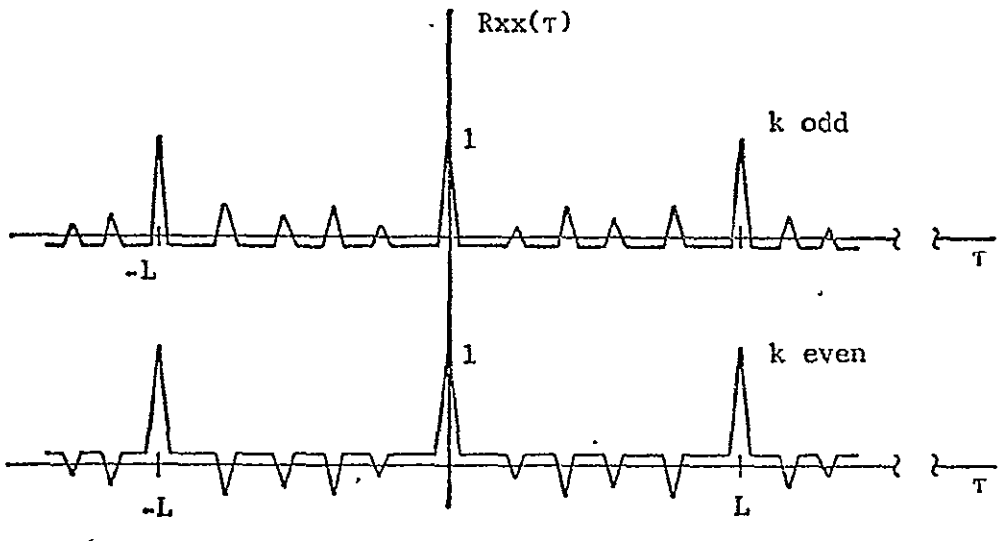

Figure V-2. Autocorrelation Function of Sum Sequencer from k Generators.

An analysis of the randomness properties can be based upon a calculation of the third moment of the filtered sequence. Figure V-3 is the impulse response of the assumed filter.

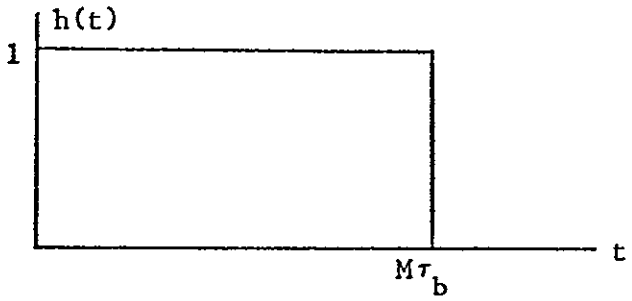

Figure V-3. Impulse Response of Filter.

The frequency response of this filter is

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\,\epsilon^{-j\omega t}dt = \frac{\mathrm{Sin}(\omega MT_b/2)}{(\omega/2)}\;\epsilon^{-j\omega MT_b/2}. \quad \text{(V-13)}$$

The time output, r(t) (See figure III-5), is

$$r(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

$$= \sum_{i=1}^{M} x(t-iT_b) \quad \text{(V-14)}$$

when x is a digital sequence. Equation (V-14) states that r(t) is equal to the sum of the last M digits that occurred before t. A set of M digits taken from a long sequence is called a "M-tuple". A study of the statistical properties of the filter output can then be based on the weight of the M-tuples where

$$S_m = \sum_{i=0}^{M-1} x_{i+m} \quad \text{(V-15)}$$

is the weight function.

S_m is the number of 1's minus the number of -1's in the M-tuple beginning at the m_{th} term of $\{x\}$ when the sequence binary characters are $\{-1, 1\}$.

In statistical theory the following moments are defined, and each indicate certain characteristics of the probability density of the variable x.

MOMENT	EQUATION	INDICATED CHARACTERISTIC
1st	$\int_{-\infty}^{\infty} xp(x)dx$	Mean Value
2nd	$\int_{-\infty}^{\infty} x^2 p(x)dx$	Variance or Central Tendancy
3rd	$\int_{-\infty}^{\infty} x^3 p(x)dx$	Skewning Tendancy

In the above equations p(x) is the probability density function.

Forming a set of M-tuple weights, S_m, with $L=2^n-1$ members for a maximal length sequence, S^P will be defined as follows:[6]

$$S^P = \frac{1}{L} \sum_{m=0}^{L-1} S_m^P \qquad \text{(V-16)}$$

The first moment is

$$S^1 = \frac{1}{L} \sum_{m=0}^{L-1} S_m = \frac{1}{L} \sum_{m=0}^{L-1} \sum_{i=0}^{M-1} x_{m+i}$$

$$= \frac{1}{L} \sum_{i=0}^{M-1} \sum_{m=0}^{L-1} x_{m+i} = \frac{-M}{L} \qquad \text{(V-17)}$$

From (V-17) the first moment depends on the number of bit periods in the filter impulse response, and on the total sequence length, but not on the particular maximum length sequence.

The second moment is

$$S^2 = \frac{1}{M} \sum_{m=0}^{L-1} S_m^2$$

$$= \frac{1}{M} \sum_{m=0}^{L-1} \sum_{i=0}^{M-1} x_{m+i}^2 \qquad \text{(V-18)}$$

Now,

$$\sum_{i=0}^{M-1} x_{m+i}^2 = M + 2 \sum_{i=0}^{M-2} \sum_{j=i+1}^{M-1} x_{m+i}x_{m+j} \qquad \text{(V-19)}$$

and

$$S^2 = \frac{1}{L} \sum_{m=0}^{L-1} M + 2 \sum_{i=0}^{M-2} \sum_{j=i+1}^{M-1} x_{m+i}x_{m+j}$$

$$= M + \frac{2}{L} \sum_{i=0}^{M-2} \sum_{j=i+1}^{M-1} \sum_{m=0}^{L-1} x_{m+i}x_{m+j} \qquad \text{(V-20)}$$

From the shift and multiply property of maximum length sequences

$$x_{m+i}x_{m+j} = x_{m+\xi} \qquad \text{(V-21)}$$

where ξ is some function of i-j.

Equation (V-20) can be simplified using (V-21),

$$S^2 = M + \frac{2}{L} \sum_{i=0}^{M-2} \sum_{j=i+1}^{M-1} (-1), \qquad \text{(V-22)}$$

because over the entire sequence length there is one more -1 digit than +1 digit.

Now

$$\sum_{i=0}^{M-2} \sum_{j=i+1}^{M-1} (-1) = -\frac{1}{2}\left[(M-1)(M-1)-(M-1)\right] - \left[M-1\right]$$

$$= -\frac{1}{2}\left[M^2-3M+2\right] - \left[M-1\right] \qquad \text{(V-23)}$$

and

$$S^2 = M - \frac{1}{L}\left[M^2-3M+2+2M-2\right]$$

$$= M - \frac{1}{L}\left[M^2-M\right]$$

$$= M\left[1 - \frac{M-1}{L}\right] . \qquad \text{(V-24)}$$

The second central moment is by definition

$$S_c^2 = \frac{1}{L} \sum_{m=0}^{L-1} (S_m - S^1)^2. \qquad \text{(V-25)}$$

Expanding (V-25)

$$S_c^2 = \frac{1}{L} \sum_{m=0}^{L-1} S_m^2 - 2S^1 \sum_{m=0}^{L-1} S_m + \sum_{m=0}^{L-1} (S^1)^2$$

$$= S^2 - (S^1)^2. \qquad \text{(V-26)}$$

From (V-24) and (V-26), the central moment is

$$S_c^2 = M \left[1 - \frac{M-1}{L}\right] - \frac{M^2}{L^2} . \qquad \text{(V-27)}$$

The characteristic of the second central moment of the weights of M-tuples from the maximum length sequence (filter output) is:

1) $S_c^2 = 0$ for M=0 and M=L.

2) S_c^2 is a maximum for

$$M = \frac{L}{2}, \qquad \text{(V-28)}$$

$$S_c^2 = \frac{L+1}{4} = 2^{n-2} \qquad \text{(V-29)}$$

when n is the number of stages in the generator.

3) The value of S_c^2 does not depend on the particular maximum length sequence.

The third moment of the M-tuple weight set, S_m, is

$$S^3 = \frac{1}{L} \sum_{m=0}^{L-1} S_m^3$$

$$= \frac{1}{L} \sum_{m=0}^{L-1} \sum_{i=0}^{M-1} x_{m+i}^3. \qquad \text{(V-30)}$$

Now

$$\sum_{i=0}^{M-1} x_{m+i}^3 = \left(\sum_{i=0}^{M-1} x_{m+i}\right)\left(M + 2 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M-1} x_{m+i}x_{m+j}\right)$$

$$= (3M-2) \sum_{i=0}^{M-1} x_{m+i}$$

$$+ 3! \sum_{i=0}^{M-3} \sum_{j+i+1}^{M-2} \sum_{k=j+1}^{M-1} x_{m+j}x_{m+k}. \qquad \text{(V-31)}$$

Substituting (V-31) into (V-30),

$$S^3 = \frac{1}{L} \sum_{m=0}^{L-1} \left[(3M-2) \sum_{i=0}^{M-1} x_{m+i}\right.$$

$$\left. + 3! \sum_{i=0}^{M-3} \sum_{j=i+1}^{M-2} \sum_{k=j+1}^{M-1} x_{m+i}x_{m+j}x_{m+k}\right]. \qquad \text{(V-32)}$$

The first double summation of (V-32) can be simplified as follows:

$$\sum_{m=0}^{L-1} \sum_{i=0}^{M-1} x_{m+i} = \sum_{i=0}^{M-1} \sum_{m=0}^{L-1} x_{m+i} = -M. \qquad \text{(V-33)}$$

The quadruple summation can be rewritten

$$\sum_{m=0}^{L-1} \sum_{i=0}^{M-3} \sum_{j=i+1}^{M-2} \sum_{k=j+1}^{M-1} x_{m+i}x_{m+j}x_{m+k} =$$

$$\sum_{i=0}^{M=3} \sum_{j=i+1}^{M-2} \sum_{k=j+1}^{M-1} \sum_{m=0}^{L-1} x_{m+i}x_{m+j}x_{m+k}. \qquad \text{(V-34)}$$

Over the range of m in (V-34), from the shift and multiply property of maximum length sequences,

$$x_{m+i}x_{m+j} = x_{m+\xi} \qquad \text{(V-35)}$$

where ξ is some function of (i-j).

Now

$$\sum_{m=0}^{L-1} x_{m+\xi} x_{m+k} = \begin{cases} N \text{ if } \xi=k \\ -1 \text{ if } \xi=k \end{cases} \tag{V-36}$$

Let B_3 be the number of ordered 3-tuples, (i, j, k), where

$$0 \leq i < j < k \leq M - 1$$

such that

$$x_{m+i} x_{m+j} = x_{m+k}. \tag{V-37}$$

Using the definition of B_3, equation (V-32) becomes

$$S^3 = -\frac{(3M-2)(M)}{L} + \frac{3!}{L}\left[\left[\binom{M}{3} - B_3\right](-1) + B_3 L\right]. \tag{V-38}$$

Now

$$\binom{M}{3} = \frac{(M)(M-1)(M-2)}{3!}, \tag{V-39}$$

and

$$S^3 = \frac{-M^3}{L} + \frac{3!(L+1)}{L} B_3. \tag{V-40}$$

The third central moment is

$$S_c^3 = \frac{1}{L} \sum_{m=0}^{L-1} (S_m - S^1)^3$$

$$= \frac{1}{L} \sum_{m=0}^{L-1} (S_m^3 - 3S_m^2 S^1 + 3S_m (S^1)^2 - (S^1)^3)$$

$$= S^3 - 3S^1 S^2 + 3(S^1)^3 - (S^1)^3, \tag{V-41}$$

or

$$S_c^3 = \frac{-M^3}{L} + \frac{3!(L+1)}{L} B_3$$

$$+ 3\left(\frac{M^2}{L}\right)\left(1 - \frac{M-1}{L}\right) - \frac{2M^3}{L^3}. \tag{V-42}$$

Observe in (V-42) if L is large and $M \ll L$

$$S_c^3 \approx 6B_3. \tag{V-43}$$

The quantity, B_3, can be related to the maximum length sequence characteristic equation. If (i, j, k) is one of the B_3 tiples such that $0 \leq i < j < k \leq M-1$ and

$$x_{m+i}x_{m+j} = x_{m+k}, \tag{V-44}$$

then this is equivalent to having a generator as shown in figure V-4 .

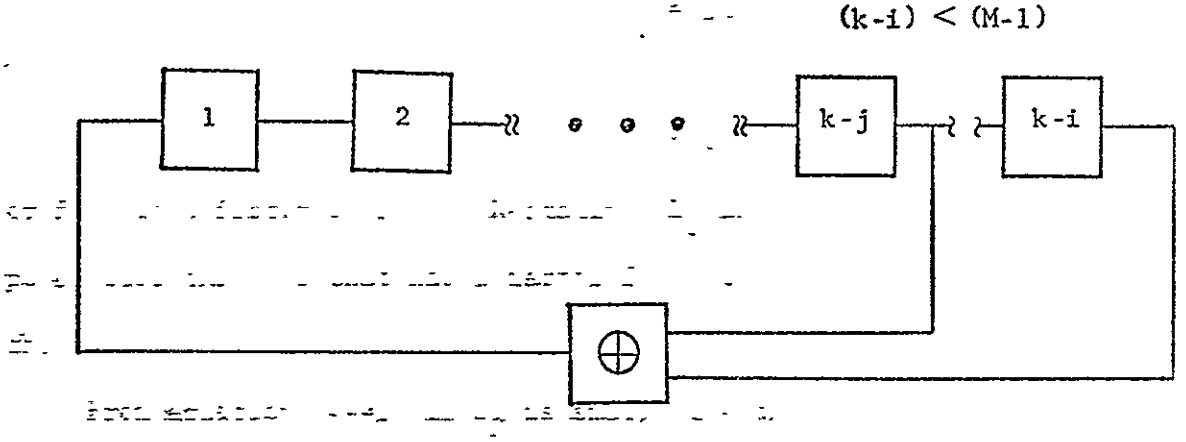

Figure V-4. Equivalent Generator of the B_3 Class.

The feedback equation is

$$[k-i,\ k-j,\ 0]$$

for the generator shown in figure V-4.

The sequence, x, must then satisfy the trinomial polynomial of the form

$$g(Z) = I + Z^{j-i} + Z^{k-i} = 0 \tag{V-45}$$

where Z is the feedback matrix (A matrix of Section I).

The sequence must satisfy the original characteristic polynomial equation,

$$f(A) = I + \sum_{i=1}^{n} C_i A^i = 0. \tag{V-46}$$

Multiplying the content vector by A^{g1} is equivalent to stepping the generator through g_1 states, and the sequence must satisfy

$$A^{g1}\ (I + \sum_{i=1}^{n} C_i A^i)$$

$$+ A^{g2}\ (I + \sum_{i=1}^{n} C_i A^i)$$

$$+ \ldots\ . \qquad = g(A). \qquad \text{(V-47)}$$

Then

$$f(A)\ (A^{g_1} + A^{g_2} + \ldots) = g(A), \qquad \text{(V-48)}$$

or f(A) is a factor of g(A). Restating, B_3 is the number of trinomials of power less than M-1 that has a factor f(A), the characteristic polynomial of the sequence.

From equation (V-43) if B_3 is small, the third central moment of the weights of M-tuples (filter output) is samll. Reference 6 gives a procedure for finding the B_3 term for maximum length sequences for given M-tuples.

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SECTION VI

APPLICATIONS AS A GAUSSIAN NOISE GENERATOR

The signal obtained by filtering a maximum length sequence has a probability density function that is approximately Gaussian. This can be intuitively justified by comparing the maximum length sequence to a random sequence of binary digits. If the random sequence digits are from the set $\{-1, 1\}$, and if during the bit interval the probability of a (-1) is B, and the probability of a (+1) is 1-B, and each bit is independent of all other bits in the sequence, then the probability that there will be m_1 (-1) bits during m_2 bit intervals is

$$P = \frac{m_2!}{m_1!(m_2-m_1)!} p^{m_1} (1-P)^{m_2-m_1} . \qquad \text{(VI-1)}$$

If (-1) and (1) are equiprobable, then equation (VI-1) becomes

$$P = \frac{m_2!}{m_1!(m_2-m_1)!} \left(\frac{1}{2}\right)^{m_2} \qquad \text{(VI-2)}$$

From (VI-2), if a long random binary sequence of equiprobable digits is examined, the following properties are evident:

(a) If there are k_1 runs of k_2 consecutive (-1) or (1) digits there should be $2k_1$ runs of k_2-1 consecutive like digits and $k_1/2$ runs of k_2+1 consecutive like digits.

(b) In a long sequence the number of (-1) digits should very nearly equal the number of (1) digits.

(c) If two long sequences are compared position for position, the number of cases where the digits agree very nearly equals the number of cases where the digits disagree. This, of course, leads to the correlation property.

In Section V it was shown that maximum length sequences satisfy the correlation property and property (c) above. A register generating a maximum length sequence goes through all but one of its possible states, therefore, (a) and (b) are also satisfied. The justification for calling a maximum length sequence a pseudo-random sequence is based on the fact that maximum length sequences do satisfy the randomness properties listed above for sequence lengths less than $L=2^n-1$ for a n stage register.

Section IV considered filtered sequences, and it was shown that the filtering process is equivalent to forming a signal by weighting and adding digits of the sequence. Likewise; it was shown that the filtering process is equivalent to adding harmonic components of the sequence up to the filter cut-off frequency. From the central limit theorem, using either view of the filtering process, the output signal density function tends toward a Gussian density function if the sequence is random, and the number of digits or harmonics added is large. This is because, for a random sequence, the density function of individual digits are independent, and the sequence harmonic components are independently distributed in phase.

The digits and harmonics components of a pseudo random maximum length sequence are not actually independent, but in many cases they approximate the random sequence well enough to approach a Gaussian distribution when filtered. The conditions from the central limit theorem can be used to qualitatively test how well a filtered maximum length sequence approaches a Gaussian distribution.

The tests are:

1) How random is the distribution of the phase components of the sequence harmonics passed by the filter.

2) If the digits are independent, the weights of M-tuples of the sequence should approach a Gaussian distribution and the third-moment of the weights should approach zero. Therefore, the value of the third moment of the weight function serves as a test for the approach to a Gaussian distribution.

To illustrate the properties of filtered maximum length sequences, the results of experimental tests and computer simulation will now be given. Figures VI-1 thru VI-6 give experimental plots of the probability density function for the filtered maximum length sequence. The sequence generator is a twenty stage unit, the filter is third order, and ratios of the clock to filter cutoff frequency,

$$B = f_c/f_d, \tag{VI-3}$$

are 5, 10, 20, 22.5, 90, and 200 respectively. From Section IV, the number of digits added to form the output is approximately B. Therefore, the statistics can be approximated by the statistic of M-tuples of length B.

From Section V, the third moment of the M-tuple weights indicates skewing, and the third moment is related to the number of trinomials of power less than M that have the sequence characteristic polynomial as a factor. From figures VI-5 and VI-6, the density function skews, therefore, there must be many trinomials of power less than 90 and 200 that have the sequence characteristic polynomial as a factor.

As another example, an eleven stage sequence generator and filtering operation was simulated by computer. The maximum length sequence generator characteristic polynomial was

$$A^{11}+A^{9}+I \tag{VI-4}$$

Figures VI-7 thru VI-21 give the time dependent filter output and density function of the waveform. The figures are for filter impulse response periods of 10, 12, 16, 20, 30, 40, 70, and 90 bits. Notice that for impulse response

periods greater than 10 (M-tuple length greater than 10) there is skewing. This indicates there exists many trinomials that have $A^{11}+A^{9}+I$ as a factor for the M-tuple lengths given.

For the purpose of comparing maximum length sequence generators and sum sequence generators, a computer simulation was made of two sequences that were formed by adding two maximum length sequences. The first sum generator forms its output by summing the outputs of maximum length sequence generators with characteristic equations (5, 2, 0) and (6, 1, 0). Figures VI-22 thru VI-35 give output waveforms and output density functions for impulse response periods of 10, 12, 14, 18, 20, 40, and 80 bits respectively. It is seen that the density functions for this generator, which uses eleven total stages, do not skew nearly as much as for the maximum length generator (11, 9, 0). This shows that the characteristic polynomial of the sum sequence is not a factor of many trinomials with power less than the M-tuple lengths (impulse response period).

It cannot be concluded that modulo-2 summed sequences always give non-skewed density function when filtered. To illustrate this, a computer simulation was run for summing and filtering the maximum length sequences with characteristic equations (5, 4, 2, 1, 0) and (6, 1, 0). Figures VI-36 thru VI-49 give the output waveforms and output density functions for impulse response periods of 10, 14, 18, 20, 40, and 80 bits respectively. Notice these density functions skew more than those for the first sum-type generators, indicating the characteristic polynomials of the generators, are factors of many trinomials of power less than the impulse response periods.

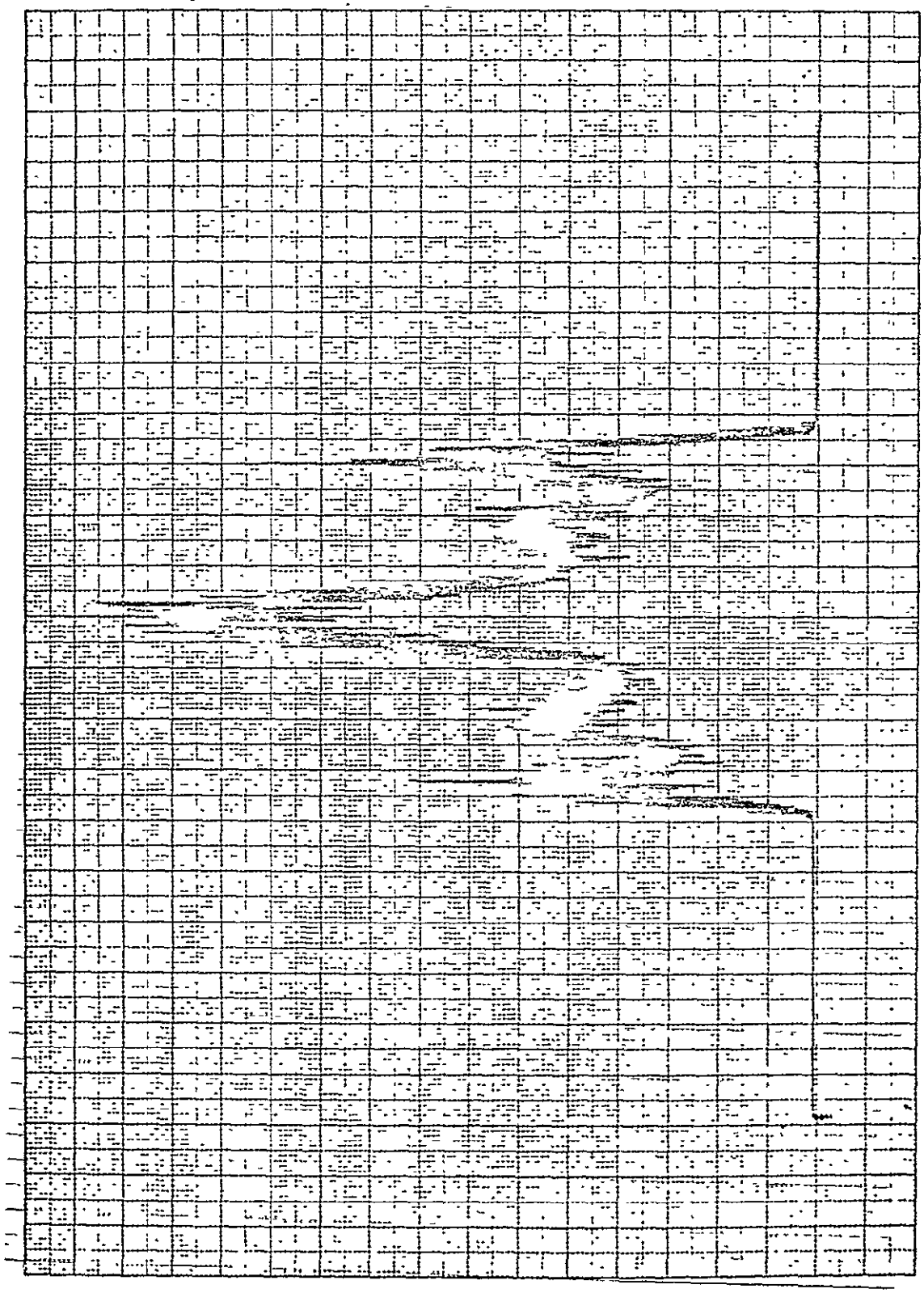
To further investigate the randomness properties of the three sequence generators used in the example, a computer calculation of the phase distributions of the Fourier transforms of the three sequences was made. Figure VI-50 is a plot of the phase distribution of the first 300 harmonics of the (11, 2, 0) maximum length sequence. Figures VI-64 and VI-74 are the phase distribution

of the first 200 harmonics of the sum sequences ((5, 2, 0) + (6, 1, 0)) and (5, 4, 2, 1, 0) + (6, 1, 0) respectively. Figures VI-51 thru VI-63 VI-65 thru VI-73, and VI-75 thru VI-83 give phase density functions for filtered versions of each sequence generator. The number of harmonics included is listed on each figure, and the relation between the ratio of clock to filter cutoff frequencies, B, and number of harmonics included is

$$\text{harmonics included} = \frac{L}{B} \qquad \text{(VI-4)}$$

where L is the sequence length, and $B = f_c/f_d$. Notice that the phase density functions for the sum generator ((5, 2, 0) + (6, 1, 0)) has the most uniform distributions, and this agrees with the previous statements about the relation between pseudo-randomness quality and the phase density function of the harmonics passed by the filter.

The phase distributions illustrated were calculated by computer from equation (III-13).



FFigure VII-1

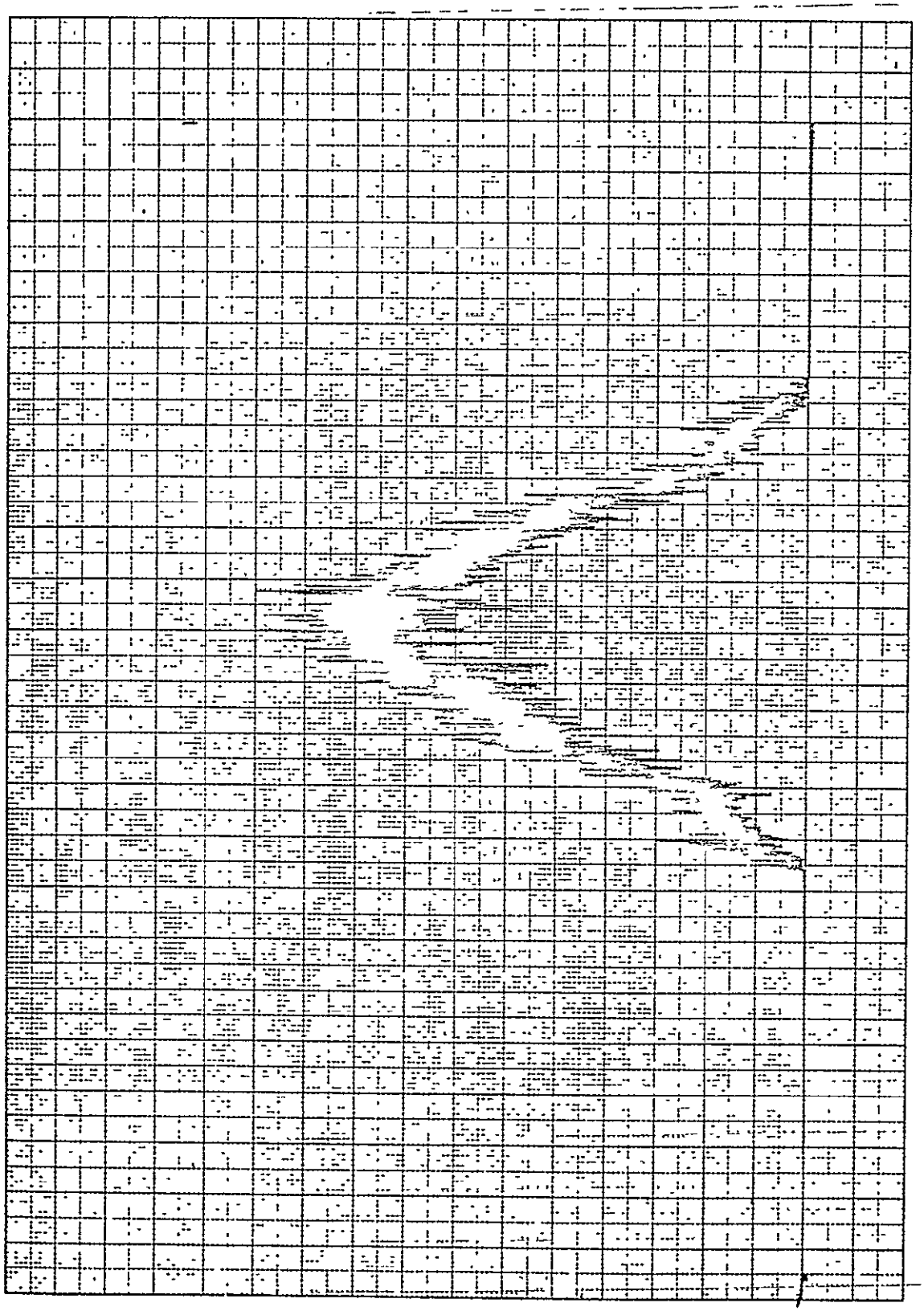


Figure VI-2

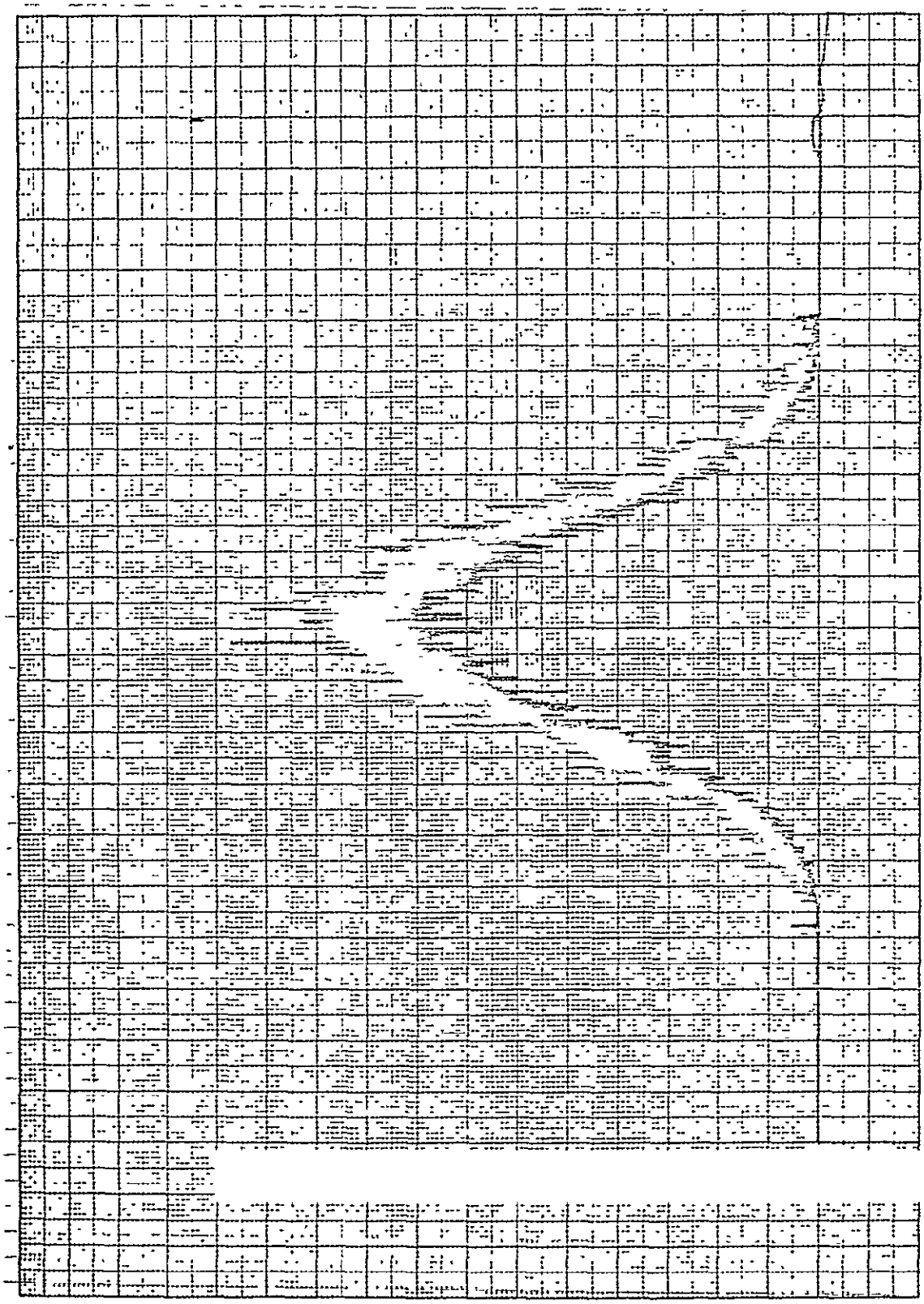


Figure VI-3

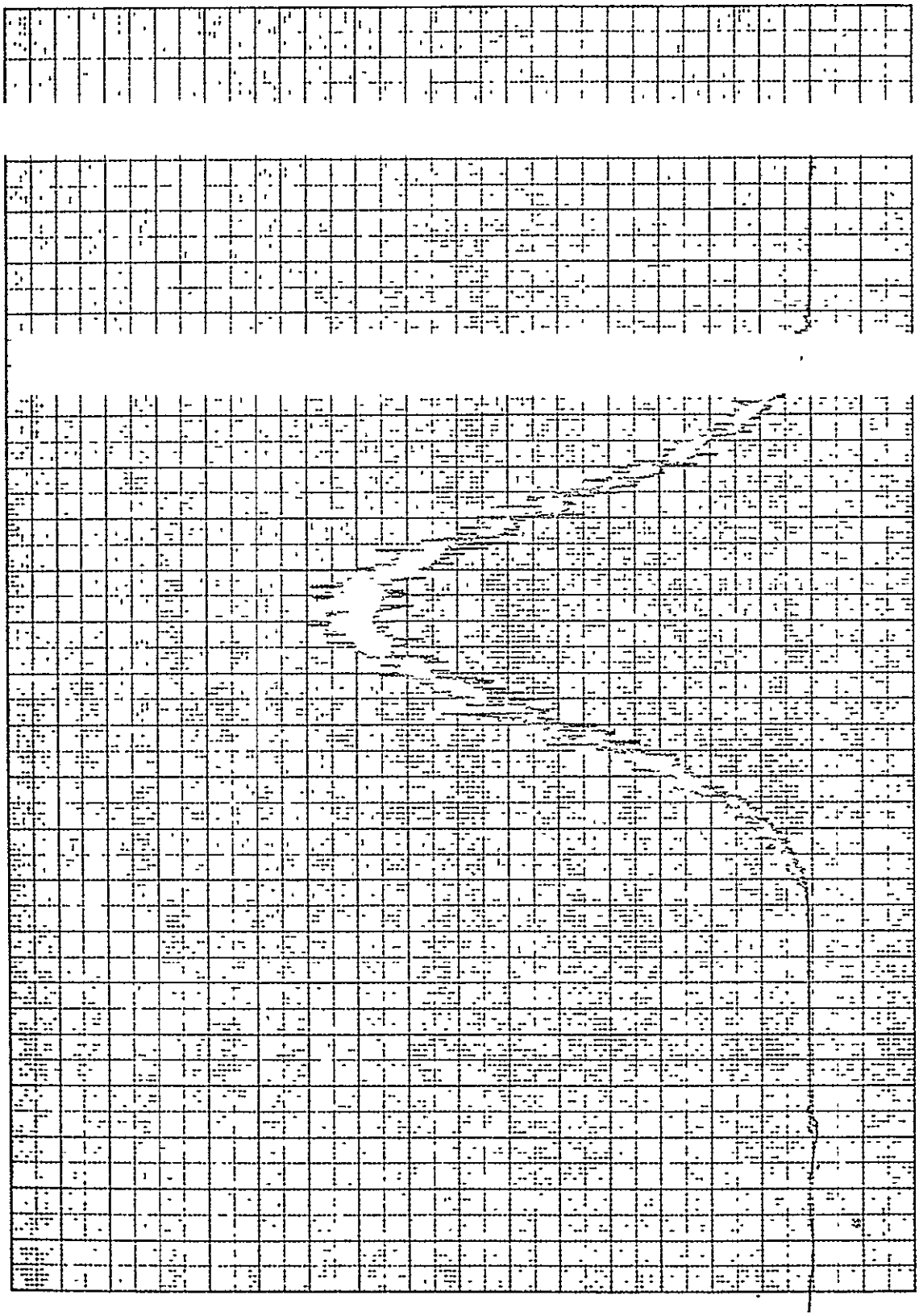


Figure VI-4

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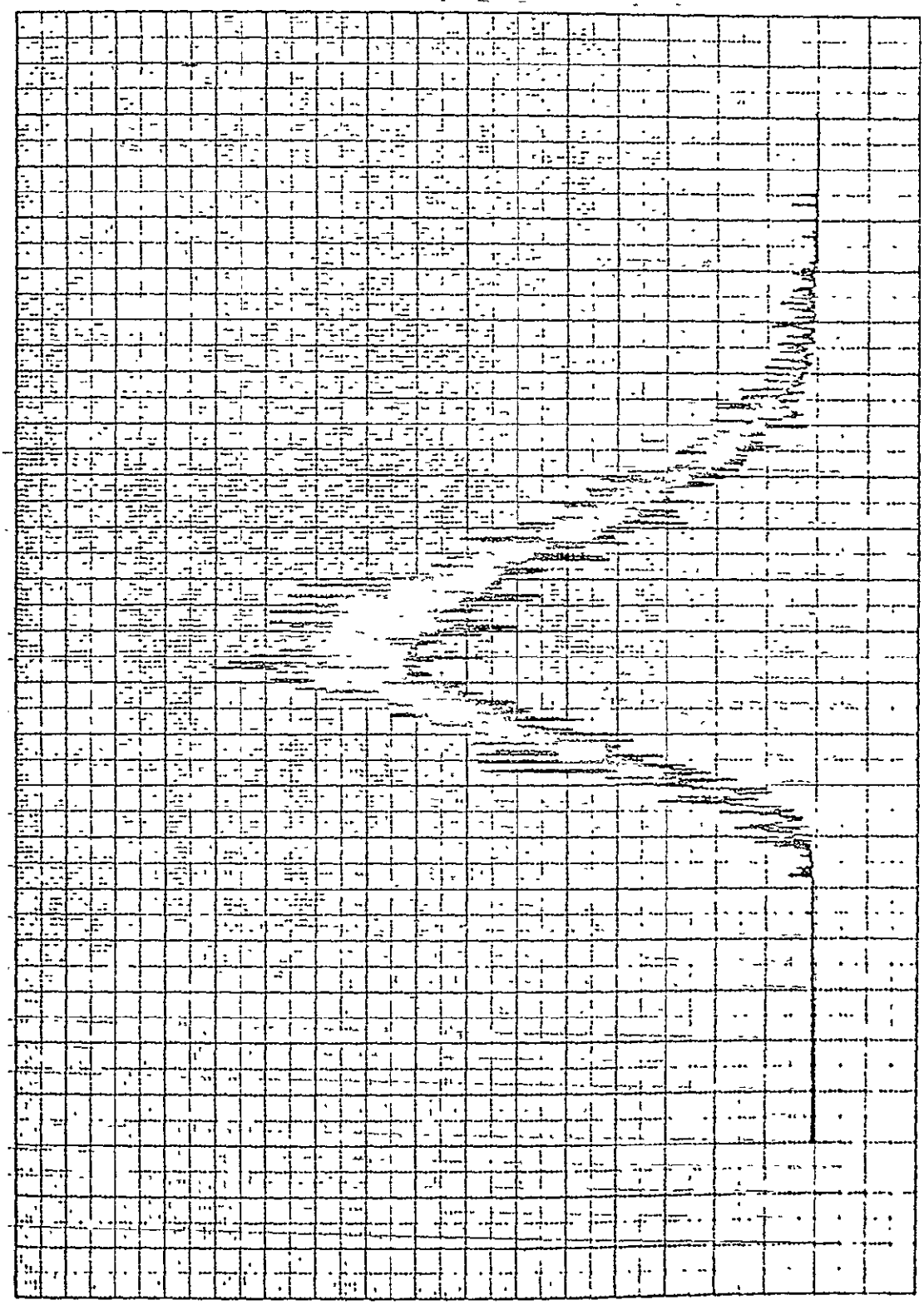


