FIRST ORDER IMPULSIVE SOLUTIONS

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ABSTRACT

This paper gives a mathematically rigorous derivation of first-order corrections to multi-impulse approximations to the solutions to space flight optimization problems with bang-bang control. The rocket is subject to an inverse square gravitational force and to a thrust force with constant magnitude. The mass decreases linearly with time. It is assumed that an optimal impulsive solution has been obtained for a problem with given initial and final conditions. The method may then be used to obtain first-order corrections to the initial values of the costate variables. Indications are given on how the theory may be extended to higher order corrections. The theory is applied to intercept and rendezvous problems.

FIRST ORDER IMPULSIVE SOLUTIONS

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Introduction. This paper is concerned primarily with first-order corrections to impulsive approximations to optimal space flight maneuvers. Theoretically, higher order corrections can be obtained in an analogous manner. An example problem is the determination of the ignition and burnout times and the time-histories of the thrust control angles which define a minimum time, exoatmospheric maneuver from a given initial orbit to a given final orbit. Several coast and powered phases may be included. Only problems with constant thrust magnitude, F, and fuel burning rate magnitude, β, on each thrust arc are considered.

The impulsive solution is defined[1] to be the limit of the bounded thrust solutions as β increases without bound, where $F = c\beta$ and c is the constant exhaust velocity. The corrections are in terms of the parameter $\varepsilon = 1/\beta$. They are the first and higher order terms of the Taylor series expansions of the variables about $\varepsilon = 0$.

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The problem of obtaining corrections to impulsive solutions has been studied independently by this author and several men[2-6] associated with Princeton University. Refs (2) and (4) are concerned with the problem of constant F/m, whereas (3) and (5) extend the work to the problem considered in this paper. The latter works make use of expansions in terms of two parameters;namely, initial thrust acceleration and the rocket jet exhaust velocity. Ref. (6) considers applications to low-thrust mission analysis.

The present paper is an extension of Ref. (7) to cover general initial conditions as well as final conditions. Moreover, the necessary conditions are given in a more explicit form. Both papers emphasize mathematical rigor in the development of the corrections.

State and Costate Equations. The basic problem is the determination of the optimal vacuum flight of a space vehicle from a prescribed set of initial conditions to a given set of final conditions. The flight has at least one coasting phase and is to be optimized with respect to payload.

The equations of motion are

$$\ddot{y} = \frac{F}{m} L(\lambda) + G(t,y), \qquad \dot{m} = -\beta$$

where $L(\lambda)$ is the optimal steering vector $\lambda/|\lambda|$. The parameters F and β are zero on coast arcs. The costate vector λ is the solution to the costate equations

$$\ddot{\lambda} = Q(t,y,\lambda).$$

Let t_k and $\bar{t}_k$ be the initial and final times on the k-th thrust arc for $k = 1,2,\ldots,N$. Let κ be the so-called "switching function" satisfying the equation

$$\dot{\kappa} = \frac{c}{m} U(\lambda, \dot{\lambda})$$

where $U = \lambda^T \dot{\lambda} / ||\lambda||$. The necessary conditions of optimality include the condition $\kappa = 0$ at times t_k and $\bar{t}_k$ for $k = 1,2,\ldots,N$ except sometimes for time t_1 and time $\bar{t}_N$ if they correspond to the initial time t_o and the final time t_f, respectively.

Impulsive Solutions. Assume that for each non-zero value of ϵ in some neighborhood of zero there is a solution $y(t,\epsilon)$, $\dot{y}(t,\epsilon)$, $\lambda(t,\epsilon)$, $\dot{\lambda}(t,\epsilon)$, $\kappa(t,\epsilon)$, $m(t,\epsilon)$, $t_k(\epsilon)$, $\bar{t}_k(\epsilon)$, $t_f(\epsilon)$ to the boundary condition problem. Here y, for example, is considered to be a function of the two arguments, t and ϵ. It is also assumed that the impulsive solution exists; i.e., that the variables approach finite limits as ϵ approaches zero. In Ref. (1) and later in this paper it is shown that, in the limit, $\bar{t}_k = t_k$, $y(\bar{t}_k,0) = y(t_k,0)$, $\lambda(\bar{t}_k,0) = \lambda(t_k,0)$, $\dot{\lambda}(\bar{t}_k,0) = \dot{\lambda}(t_k,0)$,

$$\dot{y}(\bar{t}_k,0) = \dot{y}(t_k,0) + c[\ell n \frac{m(t_k,0)}{m(\bar{t}_k,0)}] L[\lambda(t_k,0)]$$

$$\Delta V_k(0) = c \ \ell n \frac{m(t_k,0)}{m(\bar{t}_k,0)}$$

where $$\Delta V_k = |\dot{y}(\bar{t}_k,0) - \dot{y}(t_k,0)|.$$

The multi-point boundary-condition problem for the impulsive case involves the choice of y_o, $\dot{y}_o, \lambda_o, \dot{\lambda}_o, \kappa_o, t_1, t_2, \ldots, t_N$, and t_f such that $\kappa(t_k,0) = 0$ for $k = 1,2,\ldots,N$; such that $\dot{\kappa}(t_k,0) = 0$ for $k = 1,2,\ldots,N$ (except sometimes for $k = 1$ and/or $k = N$); and such that the given initial and final boundary conditions, including transversality conditions and a scaling condition upon λ, are all satisfied.

