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A SPACE-TIME TENSOR FORMULATION FOR CONTINUUM MECHANICS IN GENERAL CURVILINEAR, MOVING, AND DEFORMING COORDINATE SYSTEMS

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16. Abstract

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A SPACE-TIME TENSOR FORMULATION FOR CONTINUUM MECHANICS IN GENERAL CURVILINEAR, MOVING, AND DEFORMING COORDINATE SYSTEMS

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SUMMARY

Tensor methods are used to express the continuum equations of motion in general curvilinear, moving, and deforming coordinate systems. The space-time tensor formulation is applicable to situations in which, for example, the boundaries move and deform. Placing a coordinate surface on such a boundary simplifies the boundary-condition treatment. The space-time tensor formulation is also applicable to coordinate systems with coordinate surfaces defined as surfaces of constant pressure, density, temperature, or any other scalar continuum field function. The vanishing of the function gradient components along the coordinate surfaces may simplify the set of governing equations. In numerical integration of the equations of motion, the freedom of motion of the coordinate surfaces provides a potential for enhanced resolution of the continuum field function. The space-time tensor formulation has been applied to numerical simulations of the atmosphere and the oceans.

Two tensor expressions of inertial velocity which provide a convenient flexibility in formulating equations of motion are presented. One inertial velocity tensor, in contravariant form, has space components equal to the (nontensor) velocity relative to the coordinate system and the time component equal to an arbitrary constant (the metric tensor providing the velocity of the coordinate system). The other inertial velocity tensor, in contravariant form, has a vanishing time component. Also, a tensor of coordinate system velocity relative to inertial space exists in the space-time. Expressions for the covariant, absolute, and comoving derivatives of general space-time tensors are derived in terms of the spatial Christoffel symbols and the coordinate system inertial velocity. Expressions for the rates of change of the elements of the space-time metric tensor are presented. The equations of motion of a material continuum in the space-time tensor formalism are derived from generalized conservation principles from both Lagrangian and Eulerian viewpoints. An example problem of an incompressible, inviscid fluid with a top free surface is considered, where the surfaces of constant pressure (including the top free surface) are coordinate surfaces.

INTRODUCTION

This paper presents a space-time tensor formulation for nonrelativistic continuum mechanics which is applicable to general time-dependent geometry and coordinate systems moving with respect to each other. A space-time metric tensor which preserves the invariance of the nonrelativistic spatial distance between neighboring simultaneous events is defined. Time assumes the absolute character appropriate to Newtonian mechanics.

The space-time tensor formulation has been applied (ref. 1) to numerical simulation of atmosphere and ocean dynamics employing a coordinate system rotating with the Earth and having quasi-horizontal coordinate surfaces variable in space and time. This formulation, by virtue of its generality and compactness, has been found to be a useful device.

In general, problems which might profitably be expressed in the space-time tensor formalism include those with moving and deforming boundaries and/or those in which one may choose as coordinate surfaces, surfaces of constant pressure, temperature, or other physical parameters. In the former case, the treatment of boundary conditions is facilitated; and the latter may lead to material simplification of the governing equations. Also, tracking the chosen physical parameter with coordinate surfaces is potentially a means of improving the resolution of the parameter in numerical integration of the equations of motion, without decreasing the space and time numerical step sizes. Some examples of suitable coordinate-surface-defining functions are density, temperature, and pressure in simulations of the troposphere; and pressure (in many cases) in simulations of a general fluid in a gravitational field.

SYMBOLS

A	determinant of matrix of A_{ij}
A_{ij}, A^{ij}	covariant and contravariant elements of space-time metric tensors
$\bar{A}_{\bar{i}\bar{j}}, \bar{A}^{\bar{i}\bar{j}}$	covariant and contravariant elements of space-time metric tensors for a Riemann space-time which is a subspace of Euclidean space-time
$a^3_{(\bar{z})}$	$\bar{z}^3$ component of fluid comoving acceleration
B	$= A_{N+1,\,N+1} - 1$
$C(T)$	source term tensor, defined by equation (66)

$D(\ )$	absolute differential
f^i	space-time tensor of force per unit volume
$g_{\alpha\beta}, g^{\alpha\beta}$	covariant and contravariant elements of N-space metric tensor
$\bar{g}_{\bar{\alpha}\bar{\beta}}, \bar{g}^{\bar{\alpha}\bar{\beta}}$	covariant and contravariant elements of a subspace of Euclidean 3-space
$g, \bar{g}$	determinants of $\left[g_{\alpha\beta}\right]$ and $\left[\bar{g}_{\bar{\alpha}\bar{\beta}}\right]$
$\bar{\bar{g}}$	determinant of matrix of covariant elements of metric tensor in subspace of $y^{\bar{\bar{\alpha}}}$ coordinates
J^i	velocity of $\bar{x}$ coordinate system relative to inertial $\bar{z}$ system, $-\left(\frac{\partial \bar{x}^i}{\partial t}\right)_{\bar{z}^\phi} + \delta_4^i \lambda$
K	constant between zero and unity
N	number of spatial dimensions in Riemann space-time, $N = 1,2,3$
$\overline{N}$	number of dimensions in subspace of Euclidean 3-space, $\overline{N} = 1,2$
n_i	covariant vector normal to bottom boundary
O	origin of Riemann coordinates
p	pressure
p_o	pressure at top free surface, atmospheric pressure
S^i	inertial velocity of coordinate system, $S^i_{(y)} = -\left(\frac{\partial y^i}{\partial t}\right)_{z^\phi}$
$\bar{S}^{\bar{\phi}}$	$\bar{\phi}$ component of inertial velocity of coordinate system in subspace of Euclidean 3-space

$s(y^i)$	$\int_O^{y^i} ds$ along geodesic from point O to point y^i
ds	$= \sqrt{A_{ij}\, dy^i\, dy^j}$
$(ds)_I$	$= \sqrt{dz^{\alpha}\, dz_{\alpha}}$
$\hat{T}$	general space-time tensor
t	time
U^i	inertial velocity, $V^i + S^i$
V^i	inertial velocity, dy^i/dt
v	$(U^i - J^i)n_i$ at bottom boundary
W^i	inertial velocity of a particular coordinate system, the y system of appendix B; defined in appendix D (eqs. (D9) and (D10))
x^i	Riemann coordinates
$\bar{x}^i$	spherical polar coordinates fixed with respect to the rotating Earth; $\bar{x}^1$, $\bar{x}^2$, and $\bar{x}^3$ are south, east, and radially outward, respectively, and $d\bar{x}^4 = \lambda\, dt$
$y^i, \bar{y}^i$	general curvilinear, moving and deforming coordinates
z^i	any coordinates such that $A_{ij(z)} = \delta^i_{j(z)}$ locally
$\tilde{z}^i$	inertial Cartesian coordinate system over Euclidean space-time of four dimensions
$\bar{z}^i$	geocentric Cartesian coordinate system (approximately inertial)
δ^i_j	Kronecker delta, equal to zero for $i \neq j$ and equal to unity for $i = j$

θ	gravitational potential of Earth
λ	arbitrary nonvanishing constant
ξ	variable of integration
ρ	mass density
Ω	angular velocity of Earth's rotation relative to inertial space
ω	angular velocity of rotation of pressure gradient relative to $\bar{z}$ coordinate system
ω_{ij}	space-time vorticity, $\frac{1}{2}\left(V_{i\|j} - V_{j\|i}\right)$

Notation:

$\left\{ {i \atop j\ k} \right\}$	Christoffel symbol of the second kind in space-time
$\left\{ {\alpha \atop \beta\ \gamma} \right\}_N$	Christoffel symbol of the second kind in N-space
$\overline{\left\{ {\bar{i} \atop \bar{j}\ \bar{k}} \right\}}$	Christoffel symbol of the second kind in Riemann space-time which is a subspace of Euclidean space-time
$(\ )_{(\ )}$	upper parenthetical term is expressed in coordinate system labeled by lower parenthetical symbol (upper parentheses may be absent)
$\left[\frac{\partial(\ )}{\partial(\ )}\right]_{y^i}$	y^i coordinates are held fixed for the partial differentiation
$[\ ]_N$	bracketed expression is evaluated in N-space
$(\ )\|_O$	parenthetical expression is evaluated at point O
$(\ )\|_i$	covariant differentiation

$|[\,]|$ determinant of matrix $[\,]$

$\left[S^{N+1}{}_{|\beta}\right]_N$, $\left\{ {}_{\gamma}{}^{N+1}{}_{\beta} \right\}_N$ defined as zero

$\left\{ {}_{N+1}{}^{\gamma}{}_{\beta} \right\}_N$, $\left\{ {}_{\beta}{}^{\gamma}{}_{N+1} \right\}_N$ defined as zero

Indices:

The range of Roman indices i to u is 1 to N + 1; of Greek indices without bars, 1 to N.

The range of Roman indices $\bar{i}$ to $\bar{u}$ is 1 to $\bar{N}$ + 1; of Greek indices $\bar{\alpha}$ to $\bar{\omega}$, 1 to $\bar{N}$.

The range of Greek indices $\bar{\bar{\alpha}}$ to $\bar{\bar{\omega}}$ is 1 to $(N - \bar{N})$.

The index b refers to the bottom boundary.

The summation convention observed herein is that a lowercase letter occurring exactly twice in a term, once as a lower index and once as an upper index, denotes summation over those indices.

ANALYSIS

Introductory Remarks

The tensor formulation developed in the analysis applies to coordinate transformations of the form

$$\left.\begin{aligned} y^i &= y^i\left(\bar{y}^1, \ldots, \bar{y}^N, \bar{y}^{N+1}\right) \\ \bar{y}^j &= \bar{y}^j\left(y^1, \ldots, y^N, y^{N+1}\right) \end{aligned}\right\} \tag{1}$$

which are transformations of the coordinates of an event in space-time, where N is the number of spatial dimensions and

$$\bar{y}^{N+1} = y^{N+1} = \lambda t \tag{2}$$

The symbol λ is an arbitrary nonvanishing constant and t is the time elapsed since an arbitrary reference time. The functions y^i and $\bar{y}^j$ are required to be single valued and continuous with continuous derivatives through the third order. Such transformations and coordinate systems are called "allowed" herein.

A brief outline and description of the development sequence is given as follows:

"The Space-Time Metric Tensor"

A space-time interval in a local Cartesian coordinate system, which establishes a space-time metric tensor over a Riemann "space," is introduced. The full four-dimensional space-time is Euclidean, but the generality of Riemann space-time is retained for application to surfaces with Gaussian curvature. Appendix A develops the relationship of the Christoffel symbols in Riemann space-time to those in Euclidean space-time.

"Inertial Velocity Tensors"

Two tensor expressions of inertial velocity which provide a convenient flexibility in formulating equations of motion are presented. One inertial velocity tensor has the time component λ and the N contravariant space components equal to the (nontensor) velocity relative to the coordinate system. The other inertial velocity tensor, in contravariant form, has a vanishing time component. The difference of the latter and former tensors is the coordinate system velocity tensor (which, upon coordinate transformation, applies to the coordinate system transformed to).

"Space-Time Christoffel Symbols"

The Christoffel symbols for the Riemann space-time are expressed in terms of the coordinate system velocity and the Christoffel symbols for N-space.

"The Space-Time Tensorial Derivatives"

The covariant, absolute, and comoving derivatives of a general tensor are derived by use of the expressions for the space-time Christoffel symbols. The covariant and comoving derivatives are then specialized for first- and second-rank tensors.

"The Rates of Change of the Metric Tensors"

The partial time derivatives of the covariant and contravariant space-time metric tensors and N-space metric tensors are derived.

"Equations of Motion of a Material Continuum"

The equations of transport for a general continuum tensor field are derived from generalized conservation principles, and then specialized to various forms of the equations of motion. A space-time vorticity tensor is introduced, and the equations of motion are expressed in terms of vorticity and kinetic energy. The Eulerian and Lagrangian time derivatives of vorticity are expressed in terms of the fields of velocity, vorticity, and acceleration.

These results are applied in appendix B to an example problem, employing constant-pressure coordinate surfaces, of an ideal fluid moving over the Earth. Appendixes C and D are supplements to appendix B.

The Space-Time Metric Tensor

This section introduces a space-time metric tensor which renders invariant the spatial part of the space-time interval as well as the total space-time interval.

The square of the space-time interval $(ds)^2$ is defined by

$$(ds)^2 = dz_i \, dz^i \tag{3}$$

where the N coordinates z^α are of a local Cartesian system regarded as fixed in inertial space and $z^{N+1} = \lambda t$. Transforming the z^i coordinate increments to any allowed coordinate system y yields

$$(ds)^2 = \frac{\partial z_i}{\partial y^m} \frac{\partial z^i}{\partial y^n} dy^m \, dy^n = A_{mn} \, dy^m \, dy^n \tag{4}$$

where the spatial components of the space-time metric tensor A_{mn} can be written

$$A_{\alpha\beta} = \frac{\partial z_\gamma}{\partial y^\alpha} \frac{\partial z^\gamma}{\partial y^\beta} + \frac{\partial z_{N+1}}{\partial y^\alpha} \frac{\partial z^{N+1}}{\partial y^\beta} = \frac{\partial z_\gamma}{\partial y^\alpha} \frac{\partial z^\gamma}{\partial y^\beta} \tag{5}$$

since

$$\frac{\partial z^{N+1}}{\partial y^1} = \lambda \left(\frac{\partial t}{\partial y^1} \right)_{y^2, \ldots, y^N, t} = 0$$

and, in general,

$$\frac{\partial z^{N+1}}{\partial y^\alpha} = \frac{\partial z_{N+1}}{\partial y^\alpha} = 0 \tag{6}$$

The other components of A_{mn} are

$$A_{\alpha,N+1} = A_{N+1,\alpha} = \frac{\partial z_\gamma}{\partial y^\alpha}\frac{\partial z^\gamma}{\partial y^{N+1}} = \frac{1}{\lambda}\frac{\partial z_\gamma}{\partial y^\alpha}\left(\frac{\partial z^\gamma}{\partial t}\right)_{y^\beta} \tag{7}$$

$$A_{N+1,N+1} = \frac{\partial z_\gamma}{\partial y^{N+1}}\frac{\partial z^\gamma}{\partial y^{N+1}} + 1 = \left(\frac{1}{\lambda}\right)^2\left(\frac{\partial z_\gamma}{\partial t}\right)_{y^\beta}\left(\frac{\partial z^\gamma}{\partial t}\right)_{y^\beta} + 1 \tag{8}$$

where, it might be noted, $\left(\frac{\partial z^\gamma}{\partial t}\right)_{y^\beta}$ is the Cartesian N-vector of the velocity of the y coordinate system relative to the inertial z system. Equation (5) states that the spatial components of the covariant space-time metric tensor are the components of the covariant N-space metric tensor, or $g_{\alpha\beta}$.

The time coordinate increment transforms by invariance, from equation (2); thus, rearranging equation (3) yields

$$dz_\gamma\, dz^\gamma = (ds)^2 - dz_{N+1}\, dz^{N+1} = (ds)^2 - (\lambda)^2(dt)^2 = (ds)_I^2$$

which is invariant.

The y coordinate system velocity $\left(\frac{\partial z^\gamma}{\partial t}\right)_{y^\beta}$ occurring in equations (7) and (8) is expressed in y coordinates by

$$\left(\frac{\partial z^\gamma}{\partial t}\right)_{y^\beta} = \frac{dz^\gamma}{dt} - \frac{\partial z^\gamma}{\partial y^\alpha}\frac{dy^\alpha}{dt} = \frac{dz^\gamma}{dt} - \frac{\partial z^\gamma}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial z^\psi}\frac{dz^\psi}{dt} - \frac{\partial z^\gamma}{\partial y^\alpha}\left(\frac{\partial y^\alpha}{\partial t}\right)_{z^\phi}$$

or

$$\left(\frac{\partial z^\gamma}{\partial t}\right)_{y^\beta} = -\frac{\partial z^\gamma}{\partial y^\alpha}\left(\frac{\partial y^\alpha}{\partial t}\right)_{z^\phi} \tag{9}$$

where $-\left(\frac{\partial y^{\alpha}}{\partial t}\right)_{z^{\phi}}$ is the velocity of the y coordinate system relative to the z system, expressed in y coordinates. Let

$$S^{\alpha} = -\left(\frac{\partial y^{\alpha}}{\partial t}\right)_{z^{\phi}} \tag{10}$$

$$g_{\alpha\beta} = \frac{\partial z_{\gamma}}{\partial y^{\alpha}} \frac{\partial z^{\gamma}}{\partial y^{\beta}} \tag{11}$$

Then, from equations (5) and (7) to (11),

$$\left.\begin{aligned}
A_{\alpha\beta} &= g_{\alpha\beta} \\
A_{\alpha,N+1} &= A_{N+1,\alpha} = \frac{1}{\lambda} g_{\alpha\phi} S^{\phi} \\
A_{N+1,N+1} &= \left(\frac{1}{\lambda}\right)^{2} g_{\alpha\beta} S^{\alpha} S^{\beta} + 1
\end{aligned}\right\} \tag{12}$$

The components of the contravariant space-time metric tensor are given by

$$A^{ij} = \frac{\partial y^{i}}{\partial z_{k}} \frac{\partial y^{j}}{\partial z^{k}}$$

or

$$\left.\begin{aligned}
A^{\alpha\beta} &= g^{\alpha\beta} + \left(\frac{1}{\lambda}\right)^{2} S^{\alpha} S^{\beta} \\
A^{\alpha,N+1} &= A^{N+1,\alpha} = -\frac{1}{\lambda} S^{\alpha} \\
A^{N+1,N+1} &= 1
\end{aligned}\right\} \tag{13}$$

where $g^{\alpha\beta}$ is the contravariant N-space metric tensor.

It should be noted that the coordinate system velocity S^α given by equation (10) is not a space-time tensor, because the time component has not been defined. A space-time tensor of coordinate system velocity is introduced in the following section.

Inertial Velocity Tensors

The space-time tensor

$$V^i = \frac{dy^i}{dt} \tag{14}$$

where dy^i is the coordinate increment tensor associated with two neighboring events in space-time and $dt = \frac{1}{\lambda} dy^4$, transforms to the local Cartesian inertial z system as

$$V^j_{(z)} = \frac{dz^j}{dt}$$

the spatial components of which constitute the Cartesian N-vector of inertial velocity. The time component λ can be interpreted as a constant uniform velocity normal to all spatial directions. If the two events in equation (14) are the presence of a certain particle, then V^i is an inertial space-time velocity of the particle.