Figure VI-6
Figure VI-5

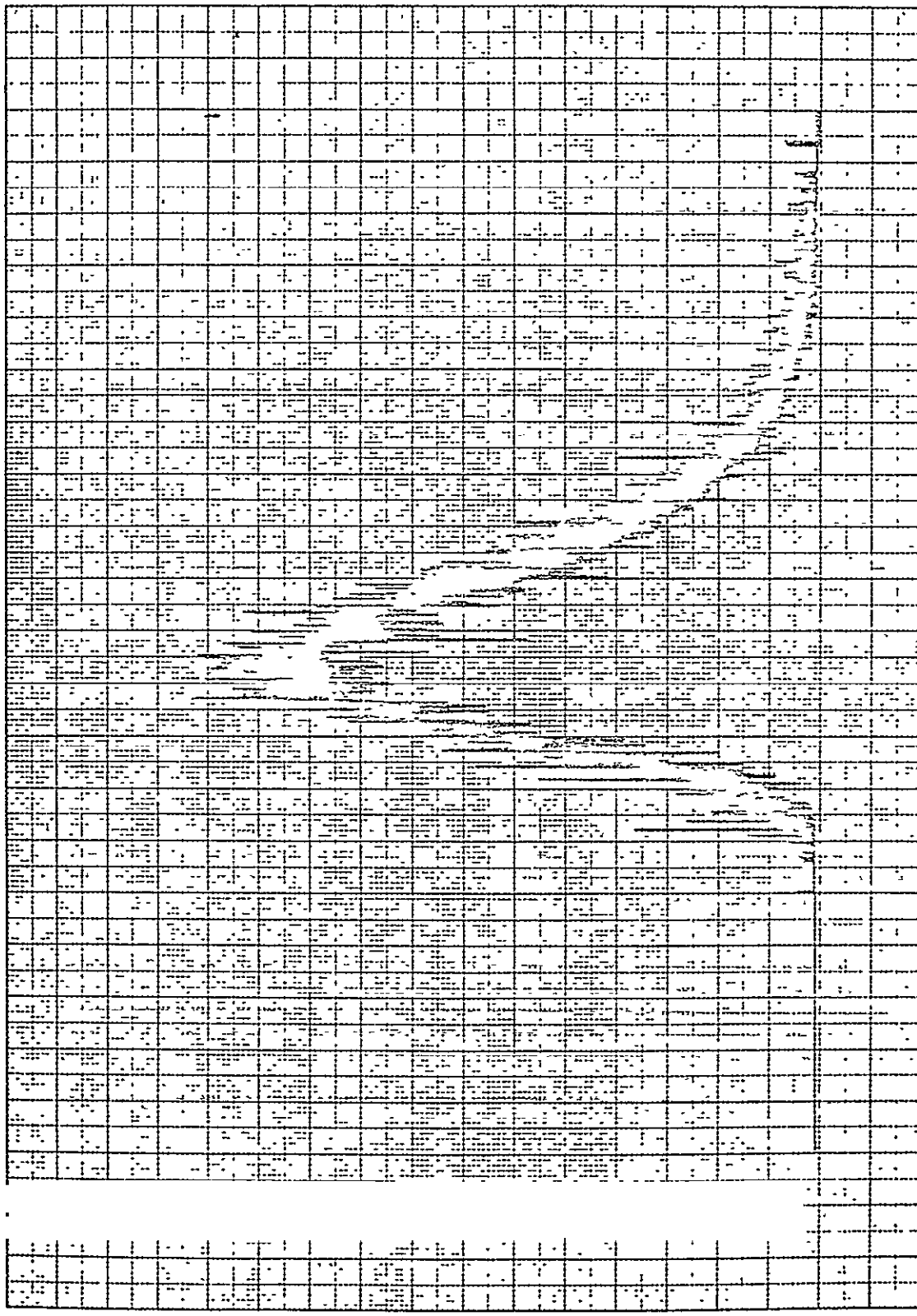
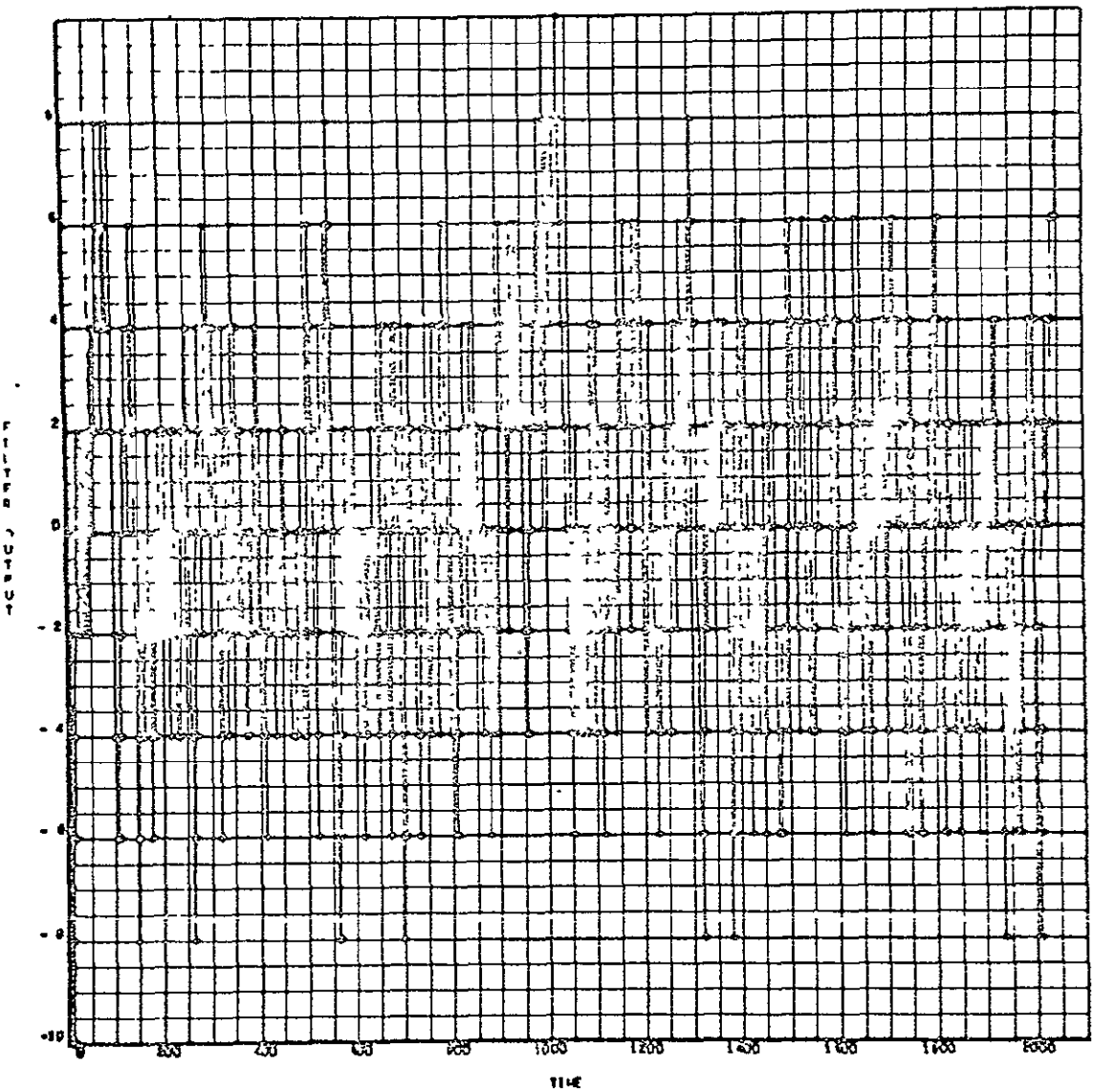


Figure VI-6



PLOT NUMBER 5

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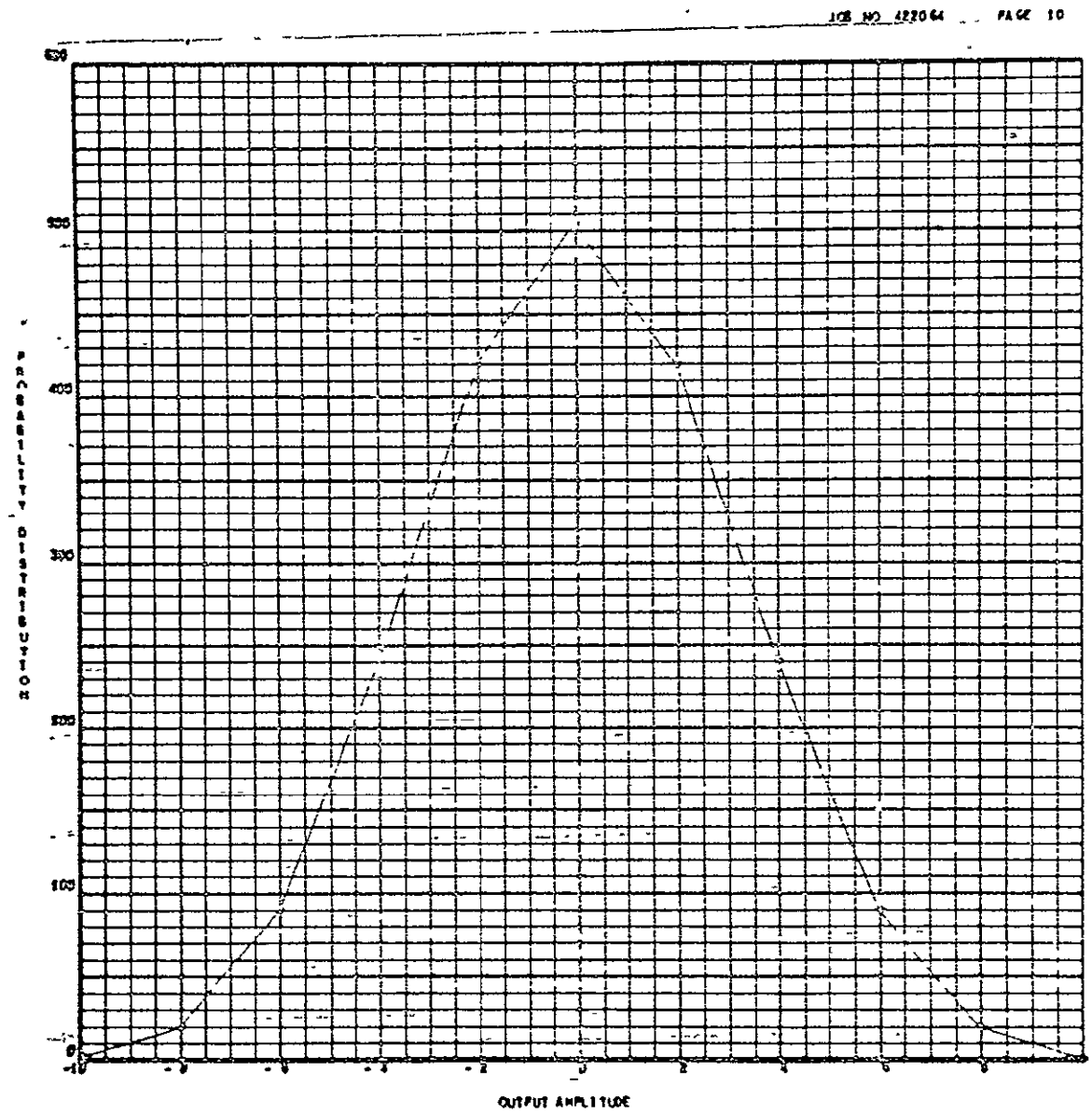
IMPULSE RESPONSE PERIOD = 10

Figure VI-7



PLOT NUMBER 5

SEQUENCE LENGTH = 2047

IMPULSE RESPONSE PERIOD = 10

Figure VI-8

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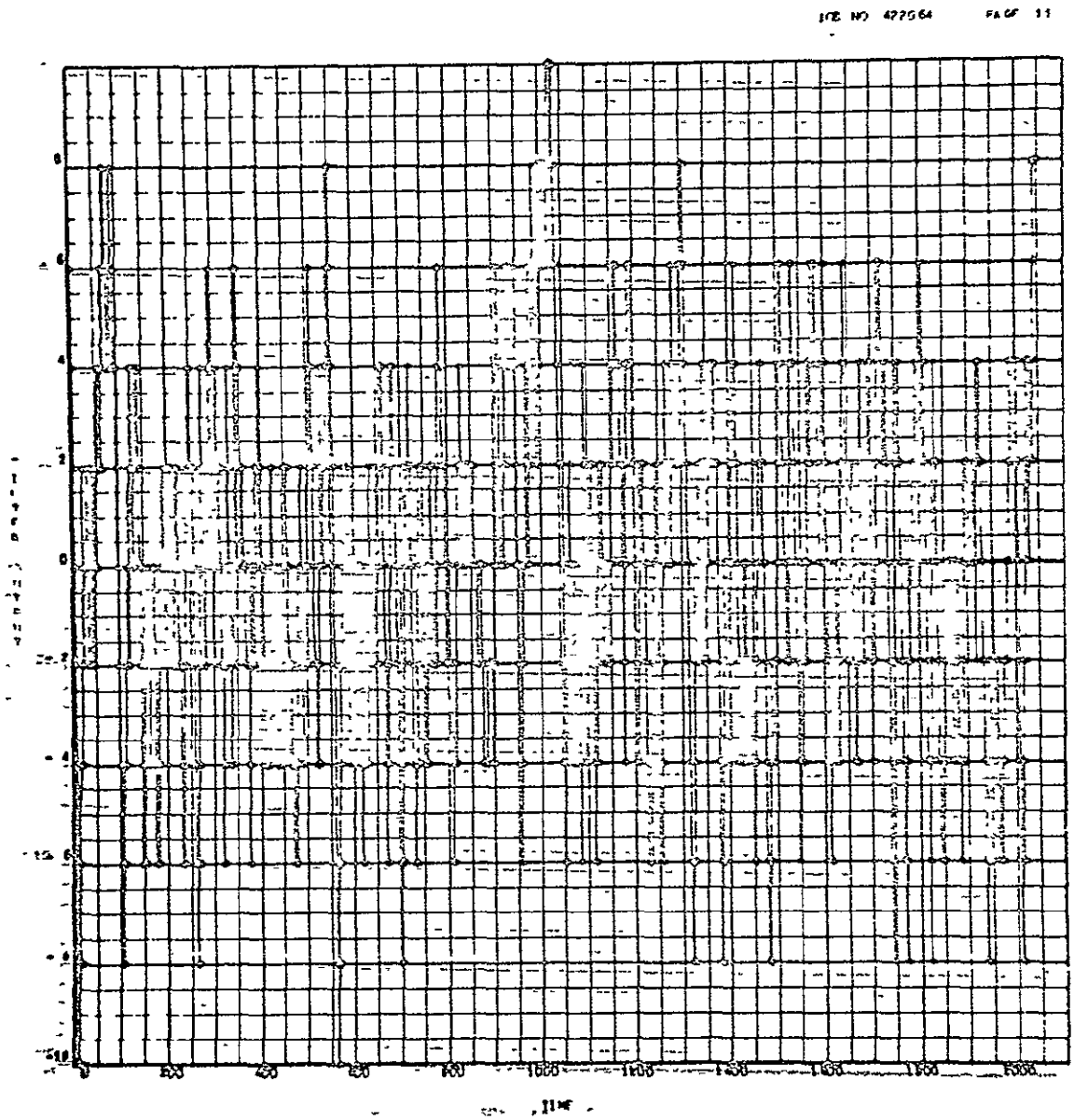

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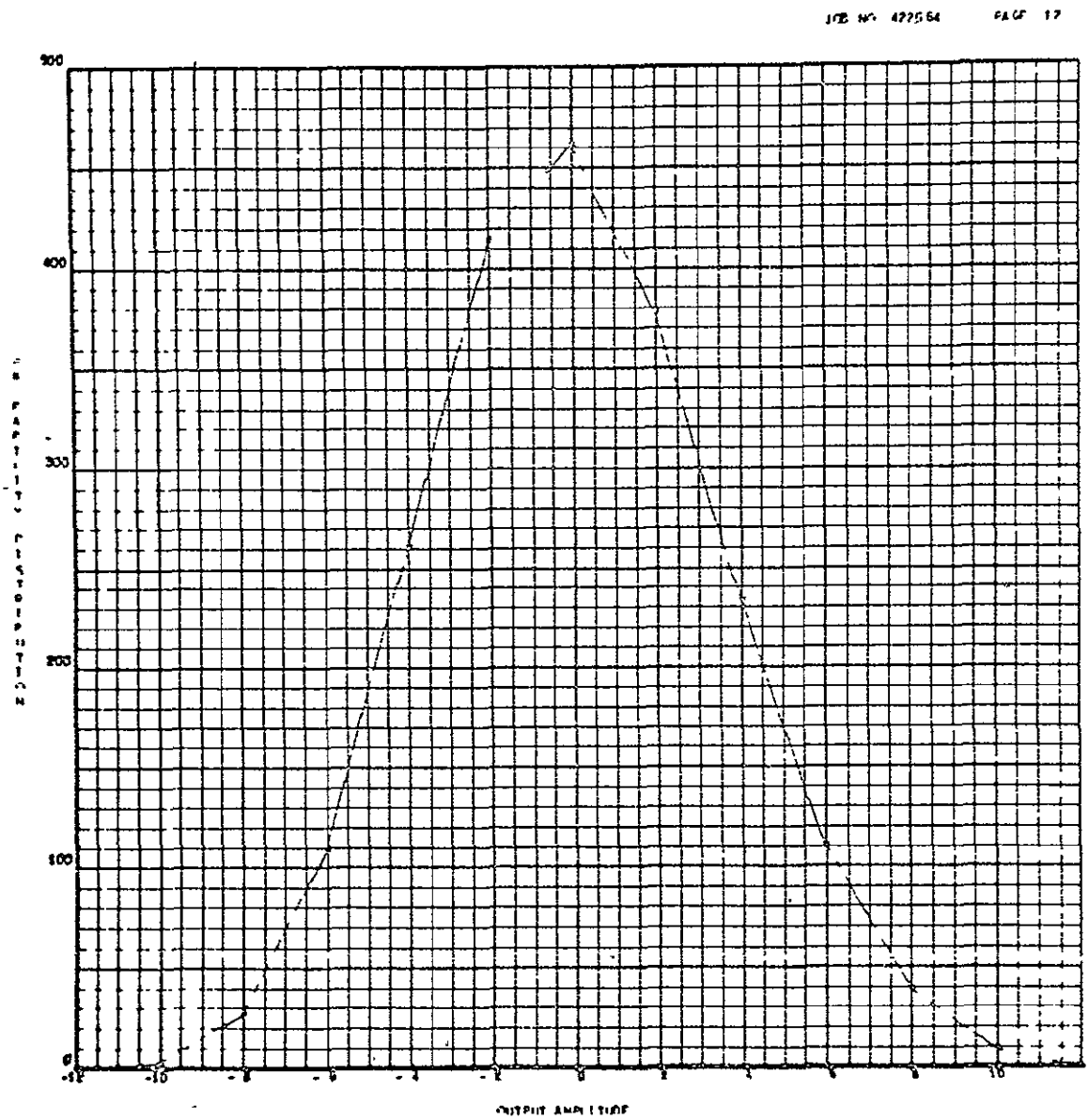
IMPULSE RESPONSE PERIOD = 12

Figure VI-9



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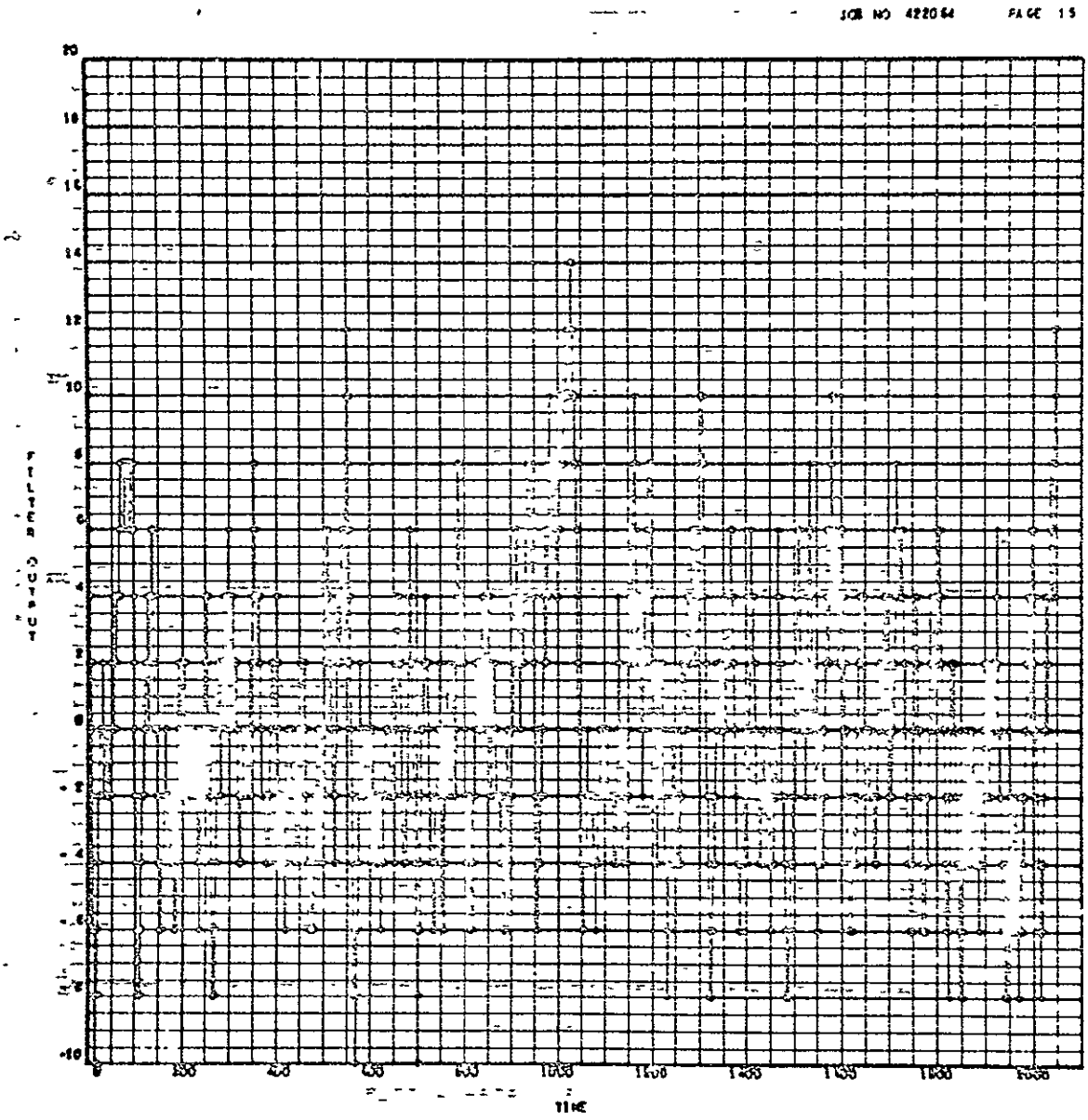
SEQUENCE LENGTH = 2047

IMPULSE RESPONSE PERIOD = 12

Figure VI-10



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IMPULSE RESPONSE PERIOD = 16

Figure VI-11

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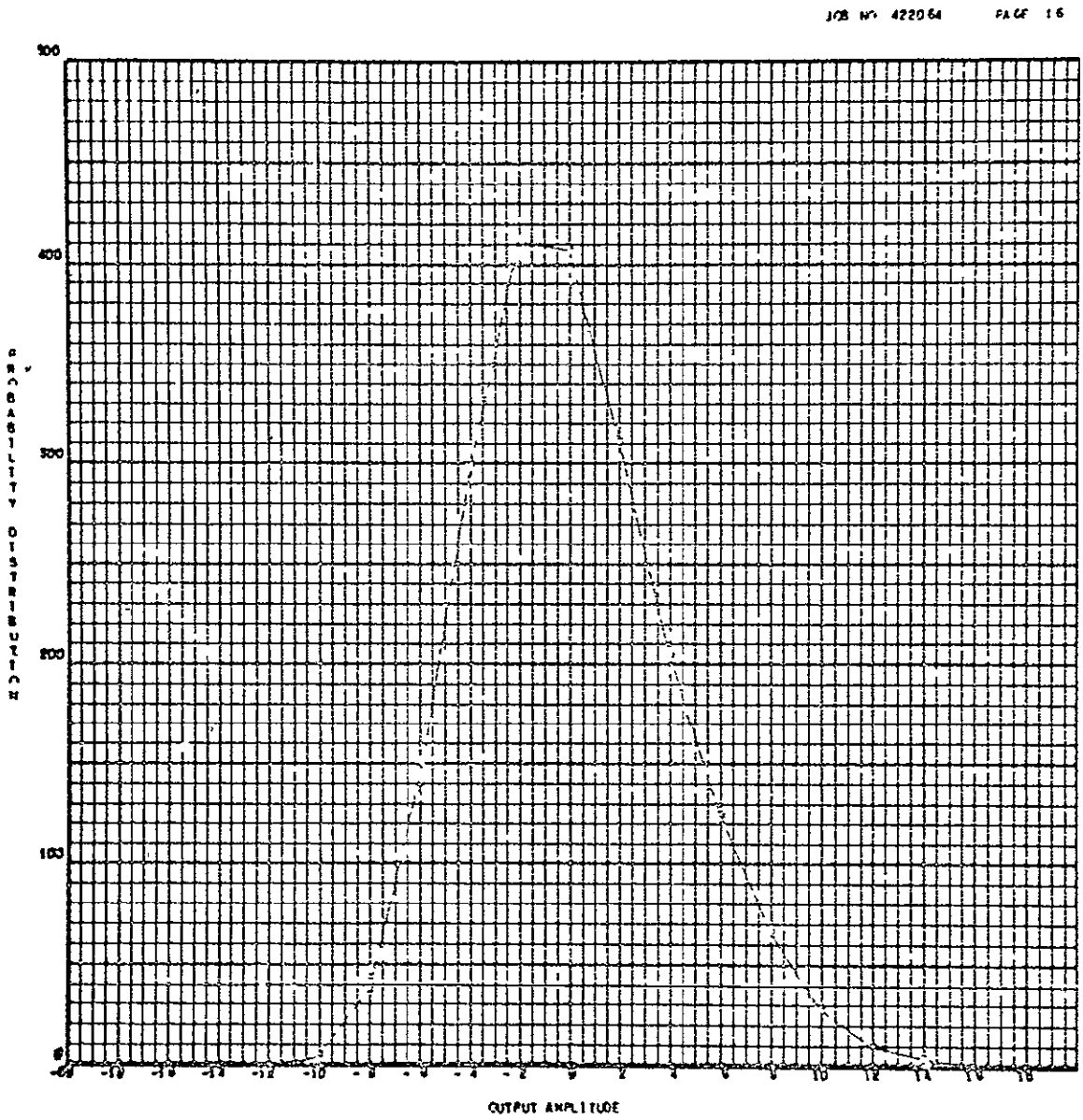

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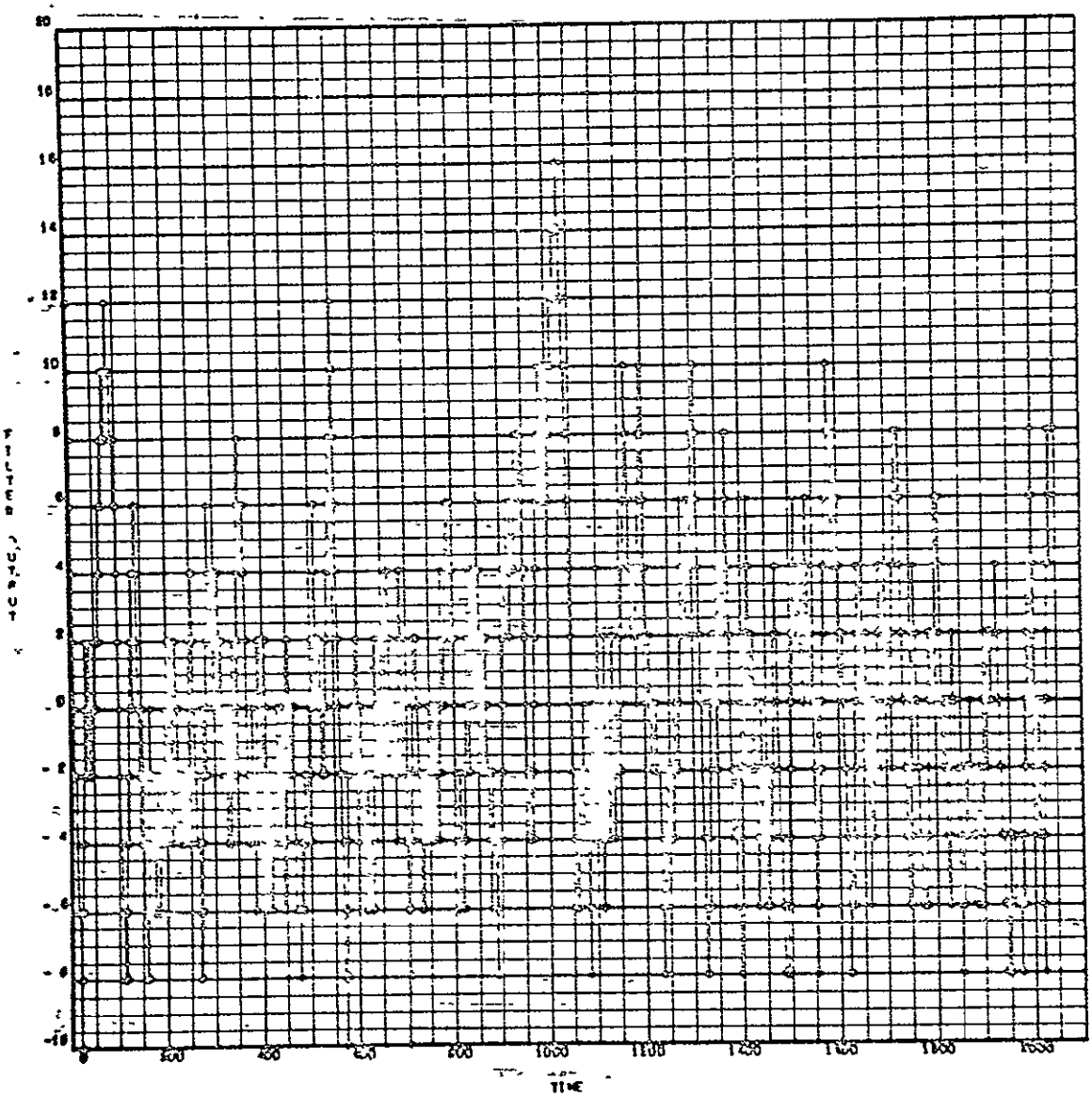
IMPULSE RESPONSE PERIOD = 16

Figure VI-12



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IMPULSE RESPONSE PERIOD = 20

Figure VI-13

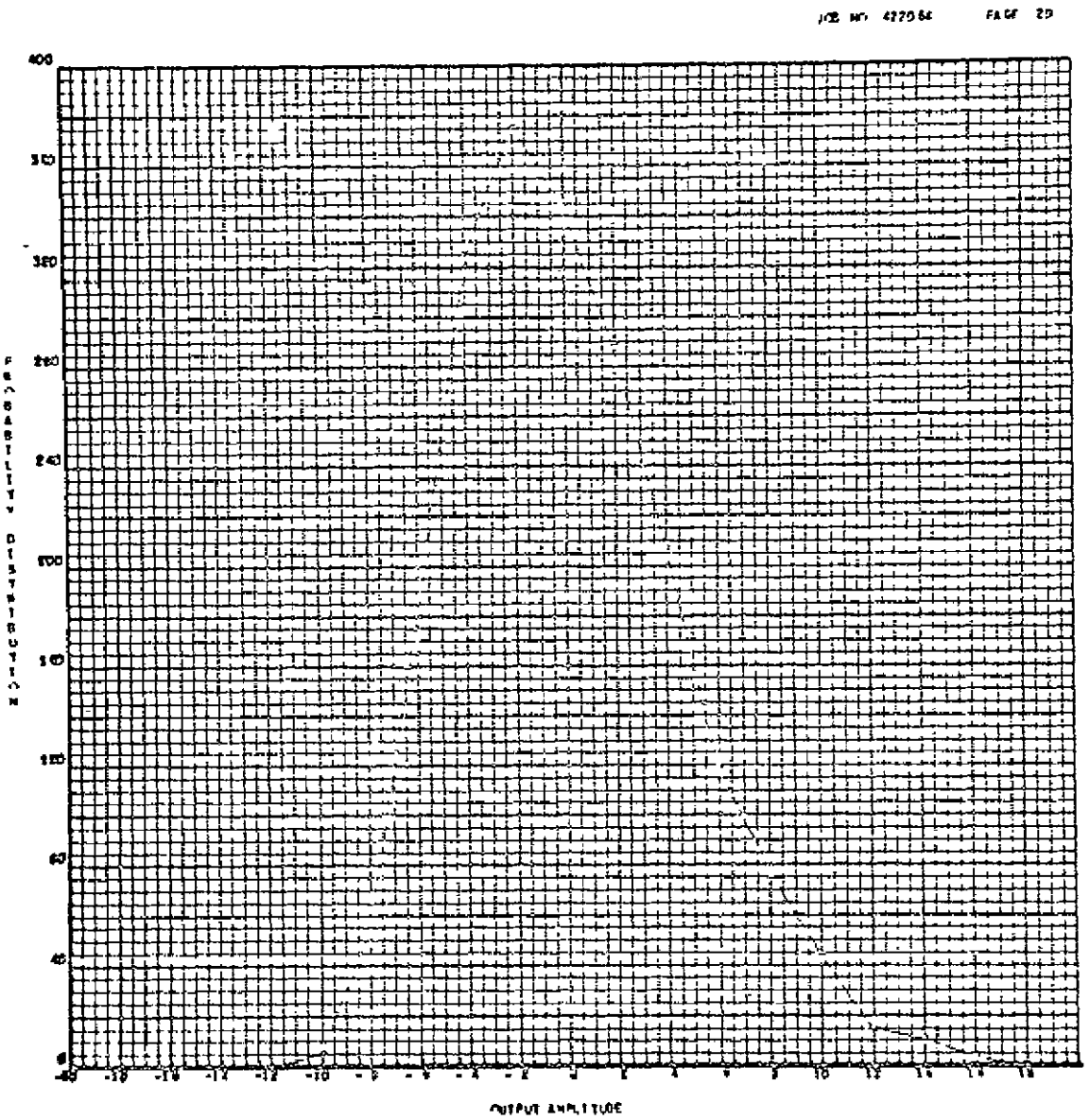

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Figure VI-13A

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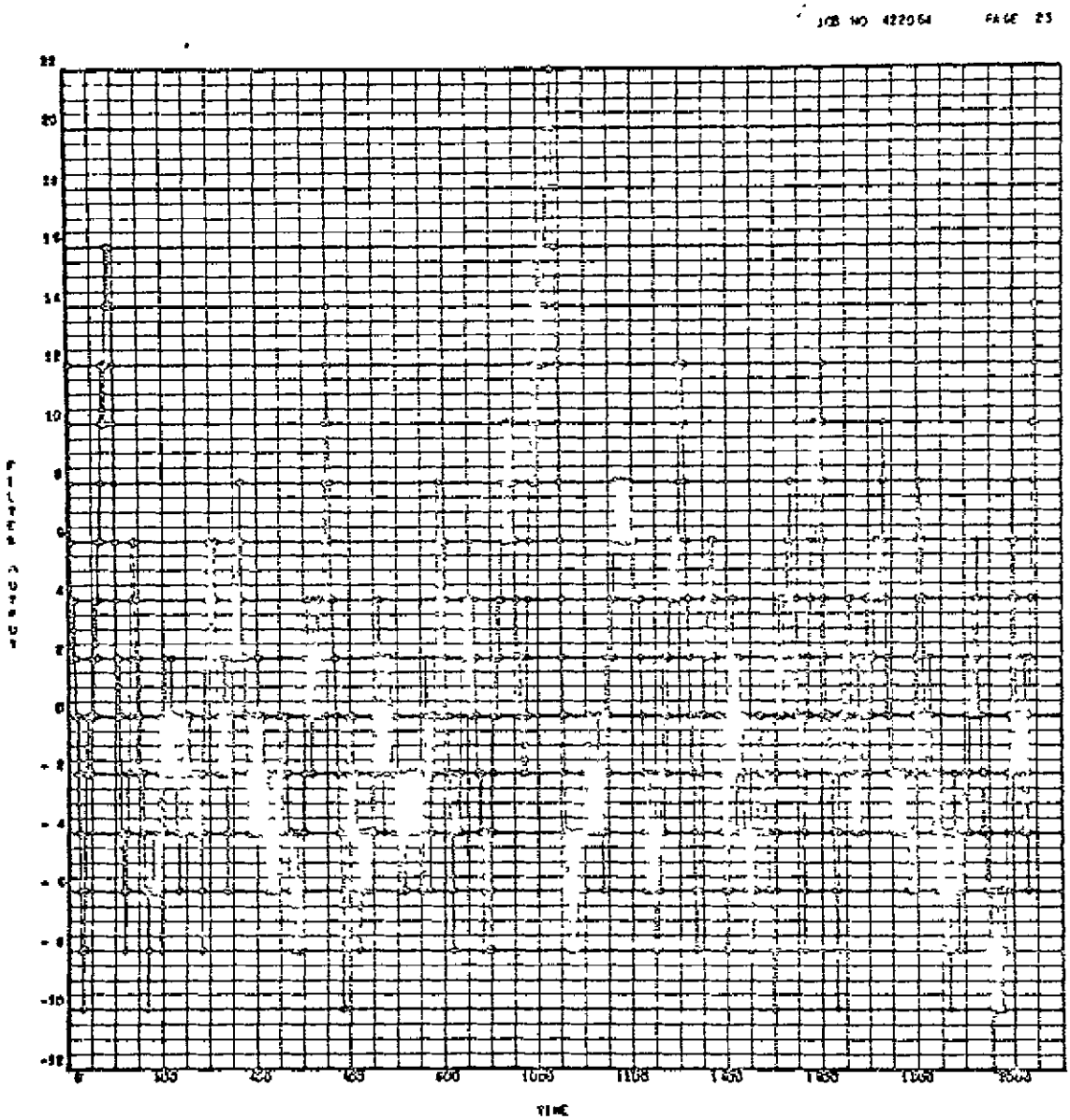

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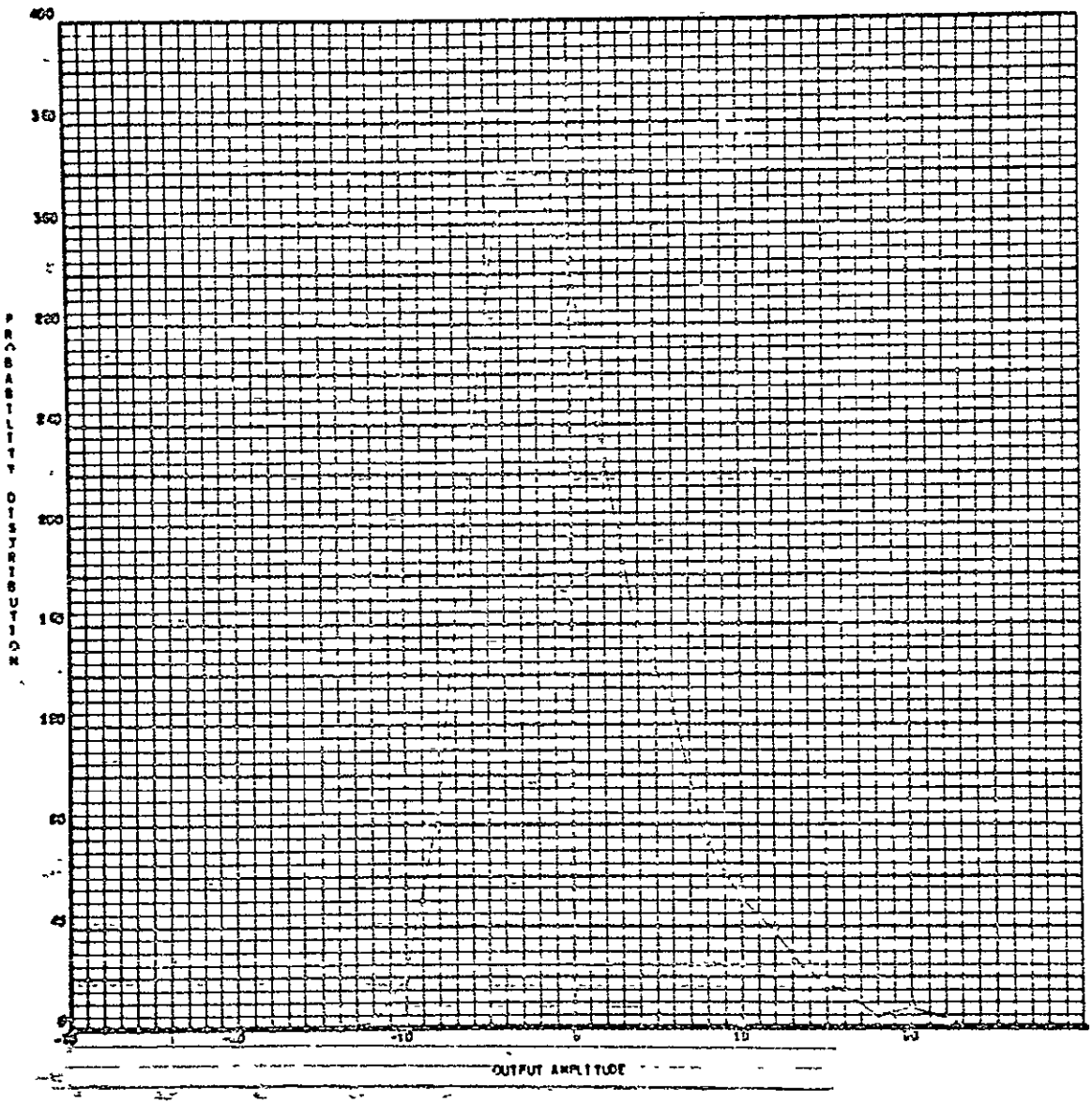
IMPULSE RESPONSE PERIOD = 30

Figure VI-14



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IMPULSE RESPONSE PERIOD = 30

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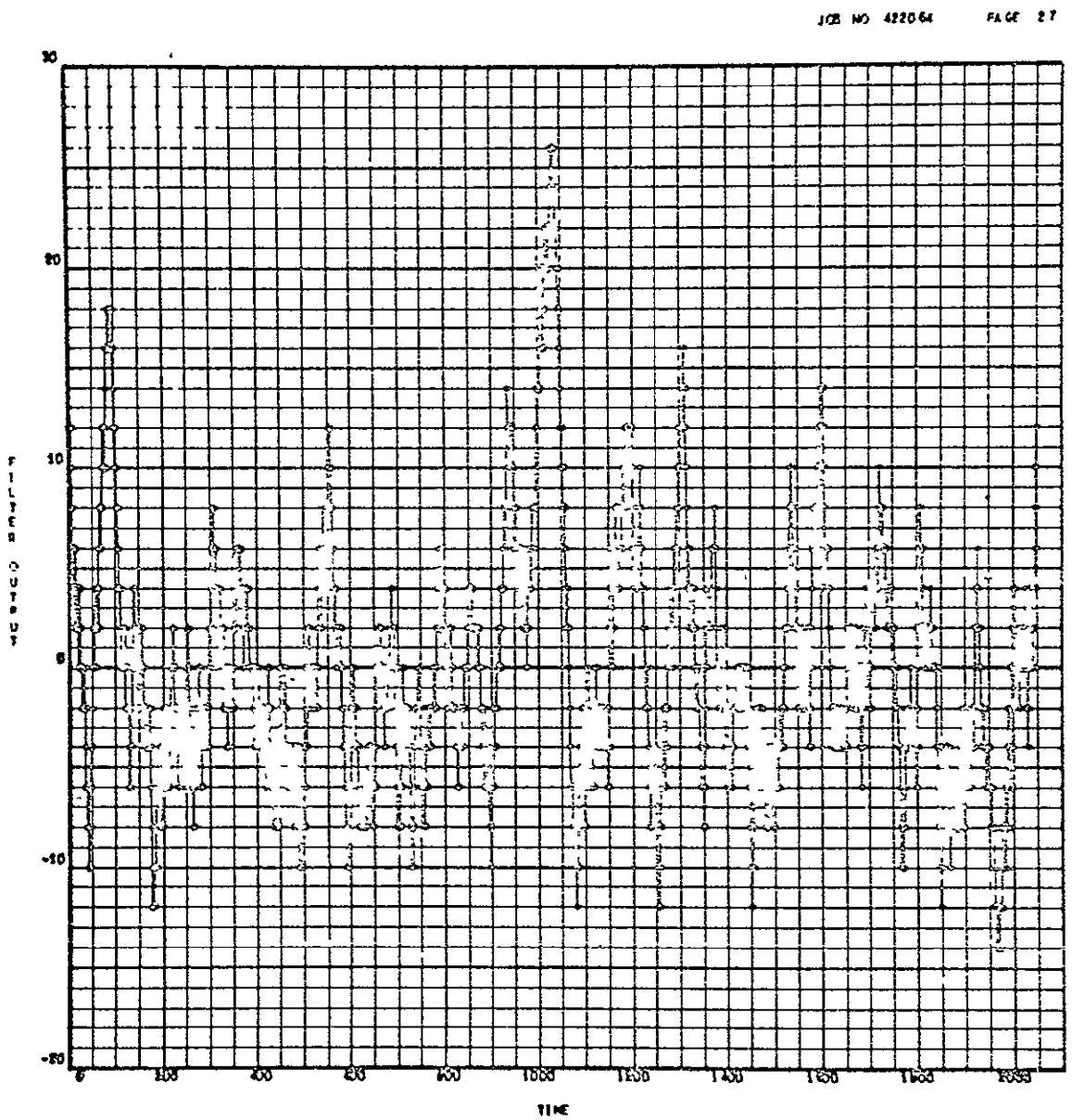
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SEQUENCE LENGTH = 2047
IMPULSE RESPONSE PERIOD = 40