Notation. Let $y_k(\varepsilon) = y[t_k(\varepsilon),\varepsilon]$, $\bar{y}_k(\varepsilon) = y[\bar{t}_k(\varepsilon),\varepsilon]$, $y_\varepsilon = \partial y/\partial\varepsilon$, $y_t = \dot{y} = \partial y/\partial t$, $y_{k\varepsilon} = y_{\varepsilon k} + t_{k\varepsilon}\dot{y}_k$, etc.
Also let $L^*(t,\varepsilon) = L[\lambda(t,\varepsilon)]$, $G^*(t,\varepsilon) = G[t(\varepsilon),y(t,\varepsilon)]$, and so on.

The symbol $\sigma(\varepsilon^n)$ will denote a finite summation of terms of the form

$$a(\varepsilon)\varepsilon^{n_1}[\Delta t_k(\varepsilon)]^{n_2}[U_k^*(\varepsilon)]^{n_3}$$

where n_1, n_2, n_3, and n are non-negative integers such that $n_1+n_2+n_3 \geq n$ and $a(\varepsilon)$ is bounded within some neighborhood of $\varepsilon = 0$. Observe that $U_k^*(0) = 0$ whenever $\dot{\kappa}_k(0) = 0$. The symbol $\tilde{\sigma}(\varepsilon^n)$ is used instead of $\sigma(\varepsilon^n)$ when each coefficient $a(\varepsilon)$ in the summation is a constant.

The General Procedure. Suppose for example that one of the constraints is $y_f = a$. Then $y_f(\varepsilon) \equiv a$ is an identity in ε. Therefore, $y_{f\varepsilon}(\varepsilon) \equiv 0$, $y_{f\varepsilon\varepsilon}(\varepsilon) \equiv 0$, etc. If y_f corresponds to $\bar{y}_N$, then $\bar{y}_{N\varepsilon} \equiv 0$. This paper gives an expression which relates $\bar{y}_{k\varepsilon}(0)$ to $y_{k\varepsilon}(0)$. Moreover, $y_{k+1,\varepsilon}$ may be written in terms of $\bar{y}_{k\varepsilon}$, $\dot{\bar{y}}_{k\varepsilon}$, $\bar{t}_{k\varepsilon}$, and $t_{k+1,\varepsilon}$. Ultimately the condition $y_{f\varepsilon}(0) = 0$ can be expressed as a linear equation in the unknowns $\lambda_{o\varepsilon}$, $\dot{\lambda}_{o\varepsilon}$, $\bar{t}_{1\varepsilon}(0)$, etc. Similarly the other boundary conditions lead to linear equations. The linear equations may be solved for the unknown derivatives. Once $\lambda_{o\varepsilon}(0)$ for example has been calculated, one may correct the impulsive value $\lambda_o(0)$ by adding the first-order correction $\varepsilon\lambda_{o\varepsilon}(0)$ with $\varepsilon = 1/\beta$.

Problems arise in the applications of the procedure just described. These are elaborated upon below.

Let $\Delta t_k = \bar{t}_k - t_k$ and $\Delta m_k = \bar{m}_k - m_k$. Since $\Delta m_k = -\Delta t_k/\varepsilon$, L'Hospital's rule gives the equation $\Delta t_{k\varepsilon}(0) = -\Delta m_k(0)$. In general,

$$\frac{d^n \Delta m_k}{d\varepsilon^n} = -\frac{1}{n+1}\,\frac{d^{n+1}\Delta t_k}{d\varepsilon^{n+1}}$$

at $\varepsilon = 0$. Thus, for example, once $t_{k\varepsilon}(0)$ is known, $\bar{t}_{k\varepsilon}(0)$ can be easily calculated. Rather than considering $t_{k\varepsilon}(0)$ and $\bar{t}_{k\varepsilon}(0)$ to be the unknowns on the k-th thrust arc, the unknowns will be $t_{k\varepsilon}$ (or $\bar{t}_{k\varepsilon}$) and $\Delta m_{k\varepsilon}$.

It will become evident that on any thrust arc for which $\dot{\kappa}_k(0) = 0$, the conditions $\kappa_{k\varepsilon}(0) = 0$ and $\bar{\kappa}_{k\varepsilon}(0) = 0$ are dependent. In order to obtain an independent condition, the condition $\bar{\kappa}_{k\varepsilon\varepsilon}(0) = 0$ will be expanded and it will be discovered that the condition will involve only first derivatives with respect to ε. In general, the condition $d^n\,\bar{\kappa}_k/d\varepsilon^n = 0$ (or $d^n\kappa_k/d\varepsilon^n = 0$) will be replaced by $d^{n+1}\,\bar{\kappa}_k/d\varepsilon^{n+1} = 0$.

Derivatives over Coast Arcs. In general

$$y_{k+1,\varepsilon} = t_{k+1,\varepsilon}y_{k+1} + [\partial y_{k+1}/\partial \bar{y}_k](\bar{y}_{k\varepsilon}-\bar{t}_{k\varepsilon}\dot{\bar{y}}_k) + [\partial y_{k+1}/\partial \dot{\bar{y}}_k](\dot{\bar{y}}_{k\varepsilon}-\bar{t}_{k\varepsilon}\bar{G}_k^*)$$

$$\dot{y}_{k+1,\varepsilon} = t_{k+1,\varepsilon}G_{k+1}^* + [\partial \dot{y}_{k+1}/\partial \bar{y}_k](\bar{y}_{k\varepsilon}-\bar{t}_{k\varepsilon}\dot{\bar{y}}_k) + [\partial \dot{y}_{k+1}/\partial \dot{\bar{y}}_k](\dot{\bar{y}}_{k\varepsilon}-\bar{t}_{k\varepsilon}G_k^*)$$