Another inertial velocity space-time tensor is defined by the contravariant vector transformation from the z coordinate system to any allowed system y of the Cartesian space-time vector

$$U^\alpha_{(z)} = V^\alpha_{(z)}$$

$$U^{N+1}_{(z)} = 0$$

Thus,

$$U^i_{(y)} = U^j_{(z)} \frac{\partial y^i}{\partial z^j}$$

$$= \frac{dz^\gamma}{dt} \frac{\partial y^i}{\partial z^\gamma}$$

$$= V^i_{(y)} - \left(\frac{\partial y^i}{\partial t}\right)_{z^\phi} \tag{15}$$

which has a vanishing time component. From equation (10), the spatial components of U^i are

$$U^{\alpha} = V^{\alpha} + S^{\alpha}$$

which expresses inertial velocity as the sum of coordinate system velocity S^{α} and velocity V^{α} relative to the coordinate system.

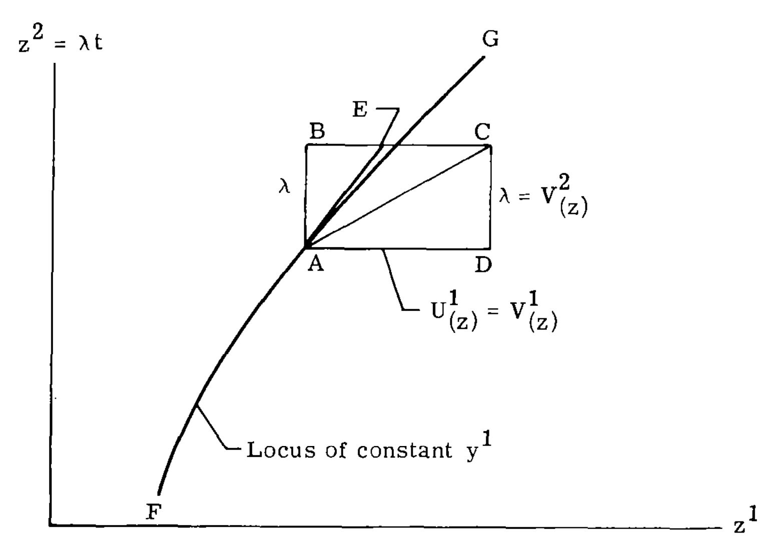
In equation (15), the term $-\left(\frac{\partial y^i}{\partial t}\right)_{z^{\phi}}$ is the difference between two space-time tensors and is therefore a space-time tensor. The tensor of coordinate system inertial velocity S^i is defined as

$$S^i_{(y)} = -\left(\frac{\partial y^i}{\partial t}\right)_{z^{\phi}} \tag{16}$$

and equation (15) becomes

$$U^i = V^i + S^i \tag{17}$$

The relationship among the velocities U^i, V^i, and S^i under coordinate transformation is illustrated in sketch (a), which represents a two-dimensional case. The curve $\widehat{FG}$ is



Sketch (a)

the path taken by a point of fixed spatial y coordinate as seen in the z coordinate system. The line $\widehat{AC}$ represents $V^i_{(z)}$ of a particle at point A. The line $\widehat{AE}$, drawn tangent to the curve $\widehat{FG}$ at point A, intersects $\widehat{BC}$ to determine the spatial component of the y coordinate system velocity, represented by $\widehat{BE}$. The point E partitions the spatial component of inertial velocity $U^1_{(z)}$ into the y coordinate system velocity $\widehat{BE}$ and the particle velocity $\widehat{EC}$ relative to the y system, expressed by

$$U^1_{(z)} = \frac{\partial z^1}{\partial y^1} V^1_{(y)} + \left(\frac{\partial z^1}{\partial t}\right)_{y^1}$$

or, in y^1 units, from equations (9) and (10),

$$U^1_{(y)} = V^1_{(y)} + S^1_{(y)}$$

The time components of U, V, and S are similarly related, since $U^2_{(y)} = 0$ and $V^2_{(y)} = -S^2_{(y)} = \lambda$:

$$U^2_{(y)} = V^2_{(y)} + S^2_{(y)}$$

Hence,

$$U^i = V^i + S^i$$

holds for any allowed transformation from the coordinate system z^1, z^2 of sketch (a).

Lowering the indices of U^i and V^i and applying equations (12) and (17) yields, for any allowed coordinate system,

$$\left.\begin{array}{ll} U_\alpha = U^\gamma g_{\gamma\alpha} & U_{N+1} = \frac{1}{\lambda} S^\phi U_\phi \\ V_\alpha = U_\alpha & V_{N+1} = U_{N+1} + \lambda \\ S_\alpha = 0 & S_{N+1} = -\lambda \end{array}\right\} \quad (18)$$

The existence of two inertial velocity tensors V and U provides a convenient flexibility in formulating the dynamical equations in curvilinear, moving coordinate systems. The space components of V^i constitute the velocity relative to the coordinate grid, which is a convenient choice of velocity representation to use in the convective terms of the dynamical equations of a material continuum, for example. On the other hand, an advantage of representing inertial velocity by U is that the time component of U^i vanishes. Inertial velocities might be represented by U at some places and by V at other places, whichever is more convenient, in the problem formulation. A given velocity representation can be tailored to one's needs by use of equation (17).

Space-Time Christoffel Symbols

In preparation for the discussion of covariant and comoving derivatives, the space-time Christoffel symbols of the second kind $\left\{ {i \atop j\ \ k} \right\}$ are derived in this section in terms of the N-space Christoffel symbols $\left\{ {\alpha \atop \beta\ \ \gamma} \right\}_N$ and the coordinate system inertial velocity.

The Christoffel symbols of the second kind, defined as

$$\left\{ {i \atop j\ \ k} \right\} = \frac{1}{2} A^{in} \left(\frac{\partial A_{kn}}{\partial y^j} + \frac{\partial A_{nj}}{\partial y^k} - \frac{\partial A_{jk}}{\partial y^n} \right) \tag{19}$$

transform from an allowed coordinate system y to any allowed system $\bar{y}$ by (p. 48, ref. 2)

$$\left\{ {r \atop m\ \ n} \right\}_{(\bar{y})} = \left\{ {j \atop p\ \ q} \right\}_{(y)} \frac{\partial \bar{y}^r}{\partial y^j} \frac{\partial y^p}{\partial \bar{y}^m} \frac{\partial y^q}{\partial \bar{y}^n} + \frac{\partial \bar{y}^r}{\partial y^j} \frac{\partial^2 y^j}{\partial \bar{y}^m \, \partial \bar{y}^n} \tag{20}$$

The second expression on the right-hand side implies that $\left\{ {i \atop j\ \ k} \right\}$ is not a tensor. In Euclidean "space," such as four-dimensional space-time, it is always possible to define a coordinate system $\tilde{z}$ such that $(ds)^2 = \sum_{i=1}^{N+1} \left(d\tilde{z}^i \, d\tilde{z}^i \right)$ everywhere (p. 583, ref. 3). The metric tensor components are then constant and the Christoffel symbols vanish in the $\tilde{z}$ coordinate system. Therefore, in Euclidean space-time, from equation (20),

$$\left\{ {m}^{\;r}{\;n} \right\}_{(\bar{y})} = \frac{\partial \bar{y}^r}{\partial \tilde{z}^j} \frac{\partial^2 \tilde{z}^j}{\partial \bar{y}^m \, \partial \bar{y}^n} \tag{21}$$

Equation (21) can be generalized for application to non-Euclidean Riemann space-time by use of Riemann coordinates x^i, which are defined by (p. 584, ref. 3)

$$x^i = sp^i \tag{22}$$

where p^i is the unit tangent vector dy^i/ds at the origin $\left(x^i = 0\right)$ of a geodesic joining the origin and the point y^j and s is the geodesic "distance" between the two points, or

$$s = \int_{x^i=0}^{x^i\left(y^j\right)} ds \quad \text{(along geodesic)}$$

From reference 3 (p. 584), Riemann coordinates can be defined in any Riemann space at any given origin, and at the origin all Christoffel symbols vanish. Therefore, from equation (20), at the origin of the x system,

$$\left\{ {m}^{\;r}{\;n} \right\}_{(\bar{y})} = \frac{\partial \bar{y}^r}{\partial x^j} \frac{\partial^2 x^j}{\partial \bar{y}^m \, \partial \bar{y}^n} \tag{23}$$

The vanishing of the Christoffel symbols in Riemann coordinates at the origin often is used to simplify mathematical arguments.

It is shown in the following paragraph that $dx^{N+1} = \lambda \, dt$, as required by equation (2) for the x coordinate system to be allowed.

A geodesic is uniquely defined by any given point on the geodesic, the direction (unit tangent vector) dy^i/ds of the geodesic at the given point, and the equation (p. 583, ref. 3)

$$\frac{d^2 y^i}{(ds)^2} + \left\{ {j}^{\;i}{\;k} \right\} \frac{dy^j}{ds} \frac{dy^k}{ds} = 0 \tag{24}$$

or

$$\frac{D}{ds}\left(\frac{dy^i}{ds}\right) = 0 \tag{25}$$

where D is the absolute differential operator. In Riemann coordinates, equation (24) becomes, by equation (22),

$$\frac{d^2x^i}{(ds)^2} = 0 \tag{26}$$

The non-Euclidean space-time of interest here is a subspace of Euclidean space-time. It is shown in equations (A19) to (A21) of appendix A that

$$\left\{ {N+1 \atop i \quad j} \right\} = 0 \tag{27}$$

$$S_{i|j} = 0 \tag{28}$$

in any allowed coordinate system over Riemann space-time. From equations (24) and (27),

$$\frac{d^2y^{N+1}}{(ds)^2} = \lambda \frac{d^2t}{(ds)^2} = 0 \tag{29}$$

or

$$\frac{dt}{ds} = \text{Constant} \tag{30}$$

and equation (26) can be written

$$\frac{d^2x^i}{(dt)^2} = 0 \tag{31}$$

which implies that the acceleration tensor $\frac{D}{dt}\left(\frac{dy^i}{dt}\right)$ vanishes along the geodesic. Equations (30) and (22) yield

$$dx^{N+1} = \left(\frac{dy^{N+1}}{ds}\right)\bigg|_O ds = \lambda\left(\frac{dt}{ds}\right)\bigg|_O ds = \lambda\ dt \tag{32}$$

where $\left(\frac{dy^{N+1}}{ds}\right)\bigg|_O$ and $\left(\frac{dt}{ds}\right)\bigg|_O$ are evaluated at the origin O of the Riemann coordinate system.

The space components of the space-time Christoffel symbols $\left\{ {\alpha \atop \beta \quad \gamma} \right\}$ reduce to the N-space Christoffel symbols, as shown in the remainder of this section. For $dt = 0$,

$$ds = \sqrt{A_{mn}\, dy^m\, dy^n} = \sqrt{g_{\sigma\phi}\, dy^\sigma\, dy^\phi} \tag{33}$$

which is the N-space interval. Equation (24) for the space-time geodesic for $i \neq N + 1$ becomes, for this case,

$$\frac{d^2y^\alpha}{(ds)^2} + \left\{ {\alpha \atop \sigma \quad \phi} \right\} \frac{dy^\sigma}{ds}\frac{dy^\phi}{ds} = 0 \tag{34}$$

and the equation for the N-space geodesic is

$$\frac{d^2y^\alpha}{(ds)^2} + \left\{ {\alpha \atop \sigma \quad \phi} \right\}_N \frac{dy^\sigma}{ds}\frac{dy^\phi}{ds} = 0 \tag{35}$$

where $\left\{ {\alpha \atop \sigma \quad \phi} \right\}_N$ is the N-space Christoffel symbol of the second kind. Equations (34) and (35) pertain to the same path, an extremal path for the integral of ds, by equation (33); therefore,

$$\left\{ {\alpha \atop \sigma \quad \phi} \right\} = \left\{ {\alpha \atop \sigma \quad \phi} \right\}_N \tag{36}$$

since dy^β/ds is arbitrary at the origin of the Riemann coordinate system, which can be anywhere in the Riemann space.

Raising the first index in equation (28) yields

$$S^i_{|j} \equiv \frac{\partial S^i}{\partial y^j} + S^k \left\{ {i \atop k \quad j} \right\} = 0 \tag{37}$$

and

$$S^{N+1} \left\{ {i \atop N+1 \quad j} \right\} = -\frac{\partial S^i}{\partial y^j} - S^\sigma \left\{ {i \atop \sigma \quad j} \right\}$$

Therefore,

$$\left\{ {}_{N+1}{}^{\alpha}{}_{\beta} \right\} = \frac{1}{\lambda}\left(\frac{\partial S^{\alpha}}{\partial y^{\beta}} + S^{\sigma} \left\{ {}_{\sigma}{}^{\alpha}{}_{\beta} \right\}\right) = \frac{1}{\lambda}\left[S^{\alpha}{}_{|\beta}\right]_N \tag{38}$$

and

$$\left\{ {}_{N+1}{}^{\alpha}{}_{N+1} \right\} = \left(\frac{1}{\lambda}\right)^2 \frac{\partial S^{\alpha}}{\partial t} + \frac{1}{\lambda} S^{\sigma} \left\{ {}_{\sigma}{}^{\alpha}{}_{N+1} \right\}$$

$$= \left(\frac{1}{\lambda}\right)^2 \left(\frac{\partial S^{\alpha}}{\partial t} + S^{\sigma}\left[S^{\alpha}{}_{|\sigma}\right]_N\right) \tag{39}$$

where

$$\left[S^{\alpha}{}_{|\beta}\right]_N \equiv \frac{\partial S^{\alpha}}{\partial y^{\beta}} + S^{\sigma} \left\{ {}_{\sigma}{}^{\alpha}{}_{\beta} \right\}_N \tag{40}$$

Finally,

$$\left\{ {}_{i}{}^{N+1}{}_{j} \right\} = 0 \tag{41}$$

which is equation (27).

The Space-Time Tensorial Derivatives

In this section, the covariant and comoving derivatives of a tensor $\hat{T}$ of any rank are expressed in terms of the N-space Christoffel symbols $\left\{ {}_{\beta}{}^{\alpha}{}_{\gamma} \right\}_N$ and the coordinate system velocity S^i. It is convenient to write the definitions

$$\left.\begin{aligned} &\left[S^{N+1}{}_{|\beta}\right]_N \equiv 0 \\ &\left\{ {}_{\gamma}{}^{N+1}{}_{\beta} \right\}_N \equiv 0 \\ &\left\{ {}_{N+1}{}^{\gamma}{}_{\beta} \right\}_N \equiv \left\{ {}_{\beta}{}^{\gamma}{}_{N+1} \right\}_N \equiv 0 \end{aligned}\right\} \tag{42}$$

The expression for the covariant derivative of a general tensor is

$$T^{r_1 \dots r_m}_{s_1 \dots s_n|p} = \frac{\partial}{\partial y^p} T^{r_1 \dots r_m}_{s_1 \dots s_n} + \left\{ {r_1 \atop q \quad p} \right\} T^{q r_2 \dots r_m}_{s_1 \dots s_n} + \dots + \left\{ {r_m \atop q \quad p} \right\}$$

$$\times T^{r_1 \dots r_{m-1} q}_{s_1 \dots s_n} - \left\{ {q \atop s_1 \quad p} \right\} T^{r_1 \dots r_m}_{q s_2 \dots s_n} - \dots - \left\{ {q \atop s_n \quad p} \right\} T^{r_1 \dots r_m}_{s_1 \dots s_{n-1} q} \tag{43}$$

and the comoving derivative is

$$\frac{D}{dt} T^{r_1 \dots r_m}_{s_1 \dots s_n} = T^{r_1 \dots r_m}_{s_1 \dots s_n|p} V^p \tag{44}$$

By use of equations (36), (38), (39), (41), and (42), equation (43) becomes, for $p \neq N + 1$,

$$T^{r_1 \dots r_m}_{s_1 \dots s_n|\beta} = \frac{\partial}{\partial y^\beta} T^{r_1 \dots r_m}_{s_1 \dots s_n} + \left(1 - \delta^{r_1}_{N+1}\right) \left\{ {r_1 \atop \gamma \quad \beta} \right\}_N T^{\gamma r_2 \dots r_m}_{s_1 \dots s_n}$$

$$+ T^{(N+1) r_2 \dots r_m}_{s_1 \dots s_n} \left(1 - \delta^{r_1}_{N+1}\right) \frac{1}{\lambda} \left[S^{r_1}{}_{|\beta}\right]_N + \dots + \left(1 - \delta^{r_m}_{N+1}\right) \left\{ {r_m \atop \gamma \quad \beta} \right\}$$

$$\times T^{r_1 \dots r_{m-1} \gamma}_{s_1 \dots s_n} + T^{r_1 \dots r_{m-1} (N+1)}_{s_1 \dots s_n} \left(1 - \delta^{r_m}_{N+1}\right) \frac{1}{\lambda} \left[S^{r_m}{}_{|\beta}\right]_N$$

$$- \left(1 - \delta^{N+1}_{s_1}\right) \left\{ {\gamma \atop s_1 \quad \beta} \right\}_N T^{r_1 \dots r_m}_{\gamma s_2 \dots s_n} - T^{r_1 \dots r_m}_{\gamma s_2 \dots s_n} \delta^{N+1}_{s_1} \frac{1}{\lambda} \left[S^{\gamma}{}_{|\beta}\right]_N - \dots$$

$$- \left(1 - \delta^{N+1}_{s_n}\right) \left\{ {\gamma \atop s_n \quad \beta} \right\}_N T^{r_1 \dots r_m}_{s_1 \dots s_{n-1} \gamma} - T^{r_1 \dots r_m}_{s_1 \dots s_{n-1} \gamma} \delta^{N+1}_{s_n} \frac{1}{\lambda} \left[S^{\gamma}{}_{|\beta}\right]_N \tag{45}$$

