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Integrals of a C^1-Compatible Triangular Surface Element

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JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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PREFACE

The work described in this report was performed by the Information Systems Division of the Jet Propulsion Laboratory.

CONTENTS

Abstract

Definite integrals are evaluated for the cardinal functions of an interpolation method which provides C^1 continuity over a triangular grid.

1. Introduction

In Ref. 1, algorithms are given for C^1-compatible interpolation over a plane triangle given values of f, f_x, and f_y at the vertices of the triangle. In particular on p. 30 of Ref. 1, functions $\alpha_i, \tilde{\beta}_i$, and $\tilde{\gamma}_i$ are defined such that with $\beta_i = \tilde{\beta}_i/2$ and $\gamma_i = \tilde{\gamma}_i/2$ the interpolated value w can be expressed as

$$(1) \qquad w := \sum_{i=1}^{3} (f_i \alpha_i + f_{x,i} \beta_i + f_{y,i} \gamma_i)$$

In this note we derive expressions for the definite integrals of the cardinal functions α_i, β_i, and γ_i, $i=1,2,3$ over the triangle. These integrals are needed, for example, if a surface representation problem includes conditions on volumes to be encompassed under the surface.

2. Preliminary integration formulas

From Ref. 2, we obtain the following formulas for integration of polynomials over a triangle:

$$(2) \qquad \int p_1 = A\, p_1[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}]$$

$$(3) \qquad \int p_2 = \tfrac{1}{3} A \{ p_2[\tfrac{1}{2}, \tfrac{1}{2}, 0] + p_2[\tfrac{1}{2}, 0, \tfrac{1}{2}] + p_2[0, \tfrac{1}{2}, \tfrac{1}{2}] \}$$

$$(4) \qquad \int p_3 = \tfrac{1}{48} A \{ 25 p_3[\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}] + 25 p_3[\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}] + 25 p_3[\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}] - 27 p_3[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}] \}$$

In the above formulas the integration is over a triangle in the xy plane. The integrand p_i denotes a polynomial of degree at most i in x and y. The symbol A denotes the area of the triangle. The notation $p_i[r_1,r_2,r_3]$ denotes the value of the polynomial p_i at the point in the triangle whose barycentric coordinates are r_1, r_2, and r_3. For discussion of the affine invariance of quadrature formulas stated in terms of barycentric coordinates see Ref. 2.

The above formulas can be used to obtain the following integrals for the fundamental linear, quadratic, and cubic functions used in Ref. 1.

(5) $\int r_i = A/3$

(6) $\int \phi_i \equiv \int r_{i+1}r_{i-1} = A/12$

(7) $\int g_i \equiv \int r_{i+1}r_{i-1}(r_{i+1}r_{i-1}) = 0$

In the following two sections we determine $\int \rho_i$ for the two different definitions of ρ_i given in Ref. 1. Formulas for the integrals of the cardinal functions are then derived in Section 5.

3. Determining $\int \rho_i$ for the rational function ρ_i

In Section 7, pp. 22-23, of Ref. 1 ρ_i is defined as the Zienkiewicz rational function

$$\rho_i = r_i r_{i+1}^2 r_{i-1}^2/[(1-r_{i+1})(1-r_{i-1})] \qquad i=1,2,3$$

Consider the particular triangle T_0 having vertices $V_1 = (0,0)$, $V_2 = (1,0)$, $V_3 = (0,1)$. In this triangle we can relate cartesian xy coordinates to barycentric coordinates as follows:

$r_1 = 1-x-y$

$r_2 = x$

$r_3 = y$

Then

$$\rho_1 = x^2y^2(1-x-y)/[(1-x)(1-y)]$$

Expand the term $(1-x-y)$ as $(1-x)+(1-y)-1$. Then $\int_{T_0} \rho_1$ is the sum of the following three integrals:

$$I_1 = \int_0^1 \frac{y^2}{1-y} \int_0^{1-y} x^2 dx\, dy$$

$$I_2 = \int_0^1 \frac{x^2}{1-x} \int_0^{1-x} y^2 dy\, dx$$

$$I_3 = -\int_0^1 \frac{x^2}{1-x} \int_0^{1-x} \frac{y^2}{1-y}\, dy\, dx$$

Evaluating I_1:

$$I_1 = \frac{1}{3} \int_0^1 \frac{y^2}{1-y} (1-y)^3\, dy$$

$$= \frac{1}{3} \int_0^1 (y^2-2y^3+y^4)\, dy$$

$$= \frac{1}{3}[\frac{1}{3} - \frac{2}{4} + \frac{1}{5}] = \frac{1}{90}$$

The integral I_2 has the same value as I_1. The inner integral of I_3 is

$$\int_0^{1-x} \frac{y^2}{1-y}\, dy = \{-\frac{(1-y)^2}{2} + 2(1-y) - \ell n(1-y)\}_0^{1-x}$$

$$= -\frac{1}{2}x^2 + 2x - \ell n\ x + \frac{1}{2} - 2$$

$$= -\frac{1}{2}(x^2-4x+3) - \ell n\ x$$

$$= -\frac{1}{2}(x-1)(x-3) - \ell n\ x$$

Thus

$$I_3 = -\frac{1}{2}\int_0^1 x^2(x-3)\ dx + \int_0^1 \frac{x^2 \ell n x}{1-x}\ dx$$

$$= \frac{3}{8} + \int_0^1 \frac{x^2 \ell n x}{1-x}\ dx$$

$$= \frac{3}{8} + \int_0^1 \frac{\ell n\ x}{1-x}\ dx - \int_0^1 \ell n\ x\ dx - \int_0^1 x \ell n x\ dx$$

$$= \frac{3}{8} - \frac{\pi^2}{6} - [x\ell n x - x + \frac{x^2}{2}\ \ell n x - \frac{x^2}{4}]_0^1$$

$$= \frac{3}{8} - \frac{\pi^2}{6} + \frac{5}{4} = (39-4\pi^2)/24$$

Combining results we obtain:

$$(8) \qquad \int_{T_0} \rho_1 = I_1 + I_2 + I_3 = \frac{1}{90} + \frac{1}{90} + \frac{39-4\pi^2}{24} = (593-60\pi^2)/360$$

Since the area of triangle T_0 is 1/2, the mean value of ρ_1 over T_0, and thus over any arbitrary triangle, is $(539-60\pi^2)/180$. Furthermore, since the integrals of ρ_1, ρ_2, and ρ_3 are the same we may drop the subscripts and simply write the mean value of any of these three functions as

$$(9) \qquad \bar{\rho} = (593-60\pi^2)/180$$

$$\doteq 0.00457\ 63107\ 47992$$

The author thanks Dr. E.W. Ng for doing the analytic integrations of this section. The result given in Eq.(9) has been corroborated by a numerical integration of ρ_1 over the triangle T_0 which gave agreement in the first seven significant digits. See Section 8 for more details on the numerical integration.

4. Determining $\int \rho_i$ for the piecewise cubic function ρ_i

In Section 8, pp. 24-26, of Ref. 1 , ρ_i is defined for $i=1$ as the Clough-Tocher piecewise cubic function

$$\rho_1 = \begin{cases} r_1[6r_2r_3+r_1(5r_1-3)]/6 & \text{if } r_1 = \min\{r_1,r_2,r_3\} \\ r_2^2(-r_2+3r_3)/6 & \text{if } r_2 = \min\{r_1,r_2,r_3\} \\ r_3^2(-r_3+3r_2)/6 & \text{if } r_3 = \min\{r_1,r_2,r_3\} \end{cases}$$

Thus ρ_1 is defined as a different cubic polynomial in each of the three subtriangles indicated in Fig. 1.

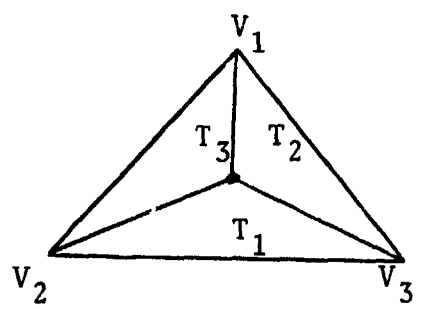

Figure 1

The common intersection point of the three subtriangles has barycentric coordinates $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$.

Within each subtriangle the integration formula, Eq. (4), can be applied. The evaluation points for this formula in subtriangles T_1 and T_2 are indicated in Fig. 2.