$$\lambda_{k+1,\varepsilon} = t_{k+1,\varepsilon}\dot{\lambda}_{k+1} + [\partial \lambda_{k+1}/\partial \bar{y}_k](\bar{y}_{k\varepsilon}-\bar{t}_{k\varepsilon}\dot{\bar{y}}_k) + [\partial \lambda_{k+1}/\partial \dot{\bar{y}}_k](\dot{\bar{y}}_{k\varepsilon}-\bar{t}_{k\varepsilon}G_k^*)$$
$$+ [\partial \lambda_{k+1}/\partial \bar{\lambda}_k](\bar{\lambda}_{k\varepsilon}-\bar{t}_{k\varepsilon}\dot{\bar{\lambda}}_k) + [\partial \lambda_{k+1}/\partial \dot{\bar{\lambda}}_k](\dot{\bar{\lambda}}_k - \bar{t}_{k\varepsilon}\bar{Q}_k^*)$$

$$\dot{\lambda}_{k+1,\varepsilon} = t_{k+1,\varepsilon}Q_{k+1}^* + [\partial \dot{\lambda}_{k+1}/\partial \bar{y}_k](\bar{y}_{k\varepsilon}-\bar{t}_{k\varepsilon}\dot{\bar{y}}_k) + [\partial \dot{\lambda}_{k+1}/\partial \dot{\bar{y}}_k](\dot{\bar{y}}_{k\varepsilon}-\bar{t}_{k\varepsilon}\bar{G}_k^*)$$
$$+ [\partial \dot{\lambda}_{k+1}/\partial \bar{\lambda}_k](\bar{\lambda}_{k\varepsilon}-\bar{t}_{k\varepsilon}\dot{\bar{\lambda}}_k) + [\partial \dot{\lambda}_{k+1}/\partial \dot{\bar{\lambda}}_k](\dot{\bar{y}}_{k\varepsilon}-\bar{t}_{k\varepsilon}\bar{Q}_k^*)$$

$$m_{k+1,\varepsilon} = \bar{m}_{k\varepsilon}$$

where the transition matrices of partial derivatives can often be expressed in closed form as on a Kepler arc or in the case of $G = -\omega^2 y$ where ω is a constant.

The derivatives of κ can be derived from the relationship $\kappa_{k+1} = \bar{\kappa}_{k\varepsilon} + (c/\bar{m}_k)(|\lambda_{k+1}|-|\bar{\lambda}_k|)$. Thus

$$\kappa_{k+1,\varepsilon} = \bar{\kappa}_{k\varepsilon} + (c/\bar{m}_k)\{t_{k+1,\varepsilon}U^*_{k+1} - \bar{t}_{k\varepsilon}\bar{U}^*_k$$

$$+ (1/|\bar{\lambda}_k|)[\lambda^T_{k+1}(\lambda_{k+1,\varepsilon}-t_{k+1,\varepsilon}\dot{\lambda}_{k+1}) - \bar{\lambda}^T_k(\bar{\lambda}_{k\varepsilon}-\bar{t}_{k\varepsilon}\dot{\bar{\lambda}}_k)]\}.$$

If $\bar{\kappa}_k \equiv 0$ and $\kappa_{k+1} \equiv 0$, the latter equation reduces to the simple condition

$$\lambda^T_{k+1}\lambda_{k+1,\varepsilon} - \lambda^T_k\bar{\lambda}_{k\varepsilon} = 0 \qquad (1)$$

applying when $\varepsilon = 0$.

Integrals which Approach Zero. Consider a function $f(\lambda,\dot{\lambda},y,\dot{y},m,t)$ which has continuous partial derivatives and such that $f^*(t,\varepsilon) = f[\lambda(t,\varepsilon),\dot{\lambda}(t,\varepsilon),y(t,\varepsilon),\dot{y}(t,\varepsilon),m(t,\varepsilon),t]$ is continuous and bounded in the interval $[t_k(\varepsilon),\ \bar{t}_k(\varepsilon)]$ within some neighborhood of $\varepsilon = 0$. Let

$$I(\varepsilon) = \int_{t_k(\varepsilon)}^{\bar{t}_k(\varepsilon)} f^*(t,\varepsilon)dt$$

By a mean value theorem for integrals,

$$I(\varepsilon) = f^*[\tilde{t}(\varepsilon),\varepsilon] \int_{t_k(\varepsilon)}^{\bar{t}_k(\varepsilon)} dt = \Delta t_k(\varepsilon)f^*[\tilde{t}(\varepsilon),\varepsilon]$$

For some $\tilde{t}$ in the interval $[t_k,\bar{t}_k]$. Since f^* is bounded, $I(\varepsilon) = o(\varepsilon)$.

Now let

$$J(\varepsilon) = \int_{t_k}^{\bar{t}_k(\varepsilon)} \int_{t_k(\varepsilon)}^{\tau} f^*(t,\varepsilon)dt\, d\tau$$

By the mean value theorem for double integrals,

$$J(\varepsilon) = f^*(\tilde{t},\varepsilon) \int_{t_k}^{\bar{t}_k} \int_{t_k}^{\tau} dt \; d\tau = \frac{1}{2}\Delta t_k^2 f^*(\tilde{t},\varepsilon) = \sigma(\varepsilon^2)$$

for some $\tilde{t}$ in the interval $[t_k, \bar{t}_k]$.