For $p = N + 1$, equation (43) becomes

$$
\begin{aligned}
T^{r_1\ldots r_m}_{s_1\ldots s_n|N+1} &= \frac{1}{\lambda}\frac{\partial}{\partial t}T^{r_1\ldots r_m}_{s_1\ldots s_n} + \left(1 - \delta^{r_1}_{N+1}\right)\frac{1}{\lambda}\left[S^{r_1}{}_{|\gamma}\right]_N T^{\gamma r_2\ldots r_m}_{s_1\ldots s_n} \\
&+ \left(1 - \delta^{r_1}_{N+1}\right)\left(\frac{1}{\lambda}\right)^2\left(\frac{\partial S^{r_1}}{\partial t} + S^{\psi}\left[S^{r_1}{}_{|\psi}\right]_N\right) T^{(N+1)r_2\ldots r_m}_{s_1\ldots s_n} + \ldots \\
&+ \left(1 - \delta^{r_m}_{N+1}\right)\frac{1}{\lambda}\left[S^{r_m}{}_{|\gamma}\right]_N T^{r_1\ldots r_{m-1}\gamma}_{s_1\ldots s_n} + \left(1 - \delta^{r_m}_{N+1}\right)\left(\frac{1}{\lambda}\right)^2 \\
&\times\left(\frac{\partial S^{r_m}}{\partial t} + S^{\psi}\left[S^{r_m}{}_{|\psi}\right]_N\right) T^{r_1\ldots r_{m-1}(N+1)}_{s_1\ldots s_n} - \left(1 - \delta^{N+1}_{s_1}\right)\frac{1}{\lambda}\left[S^{\gamma}{}_{|s_1}\right]_N \\
&\times T^{r_1\ldots r_m}_{\gamma s_2\ldots s_n} - \delta^{N+1}_{s_1}\left(\frac{1}{\lambda}\right)^2\left(\frac{\partial S^{\gamma}}{\partial t} + S^{\psi}\left[S^{\gamma}{}_{|\psi}\right]_N\right) T^{r_1\ldots r_m}_{\gamma s_2\ldots s_n} - \ldots \\
&- \left(1 - \delta^{N+1}_{s_n}\right)\frac{1}{\lambda}\left[S^{\gamma}{}_{|s_n}\right]_N T^{r_1\ldots r_m}_{s_1\ldots s_{n-1}\gamma} \\
&- \delta^{N+1}_{s_n}\left(\frac{1}{\lambda}\right)^2\left(\frac{\partial S^{\gamma}}{\partial t} + S^{\psi}\left[S^{\gamma}{}_{|\psi}\right]_N\right) T^{r_1\ldots r_m}_{s_1\ldots s_{n-1}\gamma}
\end{aligned}
\tag{46}
$$

The covariant and comoving derivatives of first- and second-rank tensors are sufficiently important to warrant special expressions. For a contravariant vector T^r, from equations (45) and (46),

$$
T^r{}_{|\beta} = \frac{\partial}{\partial y^{\beta}} T^r + \left(1 - \delta^r_{N+1}\right)\left\{{r \atop \gamma\ \beta}\right\} T^{\gamma} + \left(1 - \delta^r_{N+1}\right)\frac{1}{\lambda}\left[S^r{}_{|\beta}\right]_N T^{N+1}
\tag{47}
$$

$$
\begin{aligned}
T^r{}_{|N+1} &= \frac{1}{\lambda}\frac{\partial}{\partial t}T^r + \left(1 - \delta^r_{N+1}\right)\frac{1}{\lambda}\left[S^r{}_{|\gamma}\right]_N T^{\gamma} \\
&+ \left(1 - \delta^r_{N+1}\right)\left(\frac{1}{\lambda}\right)^2\left(\frac{\partial S^r}{\partial t} + S^{\psi}\left[S^r{}_{|\psi}\right]_N\right) T^{N+1}
\end{aligned}
\tag{48}
$$

In particular,

$$T^{N+1}{}_{|p} = \frac{\partial}{\partial y^p} T^{N+1} \tag{49}$$

$$\frac{D}{dt} T^r = T^r{}_{|p} V^p = T^r{}_{|N+1} V^{N+1} + T^r{}_{|\beta} V^\beta = T^r{}_{|N+1} \lambda + T^r{}_{|\beta} V^\beta$$

$$= \frac{\partial}{\partial t} T^r + \left(1 - \delta^r_{N+1}\right) \left[S^r{}_{|\gamma}\right]_N T^\gamma + \left(1 - \delta^r_{N+1}\right) \frac{1}{\lambda} \left(\frac{\partial S^r}{\partial t} + S^\psi \left[S^r{}_{|\psi}\right]_N\right) T^{N+1}$$

$$+ \frac{\partial T^r}{\partial y^\beta} V^\beta + \left(1 - \delta^r_{N+1}\right) \left\{ {}_\gamma{}^r{}_\beta \right\}_N T^\gamma V^\beta + \left(1 - \delta^r_{N+1}\right) \frac{1}{\lambda} \left[S^r{}_{|\beta}\right]_N T^{N+1} V^\beta \tag{50}$$

For a covariant vector T_s, from equations (45) and (46),

$$T_{s|\beta} = \frac{\partial}{\partial y^\beta} T_s - \left(1 - \delta_s^{N+1}\right) \left\{ {}_s{}^\gamma{}_\beta \right\}_N T_\gamma - \delta_s^{N+1} \frac{1}{\lambda} \left[S^\gamma{}_{|\beta}\right]_N T_\gamma \tag{51}$$

$$T_{s|N+1} = \frac{1}{\lambda} \frac{\partial}{\partial t} T_s - \left(1 - \delta_s^{N+1}\right) \frac{1}{\lambda} \left[S^\gamma{}_{|s}\right]_N T_\gamma$$

$$- \delta_s^{N+1} \left(\frac{1}{\lambda}\right)^2 \left(\frac{\partial S^\gamma}{\partial t} + S^\psi \left[S^\gamma{}_{|\psi}\right]_N\right) T_\gamma \tag{52}$$

$$\frac{D}{dt} T_s = T_{s|p} V^p = T_{s|N+1} V^{N+1} + T_{s|\beta} V^\beta = T_{s|N+1} \lambda + T_{s|\beta} V^\beta$$

$$= \frac{\partial}{\partial t} T_s - \left(1 - \delta_s^{N+1}\right) \left[S^\gamma{}_{|s}\right]_N T_\gamma - \delta_s^{N+1} \frac{1}{\lambda} \left(\frac{\partial S^\gamma}{\partial t} + S^\psi \left[S^\gamma{}_{|\psi}\right]_N\right) T_\gamma$$

$$+ \frac{\partial T_s}{\partial y^\beta} V^\beta - \left(1 - \delta_s^{N+1}\right) \left\{ {}_s{}^\gamma{}_\beta \right\}_N T_\gamma V^\beta - \delta_s^{N+1} \frac{1}{\lambda} \left[S^\gamma{}_{|\beta}\right]_N T_\gamma V^\beta \tag{53}$$

For a contravariant second-rank tensor $T^{r_1 r_2}$, from equations (45) and (46),

$$T^{r_1 r_2}{}_{|\beta} = \frac{\partial}{\partial y^\beta} T^{r_1 r_2} + \left(1 - \delta^{r_1}_{N+1}\right)\left(\left\{ {r_1 \atop \gamma \quad \beta} \right\}_N T^{\gamma r_2} + \frac{1}{\lambda}\left[S^{r_1}{}_{|\beta}\right]_N T^{(N+1)r_2}\right)$$

$$+ \left(1 - \delta^{r_2}_{N+1}\right)\left(\left\{ {r_2 \atop \gamma \quad \beta} \right\}_N T^{r_1 \gamma} + \frac{1}{\lambda}\left[S^{r_2}{}_{|\beta}\right]_N T^{r_1(N+1)}\right) \tag{54}$$

$$T^{r_1 r_2}{}_{|N+1} = \frac{1}{\lambda}\frac{\partial}{\partial t} T^{r_1 r_2} + \left(1 - \delta^{r_1}_{N+1}\right)\left[\frac{1}{\lambda}\left[S^{r_1}{}_{|\gamma}\right]_N T^{\gamma r_2}\right.$$

$$\left. + \left(\frac{1}{\lambda}\right)^2 \left(\frac{\partial S^{r_1}}{\partial t} + S^\psi \left[S^{r_1}{}_{|\psi}\right]_N\right) T^{(N+1)r_2}\right] + \left(1 - \delta^{r_2}_{N+1}\right)\left[\frac{1}{\lambda}\left[S^{r_2}{}_{|\gamma}\right]_N T^{r_1 \gamma}\right.$$

$$\left. + \left(\frac{1}{\lambda}\right)^2 \left(\frac{\partial S^{r_2}}{\partial t} + S^\psi \left[S^{r_2}{}_{|\psi}\right]_N\right) T^{r_1(N+1)}\right] \tag{55}$$

$$\frac{D}{dt} T^{r_1 r_2} = T^{r_1 r_2}{}_{|p} V^p = T^{r_1 r_2}{}_{|N+1} V^{N+1} + T^{r_1 r_2}{}_{|\beta} V^\beta = T^{r_1 r_2}{}_{|N+1} \lambda + T^{r_1 r_2}{}_{|\beta} V^\beta$$

$$= \frac{\partial}{\partial t} T^{r_1 r_2} + \left(1 - \delta^{r_1}_{N+1}\right)\left[\left[S^{r_1}{}_{|\gamma}\right]_N T^{\gamma r_2} + \frac{1}{\lambda}\left(\frac{\partial S^{r_1}}{\partial t} + S^\psi \left[S^{r_1}{}_{|\psi}\right]_N\right) T^{(N+1)r_2}\right.$$

$$\left. + V^\beta \left(\left\{ {r_1 \atop \gamma \quad \beta} \right\}_N T^{\gamma r_2} + \frac{1}{\lambda}\left[S^{r_1}{}_{|\beta}\right]_N T^{(N+1)r_2}\right)\right] + \left(1 - \delta^{r_2}_{N+1}\right)\left[\left[S^{r_2}{}_{|\gamma}\right]_N T^{r_1 \gamma}\right.$$

$$\left. + \frac{1}{\lambda}\left(\frac{\partial S^{r_2}}{\partial t} + S^\psi \left[S^{r_2}{}_{|\psi}\right]_N\right) T^{r_1(N+1)} + V^\beta \left(\left\{ {r_2 \atop \gamma \quad \beta} \right\}_N T^{r_1 \gamma} + \frac{1}{\lambda}\left[S^{r_2}{}_{|\beta}\right]_N T^{r_1(N+1)}\right)\right]$$

$$+ V^\beta \frac{\partial}{\partial y^\beta} T^{r_1 r_2} \tag{56}$$

For a covariant second-rank tensor $T_{s_1 s_2}$, from equations (45) and (46),

$$T_{s_1 s_2|\beta} = \frac{\partial}{\partial y^\beta} T_{s_1 s_2} - \left(1 - \delta_{s_1}^{N+1}\right) \left\{ {s_1}^{\ \gamma}\ \beta \right\}_N T_{\gamma s_2} - \delta_{s_1}^{N+1} \frac{1}{\lambda} \left[S^\gamma{}_{|\beta} \right]_N T_{\gamma s_2}$$

$$- \left(1 - \delta_{s_2}^{N+1}\right) \left\{ {s_2}^{\ \gamma}\ \beta \right\}_N T_{s_1 \gamma} - \delta_{s_2}^{N+1} \frac{1}{\lambda} \left[S^\gamma{}_{|\beta} \right]_N T_{s_1 \gamma} \tag{57}$$

$$T_{s_1 s_2|N+1} = \frac{1}{\lambda} \frac{\partial}{\partial t} T_{s_1 s_2} - \left(1 - \delta_{s_1}^{N+1}\right) \frac{1}{\lambda} \left[S^\gamma{}_{|s_1} \right]_N T_{\gamma s_2}$$

$$- \delta_{s_1}^{N+1} \left(\frac{1}{\lambda}\right)^2 \left(\frac{\partial S^\gamma}{\partial t} + S^\psi \left[S^\gamma{}_{|\psi} \right]_N \right) T_{\gamma s_2} - \left(1 - \delta_{s_2}^{N+1}\right) \frac{1}{\lambda} \left[S^\gamma{}_{|s_2} \right]_N T_{s_1 \gamma}$$

$$- \delta_{s_2}^{N+1} \left(\frac{1}{\lambda}\right)^2 \left(\frac{\partial S^\gamma}{\partial t} + S^\psi \left[S^\gamma{}_{|\psi} \right]_N \right) T_{s_1 \gamma} \tag{58}$$

$$\frac{D}{dt} T_{s_1 s_2} = T_{s_1 s_2|p} V^p = T_{s_1 s_2|N+1} V^{N+1} + T_{s_1 s_2|\beta} V^\beta$$

$$= T_{s_1 s_2|N+1} \lambda + T_{s_1 s_2|\beta} V^\beta$$

$$= \frac{\partial}{\partial t} T_{s_1 s_2} - \left(1 - \delta_{s_1}^{N+1}\right) T_{\gamma s_2} \left(\left[S^\gamma{}_{|s_1} \right]_N + V^\beta \left\{ {s_1}^{\ \gamma}\ \beta \right\}_N \right) - \delta_{s_1}^{N+1} \left(\frac{1}{\lambda}\right) T_{\gamma s_2}$$

$$\times \left(\frac{\partial S^\gamma}{\partial t} + S^\psi \left[S^\gamma{}_{|\psi} \right]_N + V^\beta \left[S^\gamma{}_{|\beta} \right]_N \right) - \left(1 - \delta_{s_2}^{N+1}\right) T_{s_1 \gamma} \left(\left[S^\gamma{}_{|s_2} \right]_N + V^\beta \left\{ {s_2}^{\ \gamma}\ \beta \right\}_N \right)$$

$$- \delta_{s_2}^{N+1} \left(\frac{1}{\lambda}\right) T_{s_1 \gamma} \left(\frac{\partial S^\gamma}{\partial t} + S^\psi \left[S^\gamma{}_{|\psi} \right]_N + V^\beta \left[S^\gamma{}_{|\beta} \right]_N \right) + V^\beta \frac{\partial}{\partial y^\beta} T_{s_1 s_2} \tag{59}$$

For a mixed second-rank tensor T^r_s (which can be considered to be either $T_s^{\cdot r}$ or $T^r_{\cdot s}$), from equations (45) and (46),

$$T^r_{s|\beta} = \frac{\partial}{\partial y^\beta} T^r_s + \left(1 - \delta^r_{N+1}\right)\left(\begin{Bmatrix} & r & \\ \gamma & & \beta \end{Bmatrix}_N T^\gamma_s + \frac{1}{\lambda}\left[S^r_{|\beta}\right]_N T^{N+1}_s\right)$$

$$- \left(1 - \delta^{N+1}_s\right)\begin{Bmatrix} & \gamma & \\ s & & \beta \end{Bmatrix} T^r_\gamma - \delta^{N+1}_s \left(\frac{1}{\lambda}\right)\left[S^\gamma_{|\beta}\right]_N T^r_\gamma \tag{60}$$

$$T^r_{s|N+1} = \frac{1}{\lambda}\frac{\partial}{\partial t} T^r_s + \left(1 - \delta^r_{N+1}\right)\left[\frac{1}{\lambda}\left[S^r_{|\gamma}\right]_N T^\gamma_s + \left(\frac{1}{\lambda}\right)^2\left(\frac{\partial S^r}{\partial t} + S^\psi\left[S^r_{|\psi}\right]_N\right) T^{N+1}_s\right]$$

$$- T^r_\gamma\left[\left(1 - \delta^{N+1}_s\right)\frac{1}{\lambda}\left[S^\gamma_{|s}\right]_N + \delta^{N+1}_s\left(\frac{1}{\lambda}\right)^2\left(\frac{\partial S^\gamma}{\partial t} + S^\psi\left[S^\gamma_{|\psi}\right]_N\right)\right] \tag{61}$$

$$\frac{D}{dt} T^r_s = T^r_{s|p} V^p = T^r_{s|N+1} V^{N+1} + T^r_{s|\beta} V^\beta = T^r_{s|N+1}\lambda + T^r_{s|\beta} V^\beta$$

$$= \frac{\partial}{\partial t} T^r_s + \left(1 - \delta^r_{N+1}\right)\left[\left(V^\beta \begin{Bmatrix} & r & \\ \gamma & & \beta \end{Bmatrix}_N + \left[S^r_{|\gamma}\right]_N\right) T^\gamma_s + \frac{1}{\lambda}\left(V^\beta\left[S^r_{|\beta}\right]_N\right.\right.$$

$$\left.\left. + \frac{\partial S^r}{\partial t} + S^\psi\left[S^r_{|\psi}\right]_N\right) T^{N+1}_s\right] - \left(1 - \delta^{N+1}_s\right)\left(V^\beta\begin{Bmatrix} & \gamma & \\ s & & \beta \end{Bmatrix}_N + \left[S^\gamma_{|s}\right]_N\right) T^r_\gamma - \delta^{N+1}_s\left(\frac{1}{\lambda}\right)$$

$$\times\left(V^\beta\left[S^\gamma_{|\beta}\right]_N + S^\psi\left[S^\gamma_{|\psi}\right]_N + \frac{\partial S^\gamma}{\partial t}\right) T^r_\gamma + V^\beta \frac{\partial}{\partial y^\beta} T^r_s \tag{62}$$

The Rates of Change of the Metric Tensors

The partial time derivatives of the covariant and contravariant metric tensor components are derived in this section in terms of the coordinate system velocity S^i, the metric tensors, and the N-space Christoffel symbols $\begin{Bmatrix} & \alpha & \\ \beta & & \gamma \end{Bmatrix}_N$

The covariant derivative of any form of the metric tensor vanishes identically (ref. 4, p. 43); hence, by equations (58) and (55),

$$\frac{\partial}{\partial t}\left(A_{ij}\right) = \left(1 - \delta_i^{N+1}\right)\left[S^\gamma_{|i}\right]_N A_{\gamma j} + \delta_i^{N+1}\left(\frac{1}{\lambda}\right)\left(\frac{\partial S^\gamma}{\partial t} + S^\psi\left[S^\gamma_{|\psi}\right]_N\right) A_{\gamma j}$$
$$+ \left(1 - \delta_j^{N+1}\right)\left[S^\gamma_{|j}\right]_N A_{i\gamma} + \delta_j^{N+1}\left(\frac{1}{\lambda}\right)\left(\frac{\partial S^\gamma}{\partial t} + S^\psi\left[S^\gamma_{|\psi}\right]_N\right) A_{i\gamma} \tag{63}$$

$$\frac{\partial}{\partial t}\left(A^{ij}\right) = -\left(1 - \delta^i_{N+1}\right)\left[\left[S^i_{|\gamma}\right]_N A^{\gamma j} + \left(\frac{1}{\lambda}\right)\left(\frac{\partial S^i}{\partial t} + S^\psi\left[S^i_{|\psi}\right]_N\right) A^{N+1,j}\right]$$
$$- \left(1 - \delta^j_{N+1}\right)\left[\left[S^j_{|\gamma}\right]_N A^{\gamma i} + \left(\frac{1}{\lambda}\right)\left(\frac{\partial S^j}{\partial t} + S^\psi\left[S^j_{|\psi}\right]_N\right) A^{N+1,i}\right] \tag{64}$$

Applying equations (12) and (13) yields

$$\left.\begin{aligned}
\frac{\partial}{\partial t}\left(A_{\alpha\beta}\right) &= \frac{\partial}{\partial t}\left(g_{\alpha\beta}\right) = \left[S^\gamma_{|\alpha}\right]_N g_{\gamma\beta} + \left[S^\gamma_{|\beta}\right]_N g_{\alpha\gamma} \\
\frac{\partial}{\partial t}\left(A_{N+1,\beta}\right) &= \left(\frac{1}{\lambda}\right)\left(\frac{\partial S^\gamma}{\partial t} + S^\psi\left[S^\gamma_{|\psi}\right]_N\right) g_{\gamma\beta} + \left(\frac{1}{\lambda}\right)\left[S^\gamma_{|\beta}\right]_N g_{\gamma\phi} S^\phi \\
\frac{\partial}{\partial t}\left(A_{N+1,N+1}\right) &= 2\left(\frac{1}{\lambda}\right)^2\left(\frac{\partial S^\gamma}{\partial t} + S^\psi\left[S^\gamma_{|\psi}\right]_N\right) g_{\gamma\phi} S^\phi
\end{aligned}\right\} \tag{65a}$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left(A^{\alpha\beta}\right) = -\left[S^\alpha_{|\gamma}\right]_N g^{\gamma\beta} - \left[S^\beta_{|\gamma}\right]_N g^{\alpha\gamma} + \left(\frac{1}{\lambda}\right)^2 \frac{\partial}{\partial t}\left(S^\alpha S^\beta\right) \\
&\frac{\partial}{\partial t}\left(A^{N+1,\beta}\right) = -\left(\frac{1}{\lambda}\right)\frac{\partial S^\beta}{\partial t} \qquad \frac{\partial}{\partial t}\left(A^{N+1,N+1}\right) = 0 \\
&\frac{\partial}{\partial t}\left(g^{\alpha\beta}\right) = \frac{\partial}{\partial t}\left[A^{\alpha\beta} - \left(\frac{1}{\lambda}\right)^2 S^\alpha S^\beta\right] = -\left[S^\alpha_{|\gamma}\right]_N g^{\gamma\beta} - \left[S^\beta_{|\gamma}\right]_N g^{\alpha\gamma}
\end{aligned}\right\} \tag{65b}$$

Equations of Motion of a Material Continuum

In this section, the equations of motion are derived in the Riemann space-time tensor formalism from fundamental principles. This approach, rather than that of merely restating the existing equations of motion in the tensor notation, is taken in order to demonstrate the versatility and simplicity of the Riemann space-time tensor formulation.