Figure VI-16

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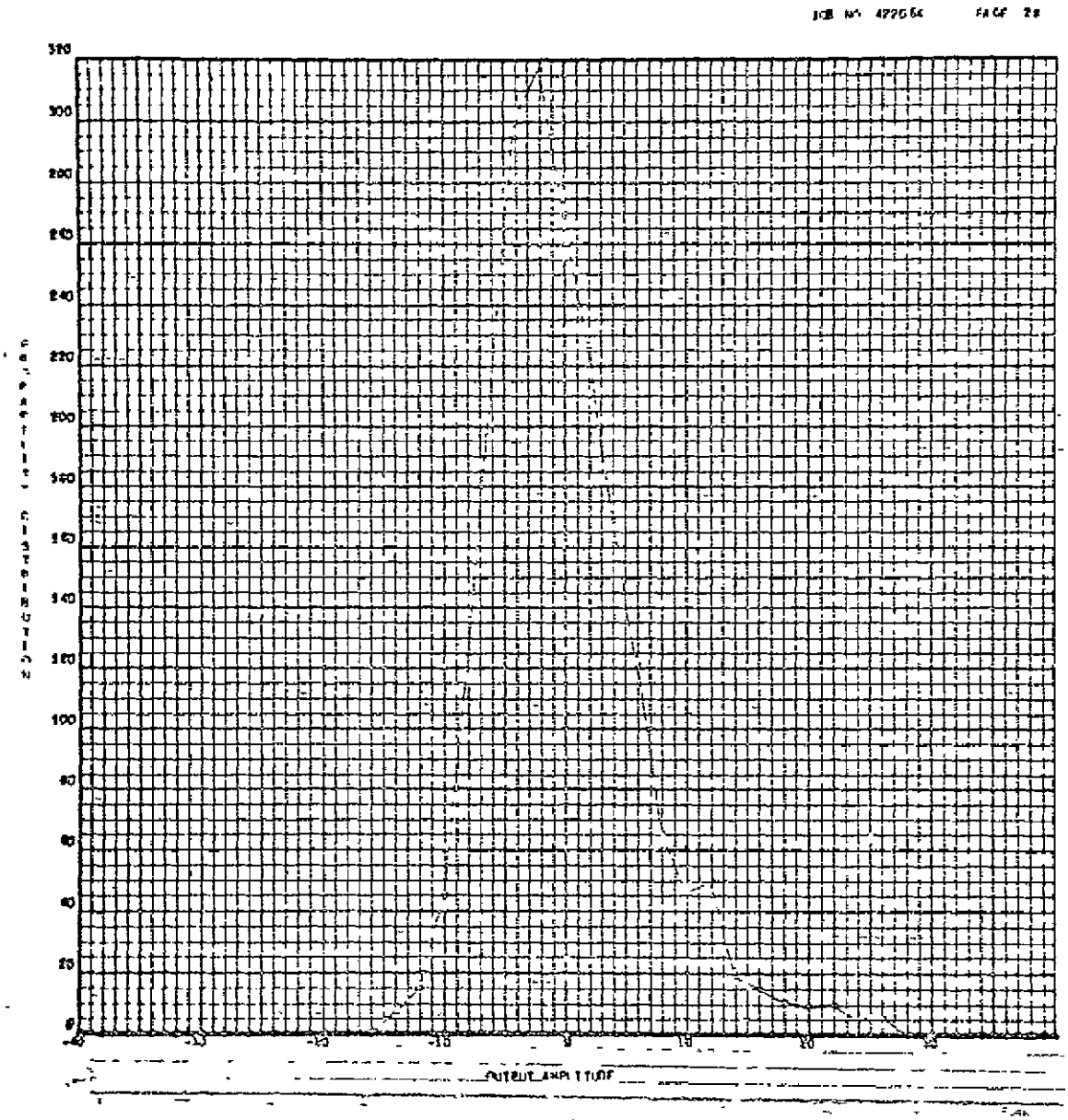
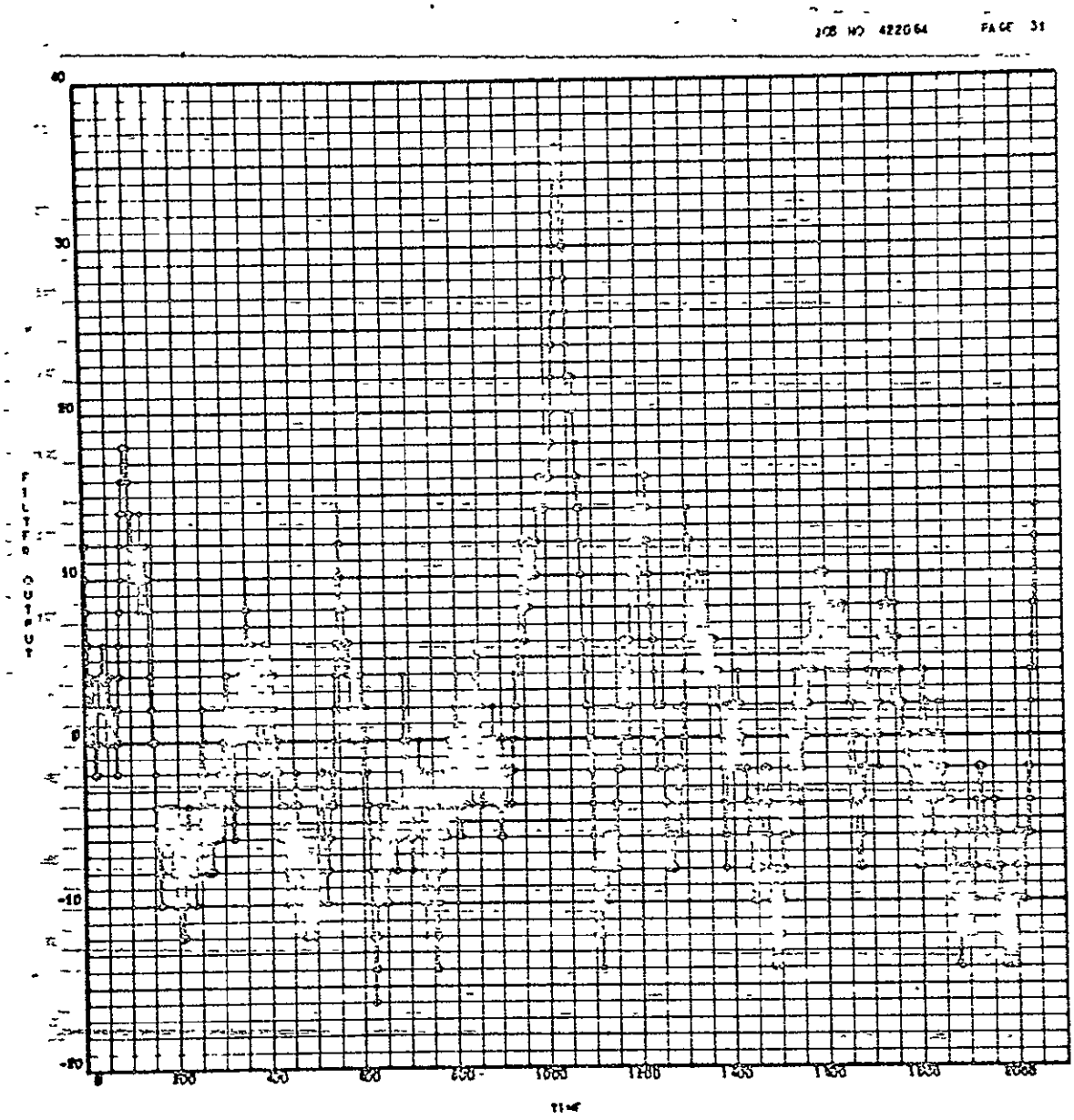


Figure VI-17



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IMPULSE RESPONSE PERIOD = 70

Figure VI-18

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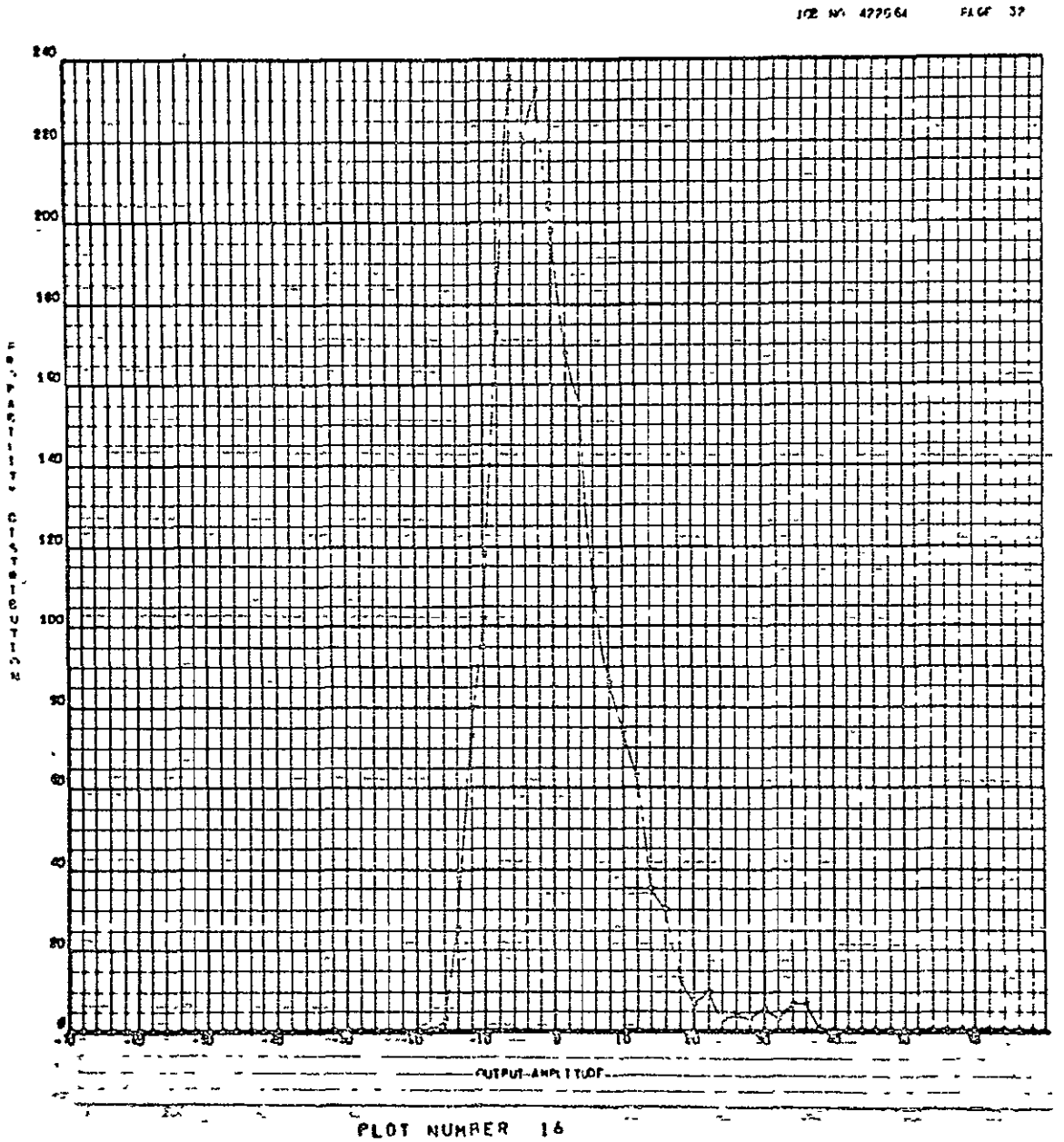

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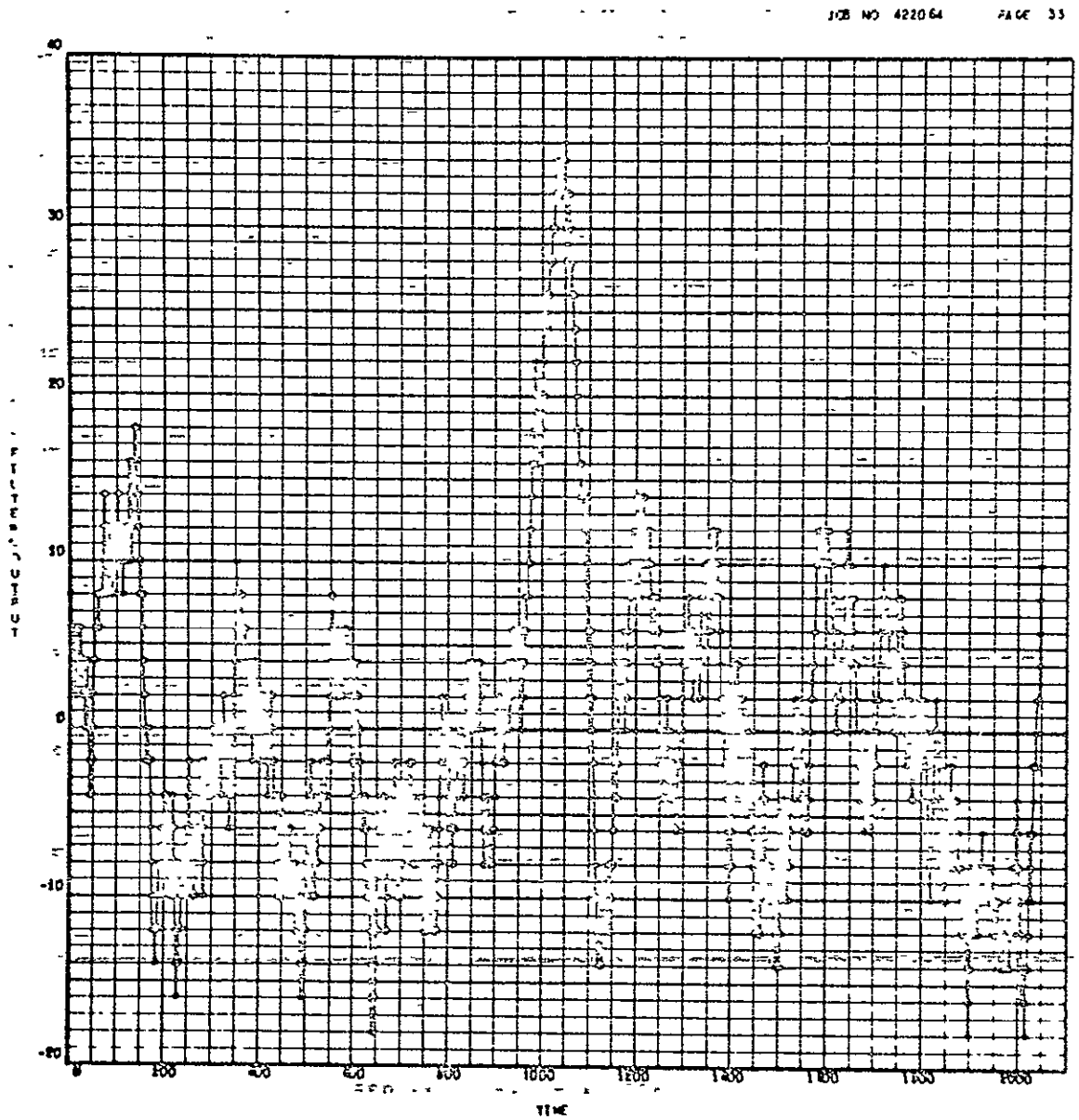

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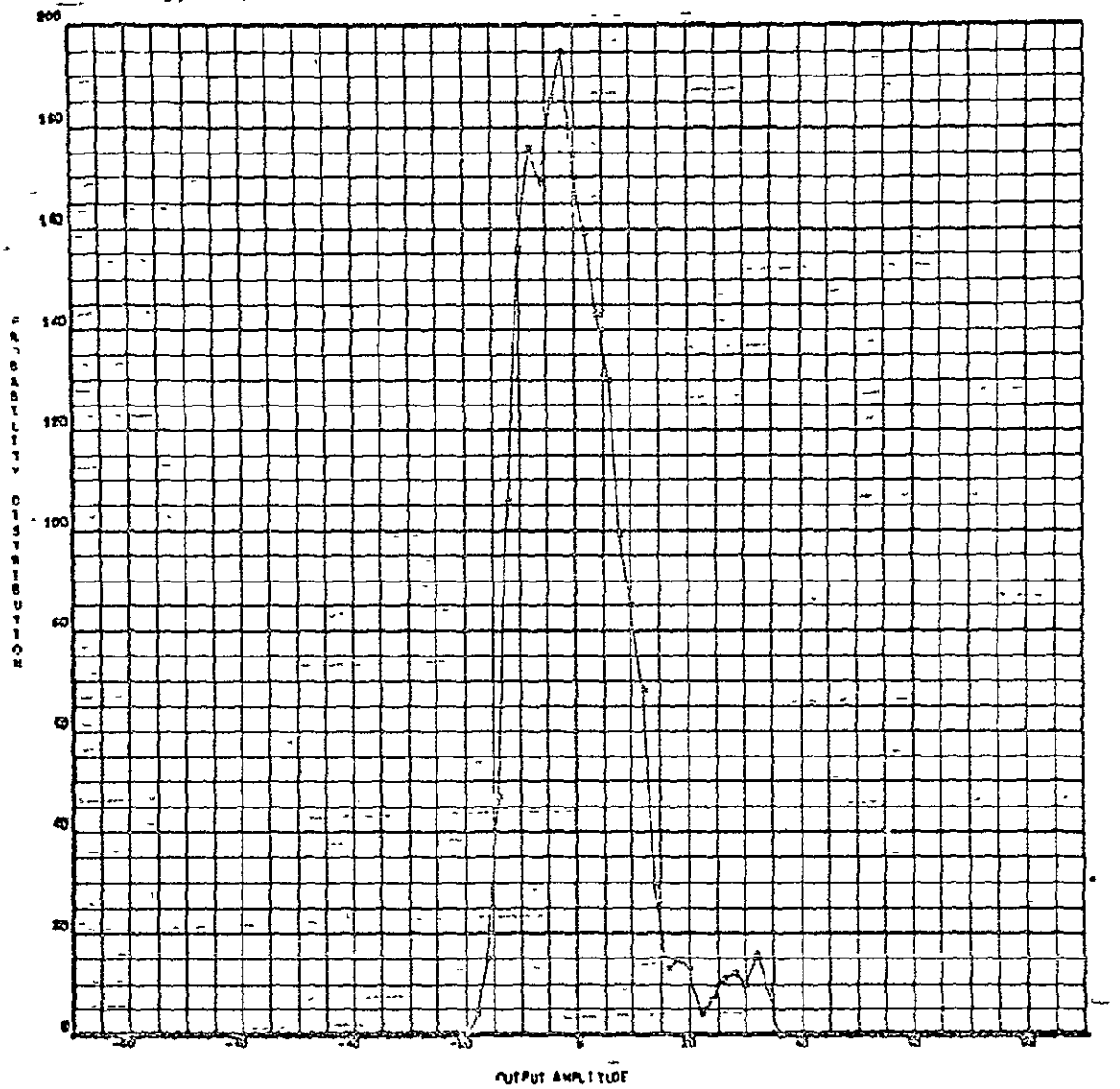
IMPULSE RESPONSE PERIOD = 90

Figure VI-20



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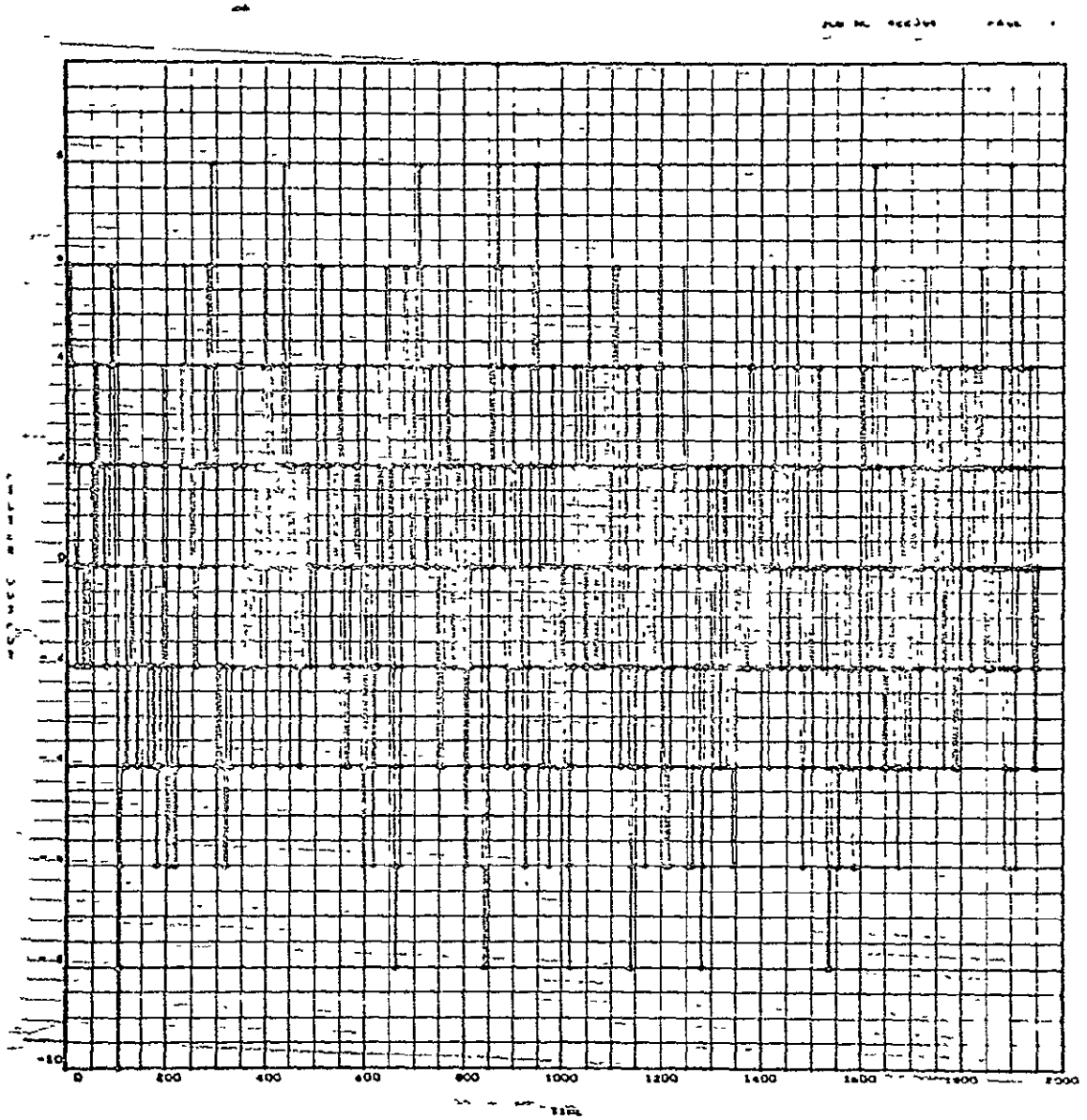
IMPULSE RESPONSE PERIOD = 90

Figure VI-21



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IMPULSE RESPONSE PERIOD = 10

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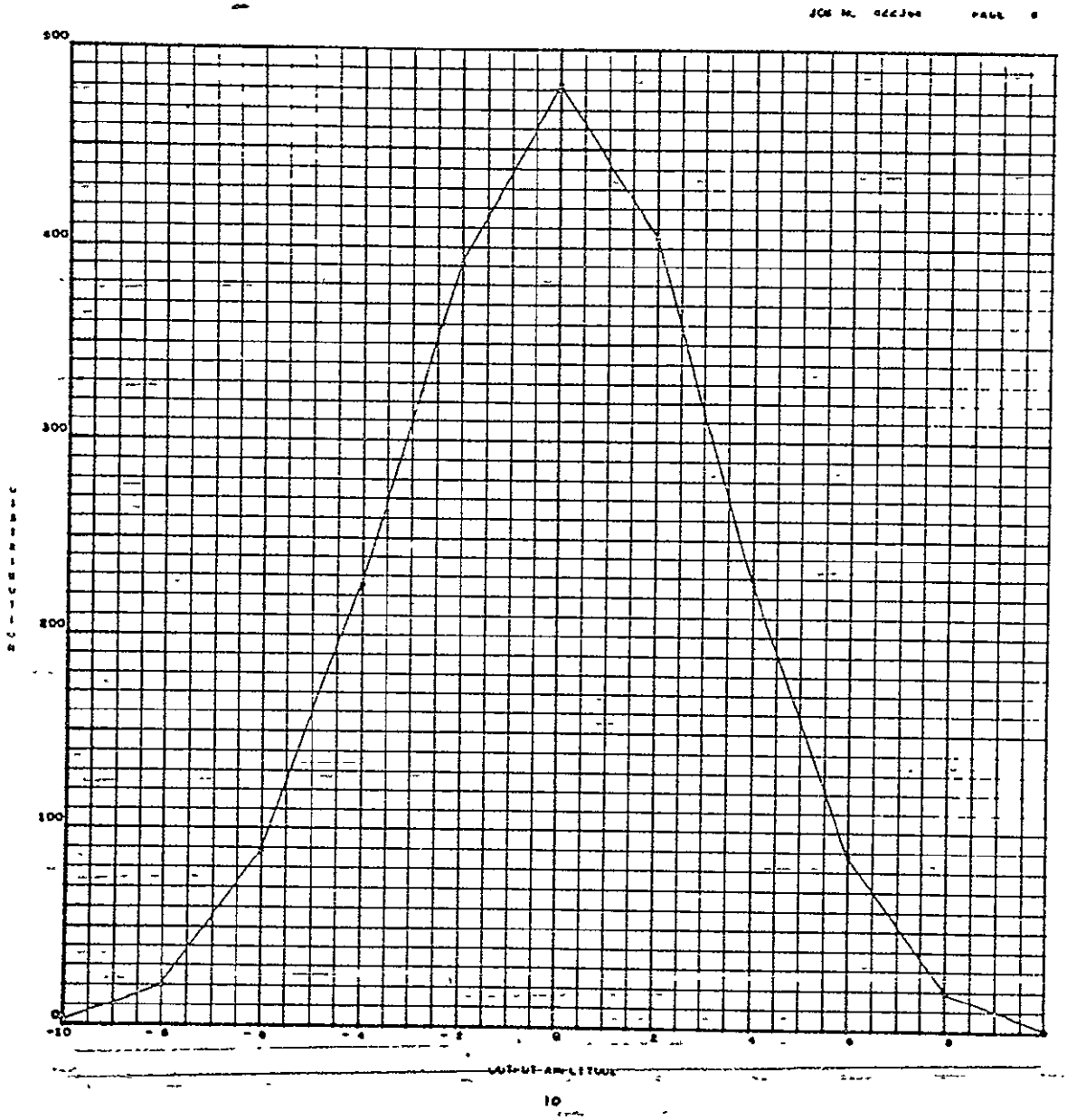

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IMPULSE RESPONSE PERIOD = 10

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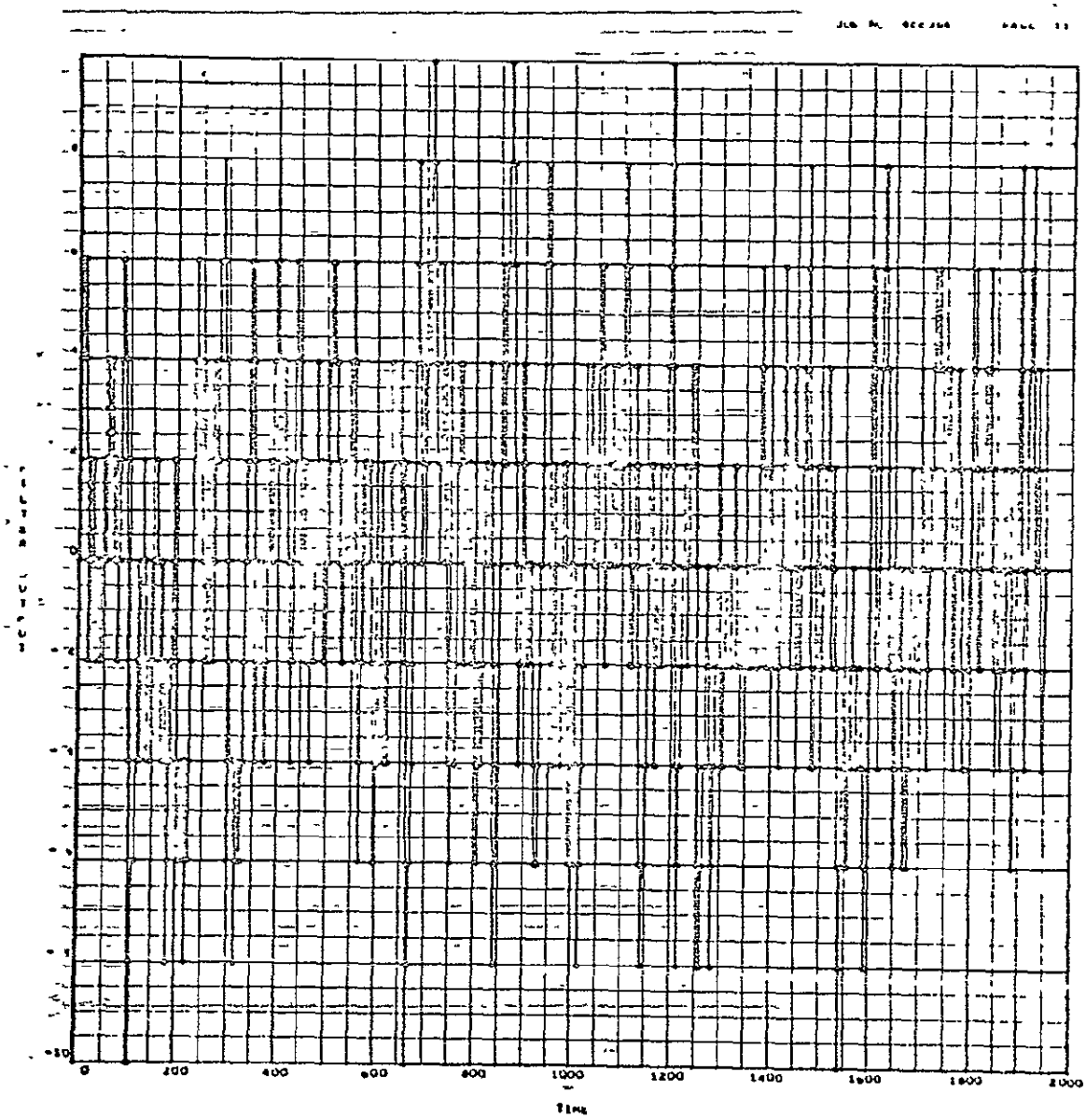

PLOT NUMBER - 6

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IMPULSE RESPONSE PERIOD = 12

Figure VI-24

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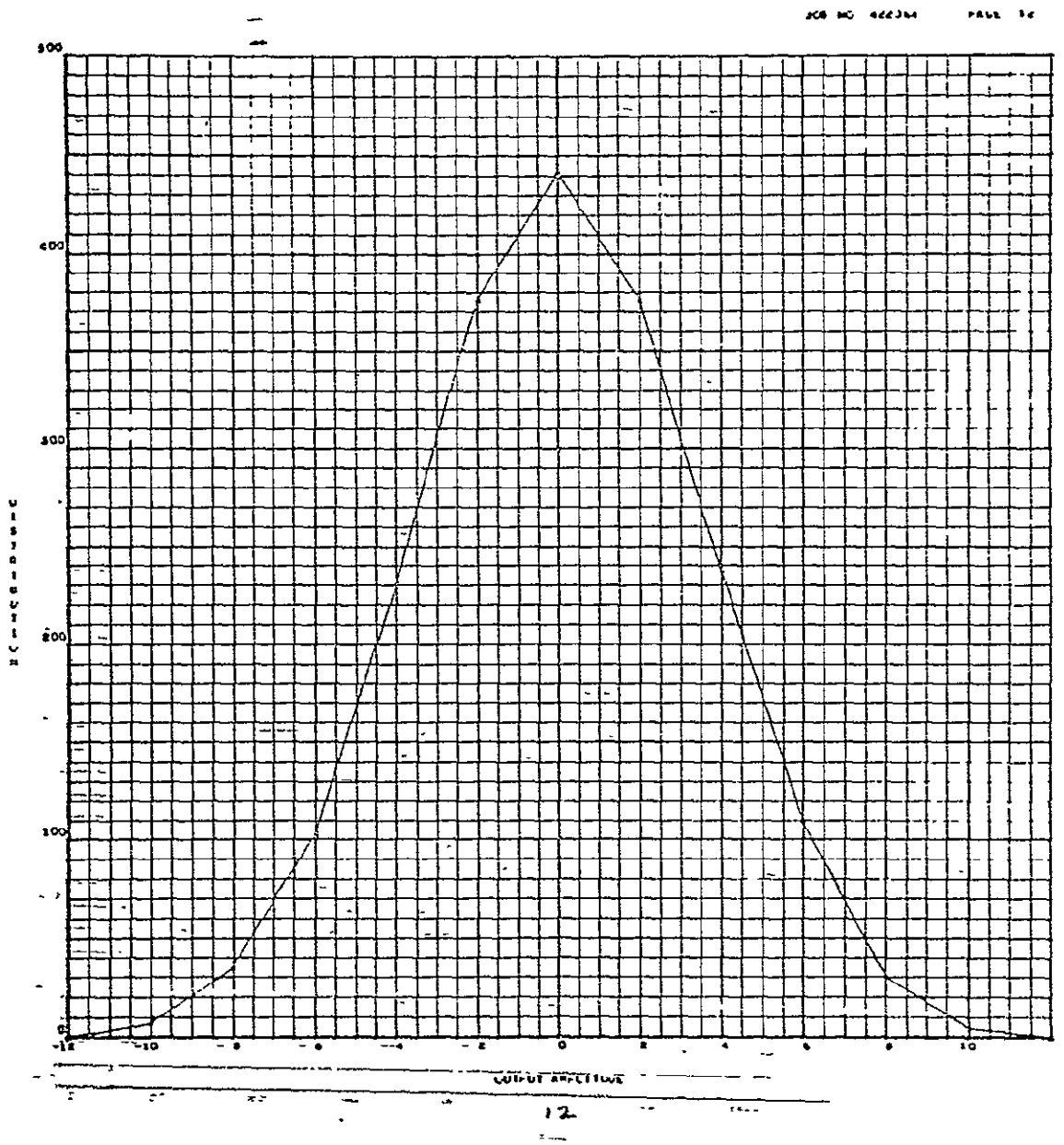

PLOT NUMBER 6

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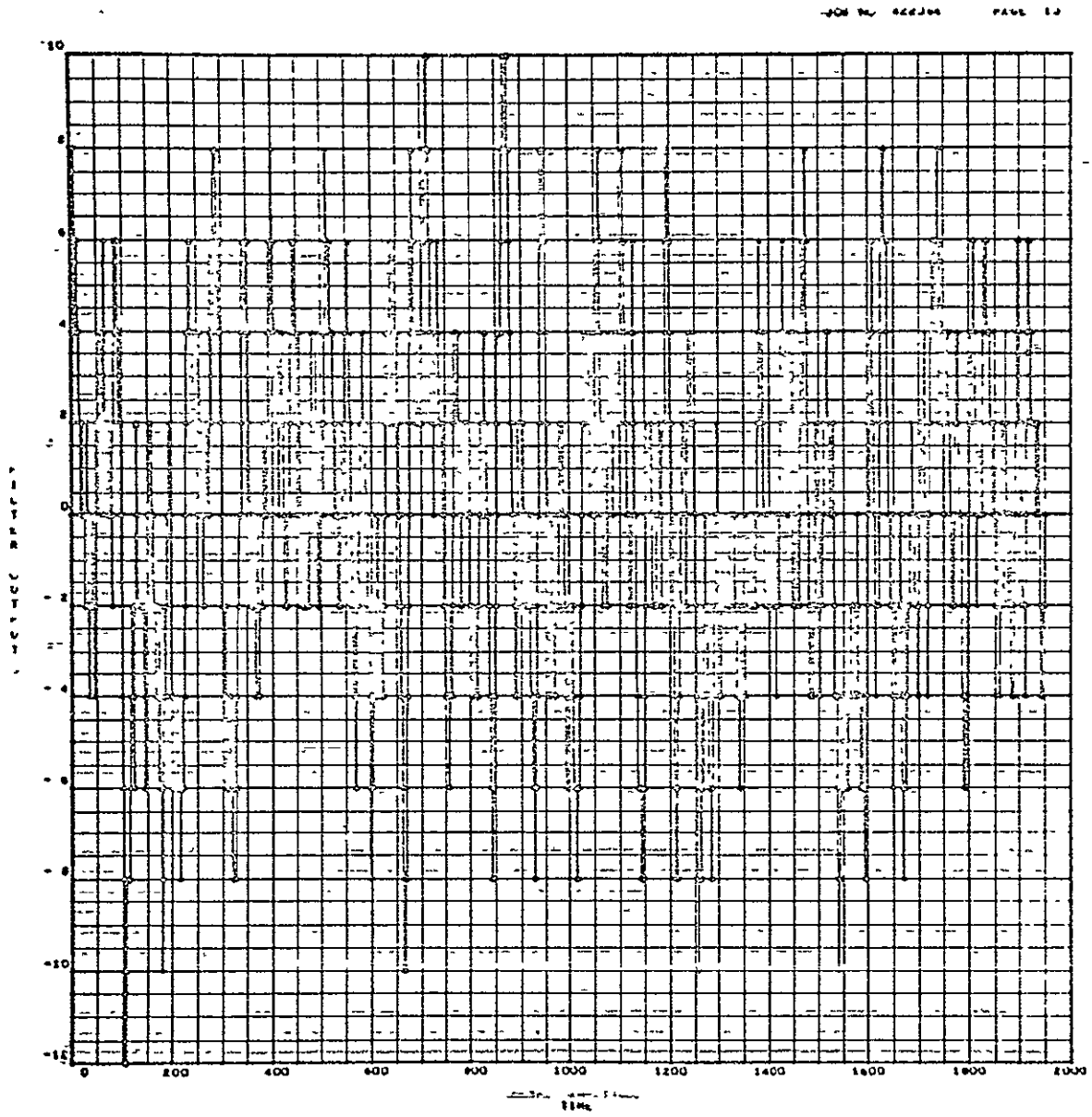
IMPULSE RESPONSE PERIOD = 12

Figure VI-25



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IMPULSE RESPONSE PERIOD = 14

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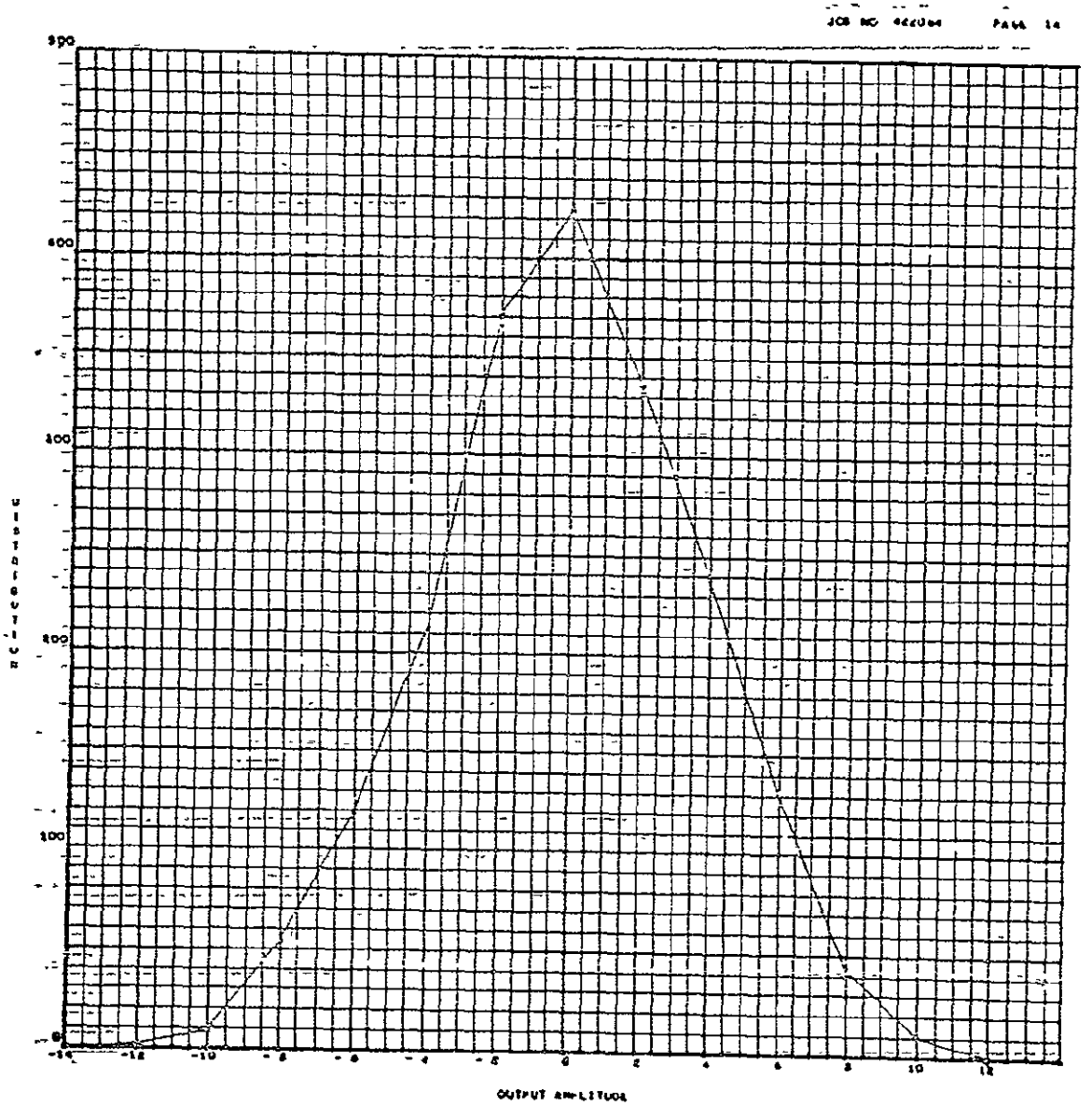

14

PLOT NUMBER 7

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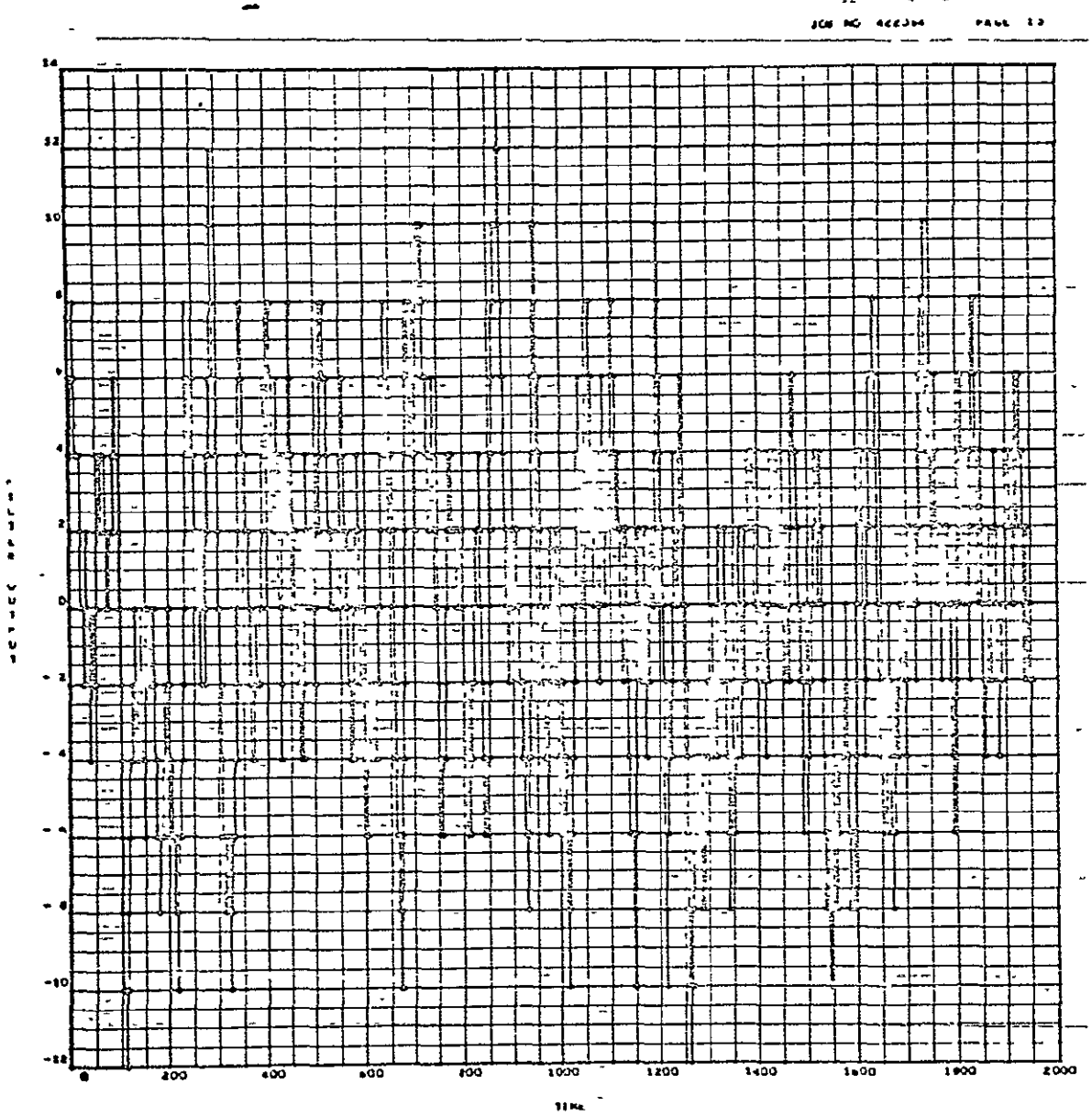
IMPULSE RESPONSE PERIOD = 14