From the rules for differentiations under an integral sign it follows that

$$\frac{d}{d\varepsilon} I(\varepsilon) = \bar{t}_{k\varepsilon} \bar{f}_k^* - t_{k\varepsilon} f_k^* + \int_{\tau_k}^{\bar{t}_k} f_\varepsilon^* \, dt$$

where

$$\int_{t_k}^{\bar{t}_k} f_\varepsilon^* \, dt = \int_{t_k}^{\bar{t}_k} (f_\lambda^{*T} \lambda_\varepsilon + f_{\dot{\lambda}}^{*T} \dot{\lambda}_\varepsilon + f_y^{*T} y_\varepsilon + f_{\dot{y}}^{*T} \dot{y}_\varepsilon + f_m^* m_\varepsilon) dt$$

If it is assumed that the components of f_λ^* and the other partial derivatives are bounded in some neighborhood of $\varepsilon = 0$, then the mean value theorem implies that

$$\int_{t_k}^{\bar{t}_k} f_\varepsilon^* dt = [f_\lambda^{*T}(\varepsilon\lambda_\varepsilon) + f_{\dot{\lambda}}^{*T}(\varepsilon\dot{\lambda}_\varepsilon) + f_y^{*T}(\varepsilon y_\varepsilon) + f_{\dot{y}}^{*T}(\varepsilon\dot{y}_\varepsilon) + f_m^*(\varepsilon m_\varepsilon)]_{t=\tilde{t}} \; \frac{\Delta t_k}{\varepsilon}$$

where $\tilde{t}$ is in $[t_k, \bar{t}_k]$. It must now be argued that the functions $\varepsilon\lambda_\varepsilon[\tilde{t}(\varepsilon),\varepsilon]$. etc. are bounded for $\varepsilon \neq 0$ within some neighborhood of $\varepsilon = 0$. For example, it will be shown that $\varepsilon m_\varepsilon$ is bounded. On a thrust arc $m_\varepsilon = 1/\varepsilon^2$. Therefore

$$\varepsilon m_\varepsilon[\tilde{t}(\varepsilon),\varepsilon] = \varepsilon m_\varepsilon[t_k(\varepsilon),\varepsilon] + \varepsilon\int_{t_k(\varepsilon)}^{\tilde{t}(\varepsilon)} \frac{1}{\varepsilon^2} dt = \varepsilon m_\varepsilon[t_k(\varepsilon),\varepsilon] + \frac{\tilde{t}(\varepsilon) - t_k(\varepsilon)}{\varepsilon}$$

But $[\tilde{t}(\varepsilon) - t_k(\varepsilon)]/\varepsilon \leq \Delta t_k(\varepsilon)$ where $\lim_{\varepsilon \to 0} [\Delta t_k(\varepsilon)/\varepsilon] = \Delta t_{k\varepsilon}(0)$. Therefore $(t - t_k)/\varepsilon$

is bounded within some neighborhood of $\varepsilon = 0$. Also $\varepsilon m_\varepsilon(t_k, \varepsilon)$ is bounded provided $\varepsilon m_\varepsilon(\bar{t}_{k-1}, \varepsilon)$ is bounded since $\varepsilon m_\varepsilon$ is constant over a coast arc. Eventually the boundedness of $\varepsilon m_\varepsilon[t_k, \varepsilon]$ depends upon whether or not $\varepsilon m_\varepsilon(t_o, \varepsilon)$ is bounded; but $\varepsilon m_\varepsilon(t_o, \varepsilon)$ is zero. Hence $\varepsilon dI/d\varepsilon = \sigma(\varepsilon)$.

Similarly it can be argued that $\varepsilon^n \partial^n m/\partial\varepsilon^n$, $\varepsilon^n \partial^n y/\partial\varepsilon^n$, etc. are bounded within some neighborhood of $\varepsilon = 0$ for $n = 1,2,\ldots$. The general idea employed is that factors of ε in the denominator of the terms in the equations of variation are removed by multiplying by factors of ε. Finally it can be concluded that $\varepsilon^n d^n I/d\varepsilon^n = \sigma(\varepsilon)$ and $\varepsilon^{n-1} d^n J/d\varepsilon^n = \sigma(\varepsilon)$.

Now it is possible to define the symbol $\theta(\varepsilon^m)$, $m \geq 1$, denoting a function $\partial(\varepsilon^m)$, a function $\tilde{\sigma}(\varepsilon^{m-1})\varepsilon^n d^n I/d\varepsilon^n$ $(n = 0,1,\ldots)$, a function $\tilde{\sigma}(\varepsilon^{m-1})\varepsilon^{n-1} d^n J/d\varepsilon^n$ $(n = 0,1,2\ldots)$, or a sum of such functions. Employing the conclusions of the preceding paragraph it can be shown that

$$\frac{d^p}{d\varepsilon^p}\,\theta(\varepsilon^m) = \theta(\varepsilon^{m-p})$$

for $m > p > 0$ and that $\theta(\varepsilon) = \sigma(\varepsilon)$. Observe that one can not state with assurance that $\frac{d}{d\varepsilon}\,\sigma(\varepsilon^m) = \sigma(\varepsilon^{m-1})$ because $\sigma(\varepsilon)$ has a factor $a(\varepsilon)$ which is bounded within some neighborhood of $\varepsilon = 0$ but whose derivatives may not exist. Little is known about $a(\varepsilon)$ because it may correspond to a function evaluated at an indeterminate time $\tilde{t}(\varepsilon)$.

Changes over Thrust Arcs. In this section expressions of the form $\bar{y}_k(\varepsilon) = y_k(\varepsilon) + Y_k(\varepsilon) + \theta(\varepsilon^3)$, $\dot{\bar{y}}_k(\varepsilon) = \dot{y}_k(\varepsilon) + \dot{Y}(\varepsilon) + \theta(\varepsilon^3), \ldots$ will be derived for the non-impulsive case. For example, the change in y over a thrust arc will be expressed as some function $Y_k(\varepsilon)$ plus a second-order term $\theta(\varepsilon^2)$. Similar expressions will be obtained for the first derivatives with respect to ε. The expressions for the higher derivatives, though complicated, can be obtained similarly.