The equations describing the motions and other properties of a material body regarded as continuous are commonly written from either of two viewpoints, the Eulerian or the Lagrangian. The Eulerian viewpoint is that of an observer at rest relative to the coordinate system; the Lagrangian viewpoint is that of an observer at rest relative to the material continuum.

An Eulerian statement of generalized conservation of a general tensor quantity $\hat{T}$ in an inertial rectangular Cartesian system $\tilde{z}$ is

$$\frac{\partial}{\partial \tilde{z}^{\gamma}}\left(\rho \hat{T} V^{\gamma}\right) + \frac{\partial}{\partial t}\left(\rho \hat{T}\right) = C\left(\rho \hat{T}\right) \tag{66}$$

where ρ is the mass density and $C\left(\rho \hat{T}\right)$ is the per unit volume rate of creation of the product of mass M and $\hat{T}$, or the source term. Here, "creation" is the increase in a quantity from any cause other than transport. The first term on the left side of equation (66) is the net per volume flux of $M\hat{T}$ transported out of a given volume element, and the second term on the left side is the observed per volume rate of increase of $M\hat{T}$ within the volume element.

Equation (66) can be written, by substituting $d\tilde{z}^{N+1} = \lambda \, dt$ and $V^{N+1} = \lambda$,

$$\frac{\partial}{\partial \tilde{z}^{i}}\left(\rho \hat{T} V^{i}\right) = C\left(\rho \hat{T}\right) \tag{67}$$

which generalizes to

$$\left(\rho \hat{T} V^{i}\right)_{|i} = C\left(\rho \hat{T}\right) \tag{68}$$

in any allowed coordinate system in Euclidean space-time. The mass continuity equation in Cartesian coordinates

$$\frac{\partial}{\partial \tilde{z}^{\gamma}}\left(\rho V^{\gamma}\right) + \frac{\partial \rho}{\partial t} = 0$$

is equation (66) for $\hat{T} = 1$ and $C(\rho) = 0$; thus,

$$\left(\rho V^{i}\right)_{|i} = 0 \tag{69}$$

in Euclidean space-time, by equations (66), (67), and (68). Applying equation (69) to equation (68) yields the Lagrangian expression of generalized conservation

$$\rho V^{i}\, \hat{T}_{|i} = \rho \frac{D}{dt}\left(\hat{T}\right) = C\left(\rho \hat{T}\right) \tag{70}$$

It follows from equations (A16) and (A17) in appendix A and equation (43) that equation (70) holds in any Riemann space-time which is a subspace of Euclidean space-time if all components of V^{i} and $\hat{T}$ not contained in the subspace vanish. (All components of S^{i} normal to the subspace vanish, by eq. (A4) in appendix A.)

Equation (69) also holds in the Riemann subspace if all components of V^{i} normal to the subspace vanish and if

$$\frac{1}{\sqrt{\bar{g}}}\frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{\bar{g}}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{g}\right) \tag{71}$$

where $\bar{i}$ is the subspace index, $\bar{g}$ is the determinant of the covariant spatial metric in the subspace, and g is the spatial metric determinant in Euclidean space-time. The proof follows from the well-known identity

$$T^{i}_{|i} = \frac{1}{\sqrt{A}}\frac{\partial}{\partial y^{i}}\left(\sqrt{A}\, T^{i}\right) \tag{72}$$

and equations (12) for the space-time covariant metric tensor. Let

$$B = A_{N+1,N+1} - 1 = \left(\frac{1}{\lambda}\right)^{2} g_{\alpha\beta}\, S^{\alpha}\, S^{\beta}$$

Then

$$A = \left|\begin{bmatrix} g_{\alpha\beta} & \begin{matrix} A_{1,N+1} \\ \vdots \\ A_{N,N+1} \end{matrix} \\ A_{1,N+1} \cdots A_{N,N+1} & B \end{bmatrix}\right| + \left|\begin{bmatrix} g_{\alpha\beta} & \begin{matrix} A_{1,N+1} \\ \vdots \\ A_{N,N+1} \end{matrix} \\ 0 \cdots 0 & 1 \end{bmatrix}\right| \tag{73}$$

In the first determinant on the right side, the (N+1)th row is a linear combination of the rows; that is, from equations (12),

$$\text{(N+1)th row} = \frac{S^1}{\lambda}\ \text{(1st row)} + \ldots + \frac{S^N}{\lambda}\ \text{(Nth row)}$$

Thus, the determinant vanishes, and

$$A = g \tag{74}$$

in any allowed coordinate system in Riemann space-time. Applying equations (74) and (72) to equation (69) yields

$$\left(\rho V^i\right)_{|i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^i}\left(\sqrt{g}\,\rho V^i\right) = 0 \tag{75}$$

or, by the assumption of V^i restricted to the subspace,

$$\begin{aligned}\left(\rho V^i\right)_{|i} &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{g}\,\rho V^{\bar{i}}\right) \\ &= \frac{\partial}{\partial y^{\bar{i}}}\left(\rho V^{\bar{i}}\right) + \rho V^{\bar{i}} \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{g}\right) \\ &= 0 \end{aligned} \tag{76}$$

which differs from the subspace divergence expression by

$$\rho V^{\bar{i}} \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{g}\right) - \rho V^{\bar{i}} \frac{1}{\sqrt{\bar{g}}} \frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{\bar{g}}\right)$$

Thus, equation (71) and the assumed restriction of V^i to the subspace are sufficient conditions for equation (69) to hold in the Riemann subspace.

Interpretation of the condition expressed by equation (71) is facilitated by letting the coordinates $y^{\bar{\bar{\alpha}}}$, held fixed to define the Riemann subspace, be normal to the subspace and, for two $y^{\bar{\bar{\alpha}}}$ coordinates, mutually orthogonal. Then

$$g = \bar{g}\,\bar{\bar{g}} \tag{77}$$

where $\bar{\bar{g}}$ is the determinant of the covariant metric tensor in the subspace of the $y^{\bar{\bar{\alpha}}}$ coordinates. Substitution of equation (77) into equation (71) yields the condition equivalent to equation (71)

$$\frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{\bar{\bar{g}}}\right) = 0 \tag{78}$$

If the subspace of $y^{\bar{\bar{\alpha}}}$ is one-dimensional, then $\sqrt{\bar{\bar{g}}}$ is the distance per unit increment of $y^{\bar{\bar{\alpha}}}$. If the subspace of $y^{\bar{\bar{\alpha}}}$ is two-dimensional, then $\sqrt{\bar{\bar{g}}}$ is the area per unit coordinate increment product $dy^2\, dy^3$ for $\bar{\bar{\alpha}} = 2,3$.

If equations (69) and (70) both hold in a Riemann space-time, then so does equation (68), since

$$\left(\rho V^i\, \hat{T}\right)_{|i} = \rho V^i\, \hat{T}_{|i} + \left(\rho V^i\right)_{|i}\, \hat{T} = C\left(\rho \hat{T}\right) \tag{79}$$

Therefore, sufficient conditions for equation (79) to hold in a Riemann space-time are

(1) All components of $\hat{T}$ and V^i not contained in the Riemann subspace vanish

(2) $\frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{\bar{\bar{g}}}\right) = 0$, or equivalently, $\frac{1}{\sqrt{\bar{g}}}\frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{\bar{g}}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial y^{\bar{i}}}\left(\sqrt{g}\right)$

For some applications of fluid mechanics, sources and sinks for mass are employed as a mathematical artifice (see, for example, ref. 5, chapter 5), which allows $C(\rho)$ to be nonzero. No sources or sinks for mass are employed herein, and $C(\rho)$ vanishes everywhere.

Generalized conservation of mass and momentum is expressible in a single tensor equation by setting $\hat{T} = V^j$ in equation (68):

$$\left(\rho V^j\, V^i\right)_{|i} = C\left(\rho V^j\right) = f^j \tag{80}$$

where, from equation (70), the force per unit volume is

$$\left.\begin{aligned} C\left(\rho V^{\alpha}\right) &= f^{\alpha} \\ C\left(\rho V^{N+1}\right) &= f^{N+1} = 0 \end{aligned}\right\} \tag{81}$$

Thus, f^j is a space-time tensor. The Lagrangian form of equation (80) is

$$\frac{D}{dt}\left(\rho V^j\right) + \rho V^j V^i_{|i} = f^j \tag{82}$$

Either equation (80) or (82) expresses generalized conservation of momentum for $j \neq N + 1$, and mass conservation for $j = N + 1$.

The force per unit volume, given by the covariant form of equation (80),

$$f_i = \left(\rho V_i V^j\right)_{|j}$$

is separated into the work-producing and velocity-turning components by

$$\begin{aligned} f_i &= \left(\rho V^j\right)_{|j} V_i + \rho V^j V_{i|j} \\ &= \frac{1}{2} \rho V^j \left[\left(V_{i|j} + V_{j|i}\right) + \left(V_{i|j} - V_{j|i}\right)\right] \\ &= \frac{1}{2} \rho \frac{DV_i}{dt} + \frac{1}{4} \rho \left(V^j V_j\right)_{|i} + \rho V^j \frac{1}{2} \left(V_{i|j} - V_{j|i}\right) \\ &= \frac{1}{2} \rho \left(V^j V_j\right)_{|i} + 2\rho V^j \omega_{ij} \end{aligned} \tag{83}$$

by use of the Eulerian mass conservation equation (69), where ω_{ij} is the "space-time vorticity" given by

$$\omega_{ij} = \frac{1}{2}\left(V_{i|j} - V_{j|i}\right) = \frac{1}{2}\left(\frac{\partial V_i}{\partial y^j} - \frac{\partial V_j}{\partial y^i}\right) \tag{84}$$

Since $S_{i|j} \equiv 0$ (by eqs. (A19) and (A21) in appendix A), an alternate expression for ω_{ij} is

$$\omega_{ij} = \frac{1}{2}\left(U_{i|j} - S_{i|j} - U_{j|i} + S_{j|i}\right) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y^j} - \frac{\partial U_j}{\partial y^i}\right) \tag{85}$$

The invariant $f_i\ V^i$ is, from equation (83),

$$f_i\ V^i = \rho\ \frac{D}{dt}\left(\frac{1}{2}\ V^j\ V_j\right) = C\left(\frac{1}{2}\ \rho V^j\ V_j\right) \tag{86}$$

by equation (70) and the skew-symmetry of ω_{ij}.

An alternative to the Eulerian momentum equation (80) in terms of kinetic energy and vorticity is obtained by partially expanding equation (83) for $i = \alpha$:

$$\begin{aligned} f_\alpha &= \rho\ \frac{\partial}{\partial y^\alpha}\left(\frac{1}{2}\ V^j\ V_j\right) + 2\rho V^\gamma\ \omega_{\alpha\gamma} + 2\rho V^{N+1}\ \omega_{\alpha,N+1} \\ &= \rho\ \frac{\partial}{\partial y^\alpha}\left(\frac{1}{2}\ V^j\ V_j\right) + 2\rho V^\gamma\ \omega_{\alpha\gamma} + 2\rho\lambda\ \frac{1}{2}\left(\frac{\partial V_\alpha}{\partial y^{N+1}} - \frac{\partial V_{N+1}}{\partial y^\alpha}\right) \end{aligned} \tag{87}$$

where, from equations (18),

$$\rho\lambda\ \frac{\partial V_{N+1}}{\partial y^\alpha} = \rho\ \frac{\partial}{\partial y^\alpha}\left(U_\beta\ S^\beta\right)$$

Thus, rearranging equation (87) gives

$$\rho\lambda\ \frac{\partial V_\alpha}{\partial y^{N+1}} = \rho\ \frac{\partial V_\alpha}{\partial t} = f_\alpha - \rho\ \frac{\partial}{\partial y^\alpha}\left(\frac{1}{2}\ V^j\ V_j\right) - 2\rho V^\gamma\ \omega_{\alpha\gamma} + \rho\ \frac{\partial}{\partial y^\alpha}\left(U_\beta\ S^\beta\right) \tag{88}$$

The Eulerian rate of change of the vorticity $\omega_{\alpha\gamma}$ in equation (88) is obtained by applying the operators $\frac{1}{2}\ \frac{\partial}{\partial y^\beta}$ and $-\frac{1}{2}\ \frac{\partial}{\partial y^\alpha}$ to $\frac{\partial V_\alpha}{\partial t}$ and $\frac{\partial V_\beta}{\partial t}$, and summing:

$$\begin{aligned} \frac{1}{2}\ \frac{\partial}{\partial y^\beta}\left(\frac{\partial V_\alpha}{\partial t}\right) - \frac{1}{2}\ \frac{\partial}{\partial y^\alpha}\left(\frac{\partial V_\beta}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\omega_{\alpha\beta}\right) \\ &= \frac{1}{2}\left[\frac{\partial}{\partial y^\beta}\left(\frac{f_\alpha}{\rho}\right) - \frac{\partial}{\partial y^\alpha}\left(\frac{f_\beta}{\rho}\right)\right] - \frac{\partial}{\partial y^\beta}\left(V^\gamma\ \omega_{\alpha\gamma}\right) + \frac{\partial}{\partial y^\alpha}\left(V^\gamma\ \omega_{\beta\gamma}\right) \end{aligned} \tag{89}$$

Thus, if the spatial components of the vorticity field vanish and the acceleration is derivable from a potential function, then the spatial components of the vorticity vanish for all time.

The Lagrangian rate of change of vorticity is

$$\frac{D}{dt}\left(\omega_{\alpha\beta}\right) = \frac{d}{dt}\left(\omega_{\alpha\beta}\right) - \omega_{\alpha i}\begin{Bmatrix} & i & \\ \beta & & j\end{Bmatrix} V^j - \omega_{i\beta}\begin{Bmatrix} & i & \\ \alpha & & j\end{Bmatrix} V^j$$

where

$$\frac{d}{dt}\left(\omega_{\alpha\beta}\right) = \frac{\partial}{\partial t}\left(\omega_{\alpha\beta}\right) + \frac{\partial}{\partial y^\gamma}\left(\omega_{\alpha\beta}\right) V^\gamma$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial y^\beta}\left(\frac{f_\alpha}{\rho}\right) - \frac{\partial}{\partial y^\alpha}\left(\frac{f_\beta}{\rho}\right)\right]$$

$$+ \left[\frac{\partial}{\partial y^\beta}\left(\omega_{\gamma\alpha}\right) + \frac{\partial}{\partial y^\alpha}\left(\omega_{\beta\gamma}\right) + \frac{\partial}{\partial y^\gamma}\left(\omega_{\alpha\beta}\right)\right] V^\gamma - \omega_{\alpha\gamma}\frac{\partial V^\gamma}{\partial y^\beta} + \omega_{\beta\gamma}\frac{\partial V^\gamma}{\partial y^\alpha}$$

by equation (89) and the skew-symmetry of $\omega_{\alpha\gamma}$. The second bracketed term on the right vanishes, by the identity

$$\frac{\partial}{\partial y^\alpha}\left(\omega_{\beta\gamma}\right) + \frac{\partial}{\partial y^\beta}\left(\omega_{\gamma\alpha}\right) + \frac{\partial}{\partial y^\gamma}\left(\omega_{\alpha\beta}\right) = 0 \tag{90}$$

Thus,

$$\frac{d}{dt}\left(\omega_{\alpha\beta}\right) = \frac{1}{2}\left[\frac{\partial}{\partial y^\beta}\left(\frac{f_\alpha}{\rho}\right) - \frac{\partial}{\partial y^\alpha}\left(\frac{f_\beta}{\rho}\right)\right] - \omega_{\alpha\gamma}\frac{\partial V^\gamma}{\partial y^\beta} + \omega_{\beta\gamma}\frac{\partial V^\gamma}{\partial y^\alpha} \tag{91}$$

and

$$\frac{D}{dt}\left(\omega_{\alpha\beta}\right) = \frac{1}{2}\left[\frac{\partial}{\partial y^\beta}\left(\frac{f_\alpha}{\rho}\right) - \frac{\partial}{\partial y^\alpha}\left(\frac{f_\beta}{\rho}\right)\right] - \omega_{\alpha\gamma} V^\gamma{}_{|\beta} + \omega_{\beta\gamma} V^\gamma{}_{|\alpha} \tag{92}$$

by the skew-symmetry of $\omega_{i\beta}$ and the vanishing of $\left\{ {}_{k}{}^{N+1}{}_{j} \right\}$ by equation (27). The source term for the vorticity-mass product

$$C\left(\rho\omega_{\alpha\beta}\right) = \rho \frac{D}{dt}\left(\omega_{\alpha\beta}\right)$$

evaluated by equation (92), can now be substituted into the generalized conservation equations (68) and (70), with $\hat{T} = \omega_{\alpha\beta}$.