Figure VI-27

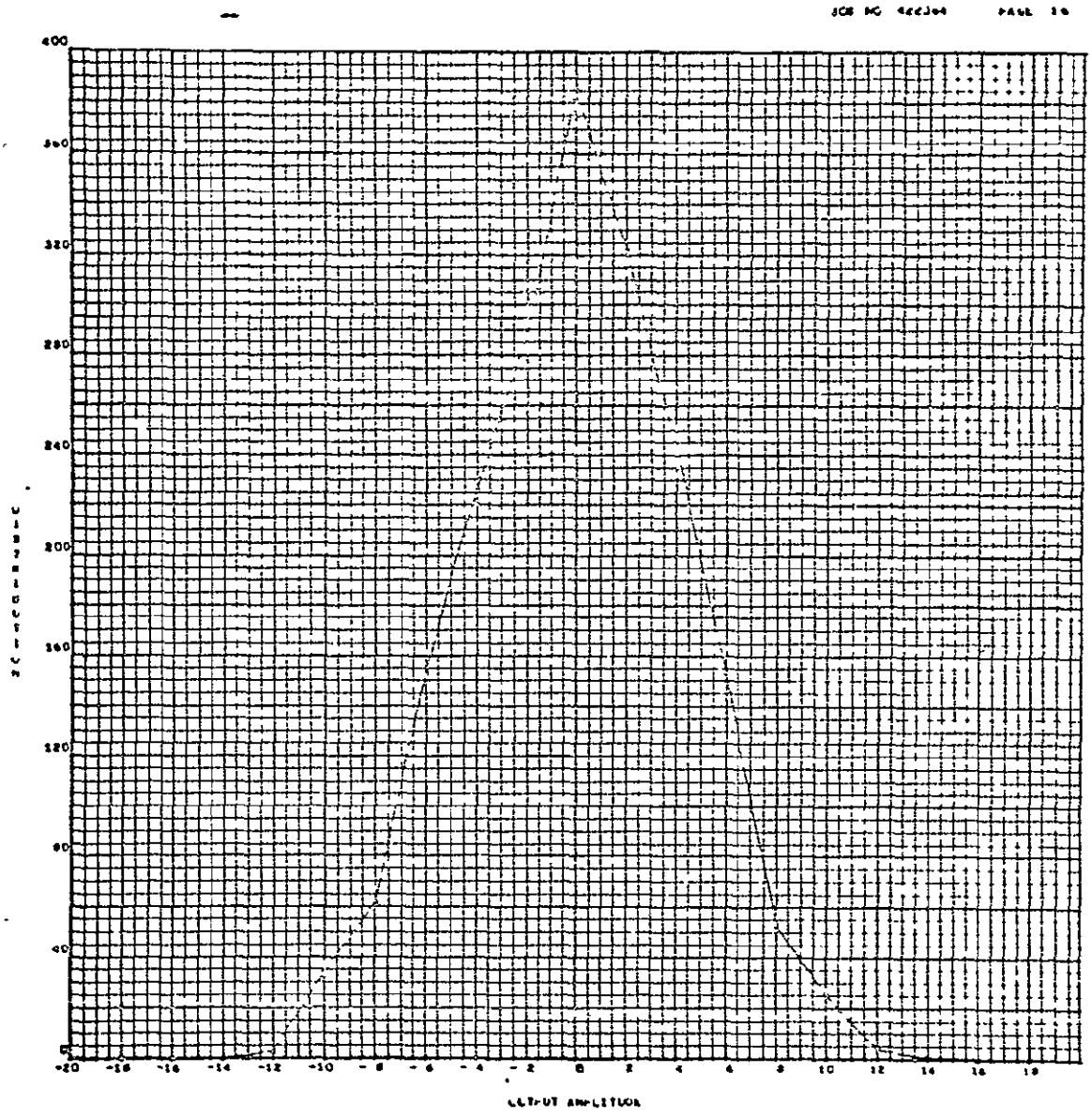


PLOT NUMBER = 8

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 18

Figure VI-28



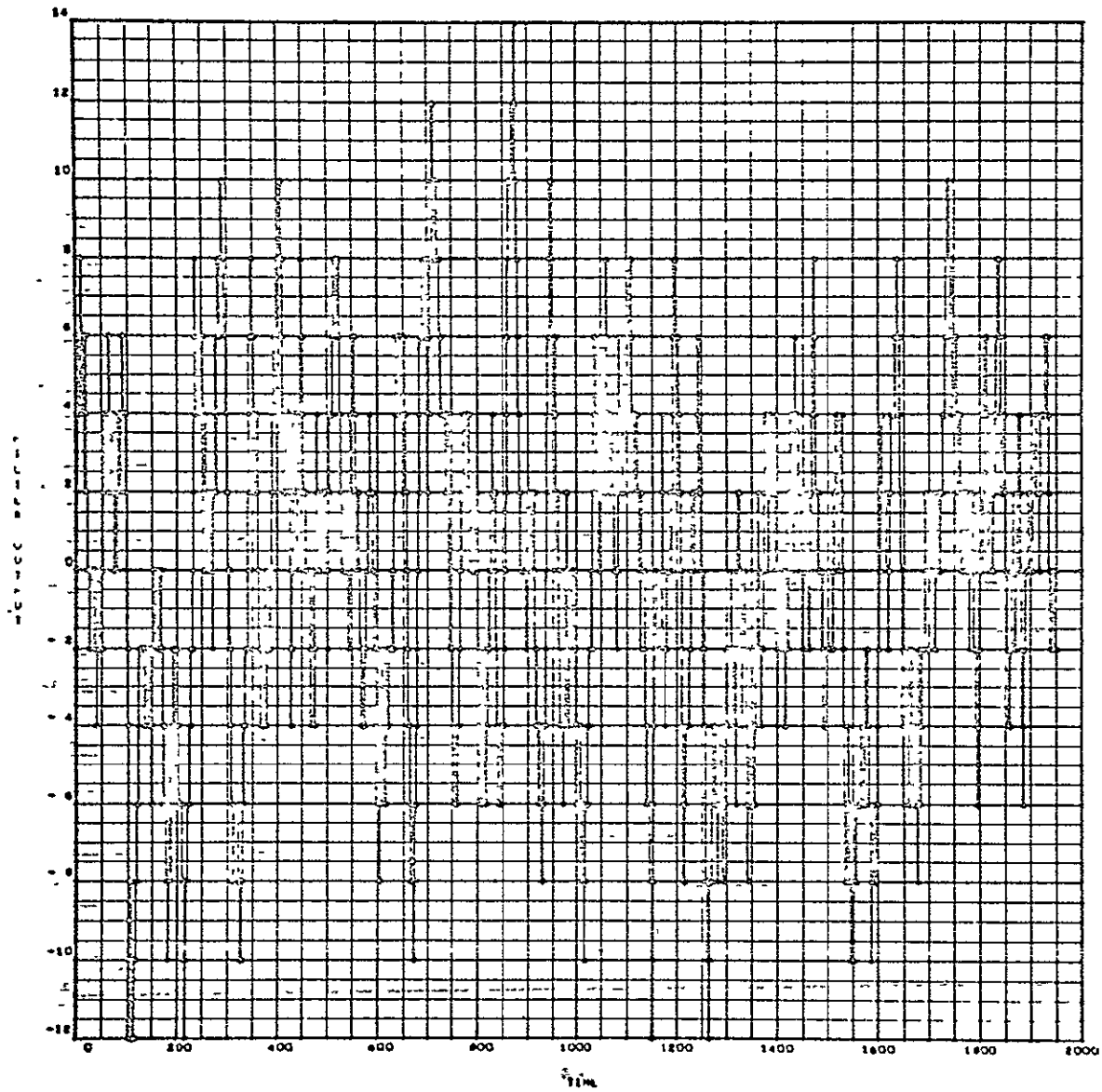
18

PLOT NUMBER 8

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 13

Figure VI-29



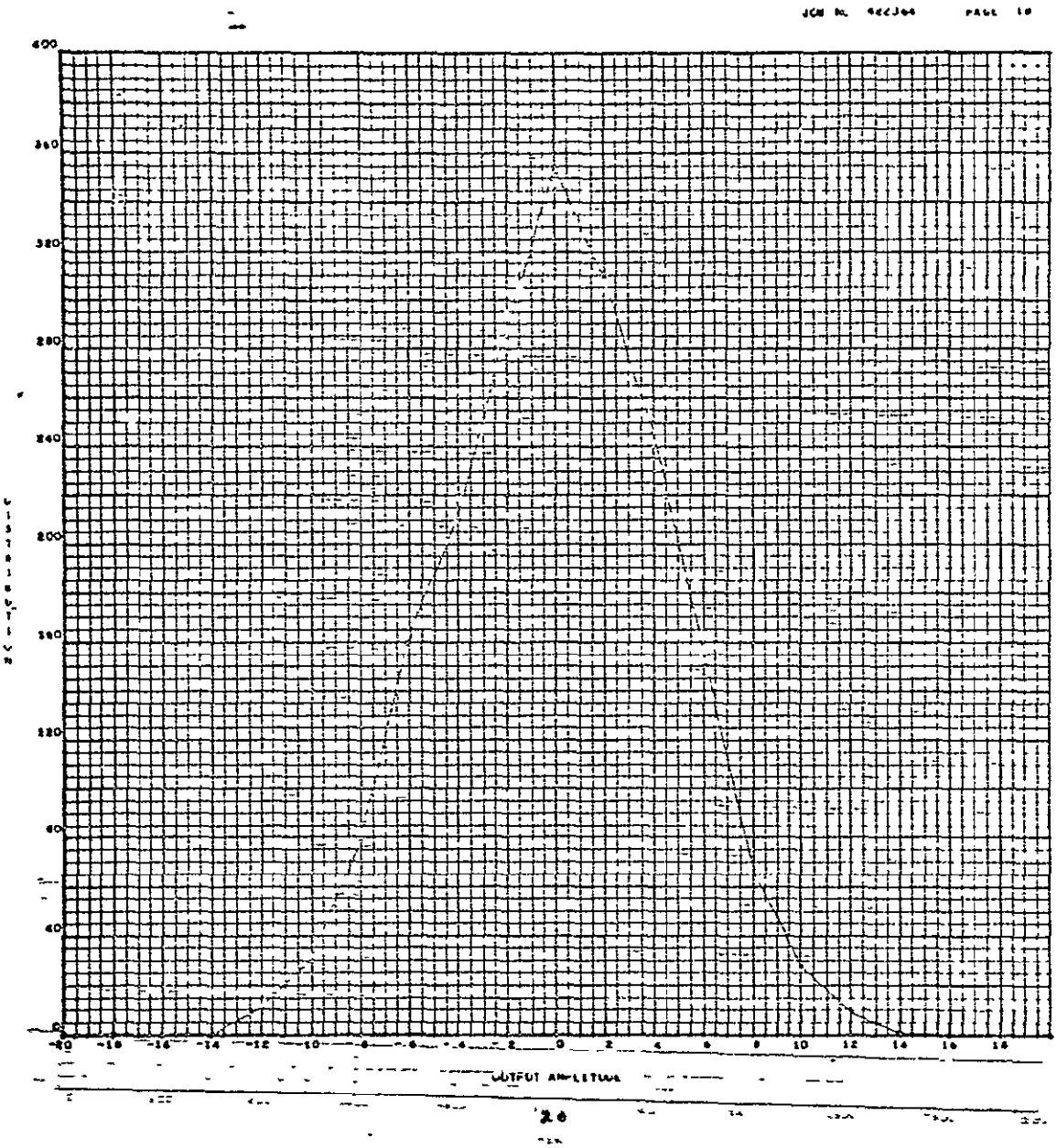
PLOT NUMBER 9

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 20

Figure VI-30

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

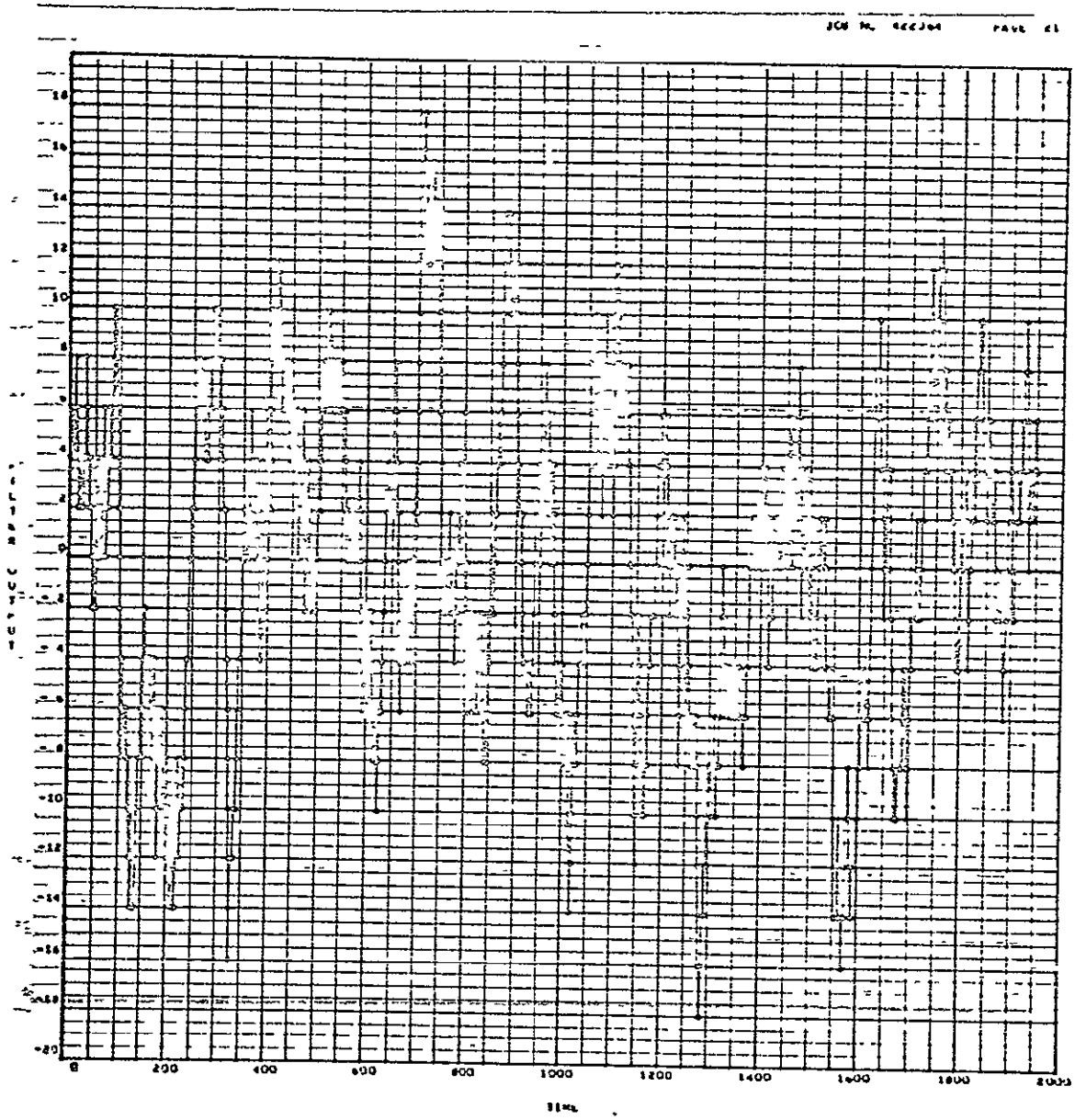


PLOT NUMBER 9

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 20

Figure VI-31



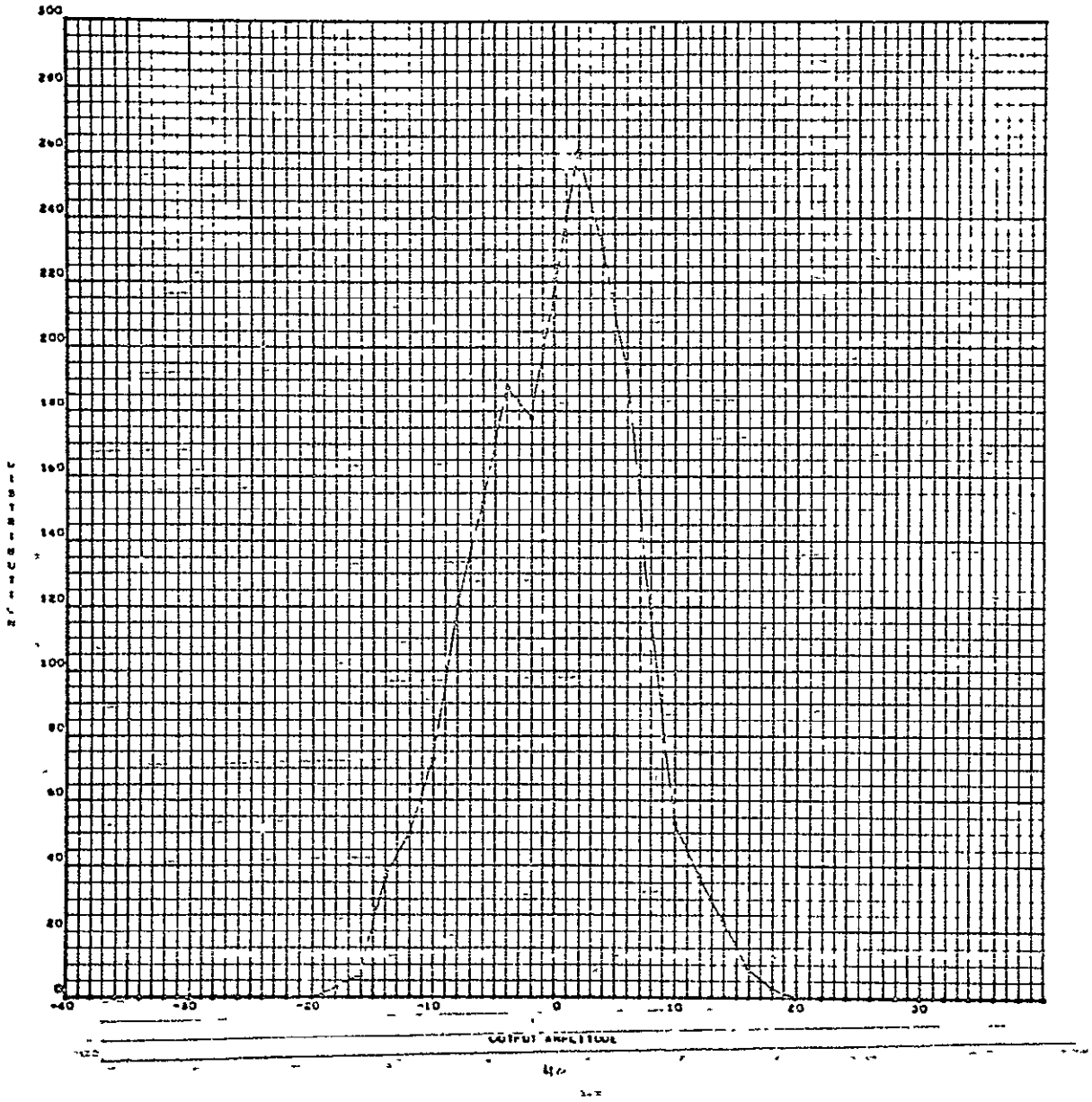
PLOT NUMBER 11

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 40

Figure VI-32

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

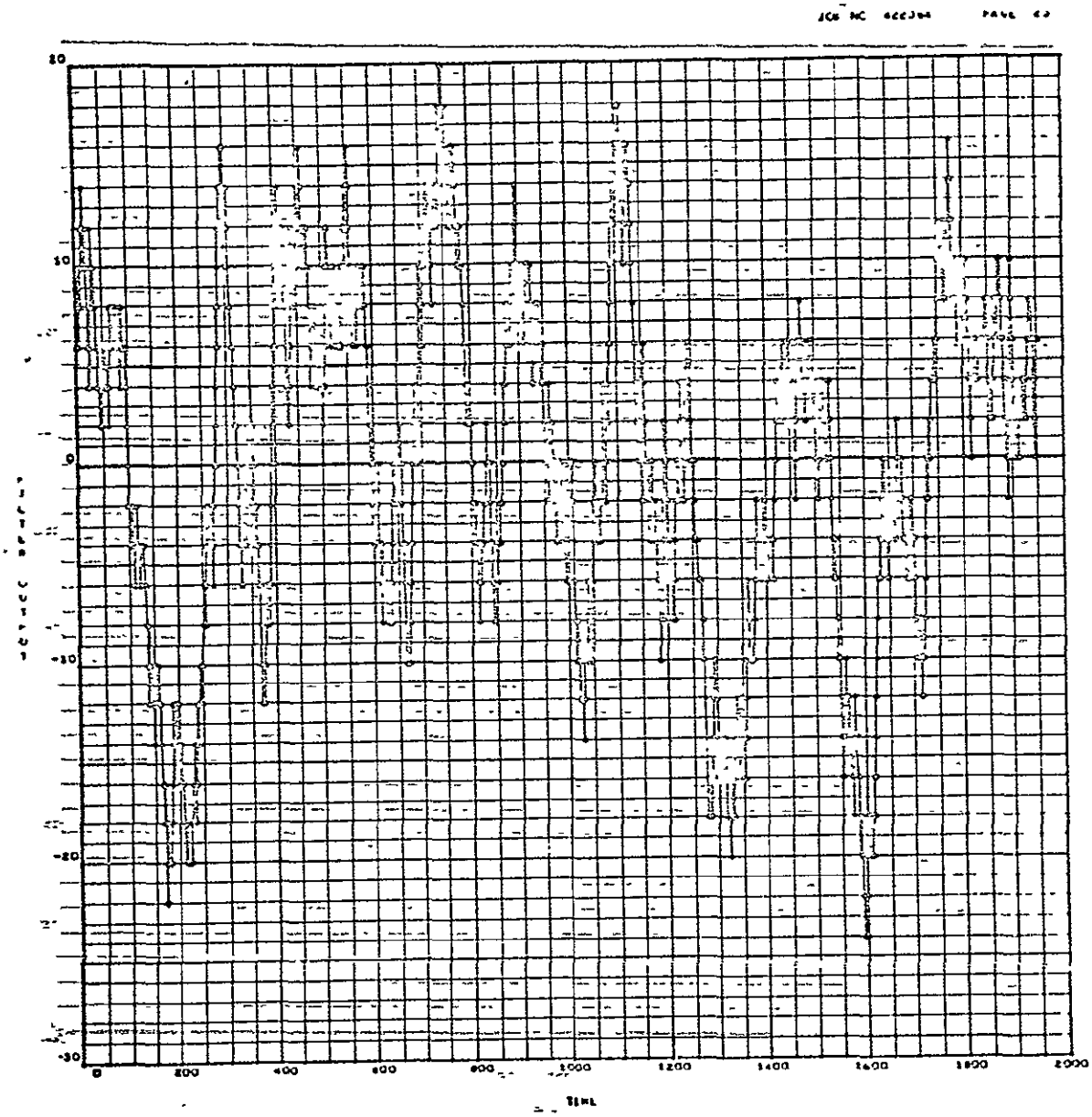


PLOT NUMBER 11

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 40

Figure VI-33

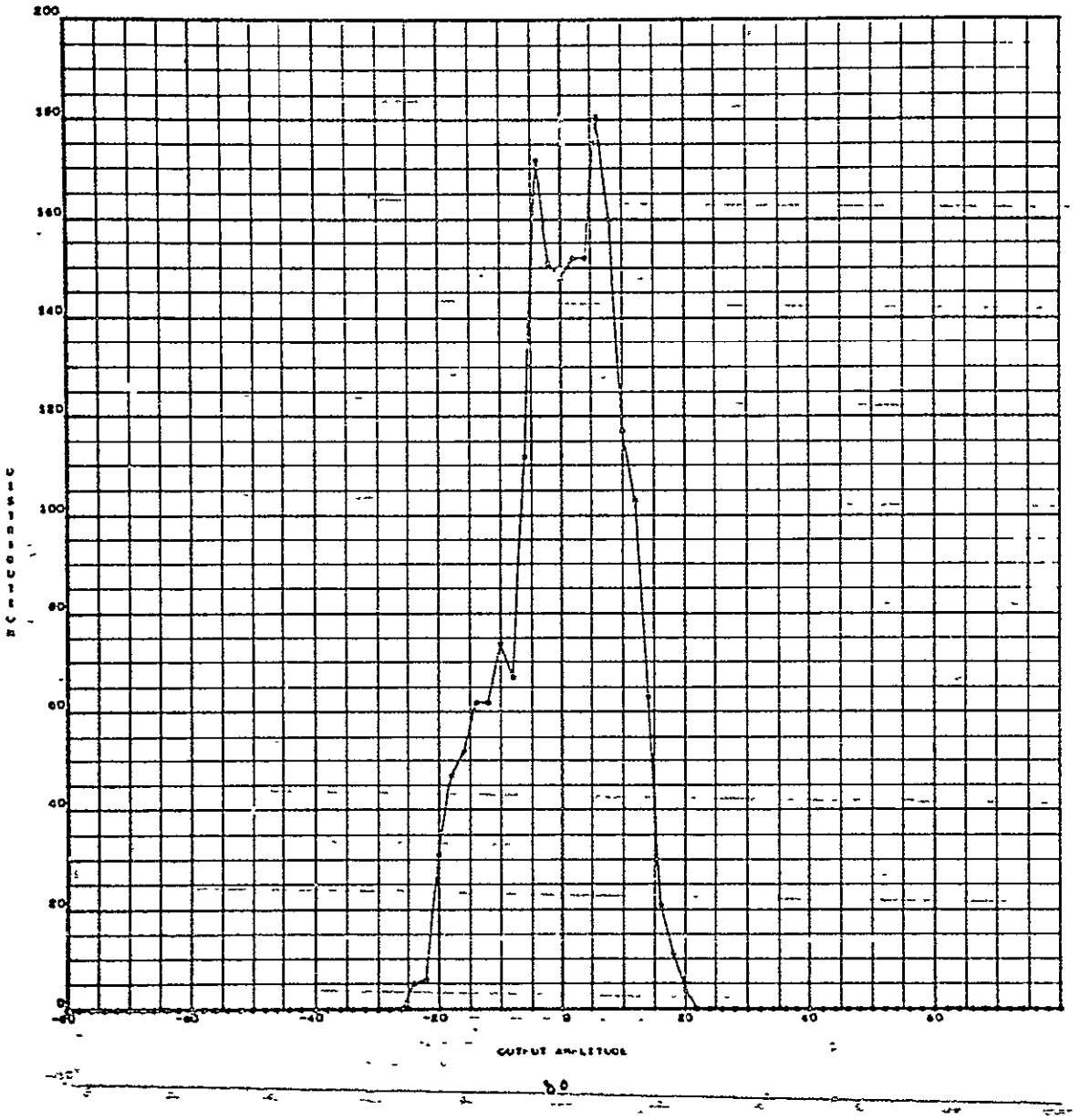


PLOT NUMBER = 12

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 80

Figure VI-34



PLOT NUMBER 12

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 80

Figure VI-3[illegible]

Figure VI-35

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

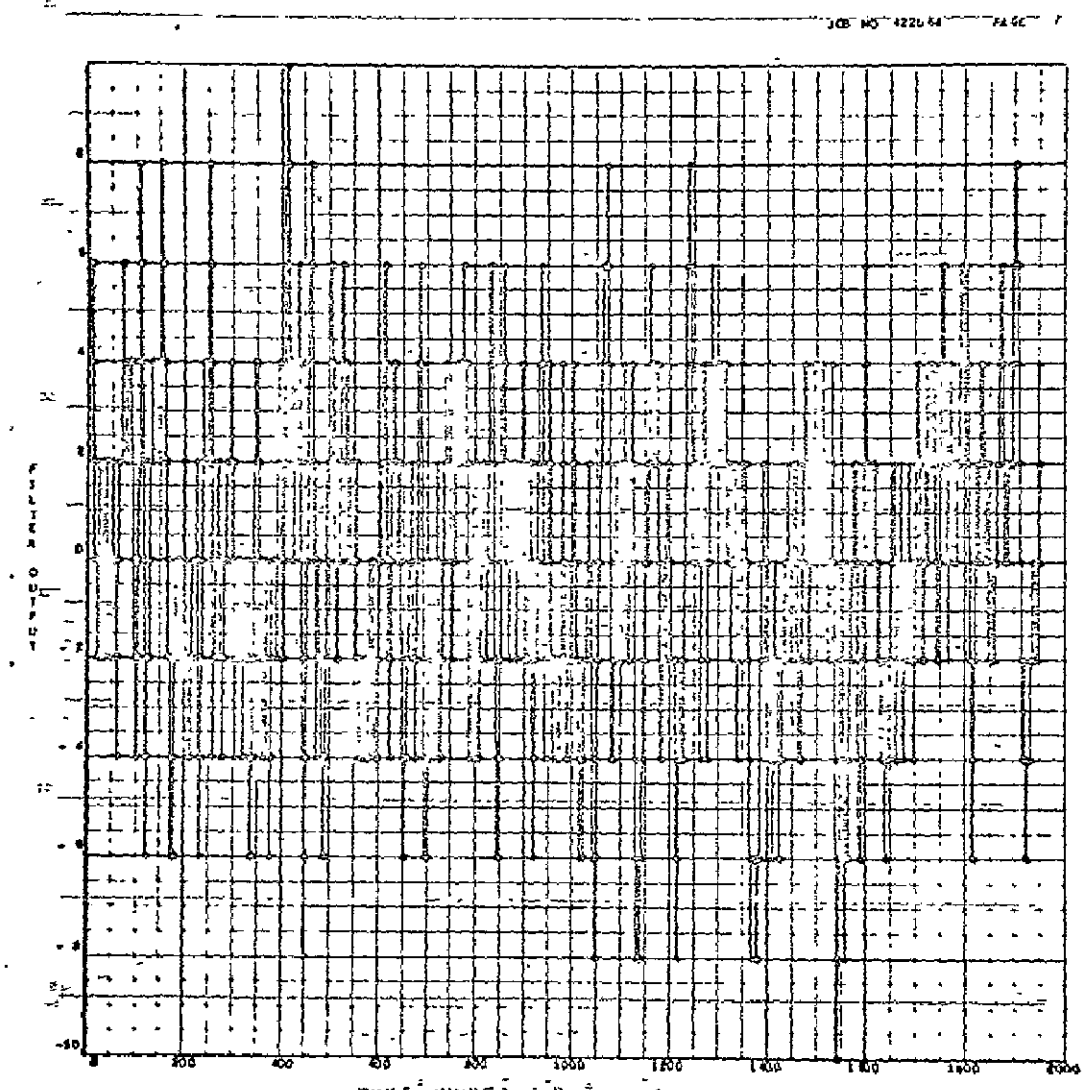
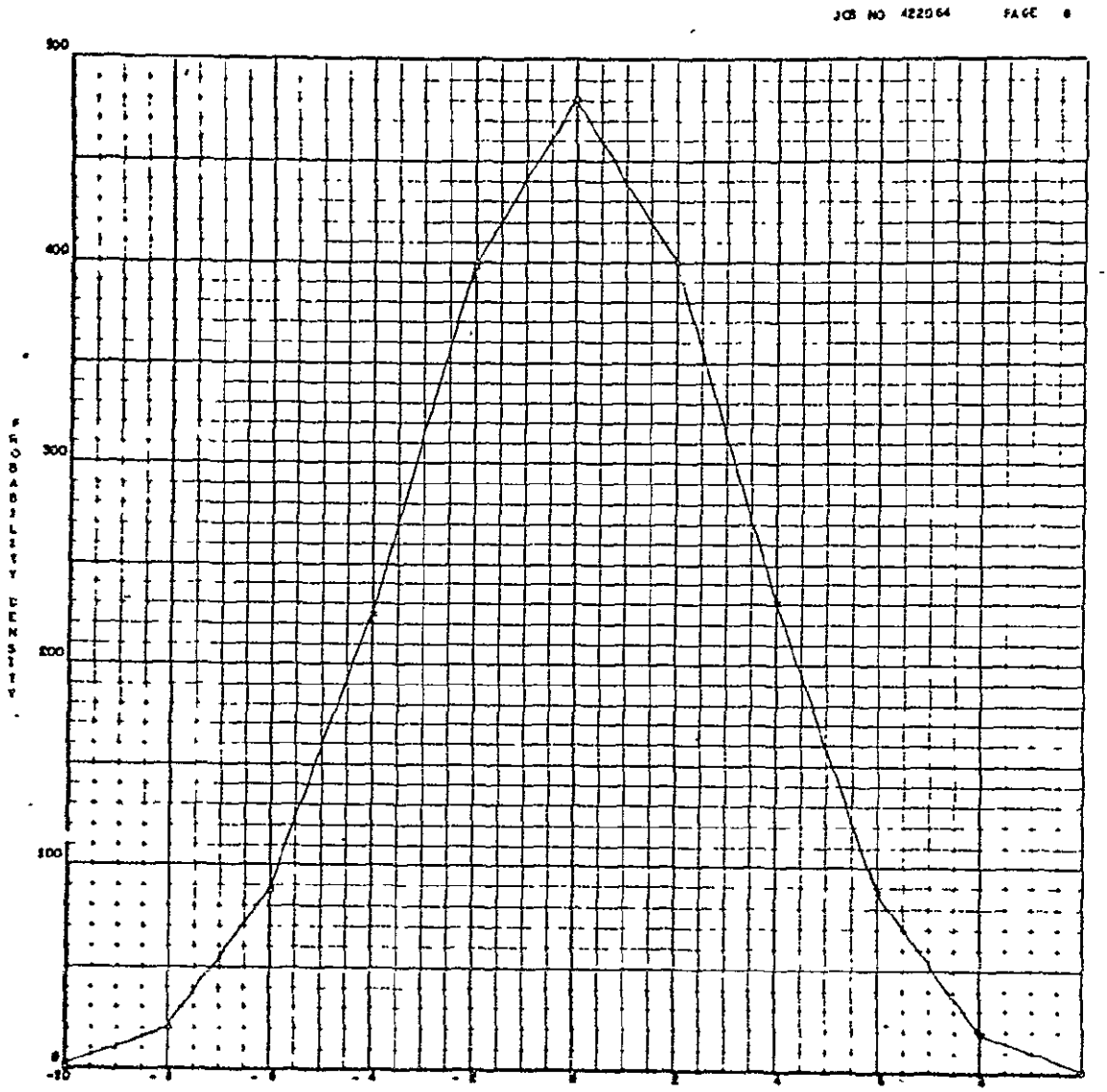


Figure VI-36


JOB NO 422064 PAGE 8
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400
300
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100
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PROBABILITY DENSITY
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-8
-6
-4
-2
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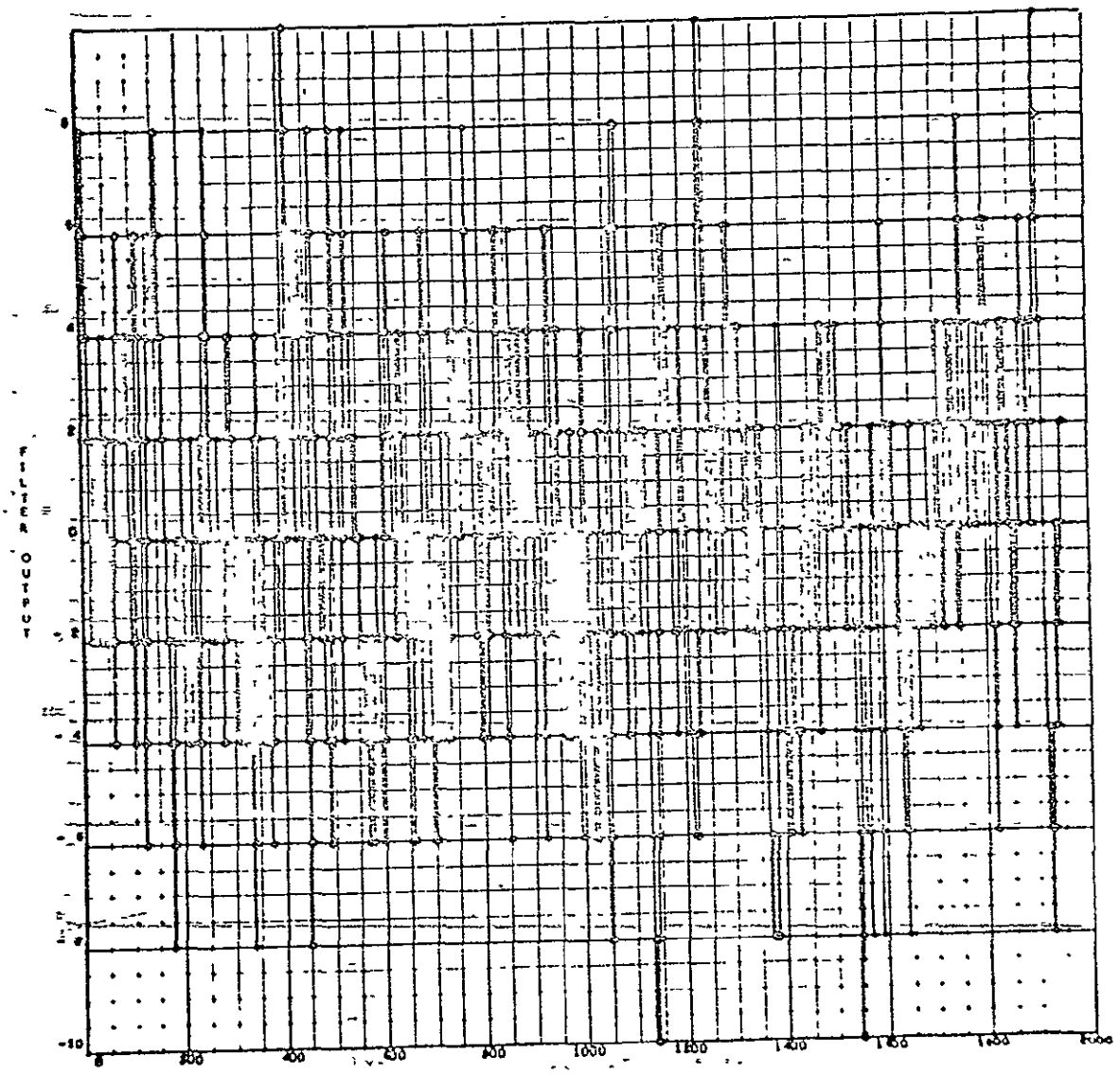
2
4
6
8
PLOT NUMBER 4
SEQUENCE LENGTH = 1953
IMPULSE RESPONSE PERIOD = 8

Figure VI-37



PLOT NUMBER 6

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 12

Figure VI-38

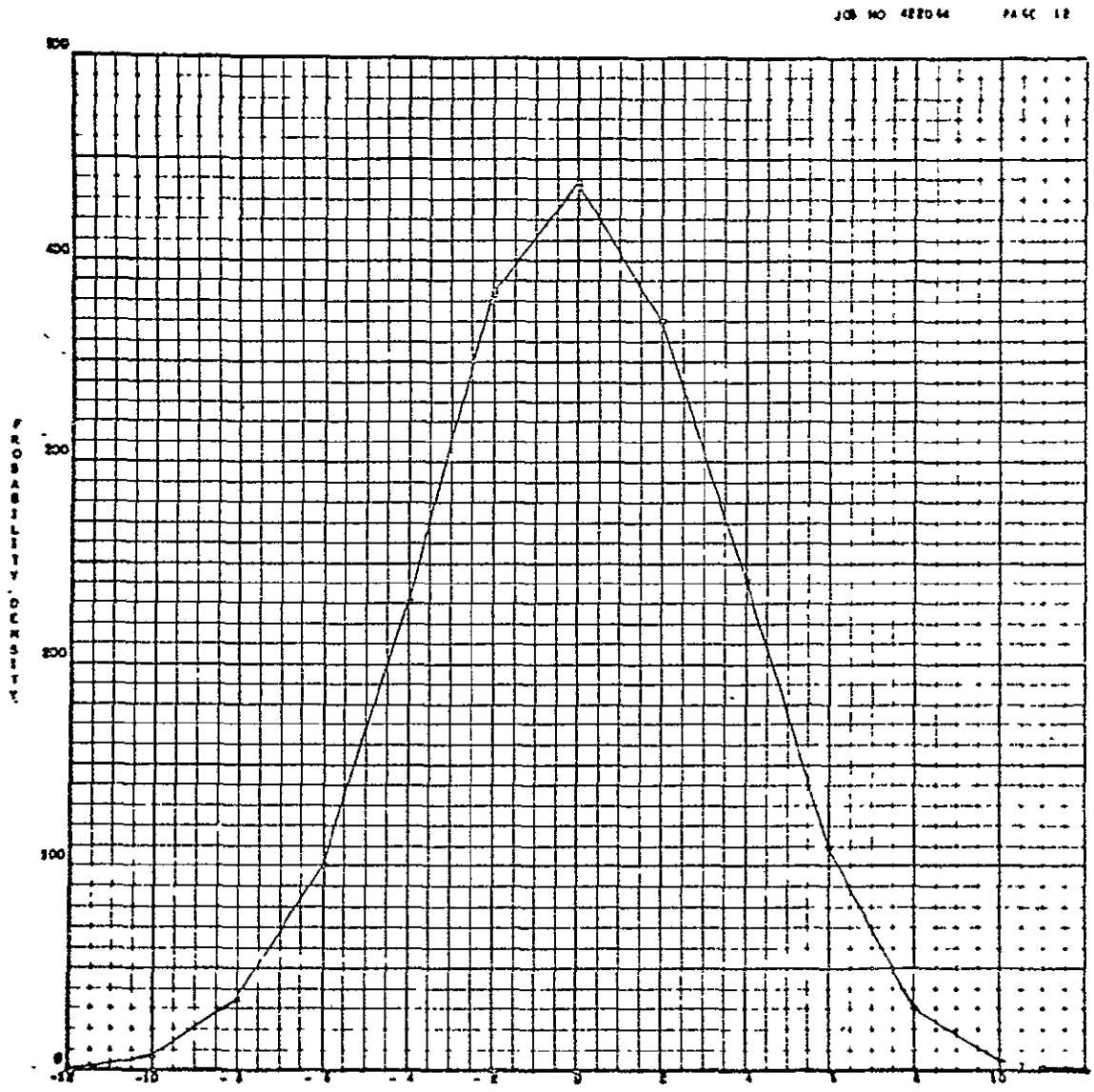
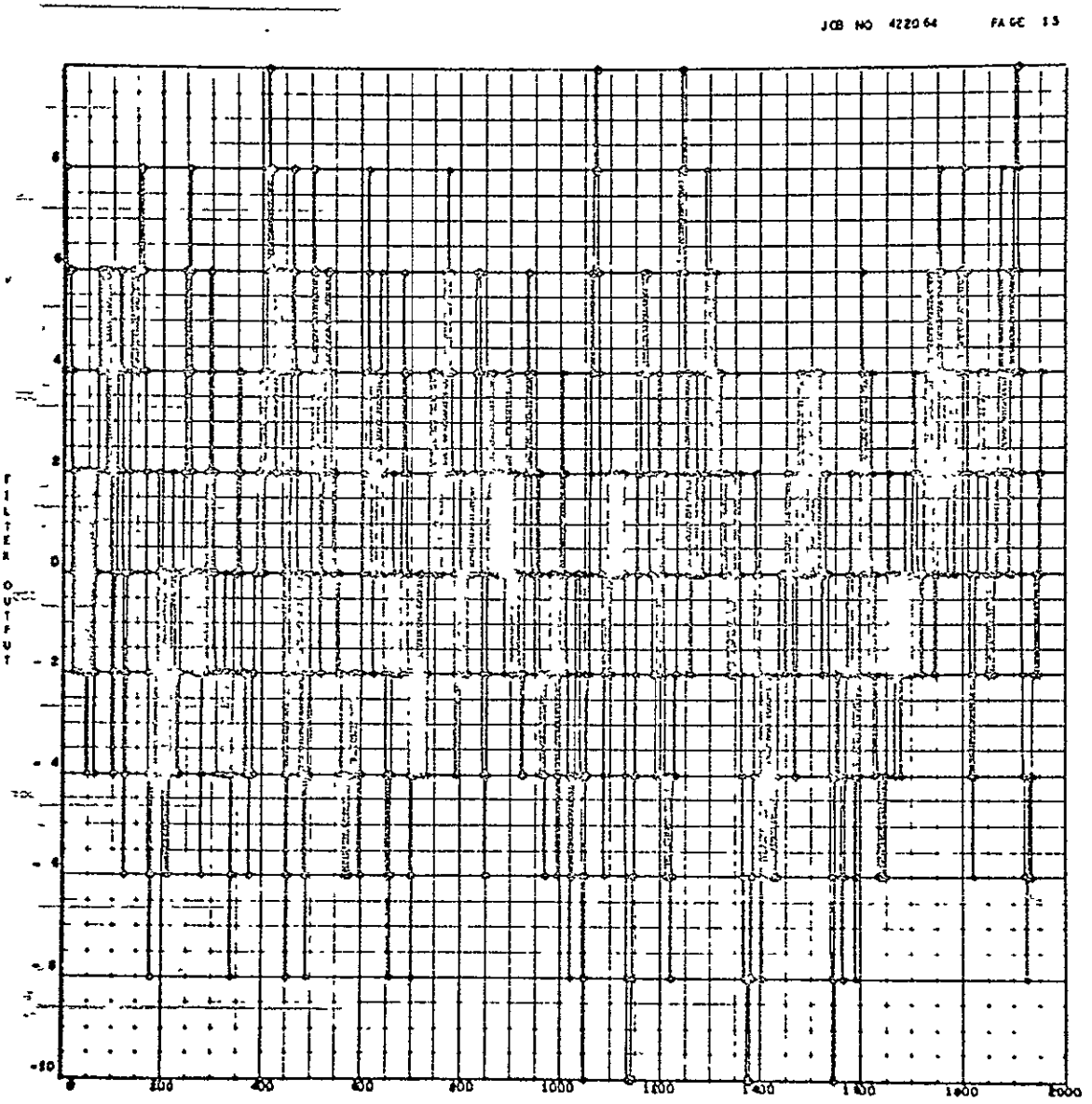

PROBABILITY DENSITY
PLOT NUMBER 6
SEQUENCE LENGTH = 1953
IMPULSE RESPONSE PERIOD = 12

Figure VI-39.