The first step is to find expressions for the integrals of G^*. Since $\dot m = -1/\varepsilon$, repeated integration by parts gives

$$\int_{t_k}^{\bar t_k} G^* dt = -\varepsilon [mG^*]_{t_k}^{\bar t_k} + \varepsilon \int_{t_k}^{\bar t_k} mG^*_t\, dt$$

$$= -\varepsilon [mG^*]_{t_k}^{\bar t_k} - \frac{1}{2}\varepsilon^2 [m^2 G^*_t]_{t_k}^{\bar t_k} - \frac{1}{4} c\varepsilon^2 [m^2 G^*_y L^*]_{t_k}^{\bar t_k} + \theta(\varepsilon^3)$$

and so on. Since the first derivatives will be of primary concern, ordinarily the expansion will only be carried out to the second-order term $\theta(\varepsilon^2)$. The double integral is simply

$$\int_{t_k}^{\bar t_k} \int_{t_k}^{\tau} G^* dt\, d\tau = \frac{1}{2}\varepsilon^2 [m^2 G^*]_{t_k}^{\bar t_k} + \varepsilon \Delta t_k m_k G^*_k + \theta(\varepsilon^3)$$

Now the changes in y and $\dot y$ over a thrusting phase will be examined. Since $\ddot y = (c\beta/m)L + G$,

$$\Delta y_k = \Delta t_k \dot y_k + \frac{c}{\varepsilon} \int_{t_k}^{\bar t_k} \int_{t_k}^{\tau} \frac{L^*}{m}\, dt\, d\tau + \int_{t_k}^{\bar t_k} \int_{t_k}^{\tau} G^* dt\, d\tau$$

$$= \Delta t_k \dot y_k - c\varepsilon \bar m_k \left(\ln \frac{m_k}{\bar m_k}\right) L^*_k - c\varepsilon [mL^*]_{t_k}^{\bar t_k} + \int_{t_k}^{\bar t_k} \int_{t_k}^{\tau} G^*\, dt\, d\tau$$

$$\Delta \dot y_k = \frac{c}{\varepsilon} \int_{t_k}^{\bar t_k} \frac{L^*}{m}\, dt + \int_{t_k}^{\bar t_k} G^*\, dt$$

$$= c\left(\ln \frac{m_k}{\bar m_k}\right) L^*_k + c\varepsilon m_k \left(\ln \frac{m_k}{\bar m_k}\right) L^*_{t_k} + c\varepsilon [mL^*_t]_{t_k}^{\bar t_k} + \int_{t_k}^{\bar t_k} G^*\, dt + \theta(\varepsilon^3)$$

Since $\ddot\lambda = Q(t,y,\lambda)$,

$$\Delta\lambda_k = \Delta t_k \dot\lambda_k + \int_{t_k}^{\bar t_k} \int_{t_k}^{\tau} Q^* dt\, d\tau$$

$$= \Delta t_k \dot\lambda_k + \theta(\varepsilon^2)$$

$$\Delta\dot\lambda_k = \int_{t_k}^{\bar t_k} Q^*\, dt$$

$$= -\ \varepsilon[mQ^*]_{t_k}^{\bar{t}_k} + \theta(\varepsilon^2)\ .$$

Since $\dot{\kappa} = \frac{c}{m} U(\lambda,\dot{\lambda})$,

$$\Delta\kappa_k = c \int_{t_k}^{\bar{t}_k} \frac{U^*}{m}\, dt = c\varepsilon U_k^* \ \ell n\ \frac{m_k}{\bar{m}_k} + c\varepsilon^2 U_{t_k}^*\ m_k\ \ell n\ \frac{m_k}{\bar{m}_k} + \theta(\varepsilon^3).$$

An alternative procedure employing series expansions in terms of ε leads to the same results. However, it is difficult or impossible to prove, in a logically rigorous way, that such series converge or that they can be differentiated.

Changes over Thrust Arcs in the Limit. Letting ε approach zero in the expressions obtained in the preceding equations, one obtains $\Delta y_k(0) = 0$, $\Delta\dot{y}_k(0) = c\ \ell n(m_k/\ \bar{m}_k)L_k^*$, $\Delta\lambda_k(0) = 0$, $\Delta\dot{\lambda}_k(0) = 0$, and $\Delta\kappa_k(0) = 0$. Since $|L_k^*| = 1$, it follows as in Ref. (1) that at $\varepsilon = 0$, $\Delta V_k = c\ \ell n(m_k/\ \bar{m}_k)$, $\bar{m}_k = m_k \exp(-\Delta V_k/c)$, and $\Delta\dot{y}_k = \Delta V_k L_k^*$.

Taking derivatives of the expressions obtained for Δy_k, etc. in the preceding section and taking the limit as ε approaches zero, one obtains

$$\Delta y_{k\varepsilon}(0) = -\Delta m_k \dot{y}_k - \bar{a}_k L_k^*$$

$$\Delta\dot{y}_{k\varepsilon}(0) = \frac{c}{m_k \bar{m}_k} (\Delta m_k m_{k\varepsilon} - m_k \Delta m_{k\varepsilon})L_k^* + \Delta V_k L_{k\varepsilon}^* - \Delta m_k Q_k^* + a_k L_{tk}^*$$

$$\Delta\lambda_{k\varepsilon}(0) = -\Delta m_k \dot{\lambda}_k,\ \Delta\dot{\lambda}_{k\varepsilon}(0) = -\ \Delta m_k Q_k^*,\ \Delta\kappa_{k\varepsilon}(0) = U_k^* \Delta V_k$$