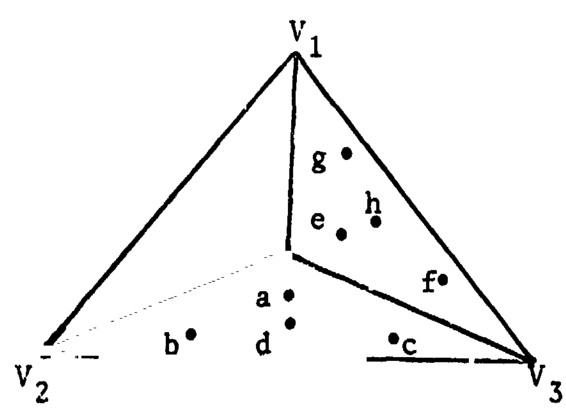

Figure 2

Table 1. Integration of ρ_1 over T_1. $\rho_1 = \frac{1}{6}r_1[6r_2r_3+r_1(5r_1-3)]$

Point	Barycentric Coordinates	Value of ρ_1	Multiplier [See Eq. (4)]	Product
a	$\frac{1}{5}, \frac{2}{5}, \frac{2}{5}$	$\frac{7}{3\cdot 125}$	25	189/(81·5)
b	$\frac{1}{15}, \frac{10}{15}, \frac{4}{15}$	$\frac{4}{81\cdot 5}$	25	100/(81·5)
c	$\frac{1}{15}, \frac{4}{15}, \frac{10}{15}$	$\frac{4}{81\cdot 5}$	25	100/(81·5)
d	$\frac{1}{9}, \frac{4}{9}, \frac{4}{9}$	$\frac{37}{81\cdot 27}$	-27	-185/(81·5)
			Sum:	204/(81·5)=68/(27·5)

Table 2. Integration of ρ_1 over T_2. $\rho_1 = \frac{1}{6}r_2^2(-r_2+3r_3)$

Point	Barycentric Coordinates	Value of ρ_1	Multiplier [See Eq. (4)]	Product
e	$\frac{2}{5}, \frac{1}{5}, \frac{2}{5}$	$\frac{1}{2\cdot 3\cdot 25}$	25	135/(2·81·5)
f	$\frac{4}{15}, \frac{1}{15}, \frac{10}{15}$	$\frac{29}{2\cdot 81\cdot 125}$	25	29/(2·81·5)
g	$\frac{10}{15}, \frac{1}{15}, \frac{4}{15}$	$\frac{11}{2\cdot 81\cdot 125}$	25	11/(2·81·5)
h	$\frac{4}{9}, \frac{1}{9}, \frac{4}{9}$	$\frac{11}{2\cdot 3^7}$	-27	-55/(2·81·5)
			Sum:	120/(2·81·5)= $\frac{4}{27}$

The arithmetic involved in applying Eq. (4) to ρ_1 in subtriangles T_1 and T_2 is summarized in Tables 1 and 2. The integral over T_3 will be the same as that over T_2.

Let A denote the area of the whole triangle. Then each subtriangle T_i has area $A/3$. It follows that

$$\int\rho_1 = \frac{A}{3\cdot 48}\left[\frac{68}{27\cdot 5} + \frac{2\cdot 4}{27}\right] = \frac{A}{180}$$

The integrals $\int\rho_2$ and $\int\rho_3$ have the same value as $\int\rho_1$. Thus the mean value of any one of these functions ρ_i is given by

(10) $$\bar{\rho} = 1/180 = 0.0055555555...$$

The result given in Eq. (10) has been corroborated by a numerical integration of ρ_1 which gave agreement in the first seven significant digits. See Section 8 for more details on the numerical integration.

5. Integrals of the cardinal functions α_i, β_i, and γ_i

Referring to p. 30 of Ref. 1, introduce $\beta = \frac{1}{2}\tilde{\beta}$ and $\gamma = \frac{1}{2}\tilde{\gamma}$ so that the final equation for w can be written simply as

$$w := \sum_{i=1}^{3} [f_i\alpha_i + f_{x,i}\beta_i + f_{y,i}\gamma_i]$$

Use a bar over the symbols p_i, q_i, α_i, β_i, and γ_i to denote the average value of each of these functions over the triangle. For example, $\bar{p}_i = \frac{1}{A}\int p_i$. In the case of the functions ρ_i the integral over the triangle is independent of i. Thus we simply write $\bar{\rho} = \frac{1}{A}\int\rho_i$ for $i=1,2,3$. From Eq. (9) and (10) respectively we have $\bar{\rho} = (593-60\pi^2)/180$ if ρ_i is the Zienkiewicz rational function and $\bar{\rho} = 1/180$ if ρ_i is the Clough and Tocher piecewise cubic function.