CONCLUSIONS

A space-time tensor formulation for nonrelativistic continuum mechanics has been presented which assures that any relationship of quantities expressed as a space-time tensor equation is equivalently expressed in all general curvilinear, moving, deforming coordinate systems. For example, D'Alambert's "force" is inherently accounted for in the tensor expression and transformation of acceleration and force.

Thus, the space-time tensor formulation applies to situations in which boundaries move and deform, and a coordinate surface is placed on the boundary in order to simplify the boundary-condition treatment.

The space-time tensor formulation also applies to situations in which one coordinate can be defined as a function of any continuum scalar (e.g., pressure, temperature, or density). During numerical integration of the equations of motion, the continuum scalar can then be represented with enhanced resolution by coordinate surfaces which move and deform as the scalar varies. Also, the vanishing of the scalar gradient along these coordinate surfaces can be a significant simplification.

The generality and compactness of the space-time tensor formulation lend it utility as a vehicle for formal manipulations as well.

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APPENDIX A

RELATIONSHIP OF CHRISTOFFEL SYMBOLS IN RIEMANN SPACE-TIME TO THOSE IN EUCLIDEAN SPACE-TIME

The Riemann space-time considered herein is the locus of constant $y^{\bar{\bar{\alpha}}}$, where $\bar{\bar{\alpha}} = 3$ or $\bar{\bar{\alpha}} = 2,3$; and where the y coordinate system is any allowed system over Euclidean four-dimensional space-time. Quantities and indices defined with respect to the Riemann subspace of Euclidean space-time are denoted by a single bar; for example:

$\bar{N}$ = Number of spatial dimensions in the subspace

$\bar{\alpha}, \bar{\beta}$, etc. $= 1,\ldots,\bar{N}$

$$d\bar{y}^{\bar{N}+1} = \lambda\, dt = dy^{N+1}$$

$$\bar{y}^{\bar{\alpha}} = y^{\bar{\alpha}}$$

For arbitrary coordinate displacements in the subspace, the relationships

$$\bar{A}_{\bar{\alpha}\bar{\beta}}\, d\bar{y}^{\bar{\alpha}}\, d\bar{y}^{\bar{\beta}} = \bar{A}_{\bar{\alpha}\bar{\beta}}\, dy^{\bar{\alpha}}\, dy^{\bar{\beta}} = A_{\bar{\alpha}\bar{\beta}}\, dy^{\bar{\alpha}}\, dy^{\bar{\beta}}$$

$$\bar{A}_{\bar{\alpha},\bar{N}+1}\, d\bar{y}^{\bar{\alpha}}\, d\bar{y}^{\bar{N}+1} = A_{\bar{\alpha}4}\, dy^{\bar{\alpha}}\, dy^4$$

$$\bar{A}_{\bar{N}+1,\bar{N}+1}\, d\bar{y}^{\bar{N}+1}\, d\bar{y}^{\bar{N}+1} = A_{44}\, dy^4\, dy^4$$

imply, with equations (12), that

$$\left.\begin{aligned} &\bar{A}_{\bar{\alpha}\bar{\beta}} = A_{\bar{\alpha}\bar{\beta}} = g_{\bar{\alpha}\bar{\beta}} = \bar{g}_{\bar{\alpha}\bar{\beta}} \\ &\bar{A}_{\bar{\alpha},\bar{N}+1} = A_{\bar{\alpha}4} \\ &\bar{A}_{\bar{N}+1,\bar{N}+1} = A_{44} \end{aligned}\right\} \tag{A1}$$

Therefore, from equations (12),

$$A_{\bar{\alpha}4} = \frac{1}{\lambda} g_{\bar{\alpha}\phi} S^{\phi} = \bar{A}_{\bar{\alpha},\bar{N}+1} = \frac{1}{\lambda} g_{\bar{\alpha}\bar{\phi}} \bar{S}^{\bar{\phi}} = \frac{1}{\lambda} g_{\bar{\alpha}\bar{\phi}} S^{\bar{\phi}}$$

or,

$$g_{\bar{\alpha}\bar{\bar{\phi}}} S^{\bar{\bar{\phi}}} = 0 \tag{A2}$$

where $\bar{S}^{\bar{\phi}} = S^{\bar{\phi}}$ follows from equation (10), and where indices with a double bar are of the y coordinates held fixed to define the subspace.

From equations (12) and (A1),

$$A_{44} = \bar{A}_{\bar{N}+1,\bar{N}+1} = \left(\frac{1}{\lambda}\right) g_{\alpha\beta} S^{\alpha} S^{\beta} + 1$$

$$= \left(\frac{1}{\lambda}\right) g_{\bar{\alpha}\bar{\beta}} S^{\bar{\alpha}} S^{\bar{\beta}} + 1$$

Therefore,

$$2g_{\bar{\bar{\alpha}}\bar{\beta}} S^{\bar{\bar{\alpha}}} S^{\bar{\beta}} + g_{\bar{\bar{\alpha}}\bar{\bar{\beta}}} S^{\bar{\bar{\alpha}}} S^{\bar{\bar{\beta}}} = 0$$

or, by equation (A2),

$$g_{\bar{\bar{\alpha}}\bar{\bar{\beta}}} S^{\bar{\bar{\alpha}}} S^{\bar{\bar{\beta}}} = 0 \tag{A3}$$

The left-hand side of equation (A3) is positive-definite; hence

$$S^{\bar{\bar{\alpha}}} = 0 \tag{A4}$$

Equation (20) expresses the transformation of the Christoffel symbols of the second kind between coordinate systems over the same space-time, and so, it is sufficient to consider the simplifying case in which the $y^{\bar{\bar{\alpha}}}$ coordinates are normal to the spatial subspace coordinates $y^{\bar{\phi}}$ and, if $\bar{N} = 1$, the y^2 coordinate is normal to the y^3 coordinate. Then,

$$\left.\begin{array}{ll} dy^3 A_{3i}\, dy^{\bar{\alpha}}\, \delta^i_{\bar{\alpha}} = 0 & (\bar{N} = 1 \text{ or } 2) \\ dy^2 A_{2i}\, dy^{\bar{\alpha}}\, \delta^i_{\bar{\alpha}} = 0 & (\bar{N} = 1) \\ dy^2 A_{2i}\, dy^3\, \delta^i_3 = 0 & (\bar{N} = 1) \end{array}\right\} \qquad \text{(A5)}$$

follow from the condition of orthogonality between the general contravariant vectors E^i and Q^i; that is, $E_i Q^i = 0$. Therefore,

$$\left.\begin{array}{ll} A_{\bar{\bar{\beta}}\bar{\alpha}} = 0 & (\bar{N} = 1 \text{ or } 2) \\ A_{23} = 0 & (\bar{N} = 1) \end{array}\right\} \qquad \text{(A6)}$$

and, by equations (12), (A4), and (A6),

$$\begin{aligned} A_{\bar{\bar{\alpha}}4} &= \frac{1}{\lambda} g_{\bar{\bar{\alpha}}\phi}\, S^{\phi} \\ &= \frac{1}{\lambda} g_{\bar{\bar{\alpha}}\bar{\phi}}\, S^{\bar{\phi}} + \frac{1}{\lambda} g_{\bar{\bar{\alpha}}\bar{\bar{\phi}}}\, S^{\bar{\bar{\phi}}} \\ &= \frac{1}{\lambda} A_{\bar{\bar{\alpha}}\bar{\phi}}\, S^{\bar{\phi}} \\ &= 0 \end{aligned} \qquad \text{(A7)}$$

In general,

$$A_{ij}\, A^{ik} = \delta^k_j$$

Thus, from equations (A6) and (A7),

$$A_{i\bar{\bar{\alpha}}}\, A^{ik} = \delta^k_{\bar{\bar{\alpha}}} = A_{\bar{\beta}\bar{\bar{\alpha}}}\, A^{\bar{\beta}k} + A_{\bar{\bar{\beta}}\bar{\bar{\alpha}}}\, A^{\bar{\bar{\beta}}k} + A_{4\bar{\bar{\alpha}}}\, A^{4k}$$

or

$$\delta^{k}_{\bar{\bar{\alpha}}} = A_{\bar{\bar{\beta}}\bar{\bar{\alpha}}} A^{\bar{\bar{\beta}}k} \tag{A8}$$

If $\overline{N} = 2$, then $\bar{\bar{\beta}} = \bar{\bar{\alpha}} = 3$, and equation (A8) yields $A_{33} \neq 0$ and

$$\left.\begin{aligned} A^{\bar{\bar{\beta}}\bar{\gamma}} &= 0 \\ A^{\bar{\bar{\beta}}4} &= 0 \end{aligned}\right\} \tag{A9}$$

If $\overline{N} = 1$, then $\bar{\bar{\beta}}, \bar{\bar{\alpha}} = 2$ or 3, and equations (A8) and (A6) yield

$$\left.\begin{aligned} A^{\bar{\bar{\beta}}\bar{\gamma}} &= 0 \\ A^{\bar{\bar{\beta}}4} &= 0 \end{aligned}\right\} \tag{A10}$$

Thus equations (A10) hold for $\overline{N} = 1$ and 2.

The spatial metric $g^{\alpha\beta}$ is uniquely determined by the $g_{\alpha\beta}$ according to the relationship

$$g^{\alpha\beta} g_{\alpha\gamma} = \delta^{\beta}_{\gamma} \tag{A11}$$

From equations (12), (A1), and (A6),

$$\begin{aligned} g^{\alpha\bar{\beta}} g_{\alpha\bar{\gamma}} &= g^{\bar{\alpha}\bar{\beta}} \bar{g}_{\bar{\alpha}\bar{\gamma}} + g^{\bar{\bar{\alpha}}\bar{\beta}} g_{\bar{\bar{\alpha}}\bar{\gamma}} \\ &= g^{\bar{\alpha}\bar{\beta}} \bar{g}_{\bar{\alpha}\bar{\gamma}} = \delta^{\bar{\beta}}_{\bar{\gamma}} \\ &= \bar{g}^{\bar{\alpha}\bar{\beta}} \bar{g}_{\bar{\alpha}\bar{\gamma}} \end{aligned}$$

Therefore,

$$\bar{g}^{\bar{\alpha}\bar{\beta}} = g^{\bar{\alpha}\bar{\beta}} \tag{A12}$$

APPENDIX A

From equations (10), (13), and (A12),

$$\left.\begin{aligned}
\bar{A}^{\bar{\alpha}\bar{\beta}} &= A^{\bar{\alpha}\bar{\beta}} \\
\bar{A}^{\bar{\alpha},\bar{N}+1} &= A^{\bar{\alpha}4} \\
\bar{A}^{\bar{N}+1,\bar{N}+1} &= A^{44}
\end{aligned}\right\} \tag{A13}$$

The Christoffel symbol of the second kind in the y coordinate system over Euclidean space-time (which also holds for Riemann space) is given by (eq. (19))

$$\left\{ {}_{j}{}^{i}{}_{k} \right\} = \frac{1}{2} A^{in} \left(\frac{\partial A_{kn}}{\partial y^{j}} + \frac{\partial A_{nj}}{\partial y^{k}} - \frac{\partial A_{jk}}{\partial y^{n}} \right) \tag{A14}$$

the components of which in the subspace of constant $y^{\bar{\bar{\alpha}}}$ are

$$\left\{ {}_{\bar{\beta}}{}^{\bar{\alpha}}{}_{\bar{\gamma}} \right\} = \frac{1}{2} A^{\bar{\alpha}n} \left(\frac{\partial A_{\bar{\gamma}n}}{\partial y^{\bar{\beta}}} + \frac{\partial A_{n\bar{\beta}}}{\partial y^{\bar{\gamma}}} - \frac{\partial A_{\bar{\beta}\bar{\gamma}}}{\partial y^{n}} \right) \tag{A15}$$

By equations (A10), $A^{\bar{\alpha}n}$ vanishes for $n = \bar{\bar{\beta}}$; therefore, from equations (A13) and (A1), the spatial components of the subspace Christoffel symbol are given by

$$\overline{\left\{ {}_{\bar{\beta}}{}^{\bar{\alpha}}{}_{\bar{\gamma}} \right\}} = \left\{ {}_{\bar{\beta}}{}^{\bar{\alpha}}{}_{\bar{\gamma}} \right\} \tag{A16}$$

where, it should be emphasized, the $y^{\bar{\bar{\alpha}}}$ coordinates (which are those held fixed to define the Riemann subspace) are normal to the spatial coordinates in the subspace and, for $\bar{N} = 1$, normal to each other. Similarly, from equations (A10), (A13), and (A1),

$$\overline{\left\{ {}_{\bar{N}+1}{}^{\bar{\alpha}}{}_{\bar{\gamma}} \right\}} = \left\{ {}_{4}{}^{\bar{\alpha}}{}_{\bar{\gamma}} \right\} \tag{A17a}$$

$$\overline{\left\{ {}_{\bar{N}+1}{}^{\bar{\alpha}}{}_{\bar{N}+1} \right\}} = \left\{ {}_{4}{}^{\bar{\alpha}}{}_{4} \right\} \tag{A17b}$$

$$\overline{\left\{ {}_{\bar{\beta}}{}^{\bar{N}+1}{}_{\bar{\gamma}} \right\}} = \left\{ {}_{\bar{\beta}}{}^{4}{}_{\bar{\gamma}} \right\} \tag{A17c}$$

$$\overline{\left\{ {}_{\overline{N}+1}{}^{\overline{N}+1}{}_{\bar{\gamma}} \right\}} = \left\{ {}_{4}{}^{4}{}_{\bar{\gamma}} \right\} \tag{A17d}$$

$$\overline{\left\{ {}_{\overline{N}+1}{}^{\overline{N}+1}{}_{\overline{N}+1} \right\}} = \left\{ {}_{4}{}^{4}{}_{4} \right\} \tag{A17e}$$

for the $y^{\bar{\bar{\alpha}}}$ coordinates normal to the spatial coordinates in the Riemann subspace and, for $\overline{N} = 1$, mutually orthogonal.

All Christoffel symbols of the form $\left\{ {}_{i}{}^{4}{}_{j} \right\}$ or $\overline{\left\{ {}_{\bar{i}}{}^{\overline{N}+1}{}_{\bar{j}} \right\}}$ in any allowed coordinate system vanish, as shown in the following equations, where $\bar{i}, \bar{j} = 1, \ldots, \overline{N} + 1$. In a Cartesian coordinate system $\tilde{z}$ over Euclidean space-time, the covariant derivative of S_i

$$S_{i|j(y)} = \frac{\partial S_i}{\partial y^j} - S_k \left\{ {}_{i}{}^{k}{}_{j} \right\} \tag{A18}$$

becomes

$$S_{i|j(\tilde{z})} = \frac{\partial S_i}{\partial \tilde{z}^j} = 0$$

by equations (18) $\left(S_{\alpha} = 0, \quad S_{N+1} = -\lambda \text{ in any allowed system} \right)$. Therefore,

$$S_{i|j(y)} = \lambda \left\{ {}_{i}{}^{4}{}_{j} \right\} = 0 \tag{A19}$$

and, by equations (A17),

$$\overline{\left\{ {}_{\bar{i}}{}^{\overline{N}+1}{}_{\bar{j}} \right\}} = 0 \tag{A20}$$

Thus,

$$\overline{\left(S_{\bar{i}|\bar{j}} \right)} = \lambda \overline{\left\{ {}_{\bar{i}}{}^{\overline{N}+1}{}_{\bar{j}} \right\}} = 0 \tag{A21}$$

APPENDIX B

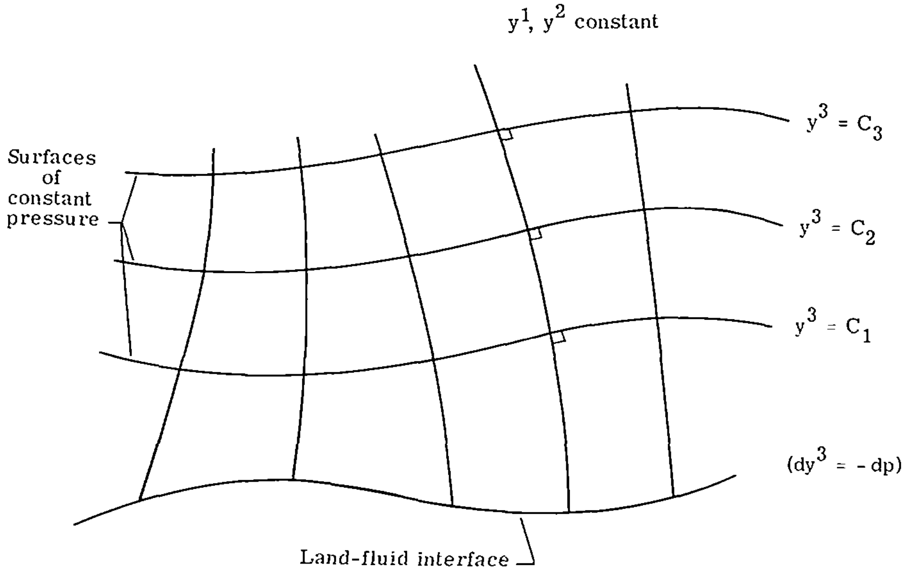
EXAMPLE PROBLEM OF IDEAL FLUID WITH TOP FREE SURFACE IN WHICH SURFACES OF CONSTANT PRESSURE ARE USED AS COORDINATE SURFACES

Coordinate System

Although the governing equations of a material continuum usually have the simplest form when expressed in Cartesian coordinates, it is often advantageous in solving the equations to set coordinate surfaces coincident with boundaries, in order to simplify the boundary treatment. If the boundaries are moving and deforming, then a moving, deforming coordinate surface is desirable.

On the other hand, it is advantageous to define coordinate surfaces as surfaces of constant pressure, density, temperature, or any other continuum scalar variable, and to define coordinate lines normal to the surfaces. The coordinate surfaces and lines would then move and deform as the continuum variable changes (with respect to a rigid coordinate system). The vanishing variable gradient components in the family of coordinate surfaces and the orthogonality of the other coordinate lines to the surfaces may simplify the set of governing equations.

These points are illustrated in the example problem of an incompressible inviscid fluid with a top free surface moving over the Earth, where, as shown in sketch (b), the



Sketch (b)

surfaces of constant y^3 are the surfaces of constant pressure. Implicit in this representation is the assumption that the pressure increase with depth, which requires only that the fluid not accelerate downward at a rate equal to or greater than the acceleration of gravity. The y^3 coordinate lines follow the pressure gradient and thus are normal to the constant-pressure surfaces. The land-fluid boundary is not, in general, a surface of constant y^3. The land-fluid boundary may rise to intersect the fluid free surface, but it is assumed that the slope of the land-fluid boundary is less than the slope of the y^3 coordinate lines, so that all y^3 coordinate lines intersect the free surface. The y^3 coordinate increments are defined as $dy^3 = -dp$.