$$\Delta\kappa_{k\varepsilon\varepsilon}(0) = \frac{2c}{m_k \bar{m}_k} U_k^*\ (\Delta m_k m_{k\varepsilon} - m_k \Delta m_{k\varepsilon}) + 2\Delta V_k U_{k\varepsilon}^* + 2a_k U_{tk}^*$$

where $\bar{a}_k = \Delta V_k \bar{m}_k + \Delta m_k c$ and $a_k = \Delta V_k m_k + \Delta m_k c$. Since the direct calculation of a_k and $\bar{a}_k$ involves subtraction of nearly equal numbers, one should employ the series expansions

$$a_k = cm_k(\Delta V_k/c)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!} (\Delta V_k/c)^n$$

$$\bar{a}_k = c\bar{m}_k(\Delta V_k/c)^2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{n+1}{(n+2)!} (\Delta V_k/c)^n$$

If $\dot{\kappa} = 0$, then $U_k^* = 0$ so that $\Delta\kappa_{k\varepsilon} = 0$. hen the condition $\kappa_{k\varepsilon}(0) = 0$ implies $\bar{\kappa}_{k\varepsilon} = 0$. In this case the condition $\bar{\kappa}_{k\varepsilon\varepsilon} = 0$ should be employed rather than one of the conditions, $\bar{\kappa}_{k\varepsilon}(0) = 0$ and $\kappa_{k\varepsilon}(0) = 0$, in obtaining the first-order corrections. The condition $\bar{\kappa}_{k\varepsilon\varepsilon} = 0$ is simply

$$\dot{\lambda}_k^T\lambda_{k\varepsilon} + \lambda_k^T\dot{\lambda}_{k\varepsilon} = -\frac{a_k}{\Delta V_k}(\dot{\lambda}_k^T\dot{\lambda}_k + \lambda_k^T Q_k^*) \qquad (2)$$

Summary of Conditions. The multi-point boundary-condition problem will be summarized as an example for problems in which the initial and final phases of the flight are thrusting phases.

Corresponding to the initial point there is the scaling condition $\lambda_1^T\lambda_1 = 1$ and initial conditions $h(y_1, \dot{y}_1, \lambda_1, \dot{\lambda}_1) = 0$, where h stands for a column vector with five or six components, some of which may represent transversality conditions. (The conditions may be simply expressions of initial values of y and $\dot{y}$.) The vector h has only five components if $\kappa_1 = 0$, in which case condition (2) also applies. As explained earlier the condition $\kappa_{1\varepsilon} = 0$ is not employed. In any case the equation $\bar{\kappa}_{1\varepsilon} = 0$ need not be included explicitly because it can be trivially satisfied by setting $\bar{\kappa}_{1\varepsilon} = 0$. Therefore, corresponding to the initial point (or initial thrust arc), we have the following seven conditions:

$$\left.\begin{array}{l}
\lambda_1^T\lambda_{1\varepsilon} = 0 \\[2ex]
\dfrac{\partial h}{\partial y_1} y_{1\varepsilon} + \dfrac{\partial h}{\partial \dot{y}_1} \dot{y}_{1\varepsilon} + \dfrac{\partial h}{\partial \lambda_1} \lambda_{1\varepsilon} + \dfrac{\partial h}{\partial \dot{\lambda}_1} \dot{\lambda}_{1\varepsilon} = 0 \\[2ex]
\dot{\lambda}_1^T\lambda_{1\varepsilon} + \lambda_1^T\dot{\lambda}_{1\varepsilon} = -\dfrac{a_1}{\Delta V_1}(\dot{\lambda}_1^T\dot{\lambda}_1 + \lambda_1^T Q_1^*) \\
\qquad\qquad \text{(applies if } \kappa_1 = 0\text{)}
\end{array}\right\}$$

There are also the 13 unknowns, $y_{1\varepsilon}$, $\dot{y}_{1\varepsilon}$, $\lambda_{1\varepsilon}$, $\dot{\lambda}_{1\varepsilon}$, $\Delta m_{1\varepsilon}$, corresponding to the initial point.

For each intermediate thrust, conditions (1) and (2) apply and there are two unknowns; namely, $t_{k\varepsilon}$ and $\Delta m_{k\varepsilon}$ on the k-th thrust arc.

Condition (1) with k = N-1, corresponding to the equation $\kappa_{N\varepsilon} = 0$ is $\lambda_N^T \lambda_{N\varepsilon} - \lambda_{N-1}^T \bar{\lambda}_{N-1,\varepsilon} = 0$. This corresponds to the final point.

Corresponding to the final point there are given conditions $g(\bar{t}_N, \bar{y}_N, \dot{\bar{y}}_N, \lambda_N, \dot{\lambda}_N, \bar{\kappa}_N) = 0$ where g has six or seven components, some of which may represent transversality conditions. The vector g has only six components in the case that $\bar{\kappa}_2 = 0$. Then the condition (2) with k = N applies. It corresponds to the condition $\bar{\kappa}_{N\varepsilon\varepsilon} = 0$. In this case the condition $\bar{\kappa}_{2\varepsilon} = 0$ or any equivalent condition must not be employed. Hence, corresponding to the final thrust arc, we have the eight conditions

$$\lambda_N^T \lambda_{N\varepsilon} - \lambda_{N-1}^T \bar{\lambda}_{N-1,\varepsilon} = 0$$

$$\dot{\lambda}_N^T \lambda_{N\varepsilon} + \lambda_N^T \dot{\lambda}_{N\varepsilon} = -\frac{a}{\Delta V_N}\left(\dot{\lambda}_N^T \lambda_N + \lambda_N^T Q_N^*\right)$$