Let

$$\lambda_i = (\ell_{i+1}^2 - \ell_{i-1}^2)/\ell_i^2$$

Then, using the notation and equations of Ref. 1, p. 30, and Eqs. (5-7) of the present paper, we obtain

$$\int \tilde{g}_i = 3\lambda_i \int p_i = 3\lambda_i \bar{\rho} A$$

$$\bar{p}_i = 3\lambda_i \bar{\rho} + \frac{1}{12}$$

$$\bar{q}_i = 3\lambda_i \bar{\rho} - \frac{1}{12}$$

$$\bar{\alpha}_i = \frac{1}{3} + 3(\lambda_{i-1} - \lambda_{i+1})\,\bar{\rho}$$

$$\bar{\beta}_i = \frac{1}{2A}\int \tilde{\beta} = \frac{1}{2}\,[u_{i-1}\bar{p}_{i-1} + u_{i+1}\bar{q}_{i+1}]$$

$$\bar{\gamma}_i = \frac{1}{2}\,[v_{i-1}\bar{p}_{i-1} + v_{i+1}\bar{q}_{i+1}]$$

Finally we have

(11) $$\int w = A \sum_{i=1}^{3} [f_i\bar{\alpha}_i + f_{x,i}\bar{\beta}_i + f_{y,i}\bar{\gamma}_i]$$

Note, as a matter of interest, that $\sum_{i=1}^{3} \bar{\alpha}_i = 1$, independent of the value of $\bar{\rho}$.

6. Example: An isosceles right triangle

Consider the isosceles right triangle of Fig. 3. whose essential geometric parameters are given in Table 3. This triangle has area $A = a^2/2$.

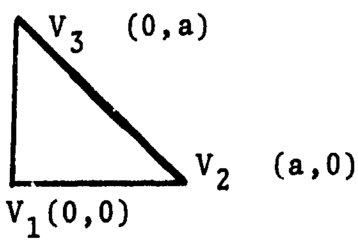

Figure 3

Table 3. Parameters of the triangle of Fig. 3.

i	u_i	v_i	ℓ_i	λ_i
1	-a	a	$2^{1/2}a$	0
2	0	-a	a	-1
3	a	0	a	1

Choosing the functions ρ_i to be the Clough-Tocher piecewise cubic functions we have $\overline{\rho} = 1/180$. Using the formulas of Section 5, we obtain the values in Table 4.

Table 4. Mean values of auxiliary and cardinal functions

i	$\overline{p}_i$	$\overline{q}_i$	$\overline{\alpha}_i$	$\overline{\beta}_i$	$\overline{\gamma}_i$
1	5/60	-5/60	22/60	6a/120	6a/120
2	4/60	-6/60	19/60	-9a/120	5a/120
3	6/60	-4/60	19/60	5a/120	-9a/120

These coefficients should provide exact quadrature for any quadratic function. As a specific example let $a=120$ and consider the function $f(x,y) = x^2$.

By the formula of Eq.(3) the integral of f over the triangle in question is given by

(12) $$I = \frac{1}{3} \frac{120^2}{2} \{60^2 + 60^2\} = 10\cdot 120^3$$

Alternatively, using Table 4, we note that only two of the nodal values are non-zero, namely $f(120,0) = 120^2$ and $f_x(120,0) = 240$. Thus using Eq.(11) and $\overline{\alpha}_2$ and $\overline{\beta}_2$ from Table 4, we compute the integral

$$J = \frac{120^2}{3}\left[\frac{120^2\cdot 19}{60} - 9\cdot 240\right]$$

$$= 10\cdot 120^3$$

which is in agreement with Eq.(12).

7. Example: An equilateral triangle

Consider the equilateral triangle of Fig. 4, whose essential geometric parameters are given in Table 5. This triangle has area $A = 3^{1/2}a^2/4$.