JOB NO 422064 PAGE 15
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-4
-6
-8
-10
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200
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1000
1200
1400
1600
1800
2000
PLOT NUMBER 7
SEQUENCE LENGTH = 1953
IMPULSE RESPONSE PERIOD = 14

Figure VI-40

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

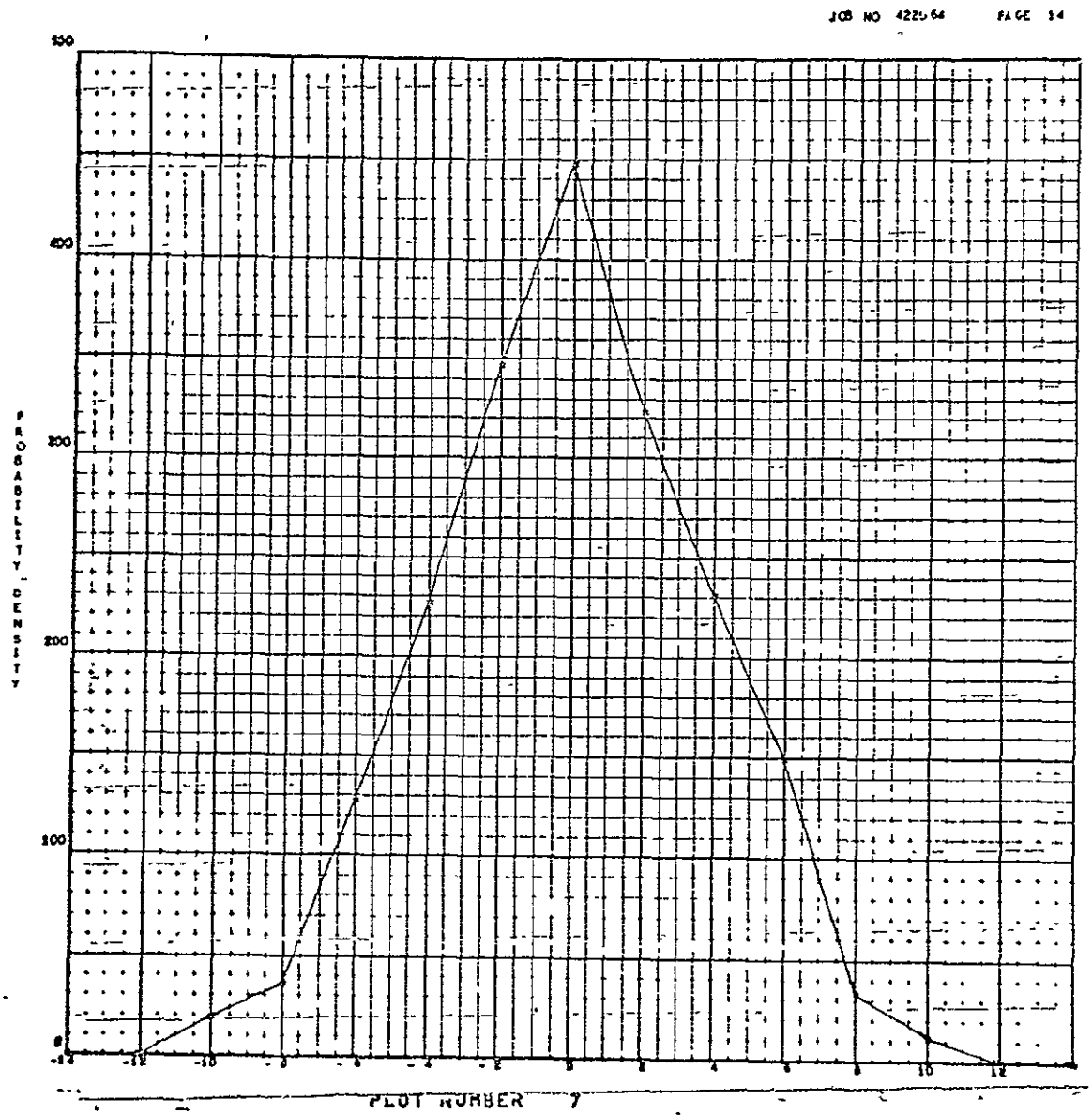


Figure VI-41

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

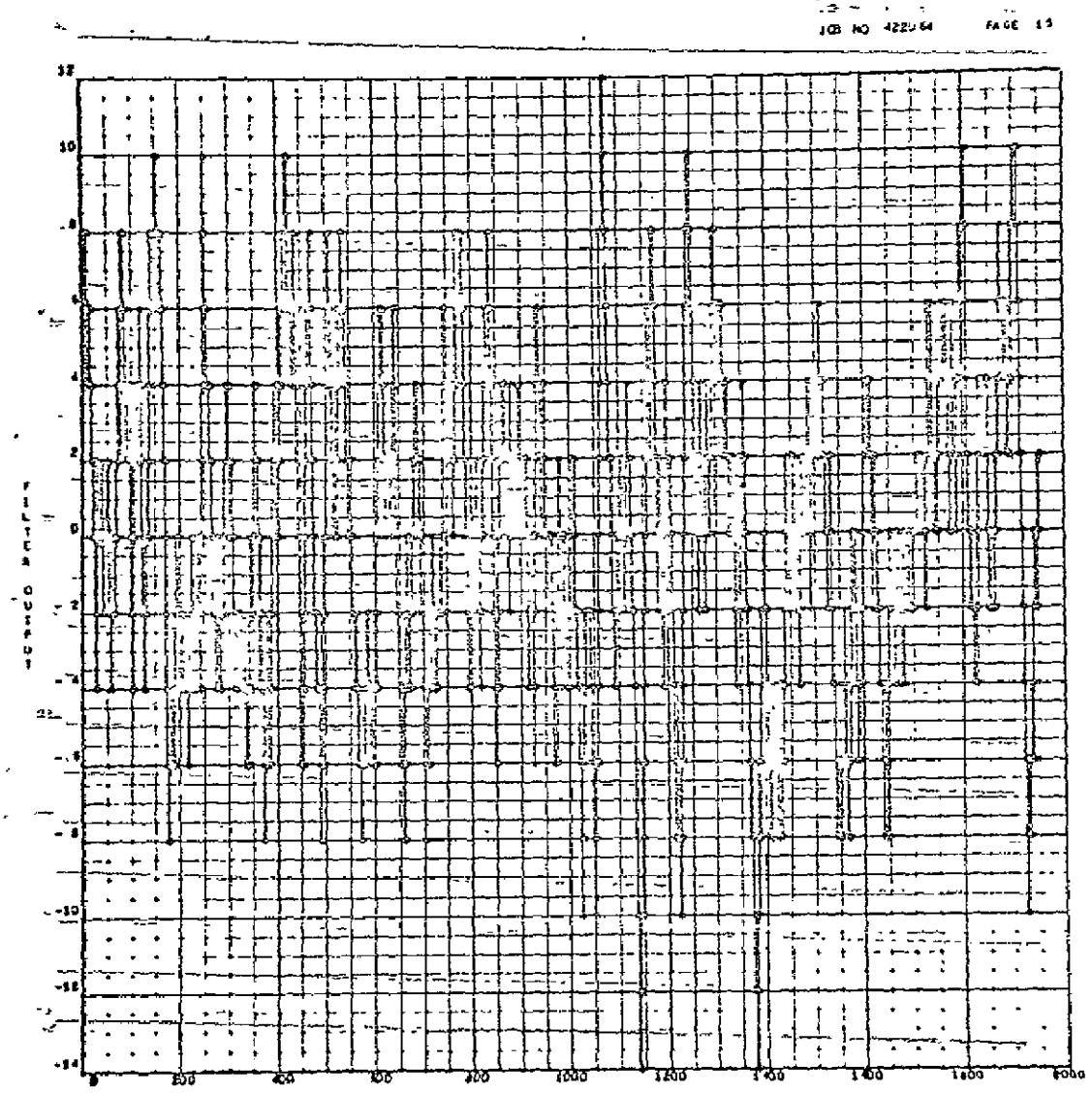


Figure VI-42

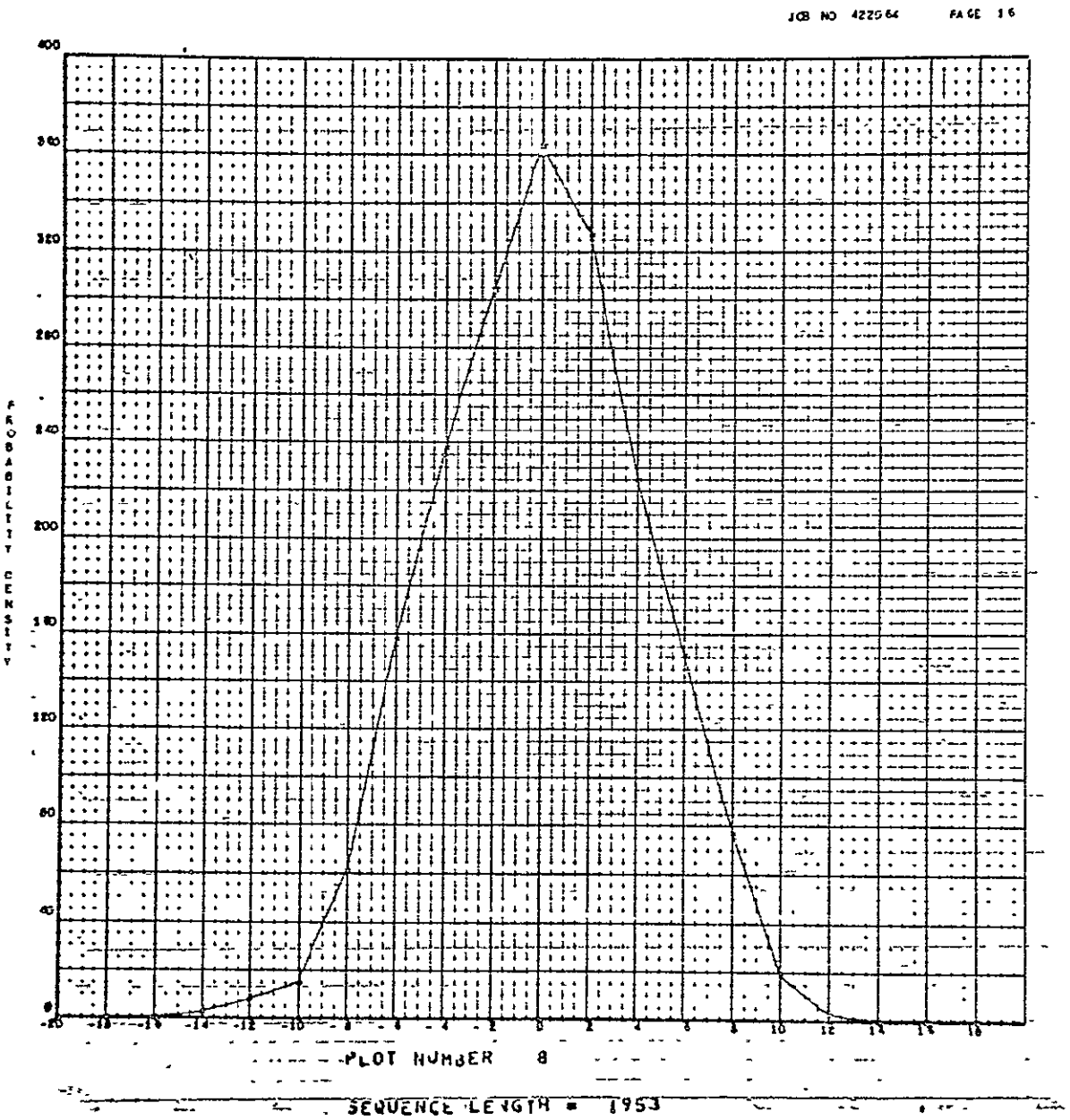

JOB NO 422564 PAGE 16
PROBABILITY DENSITY
PLOT NUMBER 8
SEQUENCE LENGTH = 1953
14PULSE RESPONSE PERIOD = 14

Figure VI-43

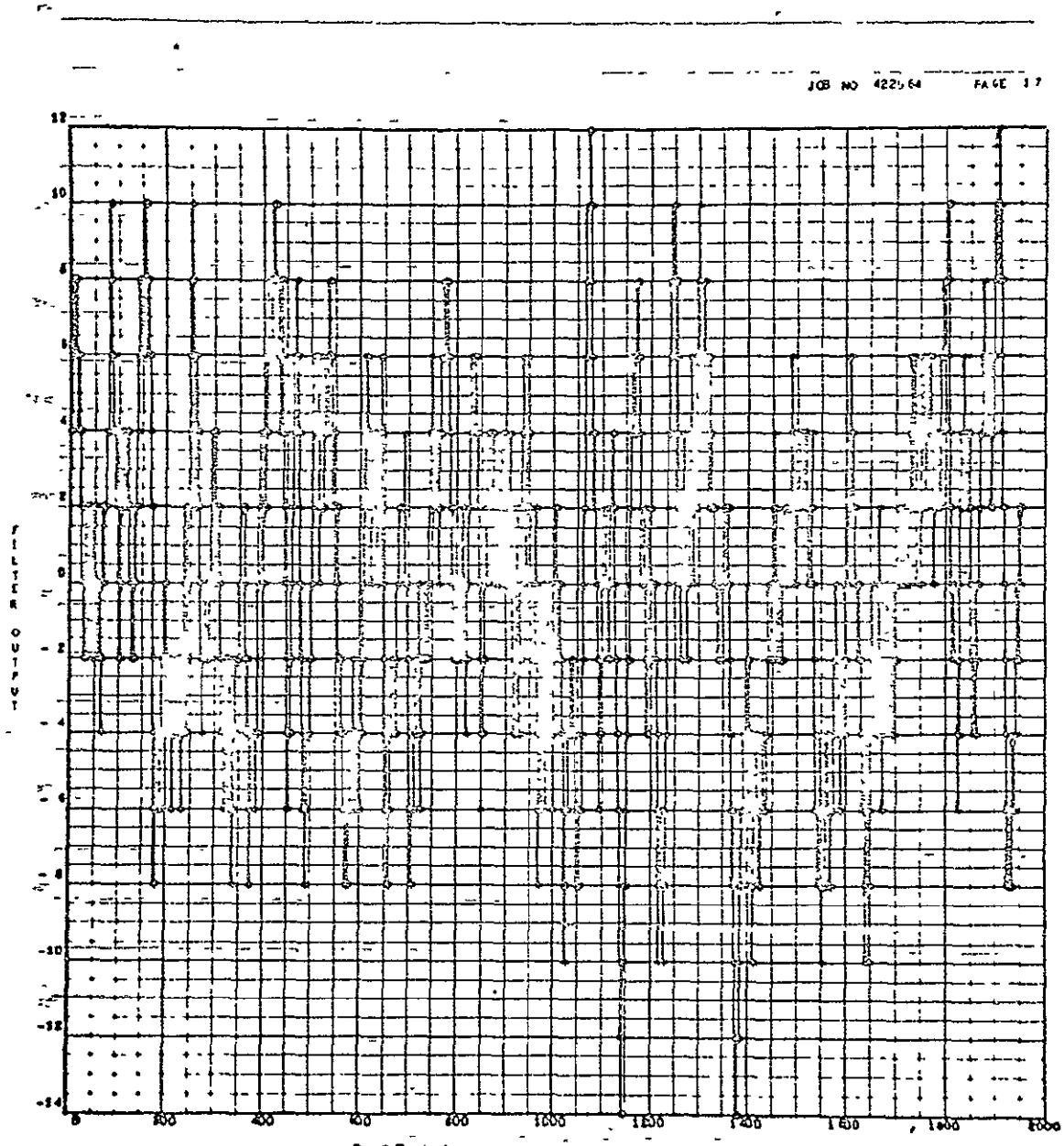

JOB NO 422564 PAGE 17
PLOT NUMBER 9
SEQUENCE LENGTH = 1953
IMPULSE RESPONSE PERIOD = 2

Figure VI-44

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

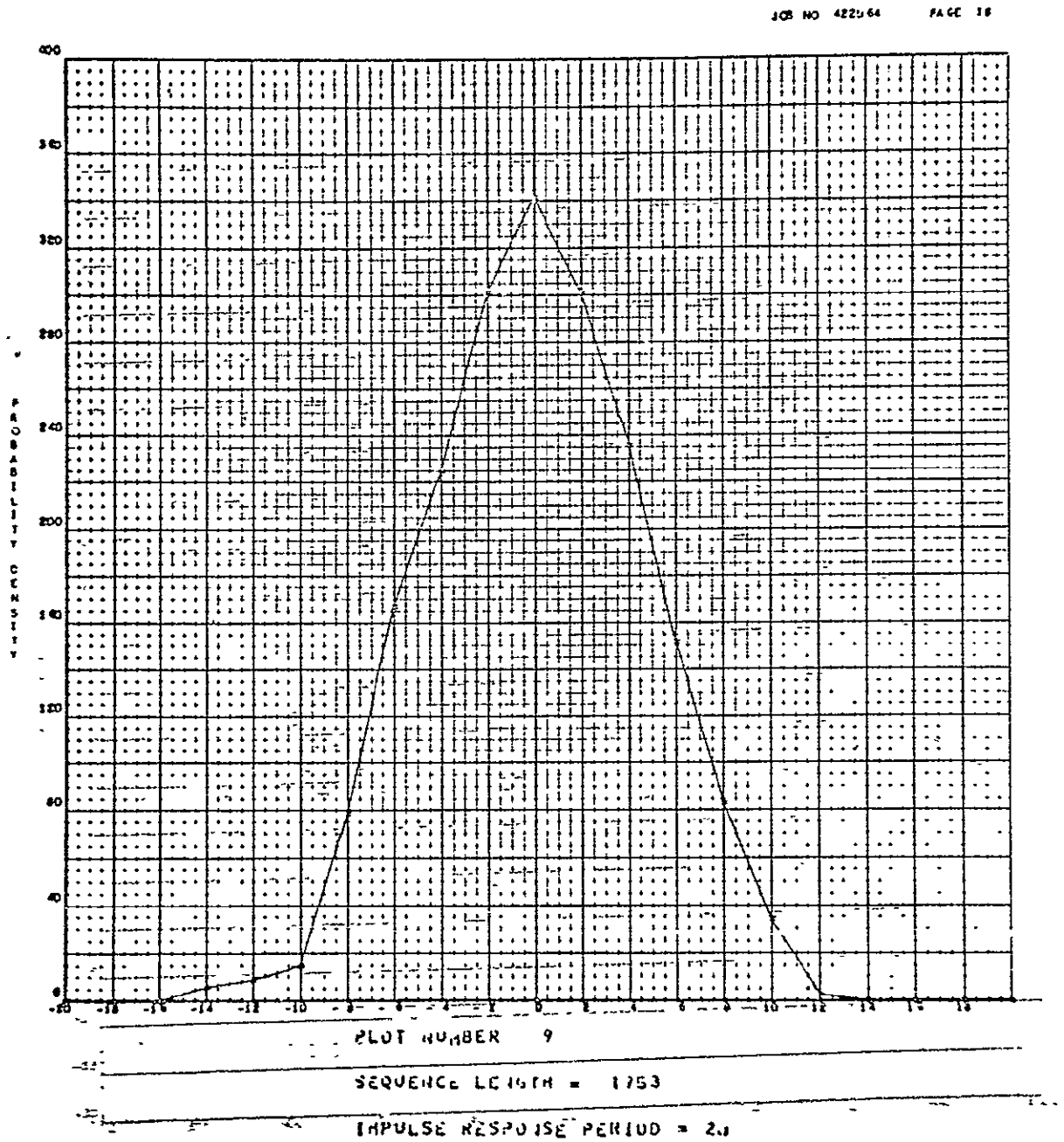


Figure VI-45

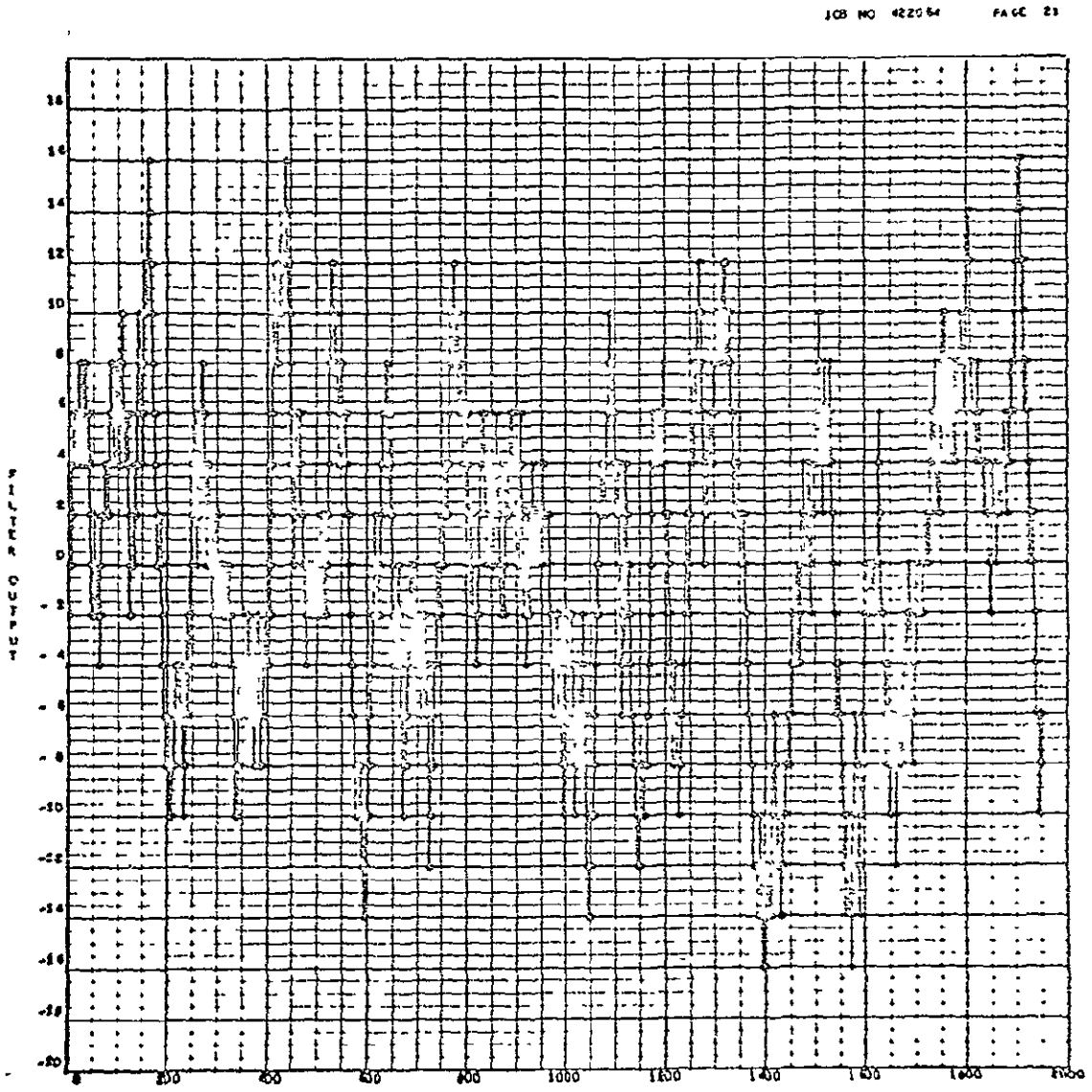
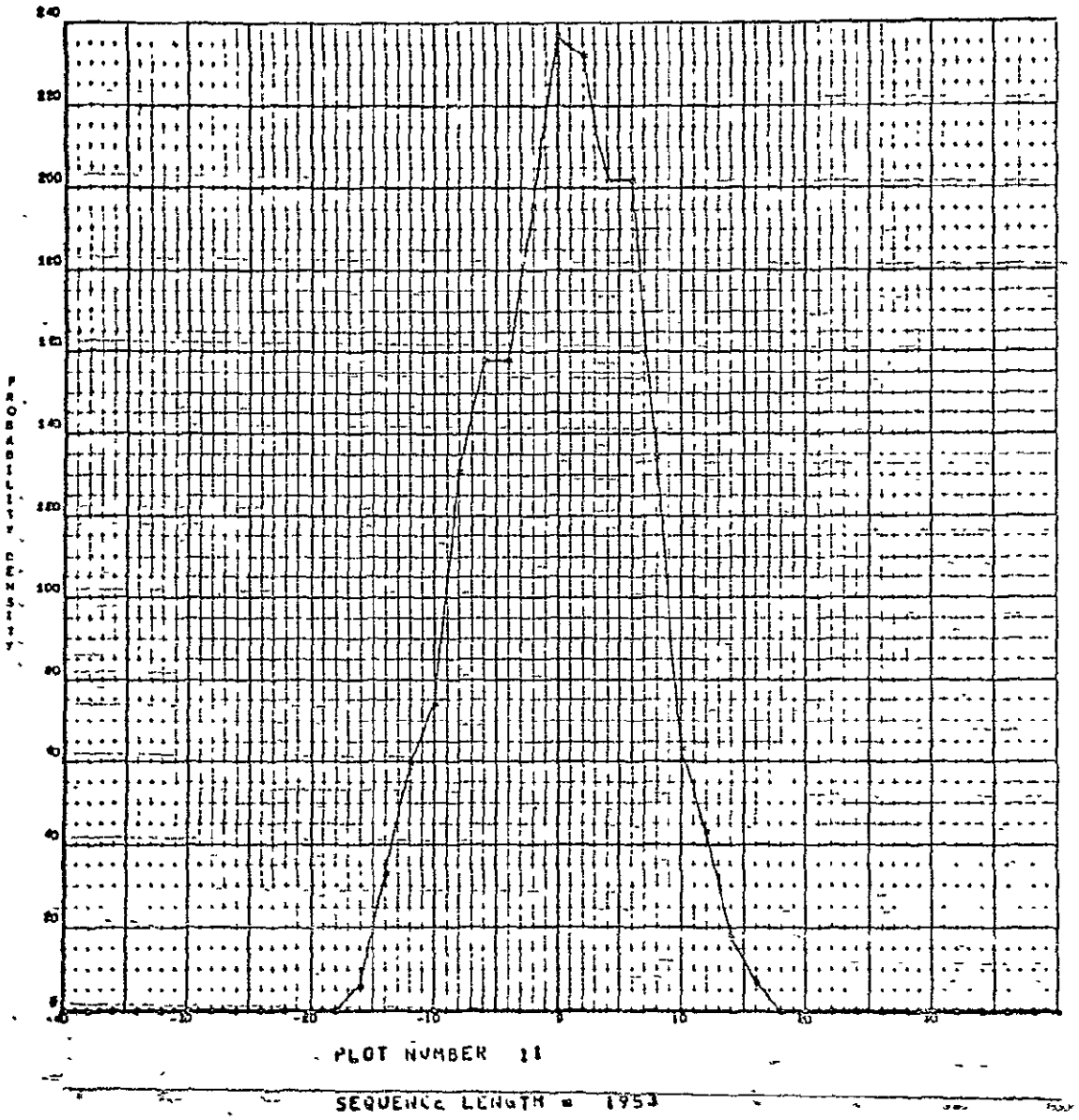

JOB NO 422064 PAGE 23
FILTER OUTPUT
PLOT NUMBER 11
SEQUENCE LENGTH = 1953
IMPULSE RESPONSE PERIOD = 43

Figure VI-46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100
80
60
40
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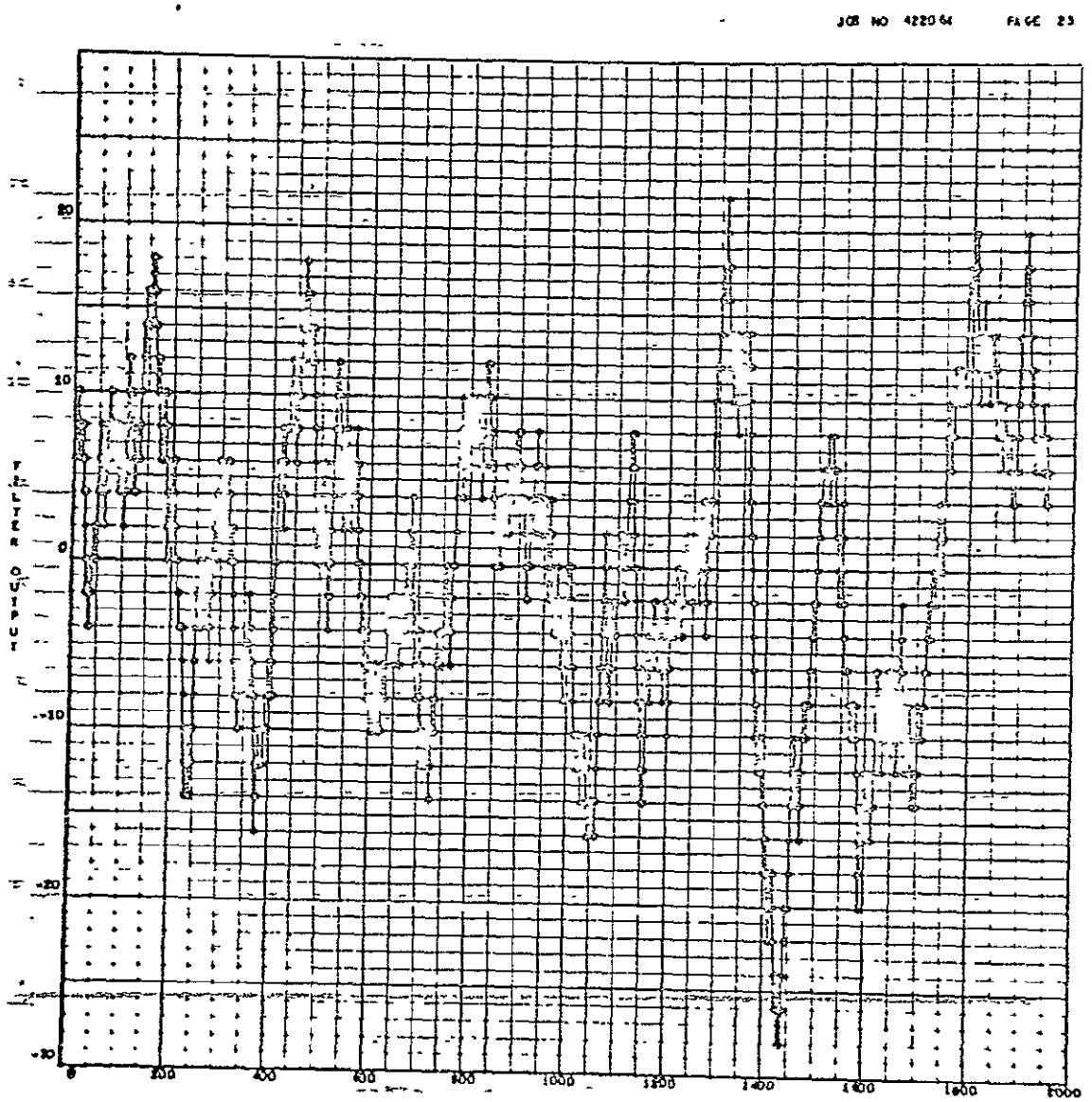
-30
-20
-10
0
10
20
30
PLOT NUMBER 11
SEQUENCE LENGTH = 1953
IMPULSE RESPONSE PERIOD = 41