(applies if $\bar{\kappa}_N = 0$)

$$\frac{\partial g}{\partial \bar{y}_N}\bar{y}_{N\varepsilon} + \frac{\partial g}{\partial \dot{\bar{y}}_N}\dot{\bar{y}}_{N\varepsilon} + \frac{\partial g}{\partial \bar{\lambda}_N}\bar{\lambda}_{N\varepsilon} + \frac{\partial g}{\partial \dot{\bar{\lambda}}_N}\bar{\lambda}_{N\varepsilon} = -\Delta V_N \dot{\lambda}_N^T \lambda_N \frac{\partial g}{\partial \bar{\kappa}_N}$$

since $\bar{\kappa}_{N\varepsilon} = \Delta V_N \lambda_N^T \lambda_N$. The two unknowns, $\bar{t}_{N\varepsilon}$ and $\Delta m_{N\varepsilon}$, correspond to the final point. (If one of the conditions is simply $\bar{t}_N$ = given number, then $\bar{t}_{N\varepsilon} = 0$.)

Therefore, for the thrust-coast-thrust problem there are 15 linear equations and 15 unknowns. For each additional intermediate thrust arc there are two additional equations and unknowns. It is understood that $\bar{y}_{N\varepsilon}$, for example, in the above equations will be expressed, by means of equations derived earlier in this paper, in terms of the indicated unknowns.

An Intercept Problem. As an example consider a thrust-coast intercept problem in which the vehicle moves from a given point in state space to a point with given position components. The time of intercept is also specified. In this problem t_o must be identified with t_1. Since $\lambda_1^T \lambda_1 = 1$,

$$\lambda_1^T \lambda_{1\varepsilon} = 0 \tag{3}$$

Since y_2 is fixed, we have $y_{2\varepsilon} = 0$ so that

$$\frac{\partial y_2}{\partial \bar{y}_1} \bar{y}_{\varepsilon 1}^{+} + \frac{\partial y_2}{\partial \dot{\bar{y}}_1} \dot{\bar{y}}_{\varepsilon 1}^{+} = 0$$

But $\bar{y}_{\varepsilon 1}^{+} = \bar{y}_{1\varepsilon} - \bar{t}_{1\varepsilon} \dot{\bar{y}}_1 = \Delta y_{1\varepsilon} + \Delta m_1 \dot{\bar{y}}_1 = -a_1 \lambda_1$ and $\dot{\bar{y}}_{\varepsilon 1}^{+} = \dot{\bar{y}}_{1\varepsilon} - \bar{t}_{1\varepsilon} \ddot{\bar{y}}_1^{+} = \Delta \dot{y}_{1\varepsilon} + \Delta m_1 G_1^* =$

$-\frac{c}{\bar{m}_1} \Delta m_{1\varepsilon} \lambda_1 + \Delta V_1 \lambda_{1\varepsilon} + a_1 (I - \lambda_1 \lambda_1^T) \dot{\lambda}_1$ at $\varepsilon = 0$.

Therefore,

$$-a_1 \frac{\partial y_2}{\partial \bar{y}_1} \lambda_1 + \frac{\partial y_2}{\partial \dot{\bar{y}}_1} \left[-\frac{c}{\bar{m}_1} \Delta m_{1\varepsilon} \lambda_1 + \Delta V_1 \lambda_{1\varepsilon} + a_1 (I - \lambda_1 \lambda_1^T) \dot{\lambda}_1 \right] = 0 \tag{4}$$

Equations (3) and (4) may be written in matrix form as

$$\begin{pmatrix} -\frac{c}{\bar{m}_1} \frac{\partial y_2}{\partial \dot{\bar{y}}_1} \lambda_1 & \Delta V_1 \frac{\partial y_2}{\partial \dot{\bar{y}}_1} \\ 0 & \lambda_1^T \end{pmatrix} \begin{pmatrix} \Delta m_{1\varepsilon} \\ \lambda_{1\varepsilon} \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}$$

where

$$\gamma = a_1 \frac{\partial y_2}{\partial \bar{y}_1} \lambda_1 - a_1 \frac{\partial y_2}{\partial \dot{\bar{y}}_1} (I - \lambda_1 \lambda_1^T) \dot{\lambda}_1$$

Therefore

$$\begin{pmatrix} \Delta m_{1\varepsilon} \\ \\ \lambda_{1\varepsilon} \end{pmatrix} = \begin{pmatrix} -\frac{\bar{m}_1}{c} \lambda_1^T \left(\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right)^{-1} & \frac{\bar{m}_1}{c} \Delta V_1 \\ \\ \frac{1}{\Delta V_1} (I - \lambda_1 \lambda_1^T) \left(\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right)^{-1} & \lambda_1 \end{pmatrix} \begin{pmatrix} \gamma \\ \\ 0 \end{pmatrix} \tag{5}$$

We observe that in general

$$\lambda_2 = \frac{\partial y_2}{\partial \bar{y}_1} \lambda_1 + \frac{\partial y_2}{\partial \dot{\bar{y}}_1} \dot{\lambda}_1$$

In an intercept problem the transversality conditions imply that $\lambda_2 = 0$. Therefore,

$$\dot{\lambda}_1 = -\left(\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right)^{-1} \frac{\partial y_2}{\partial \bar{y}_1} \lambda_1$$