Governing Equations

For an incompressible, inviscid fluid, the covariant force per unit volume is, from equations (12) and (81),

$$f_{\alpha(y)} = \left(-\frac{\partial p}{\partial \tilde{z}^\gamma} - \rho \frac{\partial \theta}{\partial \tilde{z}^\gamma}\right) \frac{\partial \tilde{z}^\gamma}{\partial y^\alpha} = -\frac{\partial p}{\partial y^\alpha} - \rho \frac{\partial \theta}{\partial y^\alpha} \tag{B1}$$

$$f_{4(y)} = f^i_{(y)} A_{i4} = \frac{1}{\lambda} f^\gamma g_{\gamma\phi} S^\phi = \frac{1}{\lambda} f_\phi S^\phi \tag{B2}$$

where p is the pressure, ρ is the density, and θ is the gravitational potential. It should be noted that θ is an invariant scalar in equation (B1); and, when θ is considered in the inertial $\tilde{z}$ coordinate system, it becomes clear that θ does not contain the centrifugal "potential" of Earth rotation which is included in the gravity (as opposed to gravitational) "potential." (Neither the centrifugal nor the gravity field is a true potential field, because neither is a scalar invariant field.) The centripetal and Coriolis accelerations appear in the acceleration tensor $\frac{D}{dt}\left(V^i\right)$, related to the force per unit volume by

$$\rho \frac{D}{dt}\left(V^i\right) = f^i \tag{B3}$$

from equations (70) and (81).

Equations (B1) and (B2) in contravariant form are, by equations (13) and (81),

$$\left.\begin{aligned} f^\beta &= \left(-\frac{\partial p}{\partial y^\alpha} - \rho \frac{\partial \theta}{\partial y^\alpha}\right) A^{\alpha\beta} + \frac{1}{\lambda} f_\phi S^\phi A^{4\beta} \\ f^\beta &= -\left(\frac{\partial p}{\partial y^\alpha} + \rho \frac{\partial \theta}{\partial y^\alpha}\right) g^{\alpha\beta} \qquad\qquad f^4 = 0 \end{aligned}\right\} \tag{B4}$$

Expanding equations (B4) by use of $\frac{\partial p}{\partial y^1} = \frac{\partial p}{\partial y^2} = 0$ and

$$g^{13} = g^{23} = g_{13} = g_{23} = 0 \tag{B5}$$

yields the expressions for f^i

$$\left.\begin{aligned} f^1 &= -\rho\left(\frac{\partial\theta}{\partial y^1} g^{11} + \frac{\partial\theta}{\partial y^2} g^{12}\right) \\ f^2 &= -\rho\left(\frac{\partial\theta}{\partial y^1} g^{12} + \frac{\partial\theta}{\partial y^2} g^{22}\right) \\ f^3 &= -\left(\frac{\partial p}{\partial y^3} + \rho\,\frac{\partial\theta}{\partial y^3}\right) g^{33} \\ f^4 &= 0 \end{aligned}\right\} \tag{B6}$$

where equation (B5) follows from the orthogonality of the y^3 coordinate lines to the y^1,y^2 surface. Equation (82) for $j = 4$ becomes, by use of the incompressibility condition $\frac{D\rho}{dt} = 0$,

$$V^i_{|i} = 0 \tag{B7}$$

From equation (28),

$$S^i_{|j} = 0 \tag{B8}$$

Thus,

$$U^i_{|i} = V^i_{|i} + S^i_{|i} = 0$$

or, by equation (49),

$$U^\gamma_{|\gamma} = 0 \tag{B9}$$

From equations (50), (B4), and (B8),

$$\frac{DU^{\beta}}{dt} = \frac{DV^{\beta}}{dt} = -\left(\frac{1}{\rho}\frac{\partial p}{\partial y^{\alpha}} + \frac{\partial \theta}{\partial y^{\alpha}}\right) g^{\alpha\beta}$$

$$= \frac{\partial U^{\beta}}{\partial t} + \frac{\partial U^{\beta}}{\partial y^{\gamma}} V^{\gamma} + \left[S^{\beta}_{|\gamma}\right]_N U^{\gamma} \left\{ {\beta \atop \gamma \quad \phi} \right\} U^{\gamma} V^{\phi}$$

$$= \frac{\partial U^{\beta}}{\partial t} + \frac{\partial U^{\beta}}{\partial y^{\gamma}} V^{\gamma} + \frac{\partial S^{\beta}}{\partial y^{\gamma}} U^{\gamma} + \left\{ {\beta \atop \gamma \quad \phi} \right\} U^{\gamma} U^{\phi} \tag{B10}$$

Boundary Conditions

The pressure p at the top surface is the atmospheric pressure p_o, which is assumed to be constant. The top surface is a y^1,y^2 coordinate surface; hence V^3 vanishes and $U^3 = S^3$ at the top surface. At the land-fluid boundary, the fluid velocity relative to the boundary $U^i - J^i$ is parallel to the boundary, or

$$\left(U^i - J^i_b\right) n_i = \left(U^{\gamma} - J^{\gamma}_b\right) n_{\gamma} = 0 \tag{B11}$$

where J^i_b is the inertial velocity of the boundary $\frac{dy^i}{dt} - \left(\frac{\partial y^i}{\partial t}\right)_{\bar{z}^{\phi}}$, and n_i is any non-vanishing covariant vector normal to the boundary surface. If the region of interest is not enclosed by physical boundaries (the top surface and the land-fluid boundary), then closure is completed by artificial boundaries, where the boundary condition

$$\left(U^{\gamma} - J^{\gamma}_b\right) n_{\gamma} = v \tag{B12}$$

is applied. Equation (B12) expresses the component of fluid velocity normal to the boundary in terms of the specified scalar v. The slope of the artificial boundaries is restricted, as is the slope of the land-fluid boundary, to be less than the slope of the y^3 coordinate lines; thus, all y^3 coordinate lines intersect the top free surface. The land-fluid boundary and the artificial boundaries are collectively termed the "bottom" boundary, and v is zero at the land-fluid boundary. In summary, the boundary conditions are

$$\left.\begin{aligned} &U^3 = S^3 \quad \text{and} \quad p = p_o && \text{(at the top surface)} \\ &\left(U^\gamma - J_b^\gamma\right) n_\gamma = v && \text{(at the bottom boundary)} \end{aligned}\right\} \tag{B13}$$

Initial Conditions

Equations (B9), (B10), and (B13) determine, in principle, the progression of the fluid state from the initial fluid state. The initial fluid state can be specified by the initial pressure contours in a rigid coordinate system $\bar{x}$ fixed in the rotating Earth and by the initial fluid velocity field $U^\alpha_{(\bar{x})}$. For this specification of the initial state, knowledge of the density field is not required for integrating the equations of motion, as shown in the next section. However, if only the top pressure p_o contour and $U^\alpha_{(\bar{x})}$ are given, then the density field must be known in order to generate the other pressure contours. A procedure for initializing the pressure field for a given top surface configuration, velocity field $U^\alpha_{(\bar{x})}$, and density field ρ is discussed in appendix C. (The vertical velocity $U^3_{(\bar{x})}$ is determined by the other two velocity components by eq. (B9) and the bottom boundary condition.) The y coordinate system is initialized concurrently with the pressure field. Subsequently, at the bottom boundary, y^1 and y^2 are dependent only on $\bar{x}^1,\bar{x}^2$.

The $\bar{x}$ coordinate system is a spherical polar system with $\bar{x}^3$ vertical, $\bar{x}^2$ eastward, and $\bar{x}^1$ southward; thus, if lunar and solar gravitational perturbations (i.e., tidal forces) are neglected, the inertial coordinates $\bar{z}^\alpha$ can have the origin at the center of the Earth, and the velocity of the $\bar{x}$ coordinate system is

$$J^1_{(\bar{x})} = J^3_{(\bar{x})} = 0 \tag{B14}$$

$$J^2_{(\bar{x})} = \Omega \tag{B15}$$

the angular velocity of Earth rotation.

The bottom boundary vector $n_{\phi(\bar{x})}$ is determined once for all time at the bottom boundary as a function of y^1 and y^2.

The quantities y^3, v, and $\left(\frac{\partial v}{\partial t}\right)_{\bar{x}^\alpha}$ are determined as initial conditions at the bottom boundary as functions of y^1 and y^2, as are the quantities

$$n_{\beta(y)} = n_{\alpha}(\bar{x}) \frac{\partial \bar{x}^{\alpha}}{\partial y^{\beta}} \tag{B16a}$$

and, from equation (B12),

$$U^3_{(y)} = J^3_{(y)} - \frac{1}{n_3}\left[\left(U^1 - J^1\right)n_1 + \left(U^2 - J^2\right)n_2 - v\right] \tag{B16b}$$

The following quantities, in addition to $g_{\alpha\beta(y)}$, $U^1_{(y)}$, and $U^2_{(y)}$, are determined as initial conditions throughout the region of interest as functions of y^1, y^2, and y^3:

$$\frac{\partial \theta}{\partial y^{\psi}} = \frac{\partial \theta}{\partial \bar{x}^3} \frac{\partial \bar{x}^3}{\partial y^{\psi}} \qquad (\psi = 1,2) \tag{B17a}$$

$$J^{\alpha}_{(y)} = \Omega \frac{\partial y^{\alpha}}{\partial \bar{x}^2} \tag{B17b}$$

$$U^3_{(y)} = U^3_b + \int_{y^3_b}^{y^3}\left[-\frac{\partial U^1}{\partial y^1} - \frac{\partial U^2}{\partial y^2} - U^{\gamma}\frac{1}{\sqrt{g}}\frac{\partial}{\partial y^{\gamma}}\left(\sqrt{g}\right)\right] d\xi \tag{B17c}$$

which follows from equation (B9) and the identity $\left\{ {\alpha \atop \alpha} \; {\atop \gamma} \right\} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial y^{\gamma}}\left(\sqrt{g}\right)$, where the integration is along y^3 coordinate lines and U^3_b and y^3_b are the bottom boundary values of $U^3_{(y)}$ and y^3, respectively,

$$S^3_{(y)}\left(y^1,y^2,y^3\right) \approx U^3\left(y^1,y^2,p_o\right) + \frac{1}{2}\int_{y^3(p_o)}^{y^3} \frac{1}{g_{33}} \frac{\partial g_{33}}{\partial y^{\gamma}} U^{\gamma} \, d\xi \tag{B17d}$$

which is derived in appendix D (eq. (D7)), where the integration is along y^3 coordinate lines,

$$S^{\beta}_{(y)} = J^{\beta}_b - \int_{y^3_b}^{y^3}\left(g^{1\beta}\frac{\partial S^3}{\partial y^1} + g^{2\beta}\frac{\partial S^3}{\partial y^2}\right) g_{33} \, d\xi \qquad (\beta = 1,2) \tag{B17e}$$

which is derived in appendix D (eq. (D29)), where the integration is along y^3 coordinate lines, and $J^{\beta}_b = J^{\beta}$ at the bottom boundary.

APPENDIX B

Time-Dependent Calculations

The equations for the time-dependent calculations, equations (B18) to (B27), are given in the order of their use in a numerical integration loop. From equations (B5) and (B10),

$$\left.\begin{aligned}\left(\frac{\partial U^1}{\partial t}\right)_{y^\psi} &= -\frac{\partial\theta}{\partial y^1} g^{11} - \frac{\partial\theta}{\partial y^2} g^{12} - \frac{\partial U^1}{\partial y^\gamma} V^\gamma - \frac{\partial S^1}{\partial y^\gamma} U^\gamma - \left\{ {1 \atop \gamma\ \phi} \right\} U^\gamma U^\phi \\ \left(\frac{\partial U^2}{\partial t}\right)_{y^\psi} &= -\frac{\partial\theta}{\partial y^2} g^{22} - \frac{\partial\theta}{\partial y^1} g^{12} - \frac{\partial U^2}{\partial y^\gamma} V^\gamma - \frac{\partial S^2}{\partial y^\gamma} U^\gamma - \left\{ {2 \atop \gamma\ \phi} \right\} U^\gamma U^\phi\end{aligned}\right\} \tag{B18}$$

where $V^\gamma = U^\gamma - S^\gamma$. During the integration time step, all comoving forces per unit mass are held fixed in the $\bar{z}$ coordinate system. From appendix D (eq. (D33)),

$$\left(\frac{\partial n_{\phi(y)}}{\partial t}\right)_{\bar{x}^\alpha} = n_{\gamma(y)} \frac{\partial}{\partial y^\phi}\left(S^\gamma_{(y)} - J^\gamma_{(y)}\right) \qquad \text{(at the bottom boundary)} \tag{B19}$$

From appendix D (eqs. (D36) and (D37)),

$$\left(\frac{\partial U^3_{(y)}}{\partial t}\right)_{y^\gamma} = -\frac{1}{n_{3(y)}}\left\{U^3\left(\frac{\partial n_3}{\partial t}\right)_{\bar{x}^\alpha} + \left[\frac{\partial}{\partial t}\left(U^1 n_1 + U^2 n_2 - v\right)\right]_{\bar{x}^\alpha}\right\}_{(y)} + \frac{\partial U^3_{(y)}}{\partial y^3}\left(S^3 - J^3\right) \tag{B20a}$$

at the bottom boundary, where

$$\left(\frac{\partial U^\beta_{(y)}}{\partial t}\right)_{\bar{x}^\alpha} = \left(\frac{\partial U^\beta_{(y)}}{\partial t}\right)_{y^\alpha} - \frac{\partial U^\beta_{(y)}}{\partial y^3}\left(S^3 - J^3\right) \tag{B20b}$$

at the bottom boundary. From equations (19), (65), (B5), and (D6), it follows that

$$\left.\begin{aligned}
&\left(\frac{\partial g_{33(y)}}{\partial t}\right)_{y^\psi} = 2g_{33}\frac{\partial S^3}{\partial y^3} + S^\phi \frac{\partial g_{33}}{\partial y^\phi} \approx -V^\phi \frac{\partial g_{33}}{\partial y^\phi} \\
&\left(\frac{\partial g_{\alpha\beta(y)}}{\partial t}\right)_{y^\psi} = g_{\gamma\beta}\left[S^\gamma{}_{|\alpha}\right]_N + g_{\alpha\gamma}\left[S^\gamma{}_{|\beta}\right]_N \qquad (\alpha,\beta,\gamma = 1,2)
\end{aligned}\right\} \qquad \text{(B21)}$$

Taking the time derivative of the expression for $U^3_{(y)}$ in equation (B17c) yields

$$\left(\frac{\partial U^3_{(y)}}{\partial t}\right)_{y^\phi} = \left(\frac{\partial U^3}{\partial t}\right)_{y^\phi} \text{(at bottom boundary)}$$

$$+ \int_{y_b^3}^{y^3}\left(-\frac{\partial}{\partial y^\psi}\left(\frac{\partial U^\psi}{\partial t}\right)_{y^\phi} - \left(\frac{\partial U^\gamma}{\partial t}\right)_{y^\phi}\frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\right)}{\partial y^\gamma} - U^\gamma\left\{\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\right)}{\partial y^\gamma}\right]\right\}_{y^\phi}\right)d\xi$$

$$\text{(along } y^3 \text{ coordinate lines)} \qquad (\psi = 1,2) \qquad \text{(B22)}$$

where $\sqrt{g} = \sqrt{g_{33}\left(g_{11}\,g_{22} - g_{12}\,g_{12}\right)}$. From equations (D31) and (D11) in appendix D,

$$\left(\frac{\partial y_b^3}{\partial t}\right)_{y^1,y^2} = \left(\frac{\partial y_b^3}{\partial t}\right)_{\bar{x}^\phi} = J^3_{(y)} - S^3_{(y)} \qquad \text{(B23)}$$

since at the bottom boundary, y^1 and y^2 are functions of $\bar{x}^1$ and $\bar{x}^2$, and $\bar{x}^3$ is a function of $\bar{x}^1$ and $\bar{x}^2$. From equation (D7) in appendix D,

$$\frac{\partial}{\partial t}S^3\left(y^1,y^2,y^3,t\right) \approx \frac{\partial}{\partial t}U^3\left(y^1,y^2,p_o,t\right) + \frac{1}{2}\int_{y^3(p_o)}^{y^3}\frac{\partial}{\partial t}\left(\frac{1}{g_{33}}\frac{\partial g_{33}}{\partial y^\gamma}U^\gamma\right)_{y^\phi} d\xi \qquad \text{(B24)}$$

$$\text{(along } y^3 \text{ coordinate lines)}$$

From equation (D41) in appendix D,

$$\left(\frac{\partial J^\psi}{\partial t}\right)_{y^\phi} = \frac{\partial J^\psi}{\partial y^\gamma}S^\gamma \qquad \text{(B25)}$$

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From equation (D29) in appendix D, for $\psi,\beta = 1,2,$

$$\left(\frac{\partial S^{\psi}_{(y)}}{\partial t}\right)_{y^{\phi}} = \left(\frac{\partial J^{\psi}}{\partial t}\right)_{y^{\phi}} \text{(at bottom boundary)} - \int_{y_b^3}^{y^3} \left[\frac{\partial}{\partial t}\left(g^{\psi\beta} \frac{\partial S^3}{\partial y^{\beta}} g_{33}\right)\right]_{y^{\phi}} d\xi \quad \text{(B26)}$$

(along y^3 coordinate lines)

From equation (D43) in appendix D,

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial y^{\beta}}\right)\right]_{y^{\phi}} = \frac{\partial}{\partial y^{\beta}}\left(S^{\gamma} \frac{\partial \theta}{\partial y^{\gamma}}\right) \qquad (\beta = 1,2) \qquad \text{(B27)}$$

Equation (B27) completes the time-dependent equations. As mentioned before, knowledge of the density field is not required for integrating the equations of motion, since ρ does not appear in the time-dependent equations. The density can be recovered by means of the three-component momentum equation; from equation (B10),

$$\frac{1}{\rho} = \frac{\partial \theta}{\partial y^3} + g_{33}\left(\frac{\partial U^3}{\partial t} + \frac{\partial U^3}{\partial y^{\gamma}} V^{\gamma} + \frac{\partial S^3}{\partial y^{\gamma}} U^{\gamma} + \begin{Bmatrix} & 3 & \\ \gamma & & \phi \end{Bmatrix} U^{\gamma} U^{\phi}\right) \qquad \text{(B28)}$$

where $\frac{\partial \theta}{\partial y^3}$ is expressed in terms of $\frac{\partial \theta}{\partial y^1}$, $\frac{\partial \theta}{\partial y^2}$, which are updated by equation (B27) by

$$\frac{\partial \theta}{\partial y^3} = \sqrt{g_{33(y)}\left[\left(\frac{\partial \theta}{\partial \bar{x}^3}\right)^2 - \frac{\partial \theta}{\partial y^{\beta}} \frac{\partial \theta}{\partial y^{\psi}} g^{\beta\psi}\right]} \qquad (\beta,\psi = 1,2) \qquad \text{(B29)}$$

This relation follows from the orthogonality of the y^3 coordinates to the y^1 and y^2 coordinates and from $\frac{\partial \theta}{\partial y^{\gamma}} \frac{\partial \theta}{\partial y^{\phi}} g^{\gamma\phi} = \left(\frac{\partial \theta}{\partial x^3}\right)^2$. (The negative radical in eq. (B29) is discarded as inconsistent with the condition that the pressure increase with depth.)