Figure VI-47



PLOT NUMBER 12

SEQUENCE LENGTH = 1953

IMPULSE RESPONSE PERIOD = 8

Figure VI-48

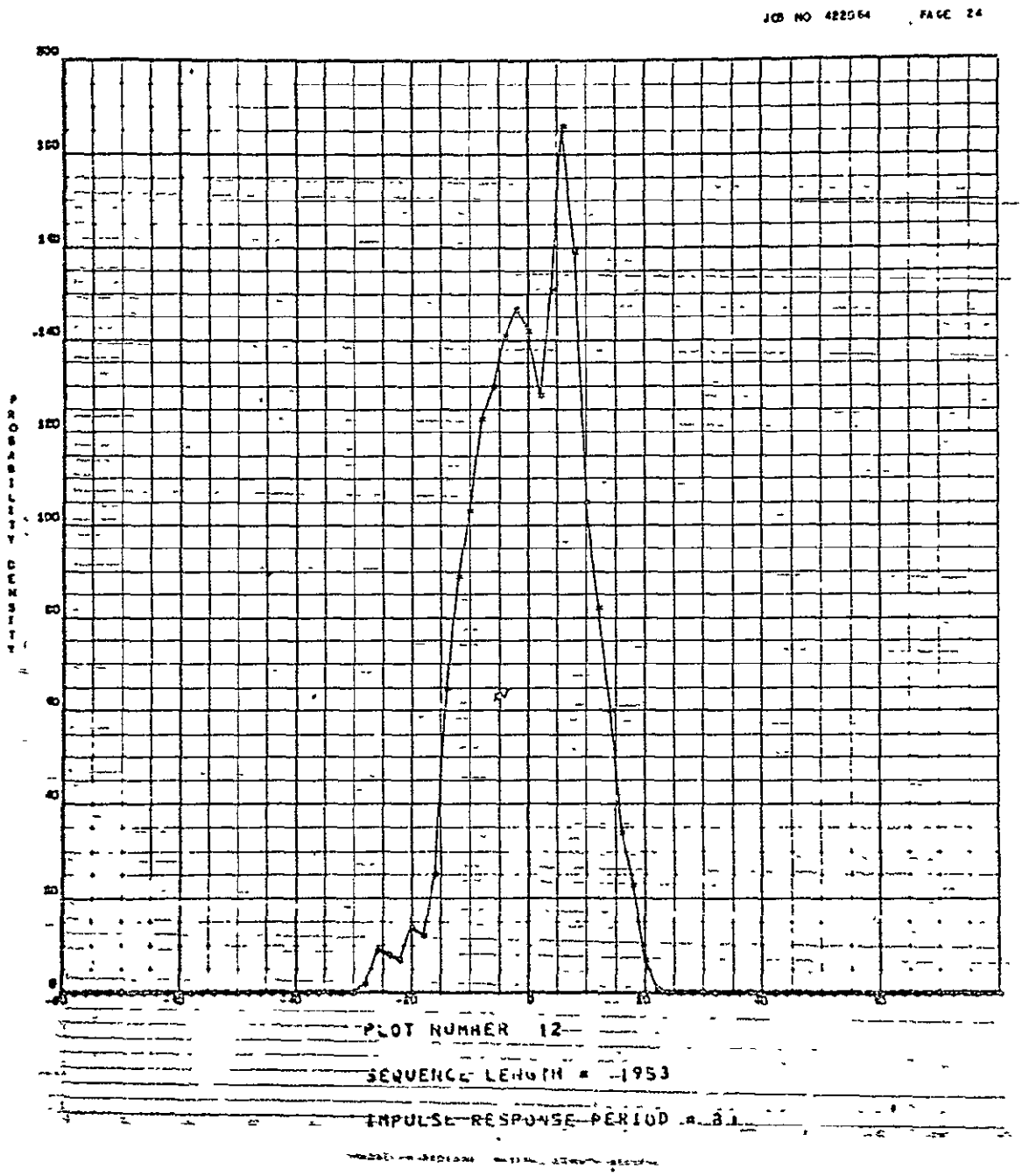

JOB NO 422064 PAGE 24
PROBABILITY DENSITY
PLOT NUMBER 12
SEQUENCE LENGTH = 1953
IMPULSE RESPONSE PERIOD = 81

Figure VI-49

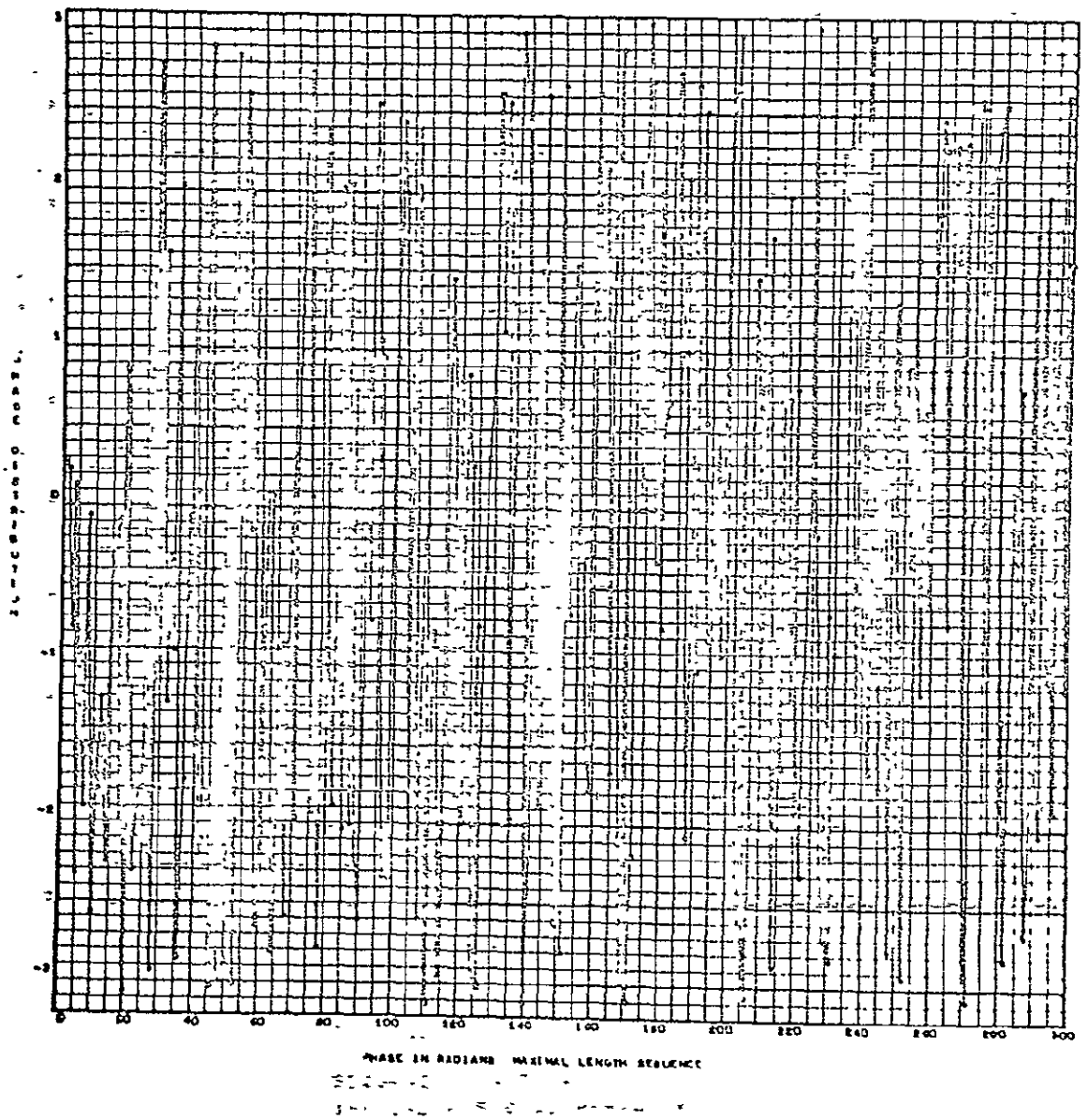


Figure VI-50

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

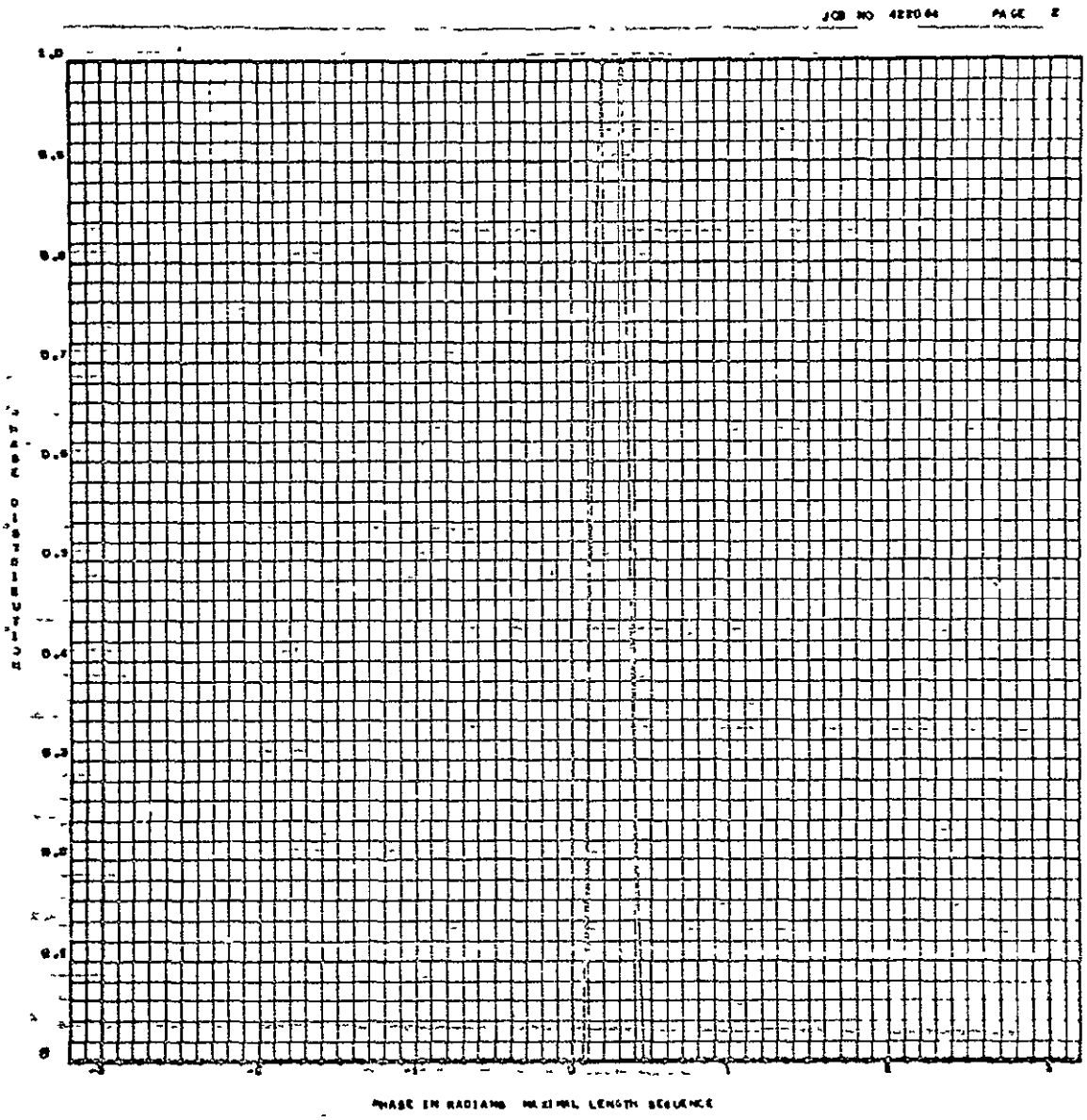
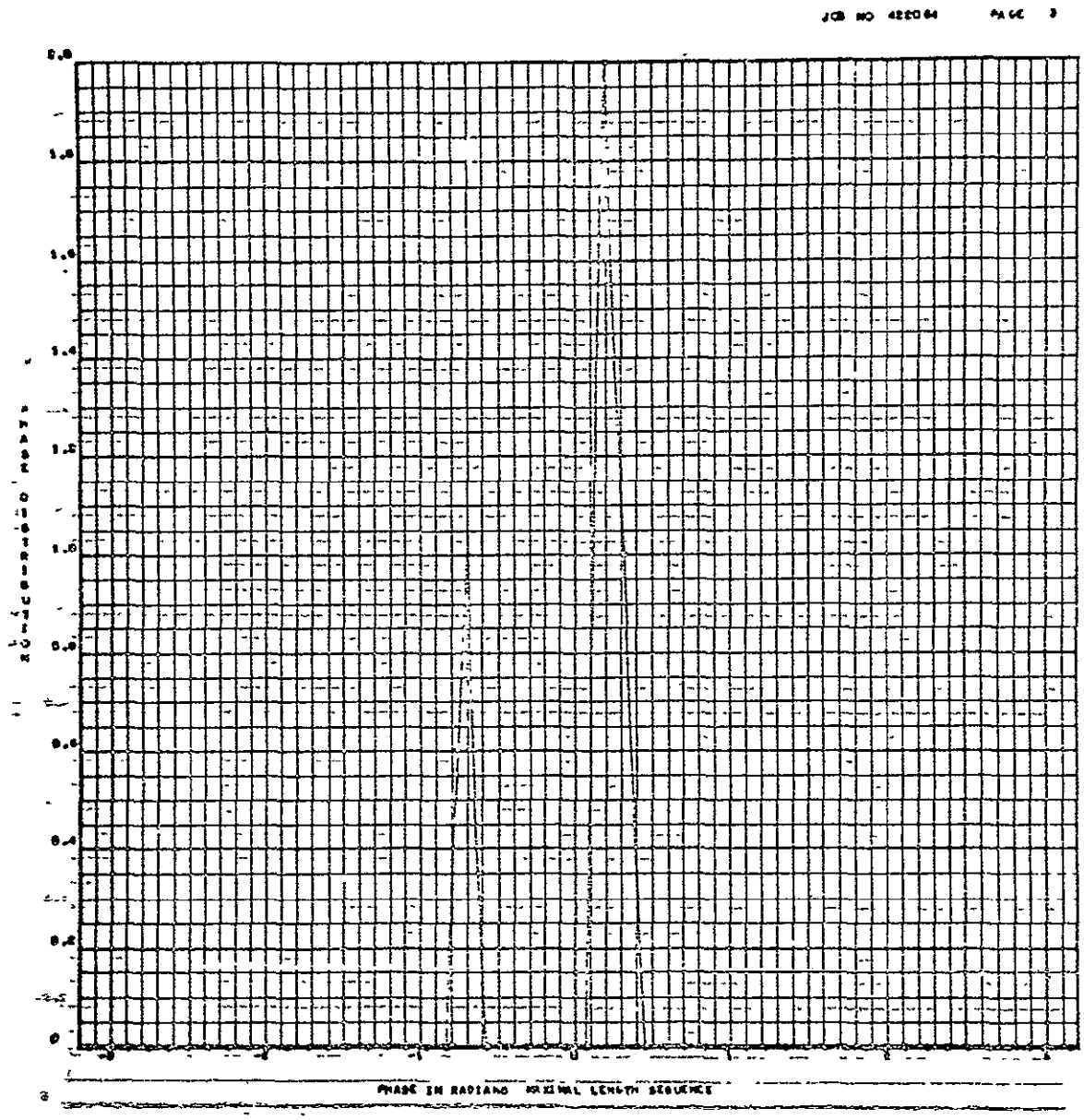