Utilizing the latter equation and equation (5), it may be shown that

$$\Delta m_{1\varepsilon} = \frac{\bar{m}_1 a_1}{c} \lambda_1^T \dot{\lambda}_1$$

$$\lambda_{1\varepsilon} = \frac{2a_1}{\Delta V_1} [(\lambda_1^T \lambda_1)\lambda_1 - \dot{\lambda}_1]$$

at $\varepsilon = 0$. Starting with the requirement that $\lambda_2 = 0$, it may also be shown that

$$\dot{\lambda}_{1\varepsilon} = \left(\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right)^{-1} \left(a_1 \frac{\partial \lambda_2}{\partial \bar{y}_1} \lambda_1 - \frac{\partial \lambda_2}{\partial \dot{\bar{y}}_1} \Delta y_{\varepsilon 1} - \frac{\partial y_2}{\partial \bar{y}_1} \lambda_{1\varepsilon}\right)$$

Let $t_1 = 0$, $t_2 = 380$, $\mu = .388 \times 10^{15}\ m^3/sec^2$, $c = 4100$ m/sec, $\beta = 22$ kg-sec/m, $m_1 = .168920 \times 10^5$ kg-sec^2/m, $y_1^T = (.287253 \times 10^7$ m, $.590785 \times 10^7$ m, $.777376 \times 10^5$ m), $\dot{y}_1^T = (-7326.35$ m/sec, 3219.14 m/sec, -474.472 m/sec), $y_2^T = (0, .655630 \times 10^7$ m, 0). A solution to the boundary condition problem gives $\bar{t}_1 = 86.00$ sec, $\lambda_1^T = (.6618, .3727, .6505)$, $\dot{\lambda}_1^T = (-.1725 \times 10^{-2}, -.1198 \times 10^{-2}, -.1596 \times 10^{-2})$.

The impulsive solution yields $\bar{t}_1 = -\Delta m_1/\beta = 76.64$ $\lambda_1^T = (.6631, .3557, .6587)$, $\dot{\lambda}_1^T = (-.1726 \times 10^{-2}, -.1149 \times 10^{-2}, -.1617 \times 10^{-2})$. A first-order correction to the impulsive solution gives $\bar{t}_1 = 84.04$, $\lambda_1^T = (.6623, .3726, .6503)$, $\dot{\lambda}_1^T = (-.1726 \times 10^{-2}, -.1198 \times 10^{-2}, -.1596 \times 10^{-2})$. The error in λ_1^T, for example, has been reduced from (.00121, .01701, .00823) to (.00041, .00004, .00013).

A Rendezvous Problem. Consider a thrust-coast-thrust rendezvous problem in which the vehicle moves from a given point in state space to a point with given position and velocity components. The time of rendezvous is specified. For simplicity we take $G = -g$ where g is constant. Therefore, $Q = \ddot{\lambda} = 0$. We identify t_o with t_1 and t_f with $\bar{t}_2$. The boundary conditions are $\bar{y}_2(\varepsilon) \equiv$ constant, $\dot{\bar{y}}_2(\varepsilon) \equiv$ constant, $\lambda_1^T(\varepsilon)\lambda_1(\varepsilon) = 1, \kappa_2(\varepsilon) = 0$.

Following a procedure similar to that of the intercept problem, we obtain

$$\Delta m_{1\varepsilon} = - \frac{\bar{m}_1}{c\Delta T} (a_1 + a_2 \lambda_1^T \lambda_2)$$

$$\Delta m_{2\varepsilon} = \frac{\bar{m}_2 (a_1 - a_2)}{c\Delta T} (\lambda_1^T \lambda_2 - 1) - \Delta m_{1\varepsilon}$$

$$\lambda_{1\varepsilon} = \frac{a_1 - a_2}{\Delta T\ V_1} [(\lambda_1^T \lambda_2)\lambda_1 - \lambda_2]$$

$$\dot{\lambda}_{1\varepsilon} = -\frac{1}{\Delta T}\lambda_{1\varepsilon} - \frac{1}{\Delta T^2}\left\{\frac{a_1 - a_2}{\Delta V_2}[\lambda_1 - (\lambda_2^T\lambda_1)\lambda_2]\right.$$

$$\left. + (\bar{m}_2 - 2\bar{m}_1 + m_1)[1 - (\lambda_2^T\lambda_1)]\lambda_2\right\}$$

where $\Delta T = \bar{t}_2 - t_1$.

Consider the numerical problem in which $t_1 = 0$, $\bar{t}_2 = 380$ sec, $m_1 = .17 \times 10^5$ kg-sec^2/m, $\beta = 22$ kg-sec/m, $c = 4100$ m/sec, $g^T = (4.63$ m/sec^2, 8.00 m/sec$^2)$, $y_1^T = (.18 \times 10^7$ m, $.63 \times 10^7$ m$)$, $y_2^T = (.413102 \times 10^7$ m, $.501462 \times 10^7$ m$)$, $\dot{y}_1^T = (.68 \times 10^4$ m/sec, $-.2 \times 10^4$ m/sec$)$, $\dot{\bar{y}}_2^T = (.528352 \times 10^4$ m/sec, $-.505836 \times 10^4$ m/sec$)$. The optimum solution is $\bar{t}_1 = 50$ sec, $t_2 = 350$, $\lambda_1^T = (.8, .6)$, $\dot{\lambda}_1^T = (-.002, -.004)$.

The impulsive solution yields $\bar{t}_1 = 46.47$, $t_2 = 352.47$, $\lambda_1^T = (.8414, .5404)$, $\dot{\lambda}_1^T = (-.001733, -.004010)$. The impulsive solution with a first order correction added is $\bar{t}_1 = 49.64$, $t_2 = 350.60$, $\lambda_1^T = (.7992, .6060)$, $\dot{\lambda}_1^T = (-.001979, -.004065)$.

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