APPENDIX C

PRESSURE FIELD INITIALIZATION FOR EXAMPLE PROBLEM OF APPENDIX B

A procedure for initializing the pressure field is discussed in this appendix. The velocities $U^1_{(\bar{x})}$, $U^2_{(\bar{x})}$; the density $\rho(\bar{x}^\alpha)$; and the configuration of the top free surface in the $\bar{x}$ coordinate system are arbitrarily specified. If the magnitude of the pressure gradient $\sqrt{g^{33}_{(y)}}$ were known everywhere, then the configuration of all of the pressure contours would be determined by the configuration of the top surface (of pressure p_o), since $-dy^3/\sqrt{g^{33}}$ is the displacement from the contour of pressure p to the contour of pressure $p - dy^3$. But, from equations (B6),

$$g^{33}_{(y)} = \frac{-f^3_{(y)}}{\frac{\partial p}{\partial y^3} + \rho \frac{\partial \theta}{\partial y^3}} = \frac{-f^3}{-1 + \rho \frac{\partial \theta}{\partial y^3}} \tag{C1}$$

where f^3 and $\rho \frac{\partial \theta}{\partial y^3}$ are not known a priori. It will be shown (eq. (C8)) that knowledge of $g^{33}_{(y)}$ at the top free surface and of the density and inertial velocity fields everywhere is sufficient to generate the pressure contours. Then an iterative procedure is suggested for determining $g^{33}_{(y)}$ at the top free surface. Once the pressure contours are determined, the y coordinate system can be established and the velocity field expressed in the y system; then the other parameters can be initialized as discussed in appendix B.

Taking the space-time divergence of f^i in equations (B6) yields

$$f^i_{|i} = f^\alpha_{|\alpha} = \left(\frac{\partial p}{\partial y^\phi} + \rho \frac{\partial \theta}{\partial y^\phi}\right)_{|\alpha} g^{\phi\alpha} = \left(\rho \frac{DU^\alpha}{dt}\right)_{|\alpha} = \frac{\partial \rho}{\partial y^\alpha} \frac{DU^\alpha}{dt} + \rho\left(\frac{DU^\alpha}{dt}\right)_{|\alpha}$$

Neglecting the self-gravitation of the fluid so that $\left[\left(\frac{\partial \theta}{\partial y^\phi}\right)_{|\alpha}\right]_N g^{\phi\alpha} = 0$, which implies $\left(\frac{\partial \theta}{\partial y^\phi}\right)_{|\alpha} g^{\phi\alpha} = 0$, yields

APPENDIX C

$$-\left(\frac{\partial p}{\partial y^\phi}\right)_{|\alpha} g^{\phi\alpha} = \frac{\partial \rho}{\partial y^\alpha}\left(\frac{\partial \theta}{\partial y^\phi} g^{\phi\alpha} + \frac{DU^\alpha}{dt}\right) + \rho\left(\frac{DU^\alpha}{dt}\right)_{|\alpha} \tag{C2}$$

Substituting $-\frac{1}{\rho}\frac{\partial p}{\partial y^\phi} g^{\phi\alpha} - \frac{\partial \theta}{\partial y^\phi} g^{\phi\alpha}$ for $\frac{DU^\alpha}{dt}$ yields

$$-\left(\frac{\partial p}{\partial y^\phi}\right)_{|\alpha} g^{\phi\alpha} = -\frac{\partial \rho}{\partial y^\alpha}\frac{1}{\rho}\frac{\partial p}{\partial y^\phi} g^{\phi\alpha} + \rho\left(\frac{DU^\alpha}{dt}\right)_{|\alpha} \tag{C3}$$

Expanding the second term on the right side yields

$$\rho\left(\frac{DU^\alpha}{dt}\right)_{|\alpha} = \rho\left(U^\alpha_{|i} V^i\right)_{|\alpha} = \rho\left(U^\alpha_{|\alpha|i} V^i + U^\alpha_{|i} V^i_{|\alpha}\right)$$

$$= \rho\, U^\alpha_{|i}\left(U^i_{|\alpha} - S^i_{|\alpha}\right) = \rho\, U^\alpha_{|\sigma} U^\sigma_{|\alpha}$$

since $U^\alpha_{|\alpha} \equiv 0 \equiv S^i_{|\alpha}$ and $U^4_{|\alpha} \equiv 0$ by equations (15), (27), (28), and (B9). Thus equation (C3) becomes

$$-\left(\frac{\partial p}{\partial y^\phi}\right)_{|\alpha} g^{\phi\alpha} = -\frac{1}{\rho}\frac{\partial \rho}{\partial y^\alpha}\frac{\partial p}{\partial y^\phi} g^{\phi\alpha} + \rho\, U^\alpha_{|\sigma} U^\sigma_{|\alpha} \tag{C4}$$

Expanding the left-hand side of equation (C4) yields

$$-\left(\frac{\partial p}{\partial y^\phi}\right)_{|\alpha} g^{\phi\alpha} = -\left[\frac{\partial}{\partial y^\alpha}\left(\frac{\partial p}{\partial y^\phi}\right) - \frac{\partial p}{\partial y^i}\begin{Bmatrix} & i & \\ \phi & & \alpha\end{Bmatrix}\right] g^{\phi\alpha}$$

$$= \frac{\partial p}{\partial y^3}\begin{Bmatrix} & 3 & \\ \phi & & \alpha\end{Bmatrix} g^{\phi\alpha}$$

$$= -\begin{Bmatrix} & 3 & \\ \phi & & \alpha\end{Bmatrix} g^{\phi\alpha} \tag{C5}$$

since $\frac{\partial p}{\partial y^\phi} = -\delta^3_\phi$ and $\begin{Bmatrix} & 4 & \\ \phi & & \alpha\end{Bmatrix} = 0$. In equation (C4),

$$\frac{\partial p}{\partial y^{\phi}} g^{\phi\alpha} = \frac{\partial p}{\partial y^3} g^{3\alpha} = -g^{33} \delta_3^{\alpha}$$

Hence,

$$-\begin{Bmatrix} & 3 & \\ \phi & & \alpha \end{Bmatrix} g^{\phi\alpha} = \frac{1}{\rho} \frac{\partial \rho}{\partial y^3} g^{33} + \rho\, U^{\alpha}_{\ |\gamma}\, U^{\gamma}_{\ |\alpha} \qquad \text{(C6)}$$

The covariant derivative of the metric tensor vanishes; thus,

$$\left[g^{3\sigma}_{\ \ |\sigma}\right]_N = 0 = \frac{\partial g^{3\sigma}}{\partial y^{\sigma}} + g^{3\phi} \begin{Bmatrix} & \sigma & \\ \phi & & \sigma \end{Bmatrix} + g^{\alpha\sigma} \begin{Bmatrix} & 3 & \\ \alpha & & \sigma \end{Bmatrix}$$

$$= -\left(g^{33}\right)^2 \frac{\partial g_{33}}{\partial y^3} + g^{33} \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\right)}{\partial y^3} + g^{\alpha\sigma} \begin{Bmatrix} & 3 & \\ \alpha & & \sigma \end{Bmatrix}$$

by use of $g^{31} \equiv g^{32} \equiv 0$, $g_{33} = \frac{1}{g^{33}}$, and $\begin{Bmatrix} & \sigma & \\ \phi & & \sigma \end{Bmatrix} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^{\phi}}\left(\sqrt{g}\right)$. Let

$$a_{12} = \sqrt{g_{11}\, g_{22} - g_{12}\, g_{12}}$$

Then

$$\sqrt{g} = \sqrt{g_{33}}\, a_{12}$$

and

$$-g^{\alpha\sigma} \begin{Bmatrix} & 3 & \\ \alpha & & \sigma \end{Bmatrix} = -\left(g^{33}\right)^2 \frac{\partial g_{33}}{\partial y^3} + g^{33} \frac{1}{\sqrt{g_{33}}} \frac{\partial\left(\sqrt{g_{33}}\right)}{\partial y^3}$$

$$+ g^{33} \frac{1}{a_{12}} \frac{\partial\left(a_{12}\right)}{\partial y^3}$$

$$= -g^{33} \frac{1}{\sqrt{g_{33}}} \frac{\partial\left(\sqrt{g_{33}}\right)}{\partial y^3} + g^{33} \frac{1}{a_{12}} \frac{\partial\left(a_{12}\right)}{\partial y^3}$$

Thus, equation (C6) can be written

$$\frac{1}{\sqrt{g_{33}}} \frac{\partial\left(\sqrt{g_{33}}\right)}{\partial y^3} = \frac{1}{a_{12}} \frac{\partial\left(a_{12}\right)}{\partial y^3} - \frac{1}{\rho} \frac{\partial \rho}{\partial y^3} - g_{33}\, \rho\, U^{\alpha}_{|\sigma}\, U^{\sigma}_{|\alpha} \tag{C7}$$

where $a_{12} = \sqrt{g_{11}\, g_{22} - g_{12}\, g_{12}}$ is the area per unit coordinate increment product $dy^1\, dy^2$. In terms of the displacement $d\bar{z}^3 = \sqrt{g_{33}}\, dy^3$ along the negative pressure gradient, equation (C7) becomes

$$\frac{\partial\left(\sqrt{g_{33}}\right)}{\partial \bar{z}^3} = \frac{\sqrt{g_{33}}}{a_{12}} \frac{\partial\left(a_{12}\right)}{\partial \bar{z}^3} - \frac{\sqrt{g_{33}}}{\rho} \frac{\partial \rho}{\partial \bar{z}^3} - g_{33}\, \rho\, U^{\alpha}_{|\sigma}\, U^{\sigma}_{|\alpha} \tag{C8}$$

where g_{33} is expressed in the y coordinate system and $\rho\, U^{\alpha}_{|\sigma}\, U^{\sigma}_{|\alpha}$ is an invariant scalar in space-time $\left(\text{equal to } \rho\, U^{i}_{|j}\, U^{j}_{|i}\right)$. The family of surfaces of constant pressure can be approximated from a finite-difference form of equation (C8) by numerical integration from the top surface to the bottom boundary, given g_{33} at the top, the density and velocity fields, and the configuration of the top surface. An iterative procedure for determining g_{33} at the top surface is described next.

The configuration of the top free surface, the velocity components $U^{1}_{(\bar{x})}$ and $U^{2}_{(\bar{x})}$, and the density ρ are specified in an Earth fixed coordinate system $\bar{x}$ (with $\bar{x}^3$ as the vertical coordinate). The vertical velocity component $U^{3}_{(\bar{x})}$ is determined by $U^{\sigma}_{|\sigma} = 0$ and the bottom boundary condition $\left(U^{\sigma} - J^{\sigma}\right) n_{\sigma} = v$. Then,

(a) The y^1 and y^2 coordinates are defined at the top surface for the initial time in $\bar{x}$ coordinates.

(b) At the top surface, $\sqrt{g_{33}}$ is initially approximated by

$$\sqrt{g_{33}} = \frac{1}{\rho\, \partial\theta / \partial \bar{z}^3}$$

where $d\bar{z}^3$ is the displacement element in the y^3 direction $\sqrt{g_{33}}\, dy^3$. This initial approximation neglects acceleration $a^{3}_{(\bar{z})}$ normal to the top surface and follows from equations (B6) with $f^3 = 0$. $\left(a^{3}_{(\bar{z})} \text{ is set equal to zero.}\right)$

(c) The y^1, y^2, and y^3 coordinates and the constant pressure surfaces are constructed in x coordinates from a finite-difference form of equation (C8). The density and velocity fields are expressed in y coordinates.

(d) The sequence of computations in appendix B consisting of the remainder of the initial conditions and the time-dependent operations through equation (B22) for $\frac{\partial}{\partial t}\left(U^3_{(y)}\right)$ are performed.

(e) $\left(a^3_{(\bar{z})}\right)'$ is set equal to $a^3_{(\bar{z})}$. The acceleration at the top surface in the $\bar{z}^3$ direction $a^3_{(\bar{z})}$ is approximated by (from eq. (B10))

$$a^3_{(\bar{z})} = \sqrt{g_{33(y)}}\left(\frac{\partial U^3}{\partial t} + \frac{\partial U^3}{\partial y^\sigma} V^\sigma + \frac{\partial S^3}{\partial y^\sigma} U^\sigma + \begin{Bmatrix} & 3 & \\ \sigma & & \phi \end{Bmatrix} U^\sigma U^\phi\right)_{(y)}$$

The next approximation to $\sqrt{g_{33}}$ at the top surface is, from equations (B6) and (B10) and the relation $g_{33} = 1/g^{33}$,

$$\sqrt{g_{33}} = \frac{1}{\left[1/\left(\sqrt{g_{33}}\right)'\right] + K\rho\left[a^3_{(\bar{z})} - \left(a^3_{(\bar{z})}\right)'\right]}$$

where $\left(\sqrt{g_{33}}\right)'$ is the previous approximation to $\sqrt{g_{33}}$ and K is a constant between zero and unity selected by trial and error for fast convergence of $\sqrt{g_{33}}$ at the top surface.

(f) Steps (c) to (e) are repeated until a chosen convergence criterion for $\sqrt{g_{33}}$ at the top surface is satisfied, then the time-dependent operations of appendix B are continued.

APPENDIX D

SUPPLEMENTARY DERIVATIONS FOR TIME-DEPENDENT CALCULATIONS OF EXAMPLE PROBLEM OF APPENDIX B

Velocity of y Coordinate System

All comoving forces per unit mass are held fixed in the $\bar{z}$ coordinate system during the integration time step; that is (from eqs. (B1) and (B2)),

$$\frac{D}{dt}\left(\frac{1}{\rho} f_{\alpha}\right) = \frac{D}{dt}\left(-\frac{1}{\rho} \frac{\partial p}{\partial y^{\alpha}} - \frac{\partial \theta}{\partial y^{\alpha}}\right) \approx 0 \tag{D1}$$

$$\frac{D}{dt}\left(\frac{1}{\rho} f_{4}\right) = \frac{1}{\lambda} \frac{D}{dt}\left(f_{\phi} S^{\phi}\right) \approx 0 \tag{D2}$$

$\left(\text{which is automatically satisfied when eq. (D1) holds, because } \frac{DS^{\phi}}{dt} \equiv 0\right)$ and

$$\frac{D}{dt}\left(-\frac{1}{\rho} \frac{\partial p}{\partial y^{\alpha}}\right) \approx 0 \tag{D3}$$

$$\frac{D}{dt}\left(-\frac{\partial \theta}{\partial y^{\alpha}}\right) \approx 0 \tag{D4}$$

Expanding equation (D3) for $\alpha = 3$ yields, by use of $dp = -dy^3$ and the incompressibility condition $\frac{D\rho}{dt} = 0$,

$$\frac{1}{\rho} \left\{ {3 \atop 3\ \ \sigma} \right\} V^{\sigma} + \frac{1}{\rho}\left(\frac{\partial S^3}{\partial y^3} + S^{\phi} \left\{ {3 \atop \phi\ \ 3} \right\}\right) \approx 0$$

or

$$\frac{\partial S^3}{\partial y^3} \approx -\left\{ {3 \atop 3\ \ \sigma} \right\} U^{\sigma} \tag{D5}$$

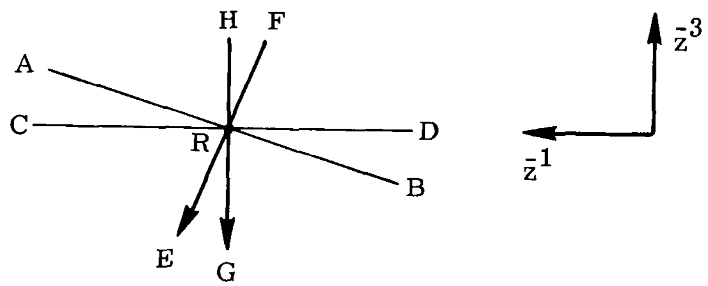
From equations (36) and (19) and the relation $g^{3\phi} = 0$ for $\phi \neq 3$, equation (D5) becomes

$$\frac{\partial S^3}{\partial y^3} \approx -\frac{1}{2}\frac{1}{g_{33}}\frac{\partial g_{33}}{\partial y^\sigma}U^\sigma \tag{D6}$$

and the top boundary condition $S^3 = U^3$ yields, by integration of equation (D6) along y^3 coordinate lines down from the top surface,

$$S^3\left(y^1,y^2,y^3,t\right) \approx U^3\left(y^1,y^2,p_o,t\right) + \frac{1}{2}\int_{y^3\left(p_o\right)}^{y^3} \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial y^\sigma}U^\sigma\, d\xi \tag{D7}$$

The y coordinate system velocity components S^1 and S^2 are considered next. Sketch (c) shows the pressure gradient and pressure contour at a point R of fixed coordinates y^ϕ at times t and $t + \Delta t$. The directions of the axes $\bar{z}^3$ and $\bar{z}^1$ are indicated; however, the point R can move relative to the origin of the $\bar{z}$ system. The



Sketch (c)

point R is fixed spatially in the y coordinate system. The pressure gradient at time t is $\overset{\frown}{HG}$. The $\bar{z}$ coordinate system is inertial Cartesian with $\bar{z}^3$ antiparallel to $\overset{\frown}{HG}$ and the $\bar{z}^3,\bar{z}^1$ plane parallel to the instantaneous plane of rotation of the pressure gradient at time t. The pressure contours $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are of the same pressure (since y^3 at R is fixed) at times $t + \Delta t$ and t, respectively. The pressure gradient at time $t + \Delta t$ is $\overset{\frown}{FE}$. The orthogonality of the pressure gradient to the pressure contours implies that $\overset{\frown}{CD}$ and $\overset{\frown}{HG}$ rotate about an axis pointing into the plane of the paper at the same angular rate ω. Therefore, at time t,

$$\omega = \frac{\partial}{\partial \bar{z}^1}\left(\frac{\partial \bar{z}^3}{\partial t}\right)_{y^\alpha} = -\frac{\partial}{\partial \bar{z}^3}\left(\frac{\partial \bar{z}^1}{\partial t}\right)_{y^\alpha} \tag{D8}$$

Let W^i be the inertial velocity of the y coordinate system defined in the $\bar{z}$ coordinate system by

$$\left.\begin{aligned} W^{\sigma}_{(\bar{z})} &= \left(\frac{\partial \bar{z}^{\sigma}}{\partial t}\right)_{y^{\alpha}} \\ W^{4}_{(\bar{z})} &= 0 \end{aligned}\right\} \tag{D9}$$

The definition of W^i is extended to any allowed coordinate system $\bar{y}$ by requiring W^i to be a space-time contravariant vector; thus,

$$W^{i}_{(\bar{y})} = W^{n}_{(\bar{z})} \frac{\partial \bar{y}^{i}}{\partial \bar{z}^{n}} = W^{\sigma}_{(\bar{z})} \frac{\partial \bar{y}^{i}}{\partial \bar{z}^{\sigma}} \tag{D10}$$

In the y coordinate system, W^i becomes, by equations (9) and (10),

$$\left.\begin{aligned} W^{\phi}_{(y)} &= \left(\frac{\partial \bar{z}^{\sigma}}{\partial t}\right)_{y^{\alpha}} \frac{\partial y^{\phi}}{\partial \bar{z}^{\sigma}} = S^{\phi}_{(y)} \\ W^{4}_{(y)} &= 0 \end{aligned}\right\} \tag{D11}$$

and, by the tensor character of W^i,

$$\left.\begin{aligned} W^{\beta}_{(\bar{y})} &= W^{i}_{(y)} \frac{\partial \bar{y}^{\beta}}{\partial y^{i}} = S^{\sigma}_{(y)} \frac{\partial \bar{y}^{\beta}}{\partial y^{\sigma}} \\ W^{4}_{(\bar{y})} &= S^{\sigma}_{(y)} \frac{\partial \bar{y}^{4}}{\partial y^{\sigma}} = 0 \end{aligned}\right\} \tag{D12}$$

From equations (D11) it can be seen that W^i is the inertial velocity $U^i = V^i + S^i$, for $V^{\alpha}_{(y)} = 0$.