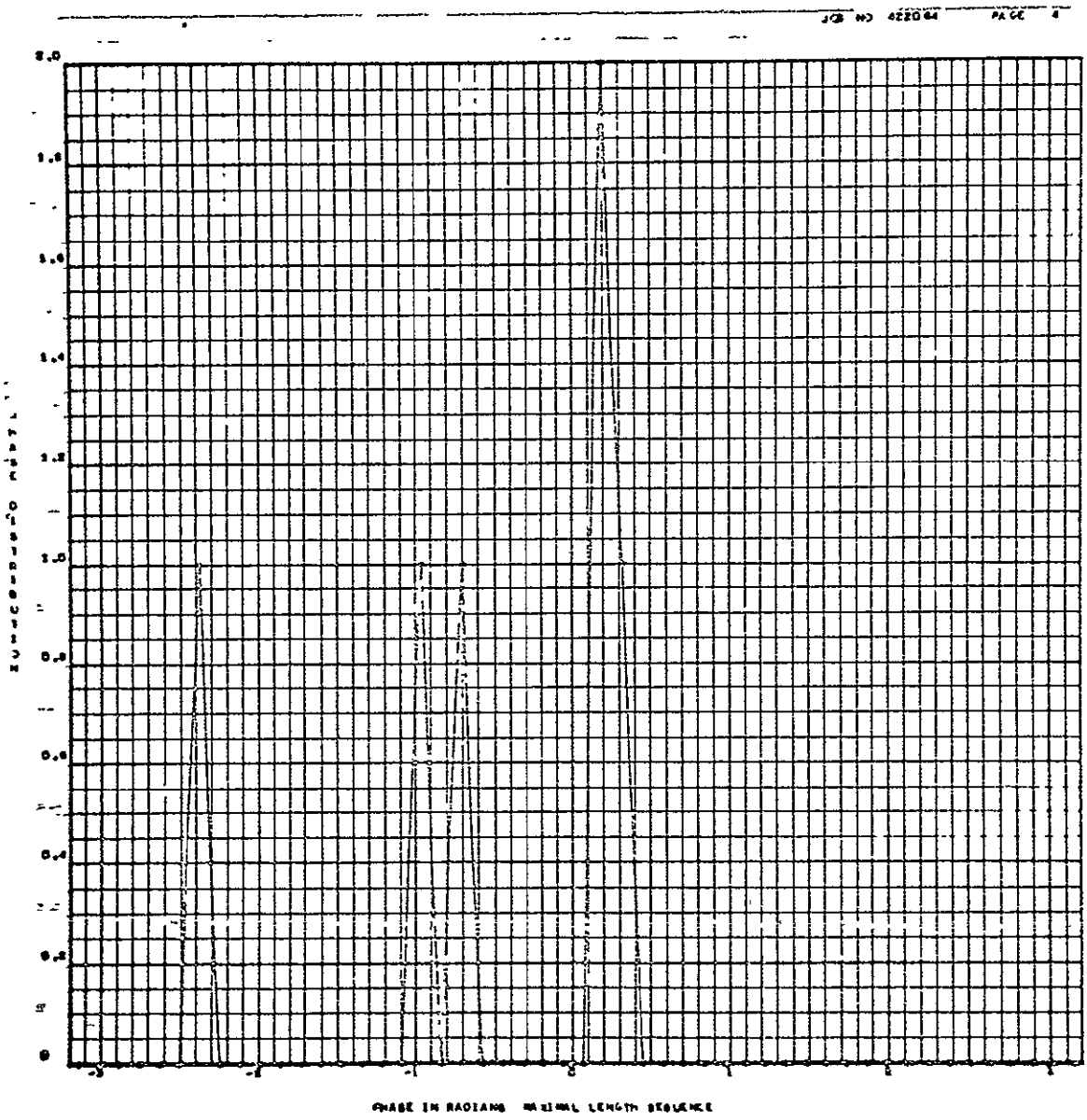
Figure VI-51



PLOT NUMBER 2

SEQUENCE LENGTH = 11
IMPULSE RESPONSE PERIOD = 4

Figure VI-52

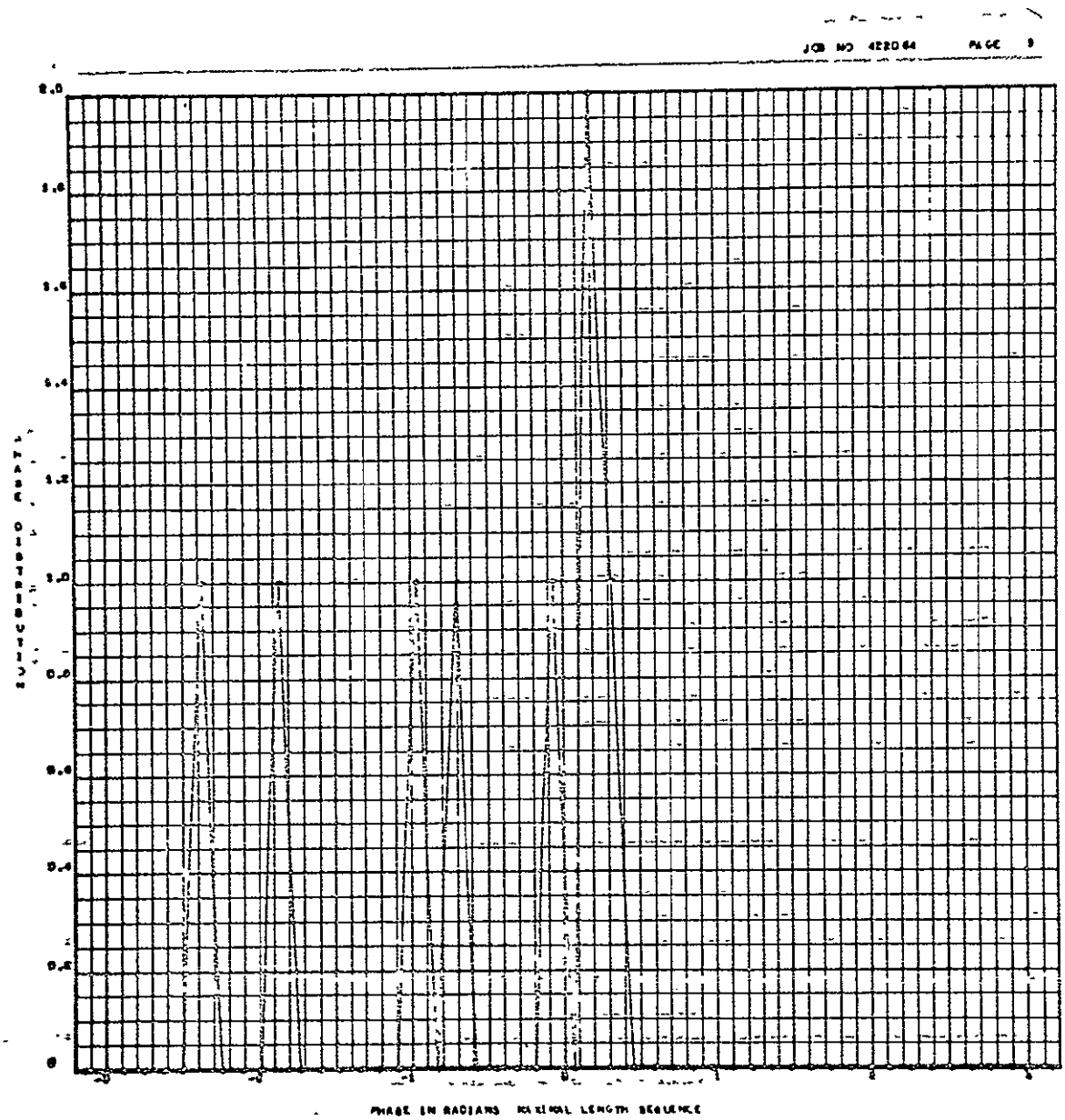


PLOT NUMBER 3

SEQUENCE LENGTH =

IMPULSE RESPONSE PERIOD = 4

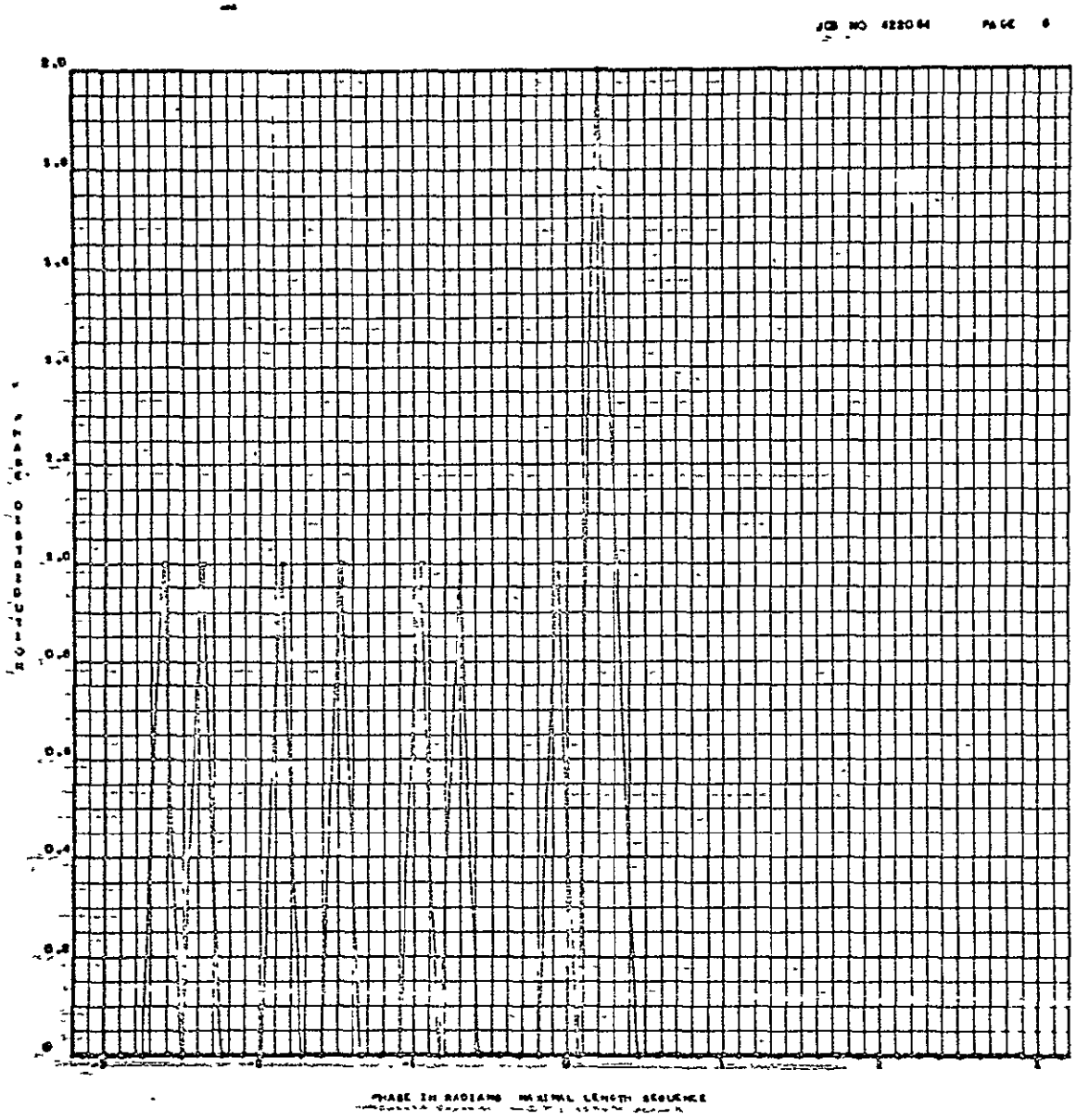
Figure VI-53



PLOT NUMBER 4

SEQUENCE LENGTH = 11
IMPULSE RESPONSE PERIOD = 8

Figure VI-54

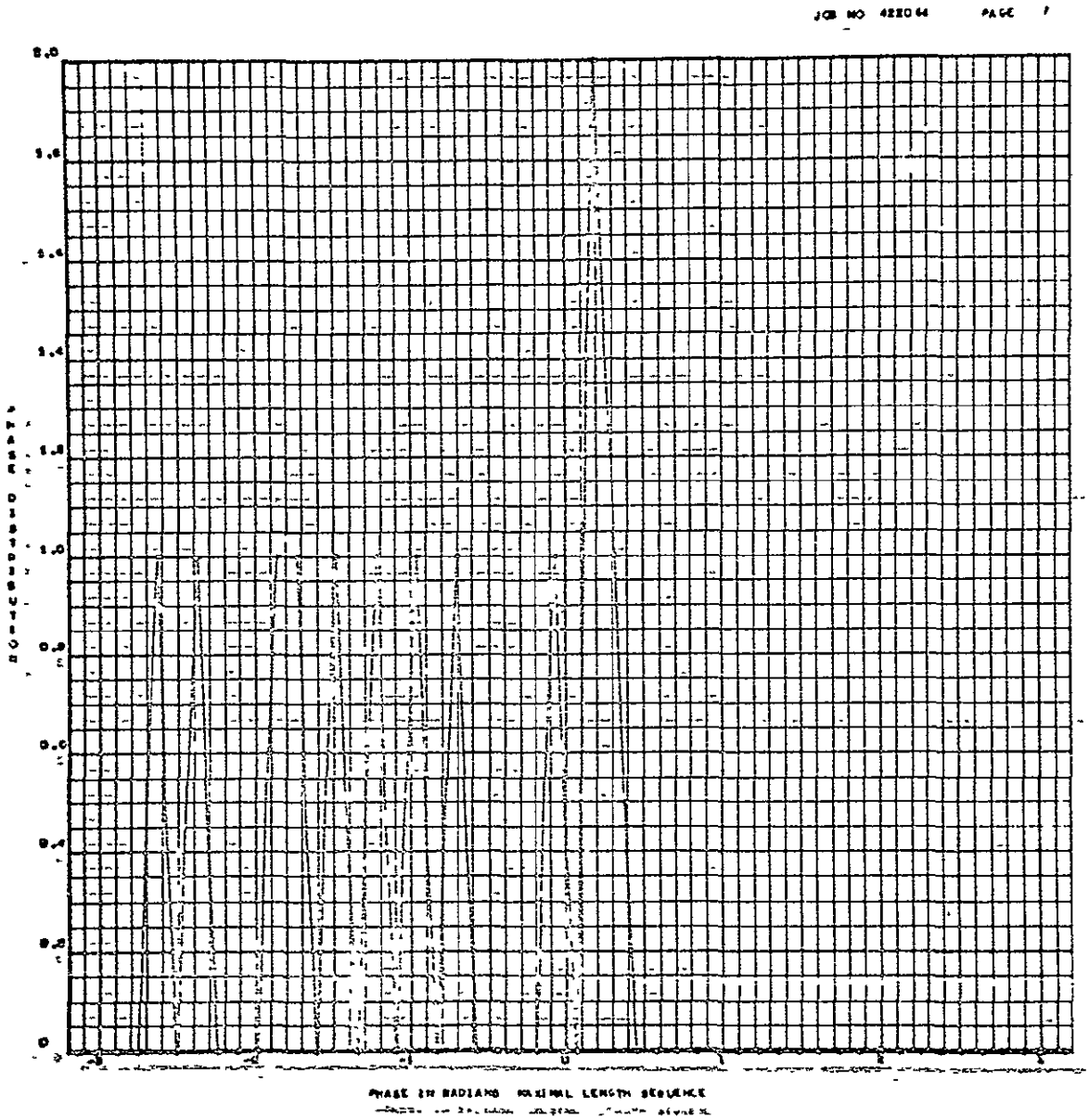


PLOT NUMBER = 5

SEQUENCE LENGTH = 11
IMPULSE RESPONSE PERIOD = 10

Figure VI-55

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

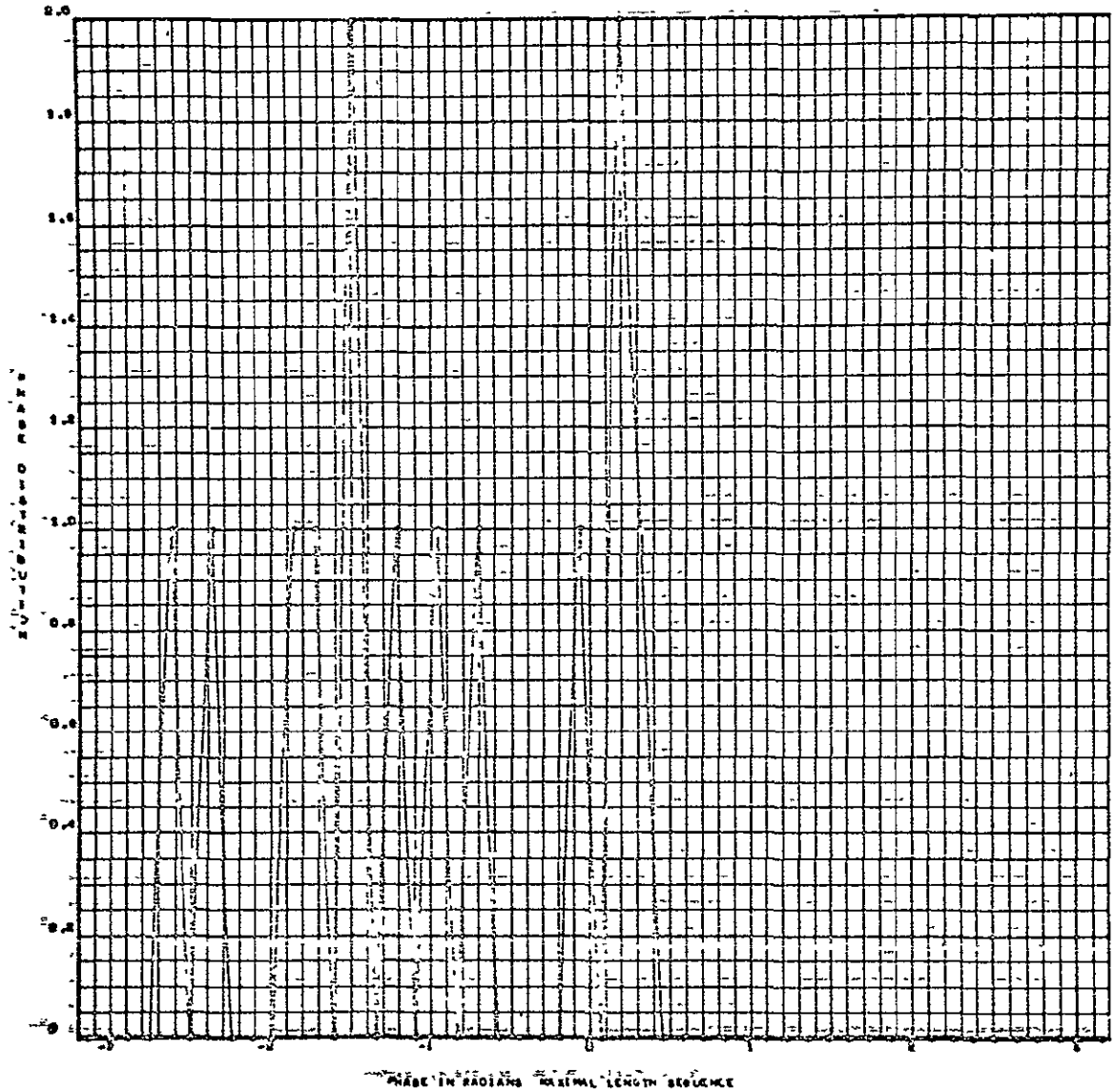


PLOT NUMBER 6

SEQUENCE LENGTH = 11

IMPULSE RESPONSE PERIOD = 12

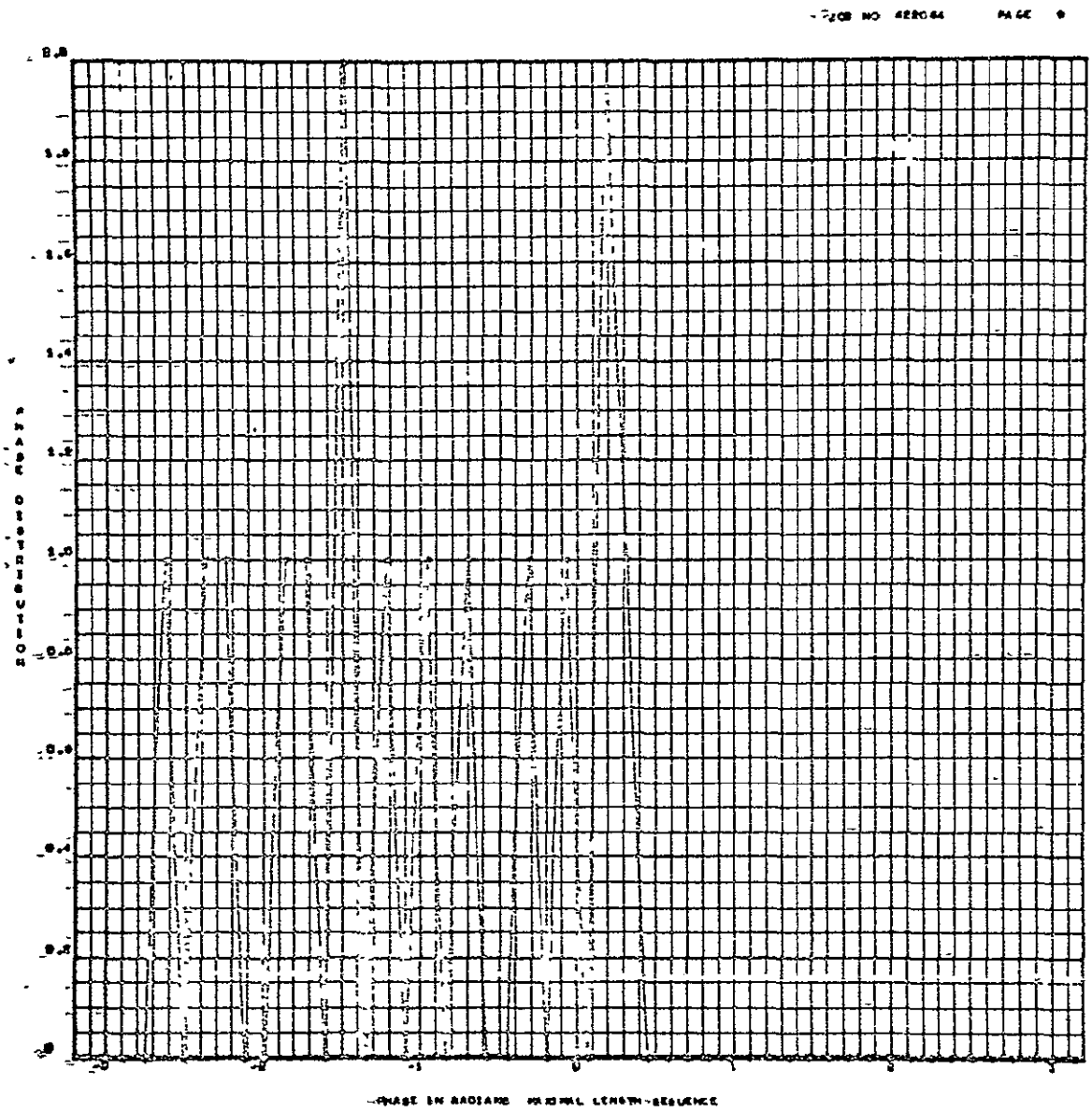
Figure VI-56-



PLOT NUMBER 7

SEQUENCE LENGTH = 11
IMPULSE RESPONSE PERIOD = 13

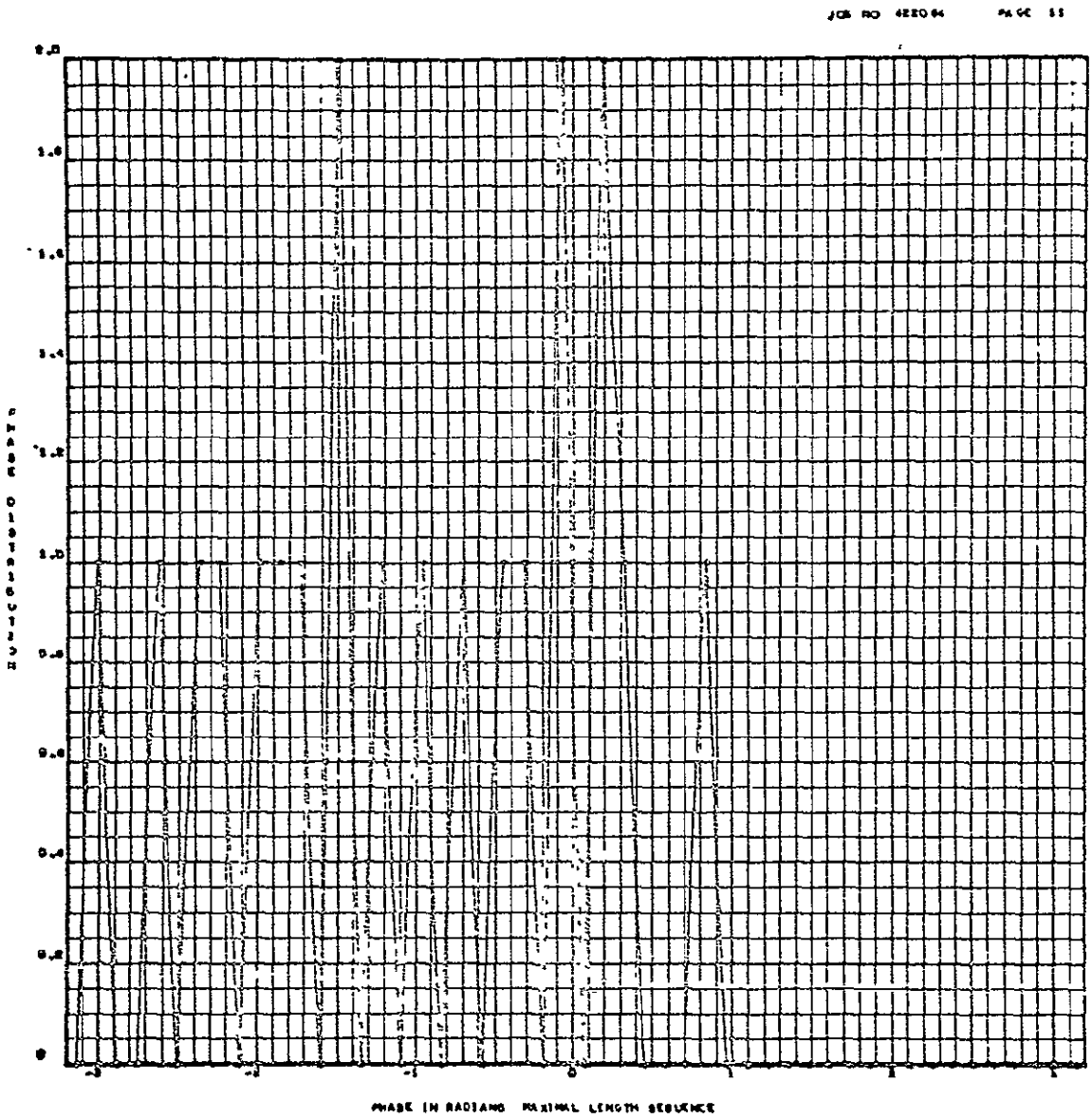
Figure VI-57



PLOT NUMBER 8

SEQUENCE LENGTH = 11
IMPULSE RESPONSE PERIOD = 15

Figure VI-58

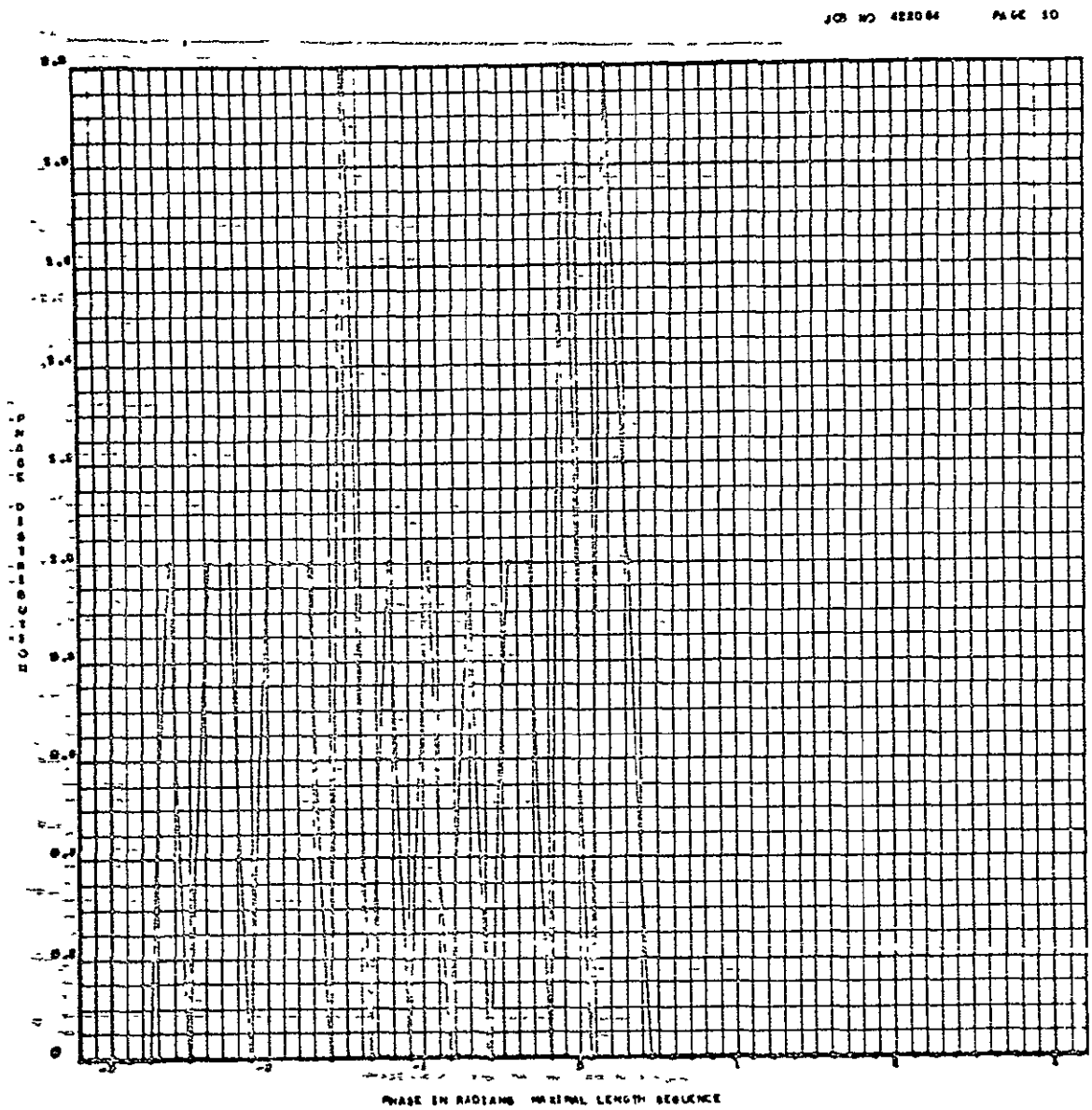


PLOT NUMBER 10

SEQUENCE LENGTH = 11
IMPULSE RESPONSE PERIOD = 20

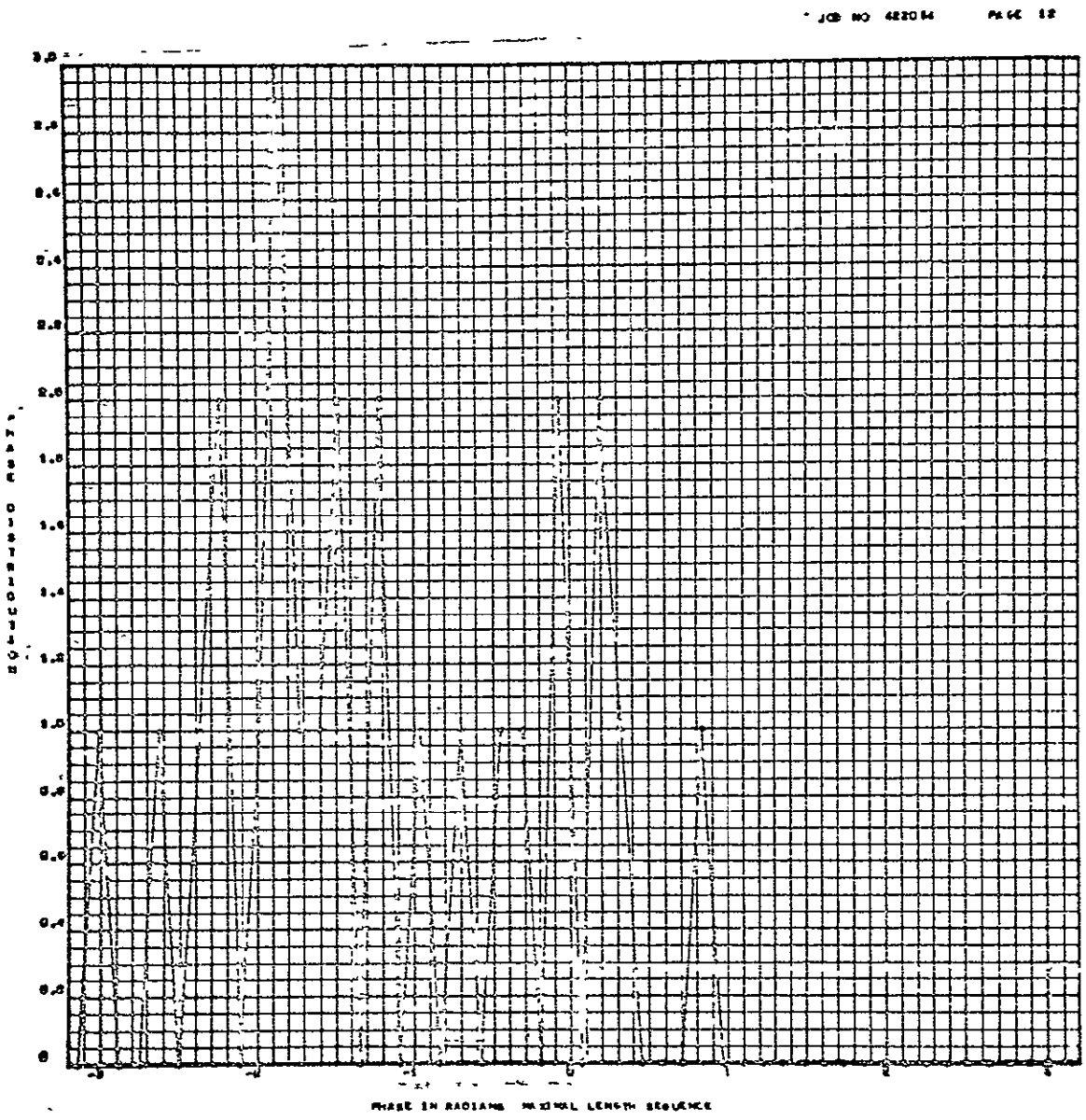
Figure VI-59

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR



PLOT NUMBER = 9
SEQUENCE LENGTH = 11
IMPULSE RESPONSE PERIOD = 18

Figure VI-60

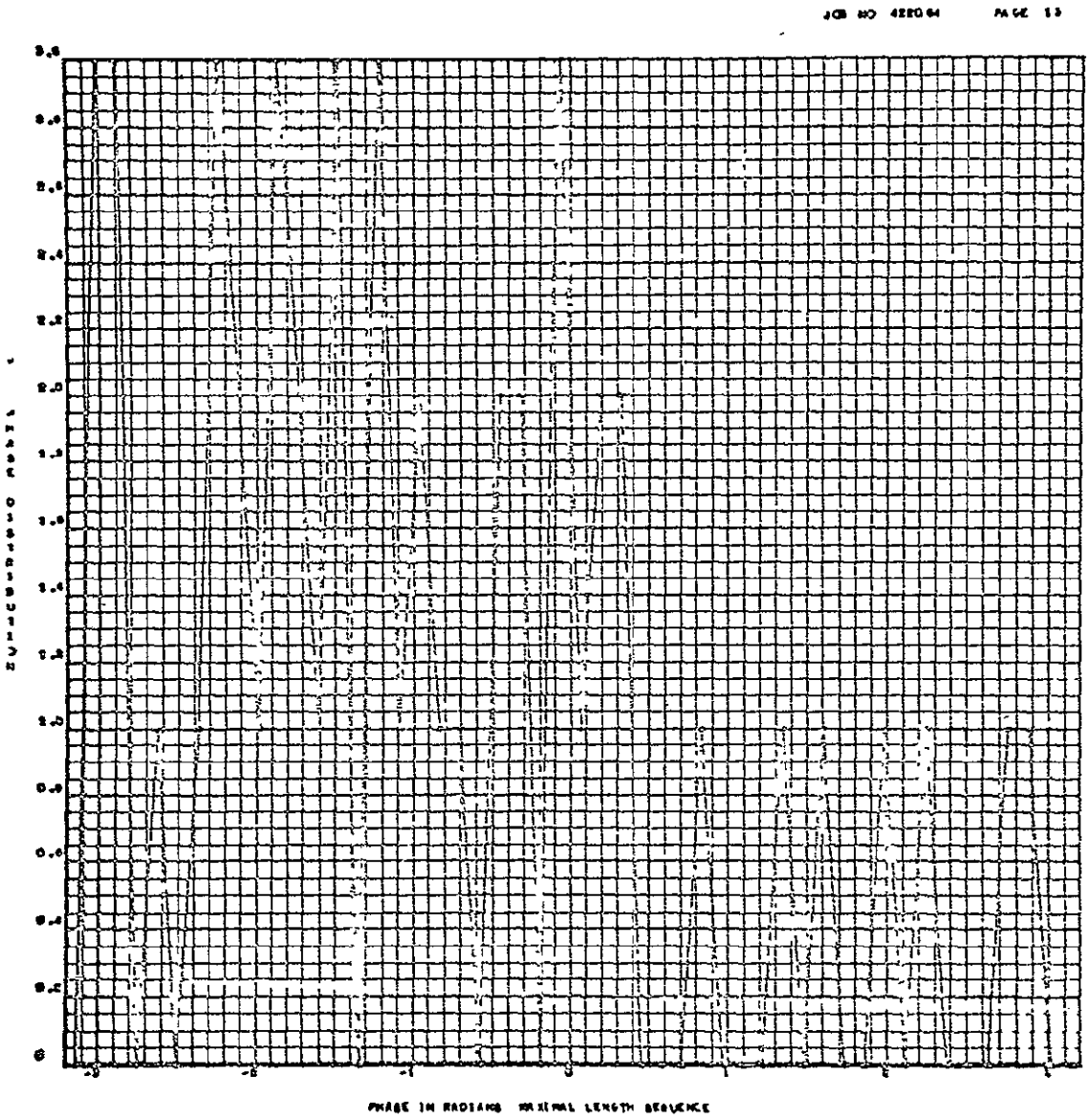


PLOT NUMBER 11

SEQUENCE LENGTH = 11

IMPULSE RESPONSE PERIOD = 25

Figure VI-61



PLOT NUMBER 12

SEQUENCE LENGTH = 11
IMPULSE RESPONSE PERIOD = 50

Figure VI-62

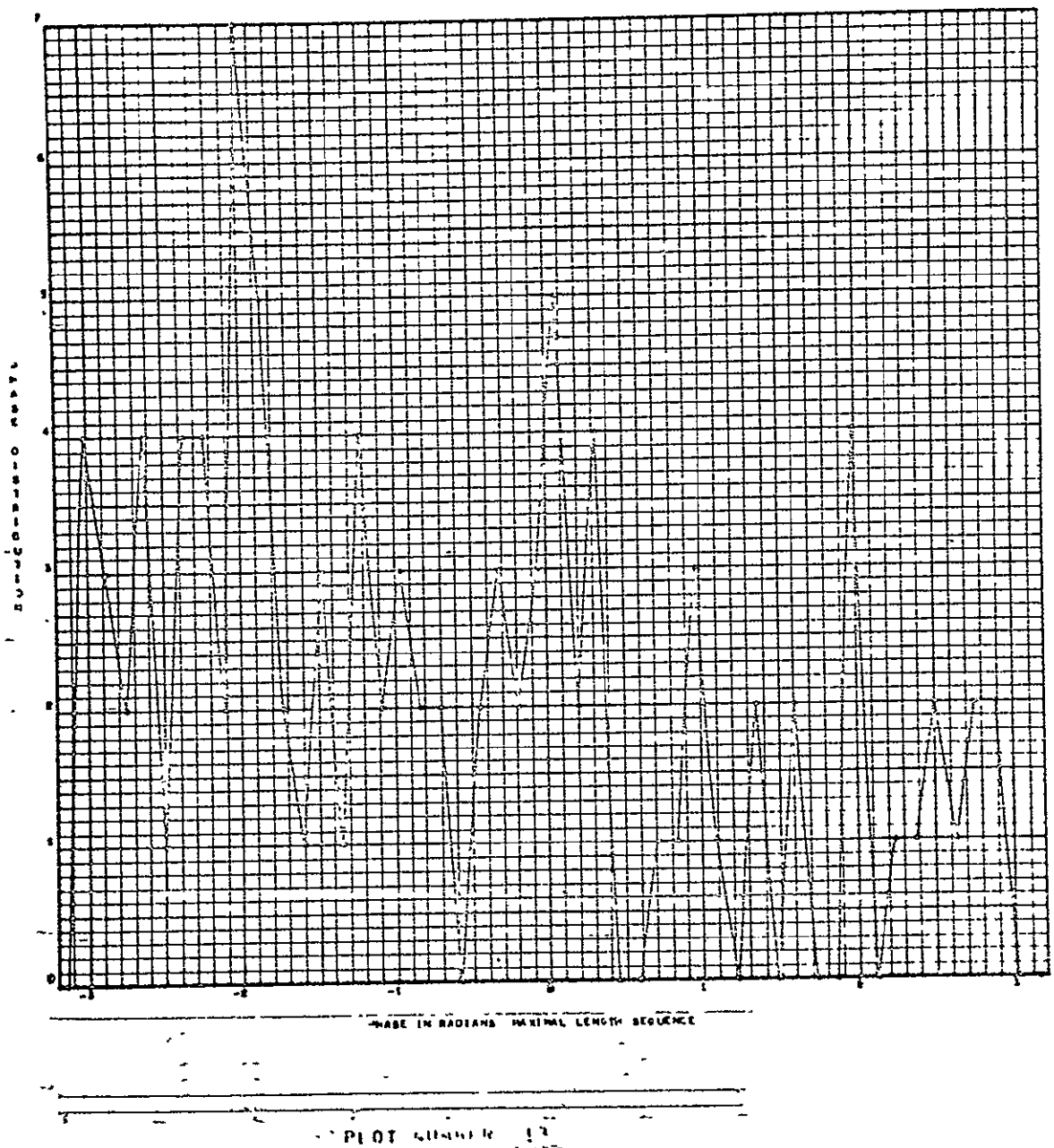


Figure VI-63

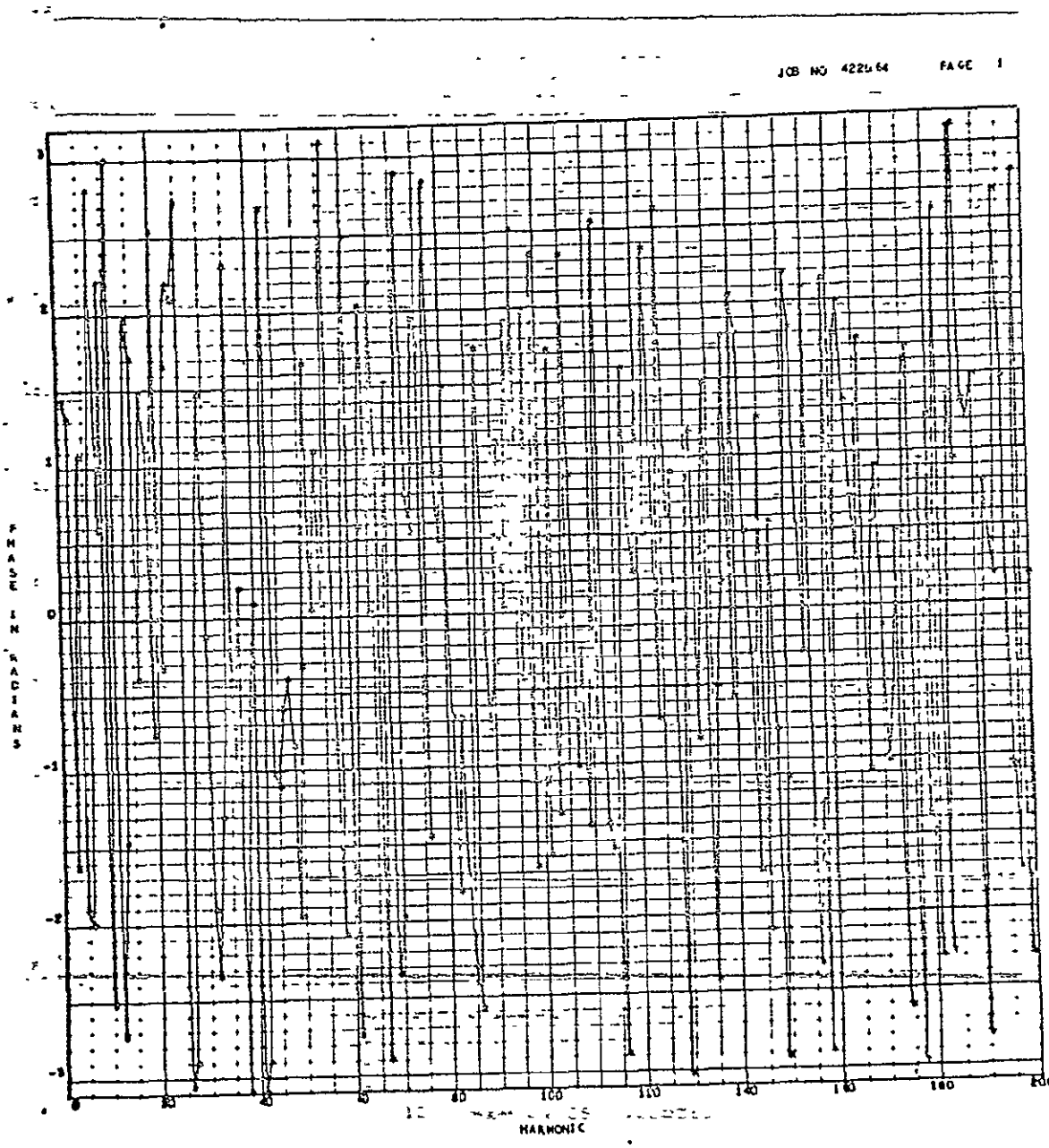
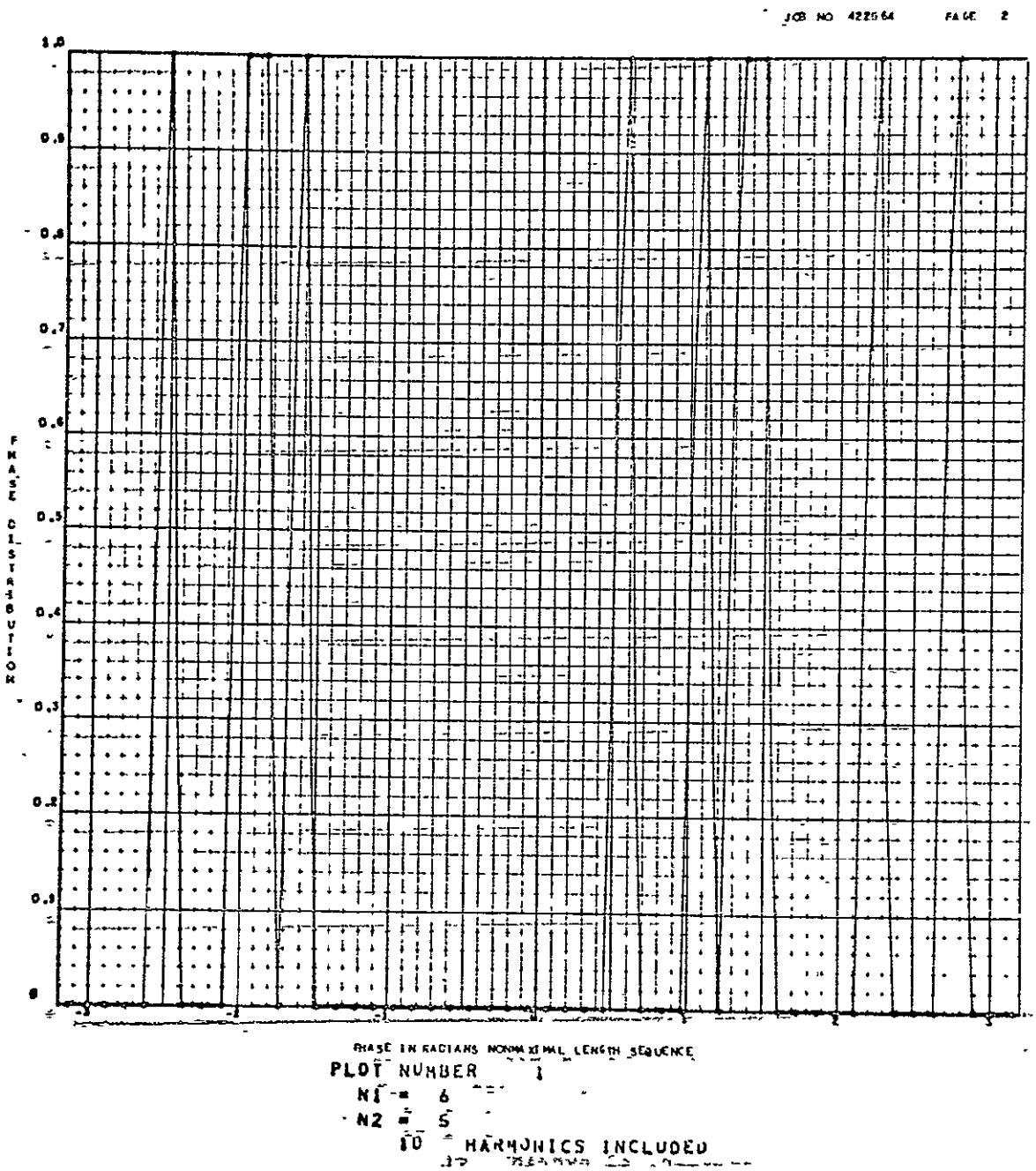


Figure VI-64


JOB NO 422564 PAGE 2
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0.6
0.5
0.4
0.3
0.2
0.1
0
PHASE DISTRIBUTION
PHASE IN RADIANS NORMALIZED LENGTH SEQUENCE
PLOT NUMBER 1
N1 = 6
N2 = 5
10 HARMONICS INCLUDED

Figure VI-65

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

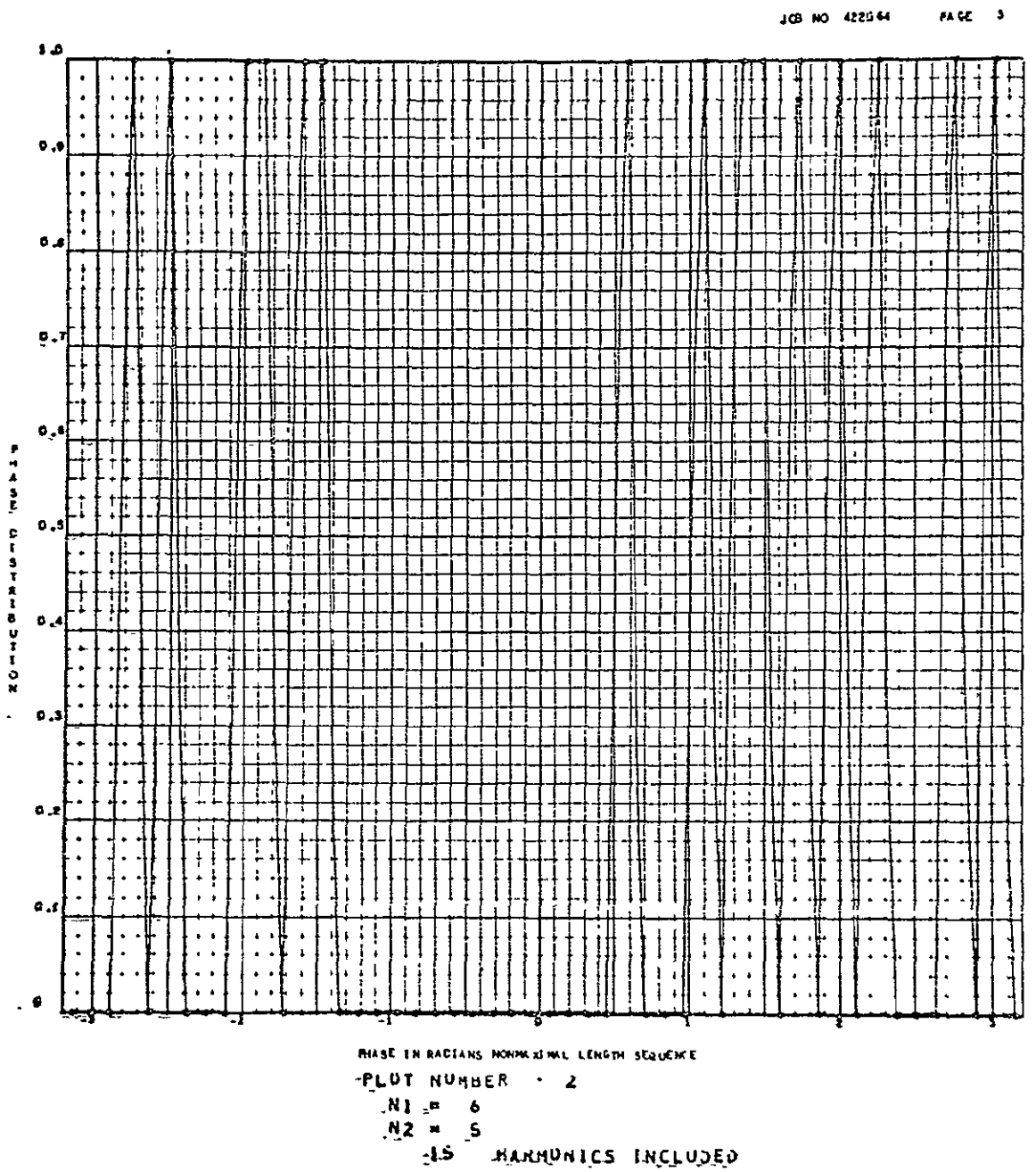


Figure VI-66

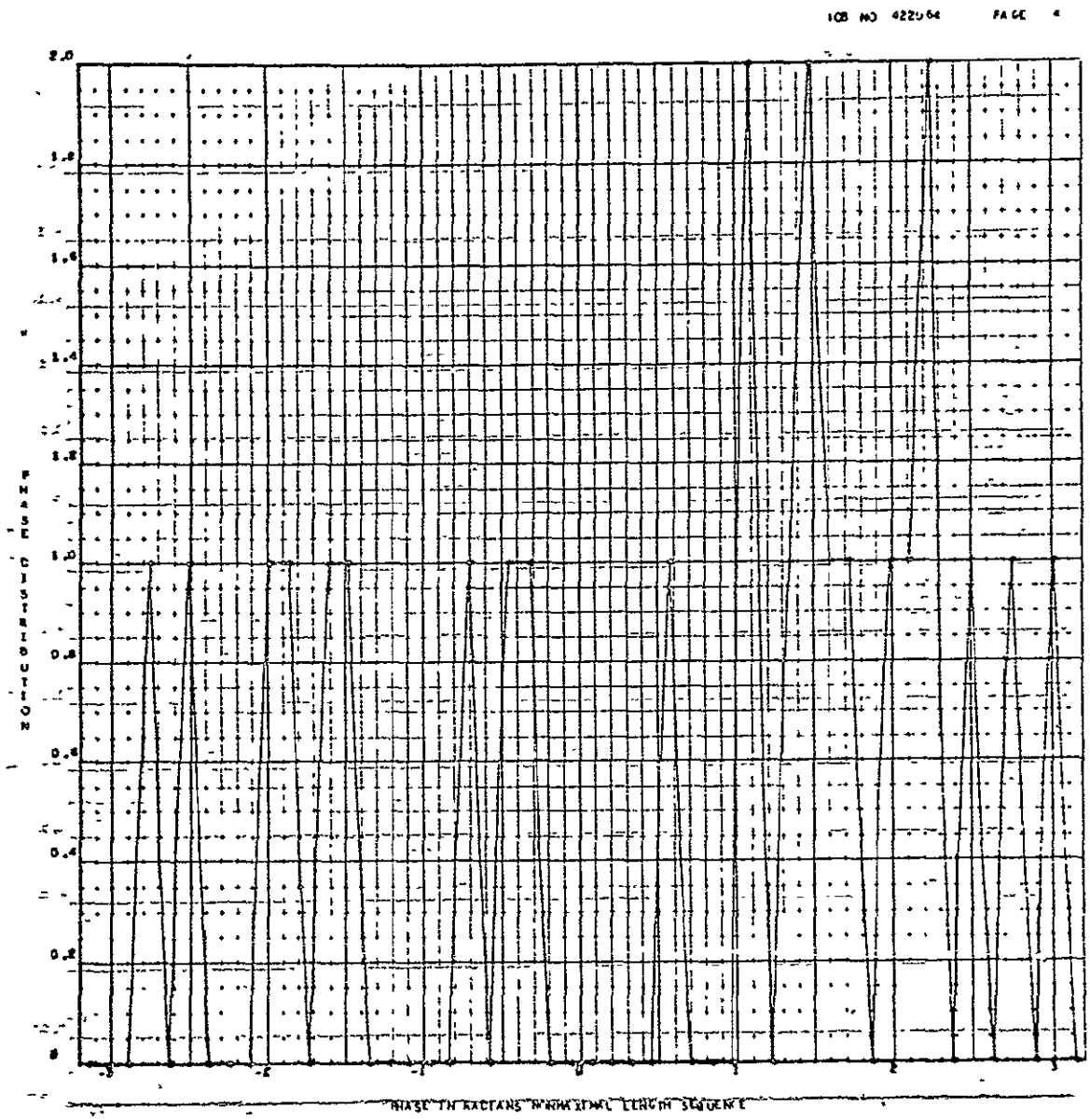

JOB NO 422064 PAGE 4
PLOT NUMBER 3
N1 = 6
N2 = 5
24 HARMONICS INCLUDED

Figure VI-67

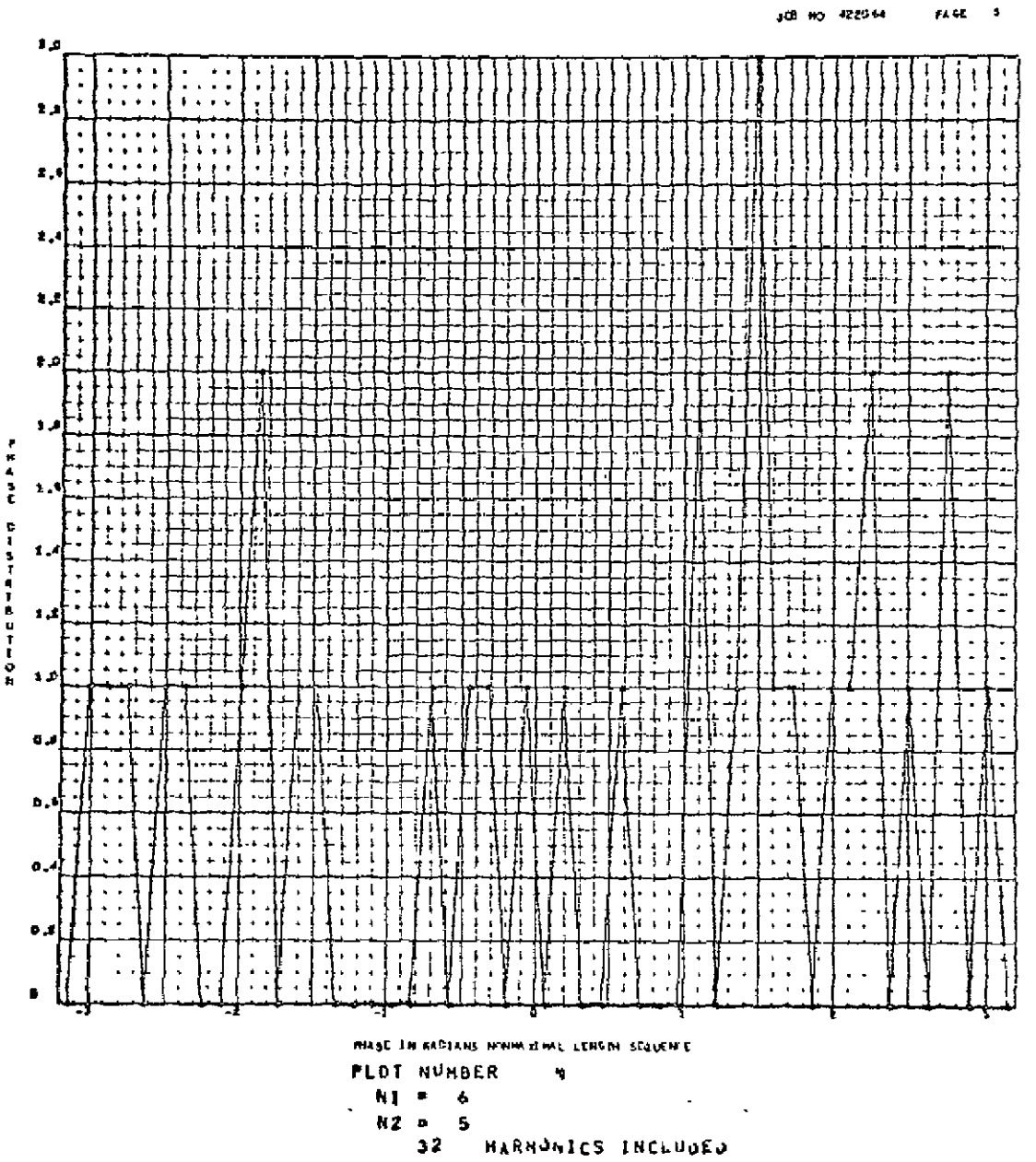

JOB NO 422044 PAGE 5
PHASE DISTRIBUTION
PHASE IN RADIANS NORMAL LENGTH SEQUENCE
PLOT NUMBER 4
N1 = 6
N2 = 5
32 HARMONICS INCLUDED

Figure VI-68

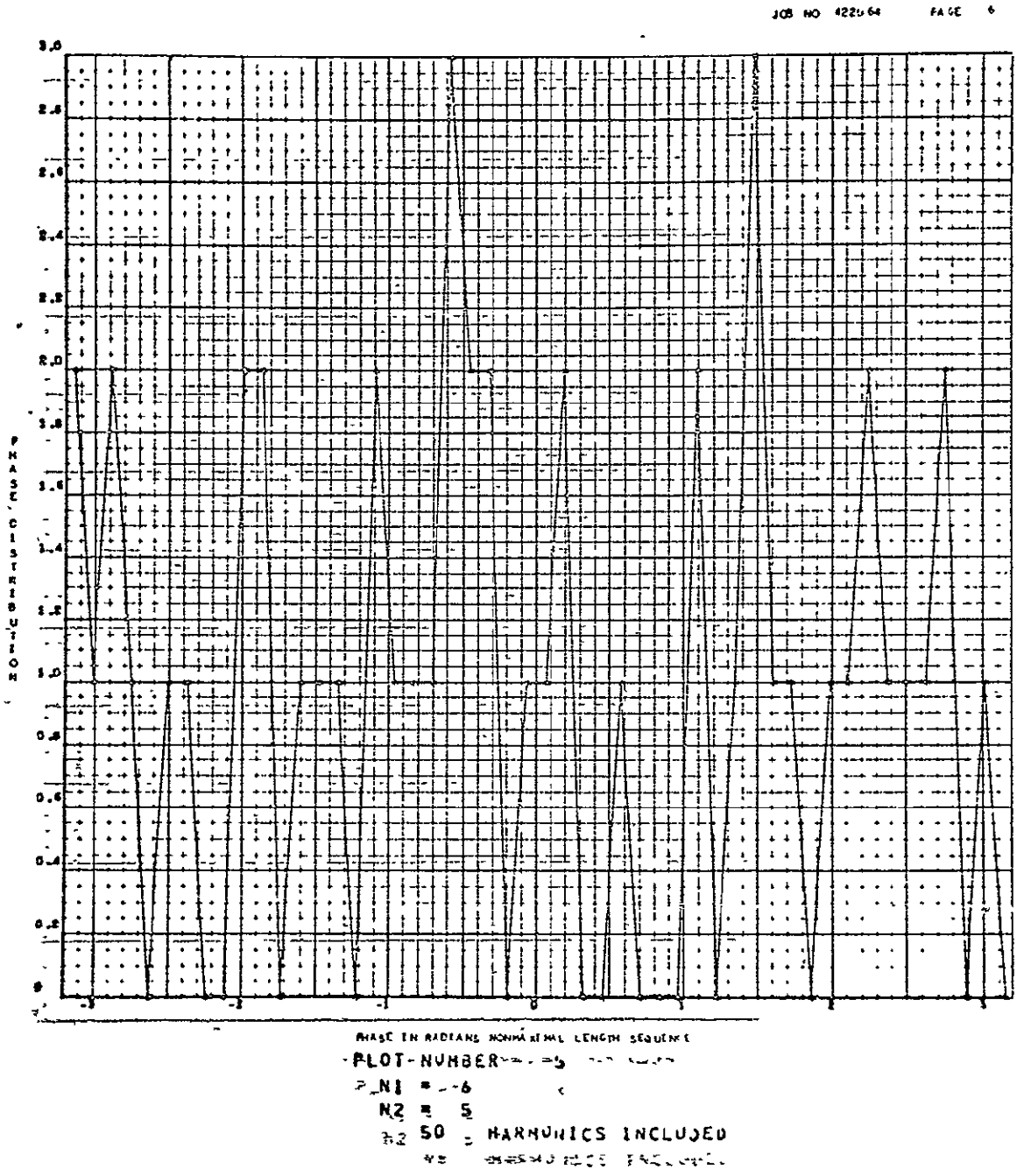

JOB NO 422064 PAGE 6
PHASE DISTRIBUTION
PHASE IN RADIANS NONMAXIMAL LENGTH SEQUENCE
PLOT-NUMBER
N1 = 6
N2 = 5
50 = HARMONICS INCLUDED

Figure VI-69

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

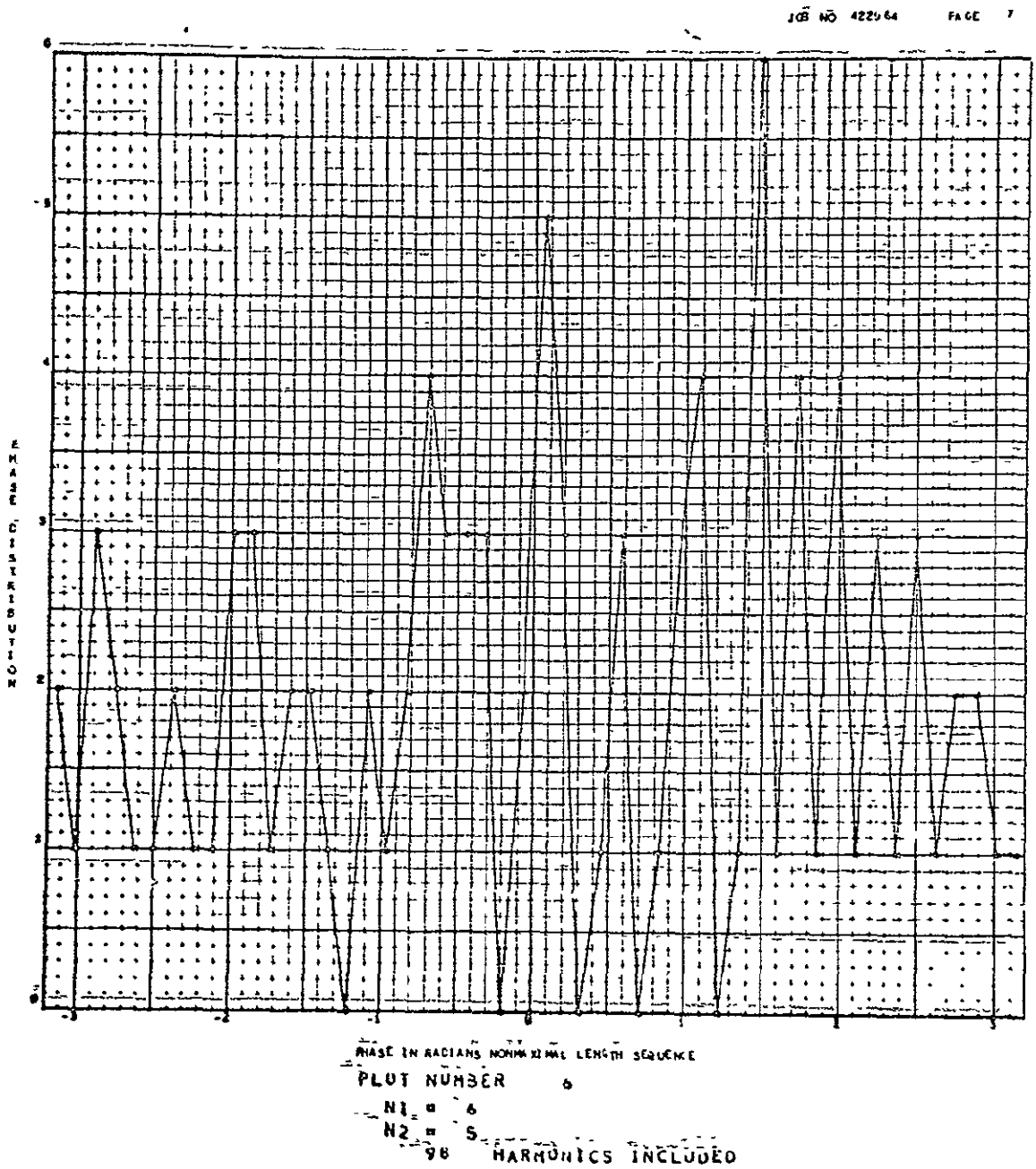


Figure VI-70.

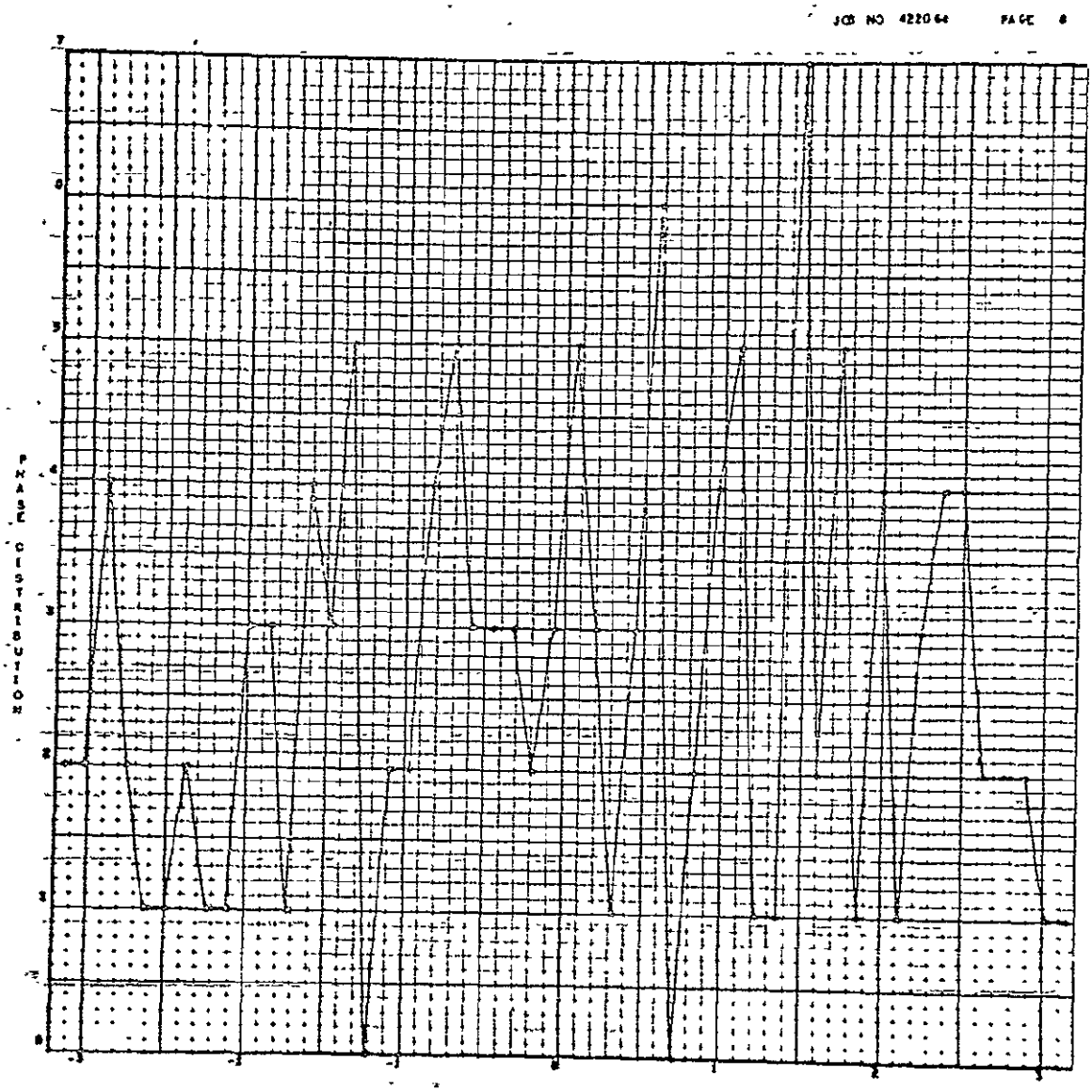
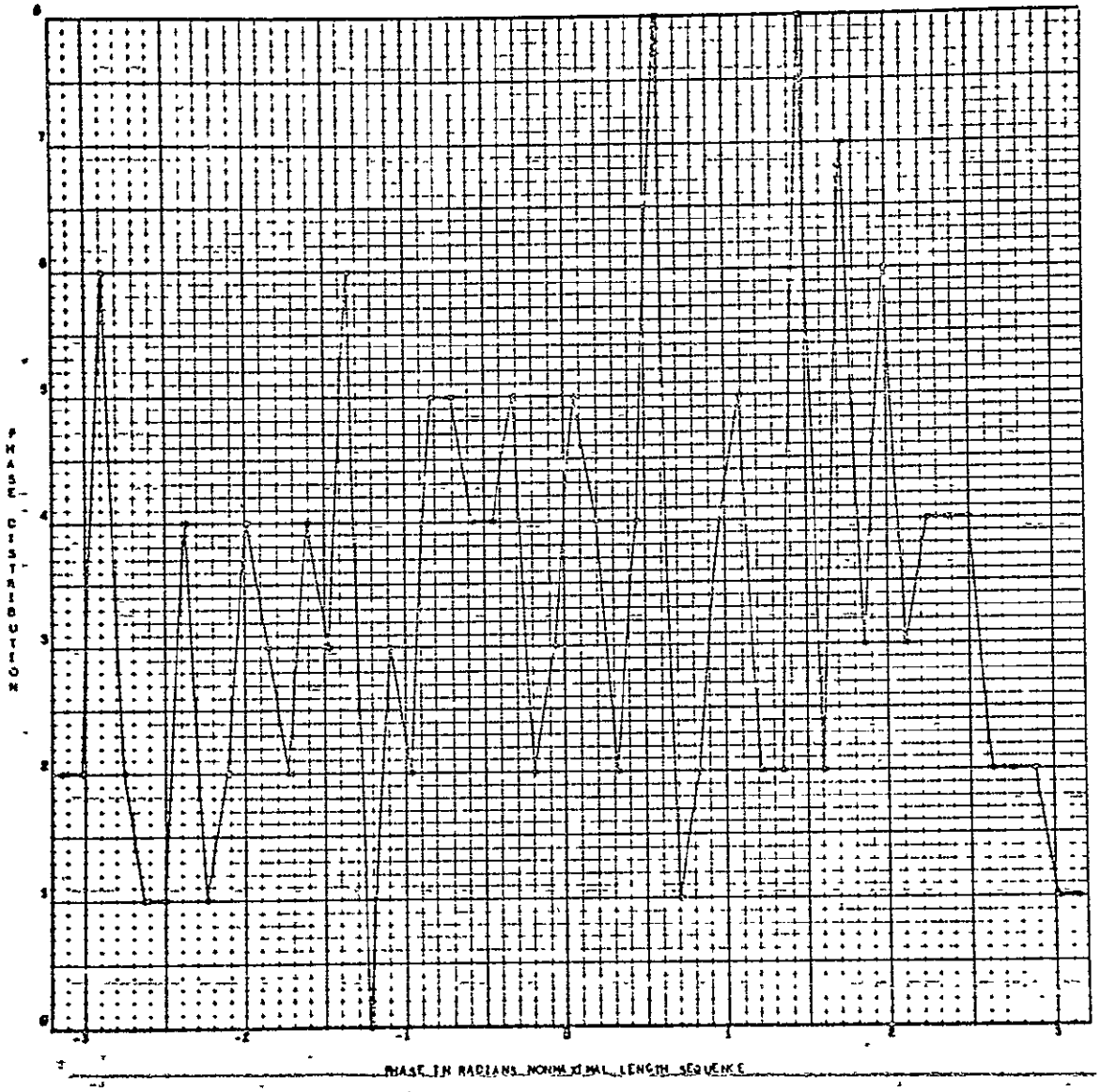

JOB NO 422064 PAGE 8
PHASE DISTRIBUTION
PHASE IN RADIANS NONMINIMAL LENGTH SEQUENCE
PLOT NUMBER 7
N1 = 6
N2 = 5
132 HARMONICS INCLUDED

Figure VI-71



PLOT NUMBER 8
N1 = 6
N2 = 5
164 HARMONICS INCLUDED

Figure VI-72

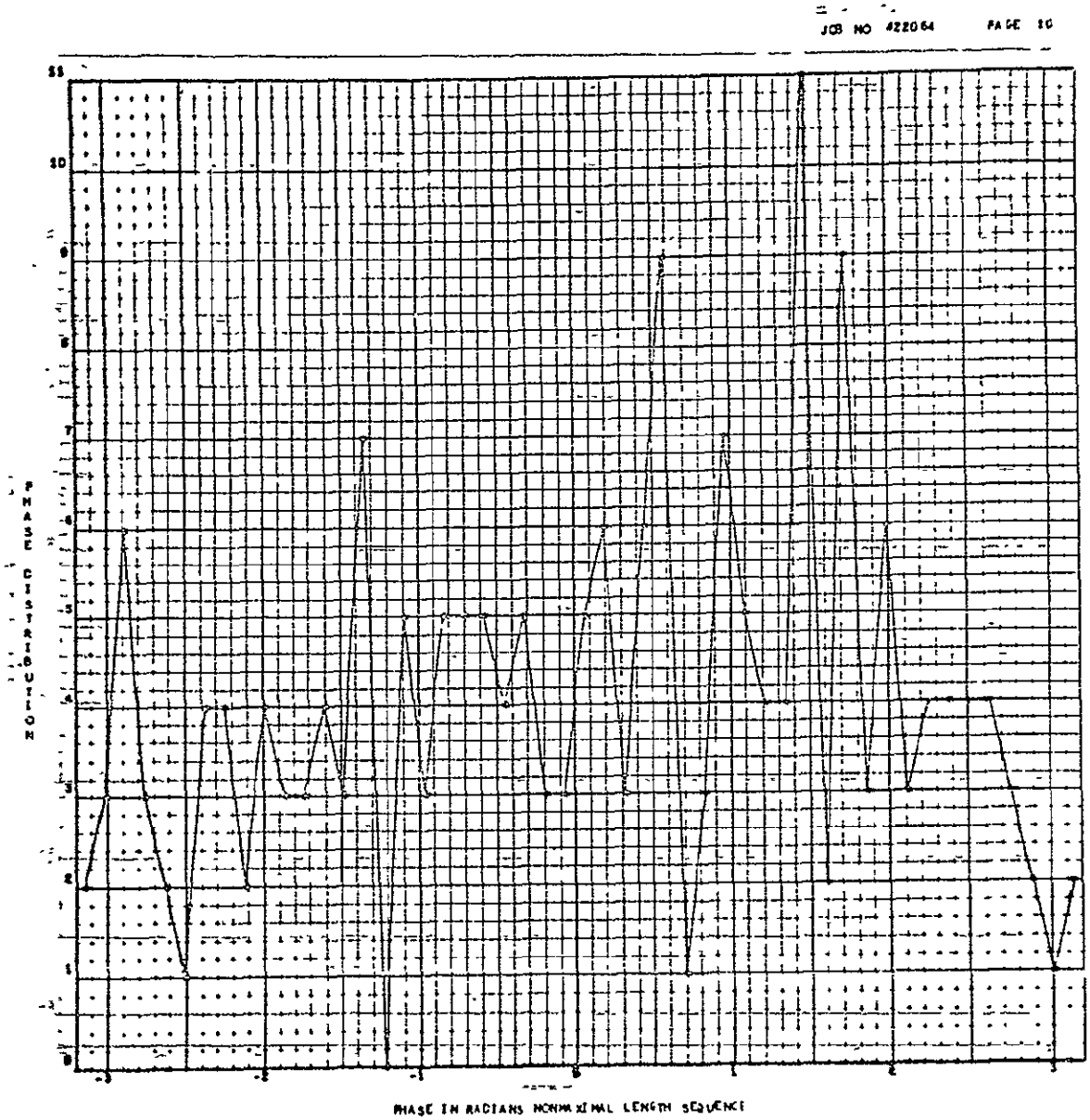

JOB NO 422064
PAGE 10
PHASE DISTRIBUTION
PHASE IN RADIANS NONMAXIMAL LENGTH SEQUENCE
PLOT NUMBER 9
N1 = 6
N2 = 5
200 HARMONICS INCLUDED