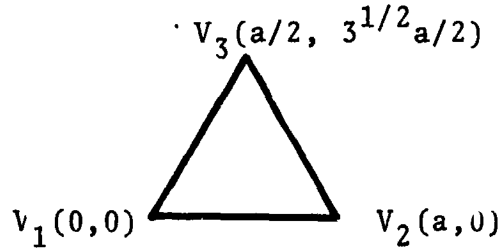

Figure 4

Table 5. Parameters of the triangle of Fig. 4.

i	u_i	v_i	ℓ_i	λ_i
1	$-a/2$	$3^{1/2}a/2$	1	0
2	$-a/2$	$-3^{1/2}a/2$	1	0
3	a	0	1	0

Since all λ_i are zero the functions ρ_i will not enter into the problem. Using the formulas of Section 5, we obtain the values in Table 6.

Table 6. Mean values of auxiliary and cardinal functions

i	$\overline{p}_i$	$\overline{q}_i$	$\overline{\alpha}_i$	$\overline{\beta}_i$	$\overline{\gamma}_i$
1	1/12	-1/12	1/3	a/16	$3^{1/2}a/48$
2	1/12	-1/12	1/3	-a/16	$3^{1/2}a/48$
3	1/12	-1/12	1/3	0	$-2\cdot 3^{1/2}a/48$

These coefficients should provide exact quadrature for any quadratic function. As a specific example let $a = 16$ and consider the function $f(x,y) = y^2$.

By the formula of Eq.(3) the integral of f over the triangle in question is given by

$$(13)\qquad I = \frac{A}{3}\{(4\cdot 3^{1/2})^2 + (4\cdot 3^{1/2})^2\}$$

$$= 32A = 2^{11}\cdot 3^{1/2}$$

Alternatively, using Table 6, we note that only two of the nodal values are non-zero, namely $f(8,8\cdot 3^{1/2}) = 3\cdot 64$ and $f_y(8,8\cdot 3^{1/2}) = 16\cdot 3^{1/2}$.

Thus using Eq.(11) and $\overline{\alpha}_3$ and $\overline{\gamma}_3$ from Table 6, we compute the integral

$$J = A \left\{\frac{3\cdot 64}{3} - \frac{16\cdot 3^{1/2}\cdot 2\cdot 3^{1/2}\cdot 16}{48}\right\}$$

$$= 32A = 2^{11} \cdot 3^{1/2}$$

which is in agreement with Eq.(13).

8. Corroboration of results by numerical integration

As a check against possible blunders in the determination of values of $\bar{\rho}$ in Sections 3 and 4, direct numerical integration was used to obtain estimates of $\bar{\rho}$ for the two different definitions of $\bar{\rho}$.

The quadrature subroutine used was the JPL Univac 1108 library subroutine RMB1/RMB2 (Ref. 3). This subroutine effects quadrature over a two-dimensional region (a triangle in the present case) by making an appropriate sequence of calls to the library's one-dimensional quadrature subroutine ROMBS/ROM2. This system of subroutines was developed by W. R. Bunton and M. R. Diethelm of JPL using an adaptive algorithm based on Romberg quadrature.

Results of these numerical integrations are summarized in Tables 7 and 8. For each of the two functions being studied, the analytically determined value of $\bar{\rho}$ is known. For each of the two functions the numerical integration was done five times with five different values of the accuracy tolerance parameter. Letting $\bar{\rho}$ denote the value obtained from the ith numerical integration, values of the difference $\bar{\rho}_i - \bar{\rho}$ are given in Tables 7 & 8.

The results of these numerical integrations are consistent with the analytically determined values of $\bar{\rho}$.

Table 7. Numerical integration of the Zienkiewicz rational function.
$\bar{\rho} = (593-60\pi^2)/180 \doteq 0.00457\ 63107\ 47992$

i	Absolute accuracy requested	Number of evaluations of integrand	$\bar{\rho}_i - \bar{\rho}$
1	10^{-3}	113	-0.00000 14834
2	10^{-4}	113	-0.00000 14834
3	10^{-5}	215	+0.00000 01164
4	10^{-6}	263	+0.00000 00059
5	10^{-7}	1413	+0.00000 00007

Table 8. Numerical integration of the Clough-Tocher piecewise cubic function.
$\bar{\rho} = 1/180 = 0.00555\ 55555\ \ldots$

i	Absolute accuracy requested	Number of evaluations of integrand	$\bar{\rho}_i - \rho$
1	10^{-3}	133	-0.00000 31391
2	10^{-4}	141	-0.00000 59111
3	10^{-5}	231	-0.00000 00555
4	10^{-6}	681	+0.00000 01064
5	10^{-7}	895	-0.00000 00055

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NASA—JPL—Coml., L.A., Calif