Figure VI-73

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

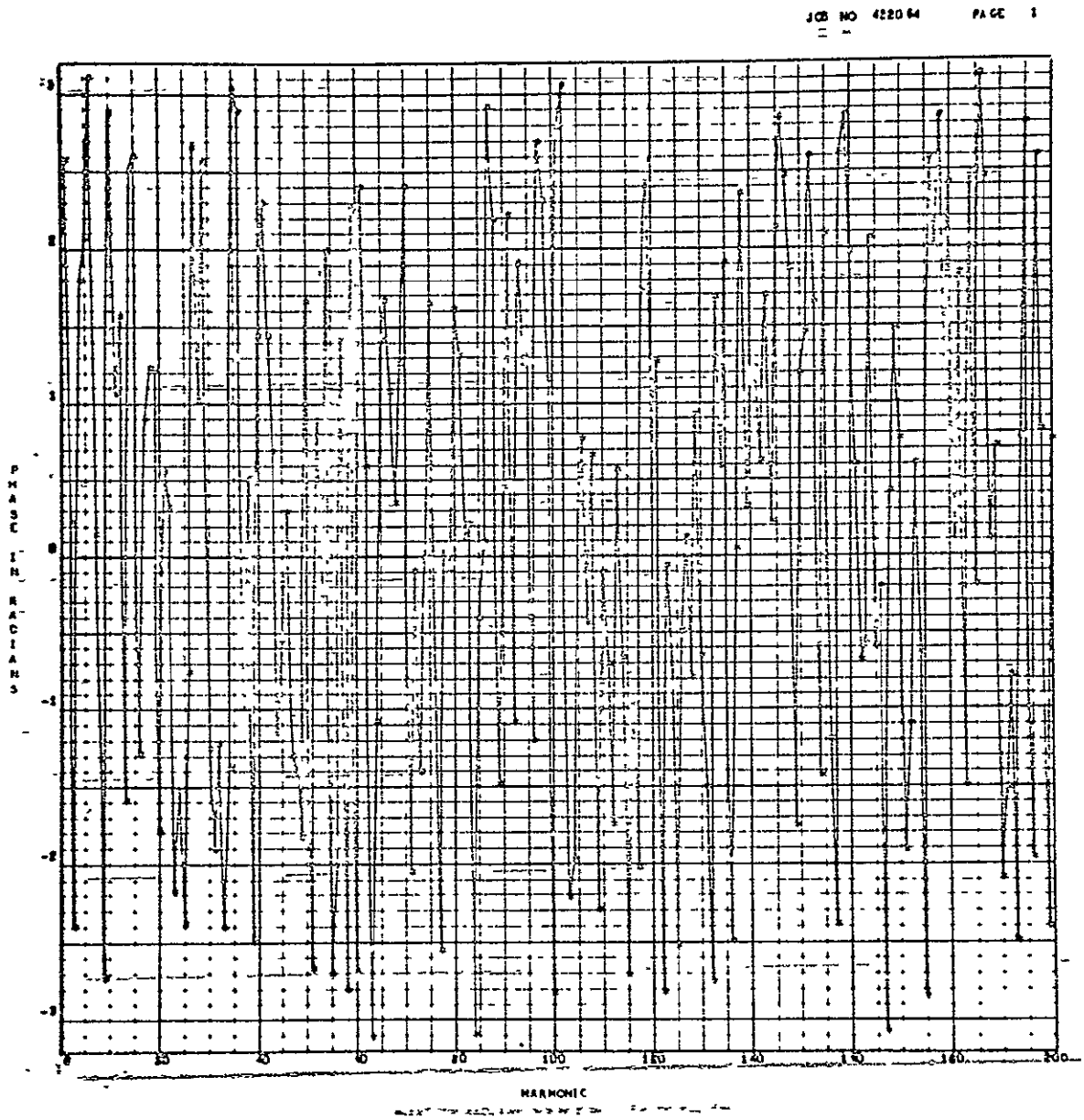


Figure V.1-74

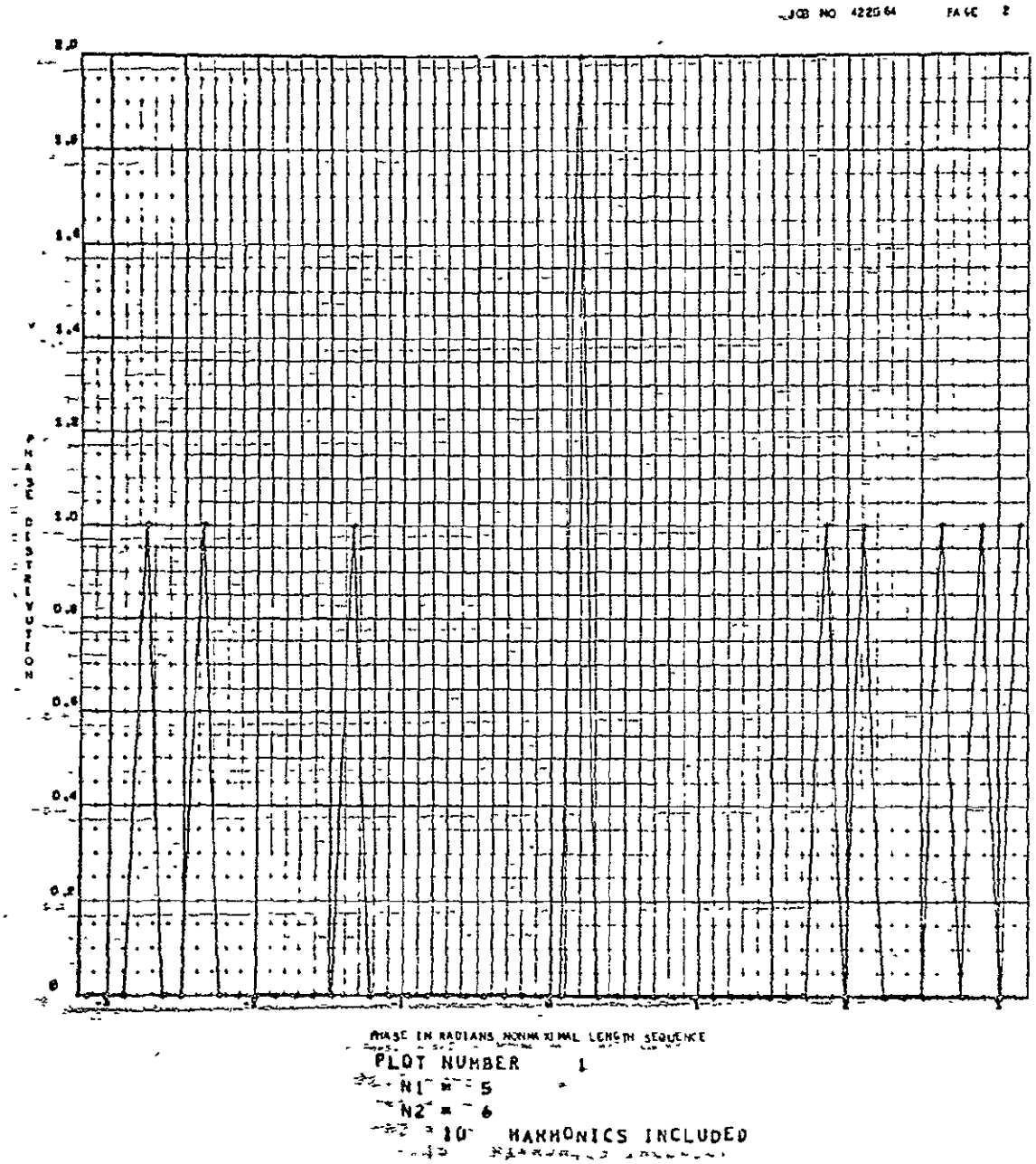

JOB NO 422564
PAGE 2
PHASE DISTRIBUTION
PHASE IN RADIANS NONMINIMAL LENGTH SEQUENCE
PLOT NUMBER 1
N1 = 5
N2 = 6
10 HARMONICS INCLUDED

Figure V.I-75

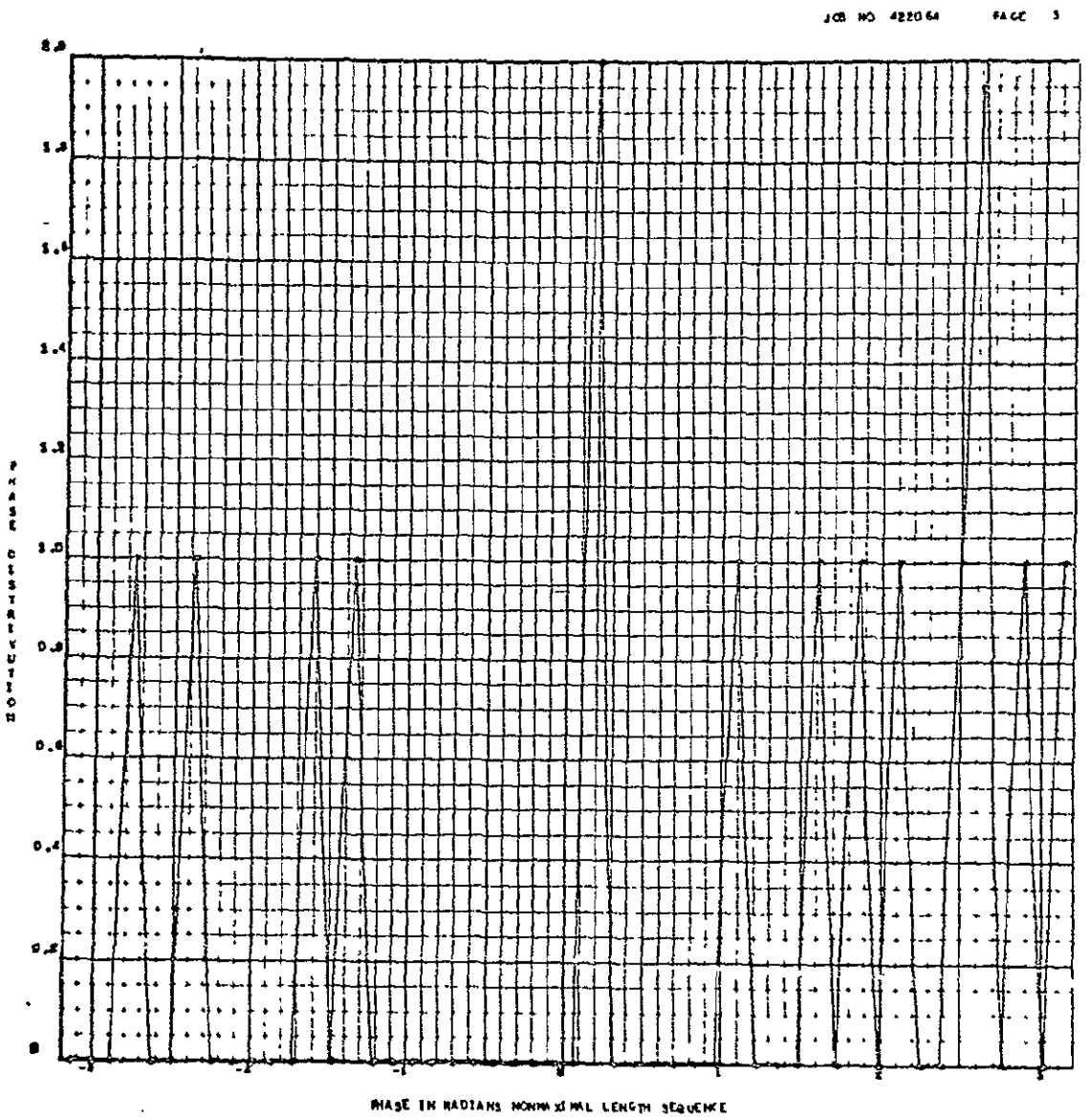
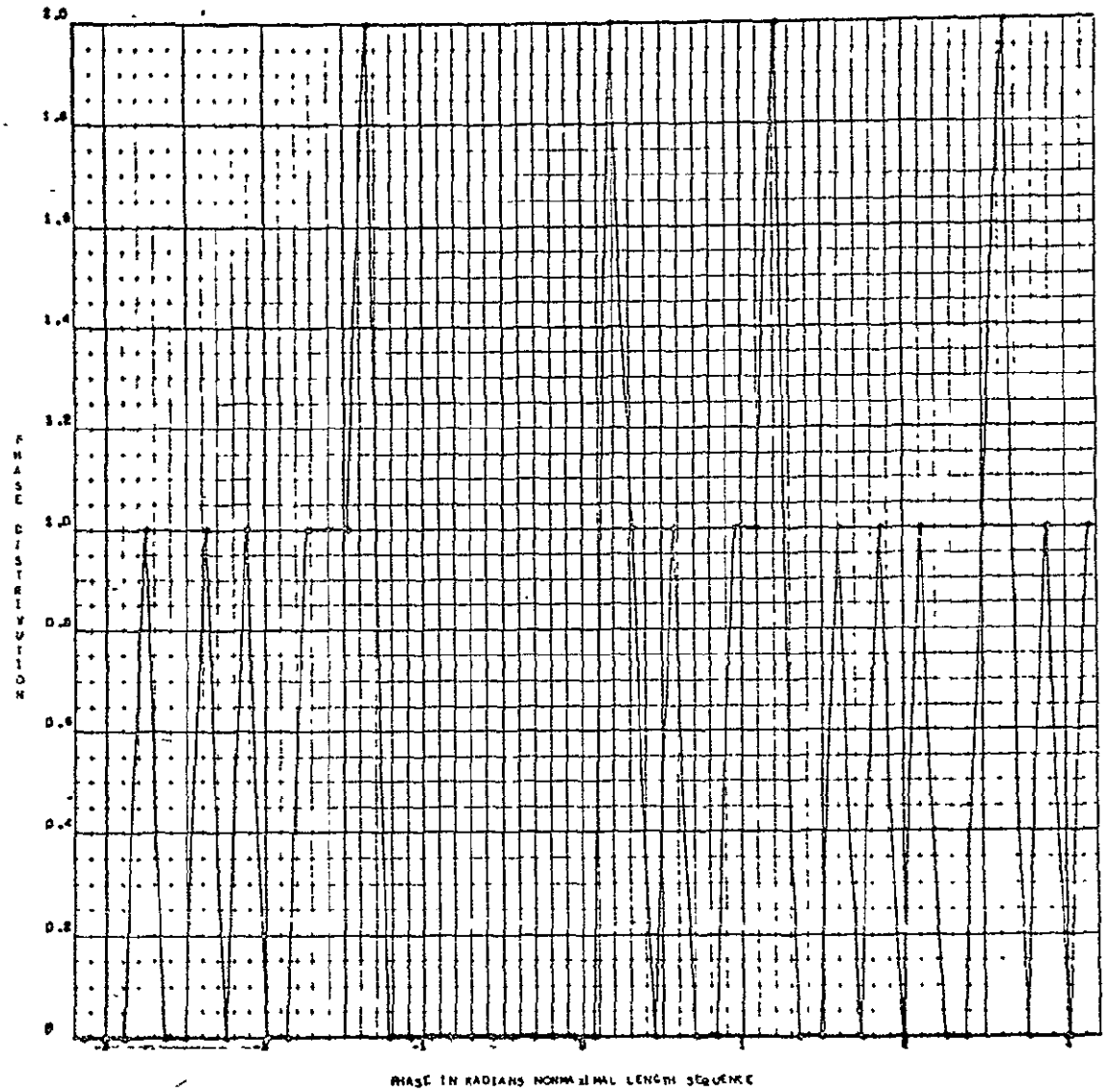

JOB NO 422064 PAGE 5
PHASE IN RADIANS NONMAXIMAL LENGTH SEQUENCE
PLOT NUMBER - 2
N1 = 5
N2 = 6
15 HARMONICS INCLUDED

Figure VI-76



PLOT NUMBER 3
N1 = 5
N2 = 6
24 HARMONICS INCLUDED

Figure VI-77

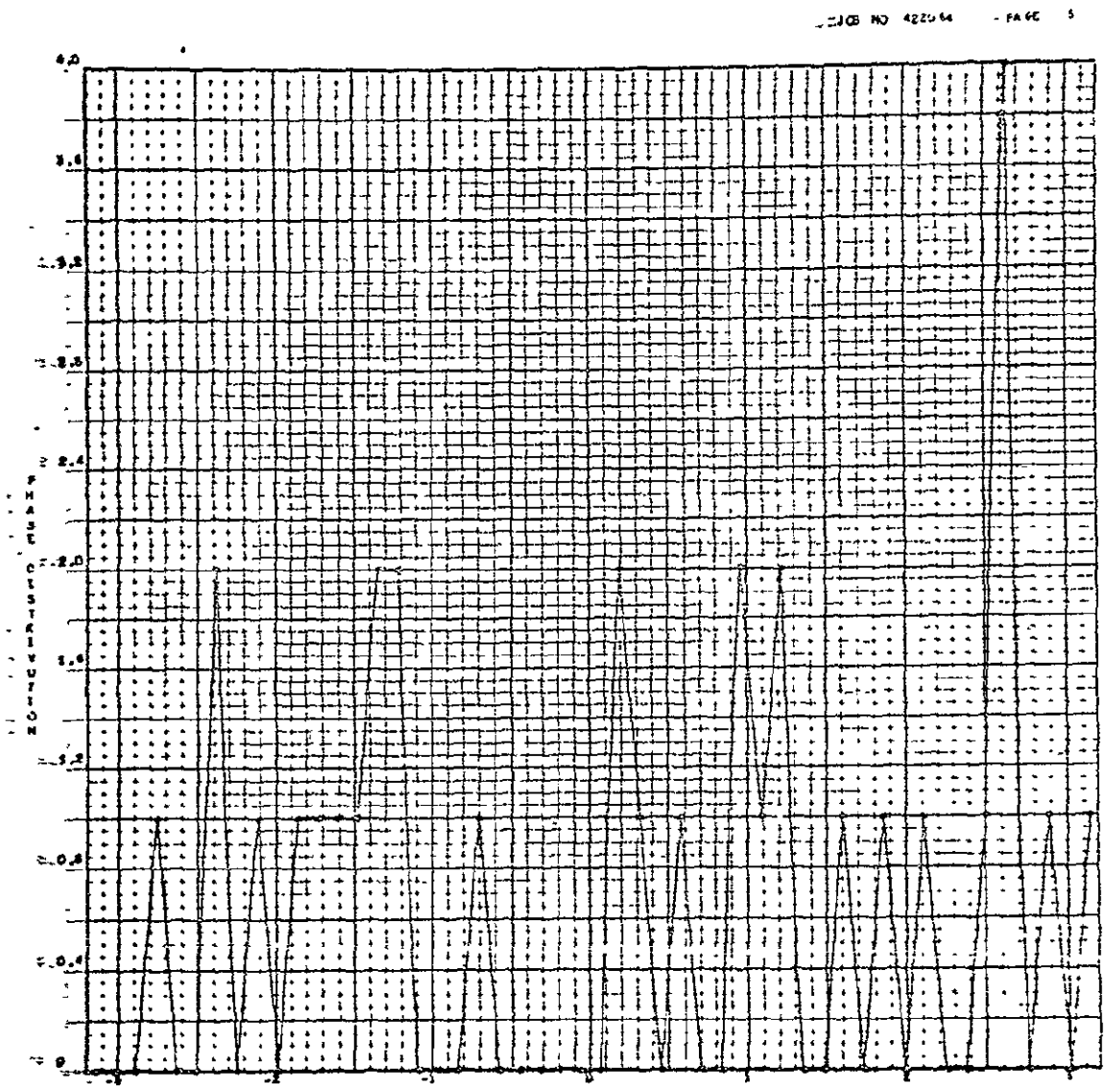

JOB NO 422064 PAGE 5
4.0
3.6
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0
NONUNIFORM PHASE
-3
-2
-1
0
1
2
3
PHASE IN RADIANS-NONMINIMAL LENGTH SEQUENCE
PLOT NUMBER 4
N1 = 5
N2 = 6
32 HARMONICS INCLUDED

Figure VI-78

REPRODUCIBILITY OF THE
ORIGINAL PAGE IS POOR

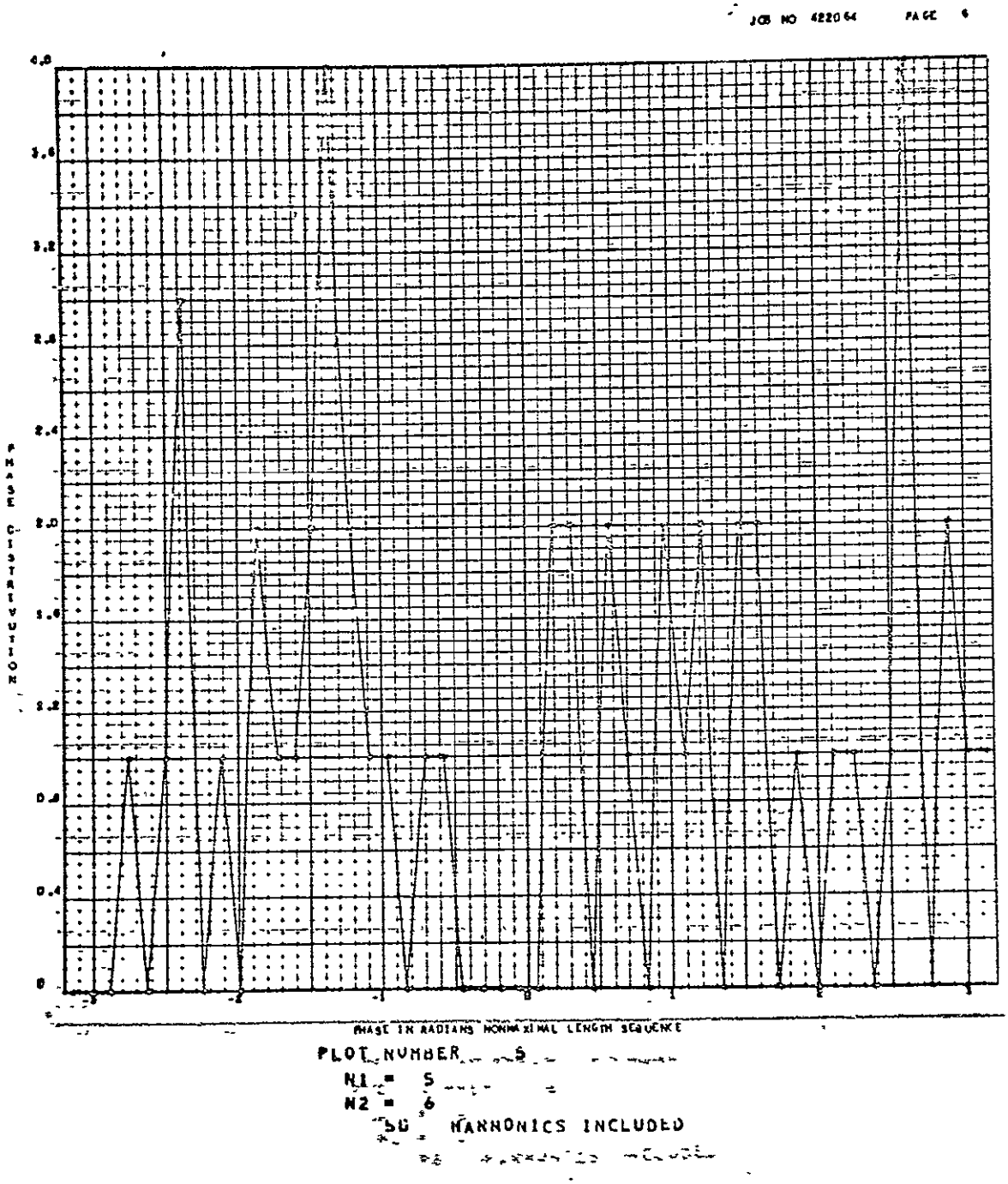


Figure VI-79

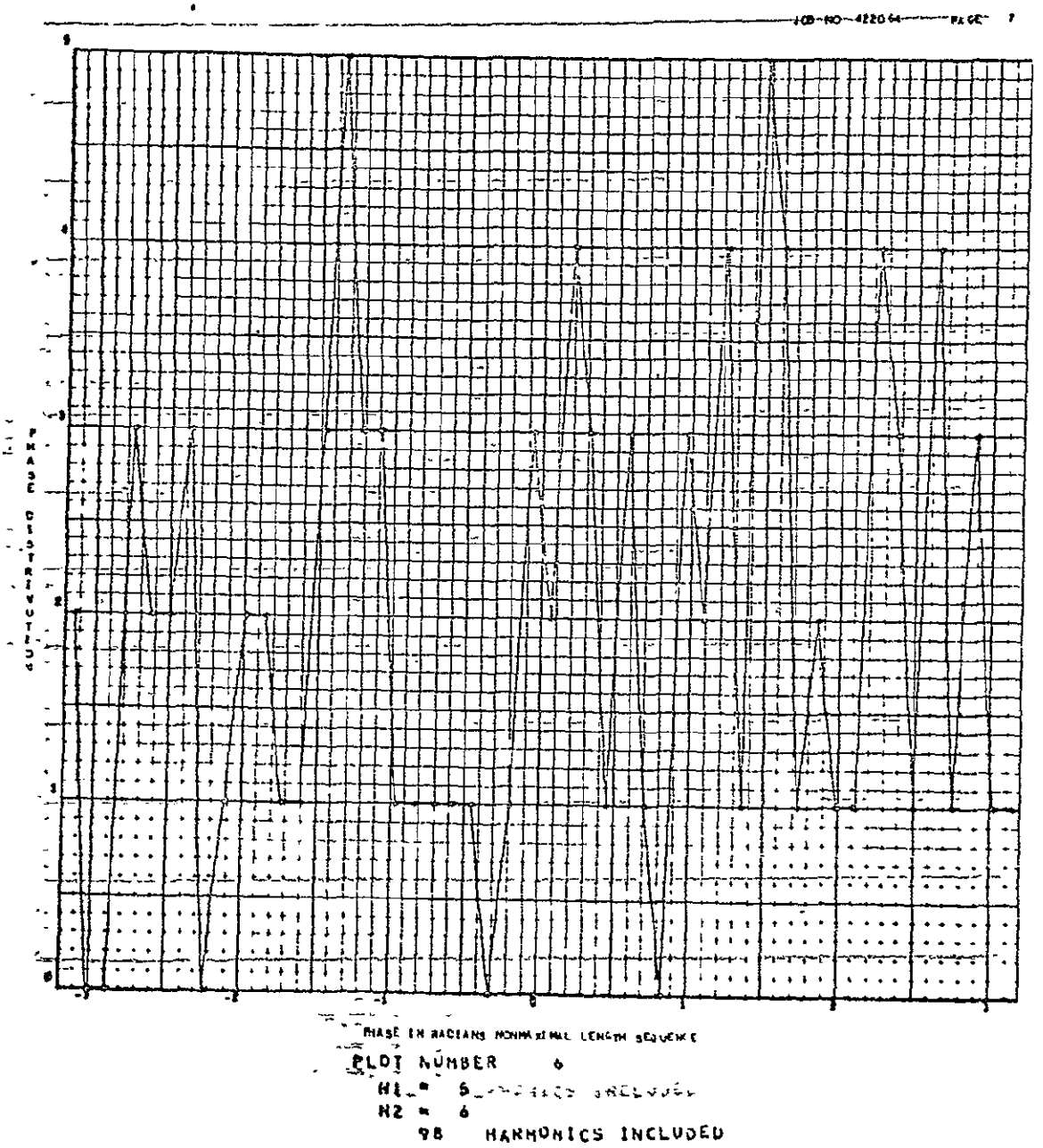

JOB NO 422064 PAGE 7
PHASE IN RADIANS NONMAXIMAL LENGTH SEQUENCE
PLOT NUMBER 6
N1 = 5
N2 = 6
98 HARMONICS INCLUDED

Figure VI-80.

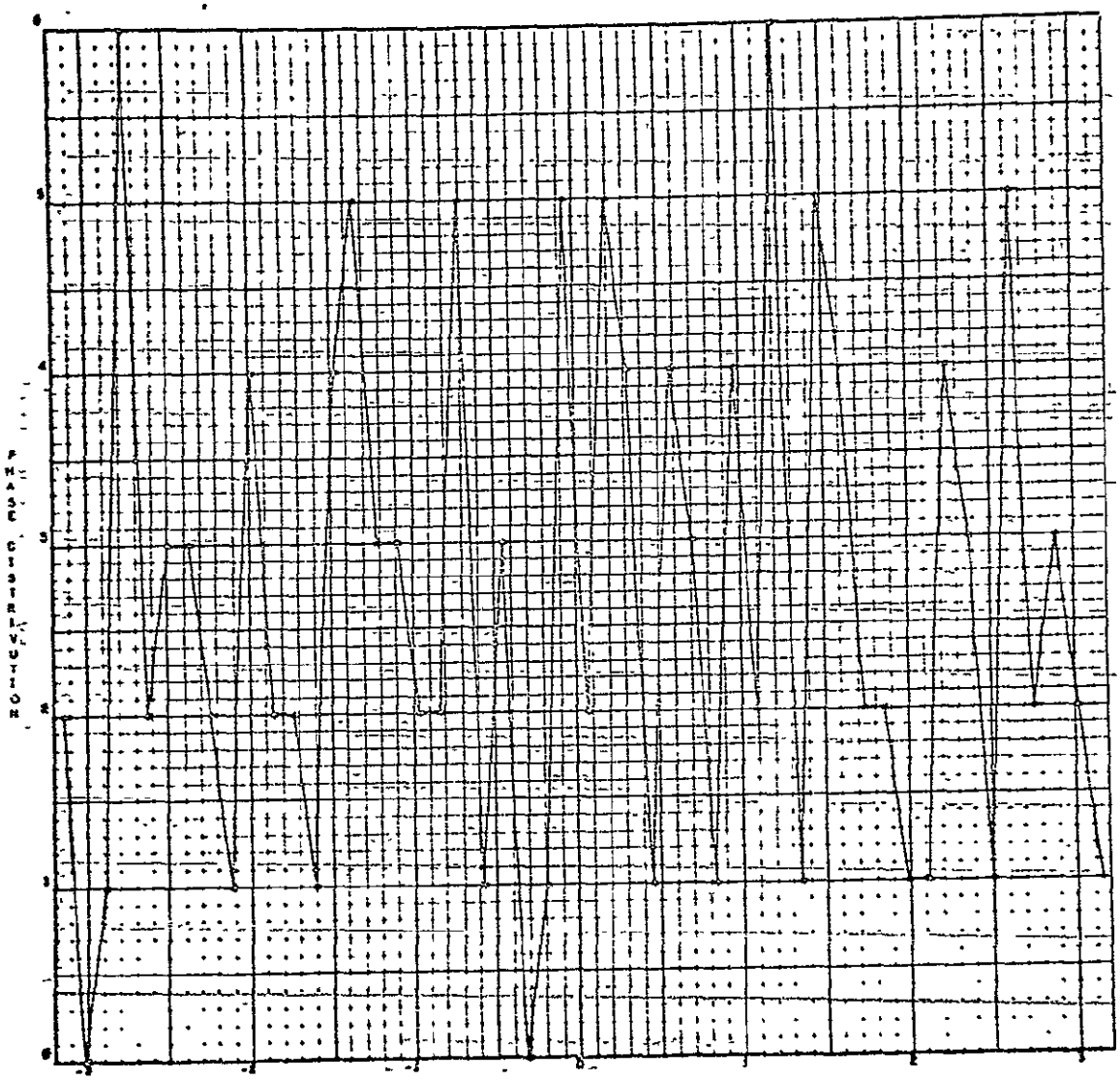
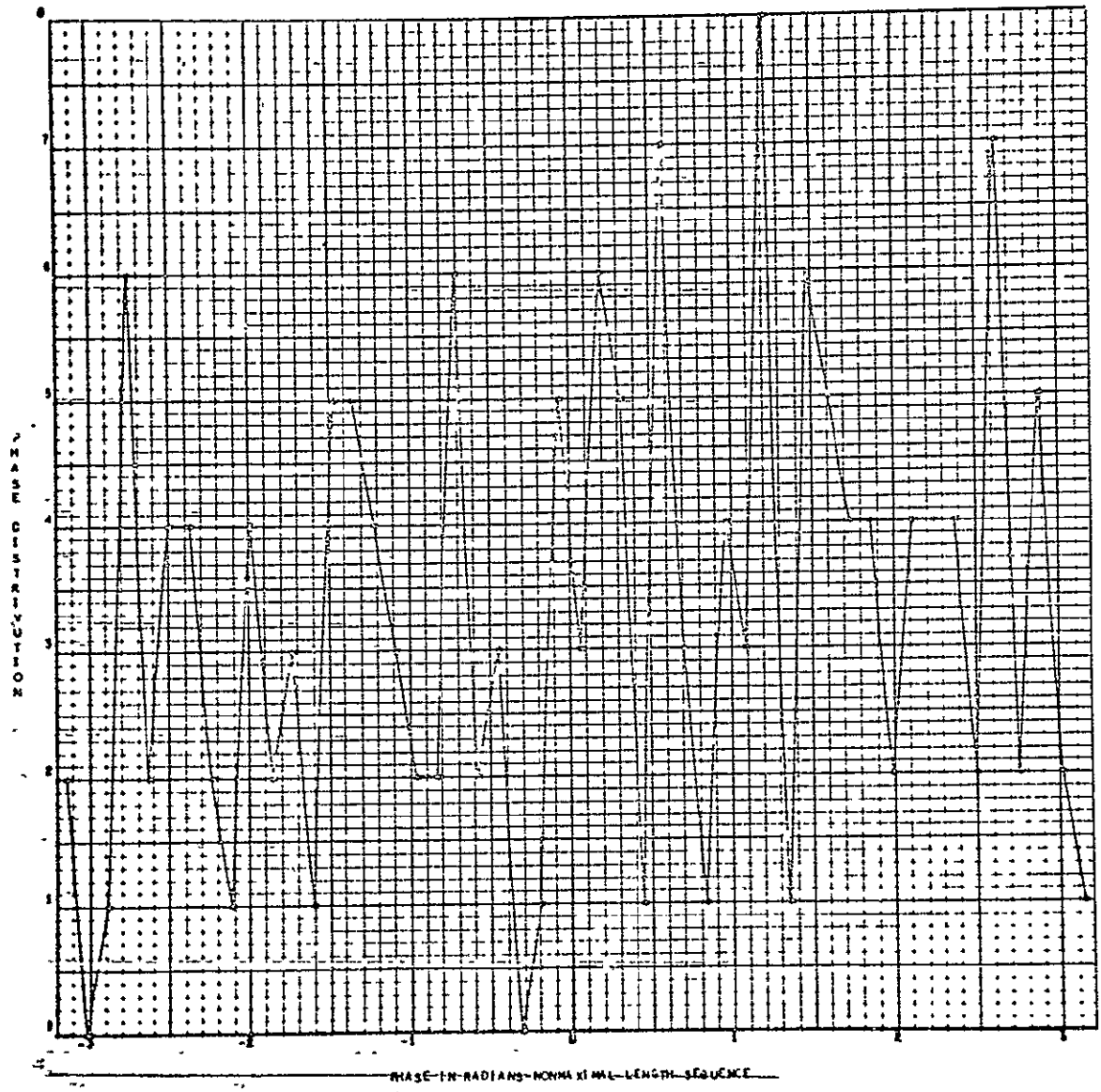
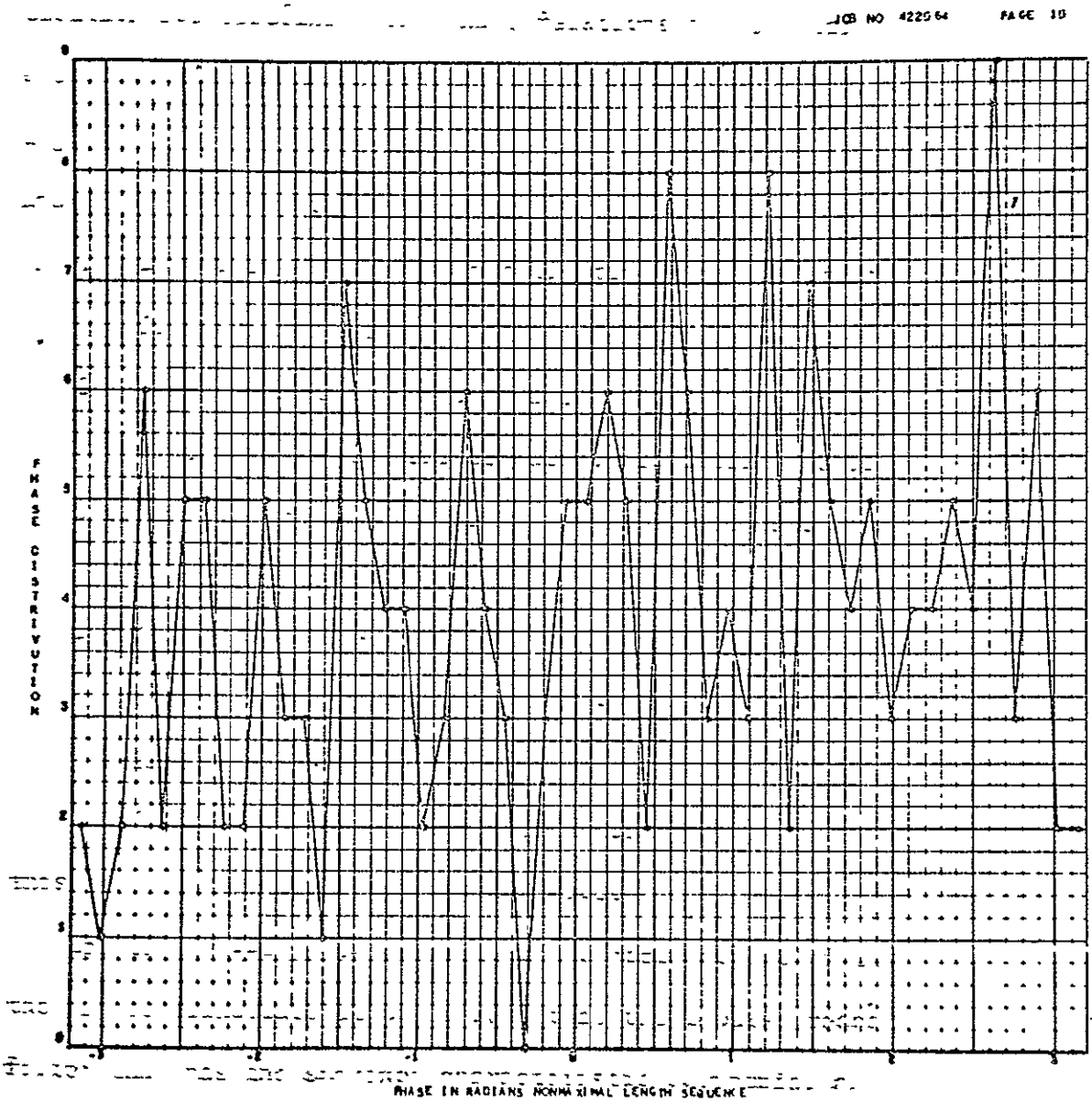

PHASE IN RADIANS NORMAL LENGTH SEQUENCE
PLOT NUMBER 1
N1 = 5
N2 = 6
132 HARMONICS INCLUDED

Figure VI-81



PLOT NUMBER 8
N1 = 5
N2 = 6
164 HARMONICS INCLUDED

Figure VI-82



[illegible] moment can oscillate [illegible] amplitude response periods. This result was confirm[illegible] ation and computer simulation for both maximum length seq[illegible] sequences formed by summing two maximum length sequences.

Figure VI-83

SECTION VII

RESULTS AND CONCLUSIONS

Section II of this report developed a method of describing a sequence generator by a characteristic polynomial equation. Section III derived the equation for describing the phase relations between harmonic components of a digital sequence, and Sections IV, V, and VI describe methods for investigating the randomness properties of filtered sequences. Four methods described and illustrated in the report are:

1) Experimental method, investigating the statistics of an actual filtered sequence.

2) Computer simulation with calculation of the time dependent output waveform, and the filter output probability density function.

3) Computer calculation of the phase distribution of the harmonic components passed by the filter.

4) Calculation of the third-moment of the filter output by finding trinomials that have the sequence characteristic polynomial as a factor.

All four methods have certain disadvantages; however, the conclusions of this section will be based on method four.

Equation (V-42) related the third central moment of the filter output to the number trinomials at power less than the impulse response period of the filter that has the sequence characteristic polynomial as a factor. Equation (V-42) shows that the third central moment can oscillate around 0 for various filter impulse response periods. This result was confirmed by both experimentation and computer simulation for both maximum length sequences and for sequences formed by summing two maximum length sequences.

An ideal sequence generator for a noise source that must produce approximately Gaussian waveforms for various ratios of the digital clock frequency to filter cutoff frequency (impulse response period in bits or M-tuple length), would be one whose characteristic polynomial is a factor of no trinomial of power up to the maximum impulse response period anticipated. Two maximum length sequences that have characteristic polynomials that are factors of no trinomial of order less than 500 are

(23, 18, 12, 6, 0)

and

(23, 17, 13, 12, 11, 9, 8, 7, 5, 3, 0).

Both these generators have twenty-three stages, and the following stages are part of the feedback:

(23, 18, 12, 6, 0) - feedback tapes on stages 23, 17, 11, and 5, (23, 17, 13, 12, 11, 9, 8, 7, 5, 3, 0) - feedback tapes on stages 23, 20, 16, 10, 15, 14, 12, 11, 10, and 6.

Impulse response periods (digital clock to filter cutoff ratio) will be less than 500 for the noise source under consideration. Therefore, the filter outputs from a noise source using one at the maximum length generators listed above will have non-skewing, approximately Gaussian, probability density functions.

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6. Lindholm, J. H., "An Analysis of the Pseudo-Randomness Properties of Subsequences of Long m-Sequences", IEEE Transactions on Information Theory, Vol. IT-14, No. 4, July 1968.