From equation (D8),

$$\frac{\partial}{\partial \bar{z}^{1}}\left(\frac{\partial \bar{z}^{3}}{\partial t}\right)_{y^{\alpha}} + \frac{\partial}{\partial \bar{z}^{3}}\left(\frac{\partial \bar{z}^{1}}{\partial t}\right)_{y^{\alpha}} = 0 \tag{D13}$$

or, by equations (D9),

$$W_{3|1(\bar{z})} + W_{1|3(\bar{z})} = 0 \tag{D14}$$

since the $\bar{z}$ system is Cartesian. Therefore,

$$\left(W_{i|j(y)} + W_{j|i(y)}\right) \frac{\partial y^i}{\partial \bar{z}^3} \frac{\partial y^j}{\partial \bar{z}^1} = 0 \tag{D15}$$

The $\bar{z}^1,\bar{z}^3$ plane is parallel to the plane of rotation of both the pressure gradient and the pressure contour, as specified in the discussion of sketch (c); thus

$$\left.\begin{aligned} \frac{\partial}{\partial \bar{z}^2}\left(\frac{\partial \bar{z}^3}{\partial t}\right)_{y^\alpha} &= 0 \\ \frac{\partial}{\partial \bar{z}^3}\left(\frac{\partial \bar{z}^2}{\partial t}\right)_{y^\alpha} &= 0 \end{aligned}\right\} \tag{D16}$$

and, by equations (D9),

$$\left.\begin{aligned} W_{3|2(\bar{z})} &= 0 \\ W_{2|3(\bar{z})} &= 0 \end{aligned}\right\} \tag{D17}$$

Therefore,

$$\left.\begin{aligned} W_{i|j(y)} \frac{\partial y^i}{\partial \bar{z}^3} \frac{\partial y^j}{\partial \bar{z}^2} &= 0 \\ W_{j|i(y)} \frac{\partial y^i}{\partial \bar{z}^3} \frac{\partial y^j}{\partial \bar{z}^2} &= 0 \end{aligned}\right\} \tag{D18}$$

Combining equations (D18) and (D15) yields

$$\left(W_{i|j} + W_{j|i}\right)_{(y)} \frac{\partial y^i}{\partial \bar{z}^3} \frac{\partial y^j}{\partial \bar{z}^\beta} = 0 \qquad (\beta = 1,2) \tag{D19}$$

Equation (D19) is expressed in terms of the contravariant vector W^k by

$$\left(A_{in}\, W^n_{|j} + A_{jn}\, W^n_{|i}\right)_{(y)} \frac{\partial y^i}{\partial \bar{z}^3} \frac{\partial y^j}{\partial \bar{z}^\beta} = 0 \qquad (\beta = 1,2) \qquad \text{(D20)}$$

Or, applying

$$\frac{\partial y^i}{\partial \bar{z}^3} = \delta^i_3 \sqrt{g_{33}}$$

$$\sqrt{g_{33}} \neq 0$$

$$\frac{\partial y^j}{\partial \bar{z}^\beta} = 0 \qquad (j = 3,4)$$

yields

$$\left(A_{3n}\, W^n_{|\phi} + A_{\phi n}\, W^n_{|3}\right)_{(y)} \frac{\partial y^\phi}{\partial \bar{z}^\beta} = 0 \qquad (\beta,\phi = 1,2) \qquad \text{(D21)}$$

From equations (49) and (D12),

$$W^4_{|\alpha} = 0 \qquad \text{(D22)}$$

Equations (12) and the orthogonality of dy^3 to dy^1 and dy^2 imply

$$A_{3\alpha} = g_{33}\, \delta^3_\alpha$$

Thus equation (D21) becomes

$$\left(g_{33}\, W^3_{|\phi} + g_{\phi\psi}\, W^\psi_{|3}\right)_{(y)} \frac{\partial y^\phi}{\partial \bar{z}^\beta} = 0 \qquad (\psi,\beta,\phi = 1,2) \qquad \text{(D23)}$$

which are two equations in the unknowns $\left(g_{33}\, W^3_{|1} + g_{1\psi}\, W^\psi_{|3}\right)$ and $\left(g_{33}\, W^3_{|2} + g_{2\psi}\, W^\psi_{|3}\right)$. The determinant of the coefficients $\partial y^\phi / \partial \bar{z}^\beta$ is the Jacobian for the nonsingular transformation from y^ϕ coordinates to $\bar{z}^\beta$ coordinates, and hence is nonvanishing. Thus, the trivial solution is the only solution, and

$$g_{\beta\psi(y)}\, W^\psi_{|3(y)} = -g_{33(y)}\, W^3_{|\beta(y)} \qquad (\beta,\psi = 1,2) \qquad \text{(D24)}$$

or

$$\begin{aligned} g^{\phi\beta}\, g_{\beta\psi}\, W^\psi_{|3} &= \left(\delta^\phi_\psi - g^{\phi 3}\, g_{3\psi}\right) W^\psi_{|3} \\ &= W^\phi_{|3}\left(1 - \delta^\phi_3\right) \\ &= -g^{\phi\beta}\, g_{33}\, W^3_{|\beta} \end{aligned}$$

Thus,

$$W^\psi_{|3} = -g^{\psi\beta}\, g_{33}\, W^3_{|\beta} \qquad (\beta,\psi = 1,2) \qquad \text{(D25)}$$

Expanding the left-hand side of equation (D25) and then the right-hand side yields, by equations (19), (B5), and (D11),

$$\begin{aligned} \frac{\partial W^\psi}{\partial y^3} + W^\sigma \left\{ {}_\sigma{}^\psi{}_3 \right\} &= -g^{\psi\beta}\, g_{33}\left(\frac{\partial W^3}{\partial y^\beta} + W^\sigma \left\{ {}_\sigma{}^3{}_\beta \right\}\right) \\ &= \frac{\partial W^\psi}{\partial y^3} + \frac{1}{2}\, W^\sigma\, g^{\psi\phi}\left(\frac{\partial g_{\phi\sigma}}{\partial y^3} - \frac{\partial g_{3\sigma}}{\partial y^\phi}\right) \\ &= -g^{\psi\beta}\, g_{33}\left[\frac{\partial W^3}{\partial y^\beta} + \frac{1}{2}\, W^\sigma\, g^{33}\left(\frac{\partial g_{3\sigma}}{\partial y^\beta} - \frac{\partial g_{\sigma\beta}}{\partial y^3}\right)\right] \\ &= -g^{\psi\beta}\, g_{33}\, \frac{\partial W^3}{\partial y^\beta} - \frac{1}{2} W^3\, g^{\psi\beta}\, \frac{\partial g_{33}}{\partial y^\beta} + \frac{1}{2}\, W^\phi\, g^{\psi\beta}\, \frac{\partial g_{\phi\beta}}{\partial y^3} \end{aligned}$$

or

$$\frac{\partial W^{\psi}}{\partial y^3} = -g^{\psi\beta} g_{33} \frac{\partial W^3}{\partial y^{\beta}} \qquad (\beta,\psi = 1,2) \qquad \text{(D26)}$$

By equations (D11) and (D26),

$$\frac{\partial S^{\psi}}{\partial y^3} = -g^{\psi\beta} g_{33} \frac{\partial S^3}{\partial y^{\beta}} \qquad (\beta,\psi = 1,2) \qquad \text{(D27)}$$

At the bottom boundary $\left(y^3 = y_b^3\right)$, $y^{\psi} = y^{\psi}\left(\bar{x}^1, \bar{x}^2\right)$; therefore

$$J^{\psi} - S^{\psi} = \left(\frac{\partial y^{\psi}}{\partial t}\right)_{\bar{x}^{\phi}} = 0 \qquad \text{(D28)}$$

Hence,

$$S^{\psi} = J_b^{\psi} - \int_{y_b^3}^{y^3} \left(g^{\psi 1} \frac{\partial S^3}{\partial y^1} + g^{\psi 2} \frac{\partial S^3}{\partial y^2}\right) g_{33}\, d\xi \qquad (\psi = 1,2) \qquad \text{(D29)}$$

(along y^3 coordinate lines)

Rate of Change of Vector Normal to Bottom Boundary

The rate of change of the covariant vector n_{ϕ} normal to the bottom boundary, which enters into the application of the bottom boundary conditions, is derived in this section. At the bottom boundary,

$$\left(\frac{\partial n_{\phi(y)}}{\partial t}\right)_{\bar{x}^{\alpha}} = \left[\frac{\partial}{\partial t}\left(n_{\sigma}(\bar{x}) \frac{\partial \bar{x}^{\sigma}}{\partial y^{\phi}}\right)\right]_{\bar{x}^{\alpha}} = n_{\sigma}(\bar{x}) \left[\frac{\partial}{\partial t}\left(\frac{\partial \bar{x}^{\sigma}}{\partial y^{\phi}}\right)\right]_{\bar{x}^{\alpha}}$$

$$= n_{\sigma}(\bar{x})\, \lambda \frac{\partial}{\partial \bar{x}^4}\left(\frac{\partial \bar{x}^{\sigma}}{\partial y^{\phi}}\right) = n_{\sigma}(\bar{x})\, \lambda \frac{\partial}{\partial y^i}\left(\frac{\partial \bar{x}^{\sigma}}{\partial y^{\phi}}\right) \frac{\partial y^i}{\partial \bar{x}^4}$$

$$= n_{\sigma}(\bar{x})\, \lambda \left(\frac{\partial^2 \bar{x}^{\sigma}}{\partial y^{\psi}\, \partial y^{\phi}} \frac{\partial y^{\psi}}{\partial \bar{y}^4} + \frac{\partial^2 \bar{x}^{\sigma}}{\partial y^{\phi}\, \partial y^4}\right) = n_{\sigma}(\bar{x}) \left[\frac{\partial^2 \bar{x}^{\sigma}}{\partial y^{\psi}\, \partial y^{\phi}} \left(\frac{\partial y^{\psi}}{\partial t}\right)_{\bar{x}^{\alpha}} + \frac{\partial}{\partial y^{\phi}}\left(\frac{\partial \bar{x}^{\sigma}}{\partial t}\right)_{y^{\beta}}\right]$$

(D30)

The expression $\left(\frac{\partial y^{\psi}}{\partial t}\right)_{\bar{x}^{\alpha}}$ is the rate of change of the coordinates y^{ψ} of a point fixed in the spatial $\bar{x}$ system, or the velocity of the $\bar{x}$ system relative to the y system expressed in y coordinates; thus, by equations (D11),

$$\left(\frac{\partial y^{\psi}}{\partial t}\right)_{\bar{x}^{\alpha}} = J^{\psi}_{(y)} - W^{\psi}_{(y)} \tag{D31}$$

Similarly,

$$\left(\frac{\partial \bar{x}^{\sigma}}{\partial t}\right)_{y^{\beta}} = W^{\sigma}_{(\bar{x})} - J^{\sigma}_{(\bar{x})} \tag{D32}$$

and equation (D30) becomes

$$\begin{aligned}
\left(\frac{\partial n_{\phi(y)}}{\partial t}\right)_{\bar{x}^{\alpha}} &= n_{\sigma(\bar{x})}\left[\frac{\partial^2 \bar{x}^{\sigma}}{\partial y^{\psi}\,\partial y^{\phi}}\left(J^{\psi}_{(y)} - W^{\psi}_{(y)}\right) + \frac{\partial}{\partial y^{\phi}}\left(W^{\sigma}_{(\bar{x})} - J^{\sigma}_{(\bar{x})}\right)\right] \\
&= n_{\sigma(\bar{x})}\left\{\frac{\partial^2 \bar{x}^{\sigma}}{\partial y^{\psi}\,\partial y^{\phi}}\left(J^{\psi}_{(y)} - W^{\psi}_{(y)}\right) + \frac{\partial}{\partial y^{\phi}}\left[\frac{\partial \bar{x}^{\sigma}}{\partial y^{\psi}}\left(W^{\psi}_{(y)} - J^{\psi}_{(y)}\right)\right]\right\} \\
&= n_{\sigma(\bar{x})}\,\frac{\partial \bar{x}^{\sigma}}{\partial y^{\psi}}\,\frac{\partial}{\partial y^{\phi}}\left(W^{\psi}_{(y)} - J^{\psi}_{(y)}\right) \\
&= n_{\psi(y)}\,\frac{\partial}{\partial y^{\phi}}\left(S^{\psi}_{(y)} - J^{\psi}_{(y)}\right)
\end{aligned} \tag{D33}$$

Rate of Change of Velocity Component $U^3_{(y)}$ at Bottom Boundary

From equation (B12) or equations (B16), the velocity component $U^3_{(y)}$ at the bottom boundary is

$$U^3_{(y)} = J^3 - \frac{1}{n_3}\left[\left(U^1 - J^1\right) n_1 + \left(U^2 - J^2\right) n_2 - v\right]$$

$$= \frac{1}{n_3} n_\sigma J^\sigma - \frac{1}{n_3}\left(U^1 n_1 + U^2 n_2 - v\right) \tag{D34}$$

The invariant scalar $n_\sigma J^\sigma$ is, by equations (B14) and (B15),

$$n_\sigma J^\sigma = n_{2(\bar{x})}\ \Omega$$

Thus,

$$\left[\frac{\partial}{\partial t}\left(n_\sigma J^\sigma\right)\right]_{\bar{x}\phi} = 0 \tag{D35}$$

Applying equation (D35) to equation (D34) and differentiating yields

$$\left(\frac{\partial U^3_{(y)}}{\partial t}\right)_{\bar{x}\phi} = \left[\frac{\partial}{\partial t}\left(\frac{1}{n_3}\right)\right]_{\bar{x}\phi} U^3 n_3 - \frac{1}{n_3}\left[\frac{\partial}{\partial t}\left(U^1 n_1 + U^2 n_2 - v\right)\right]_{\bar{x}\phi} \tag{D36}$$

The quantities $\left(\frac{\partial U^1}{\partial t}\right)_{\bar{x}\phi}$ and $\left(\frac{\partial U^2}{\partial t}\right)_{\bar{x}\phi}$ in equation (D36) are evaluated by

$$\left(\frac{\partial U^\beta_{(y)}}{\partial t}\right)_{\bar{x}\phi} = \lambda \frac{\partial U^\beta_{(y)}}{\partial \bar{x}^4} = \lambda \frac{\partial U^\beta_{(y)}}{\partial y^i} \frac{\partial y^i}{\partial \bar{x}^4}$$

$$= \frac{\partial U^\beta_{(y)}}{\partial y^\sigma}\left(\frac{\partial y^\sigma}{\partial t}\right)_{\bar{x}\alpha} + \left(\frac{\partial U^\beta_{(y)}}{\partial t}\right)_{y\phi}$$

or, by equations (D31), (D11), and (D28),

$$\left(\frac{\partial U^\beta_{(y)}}{\partial t}\right)_{\bar{x}\phi} = \frac{\partial U^\beta}{\partial y^3}\left(J^3 - S^3\right) + \left(\frac{\partial U^\beta}{\partial t}\right)_{y\phi} \tag{D37}$$

APPENDIX D

Rate of Change of Earth-Fixed $\bar{x}$ Coordinate System

Velocity J^α in y Coordinates

The inertial Cartesian coordinate system $\bar{z}$, shown in sketch (c), has the origin at the center of the Earth; therefore, the velocity of the $\bar{x}$ coordinate system $J^\alpha_{(\bar{z})}$ at a fixed point in $\bar{z}^\phi$ coordinates does not change with time; or

$$\left(\frac{\partial J^\alpha_{(\bar{z})}}{\partial t}\right)_{\bar{z}^\phi} = 0$$

Hence, the comoving time derivative of J^α at a point fixed in the y coordinate system is

$$\left(\frac{DJ^\alpha}{dt}\right)_{(\bar{z})} = J^\alpha_{|i(\bar{z})} \frac{d\bar{z}^i}{dt} = J^\alpha_{|\sigma(\bar{z})} \frac{d\bar{z}^\sigma}{dt} = J^\alpha_{|\sigma(\bar{z})} \left(\frac{\partial \bar{z}^\sigma}{\partial t}\right)_{y^\psi}$$

or, by equations (D9),

$$\left(\frac{DJ^\alpha}{dt}\right)_{(\bar{z})} = J^\alpha_{|\sigma(\bar{z})} W^\sigma_{(\bar{z})} = J^\alpha_{|i(\bar{z})} W^i_{(\bar{z})}$$

Thus, by equations (D11),

$$\left(\frac{DJ^\alpha}{dt}\right)_{(y)} = J^\alpha_{|i(y)} W^i_{(y)} = J^\alpha_{|\sigma(y)} W^\sigma_{(y)} = \frac{dJ^\alpha}{dt_{(y)}} + J^\phi \begin{Bmatrix} \alpha \\ \phi \quad \sigma \end{Bmatrix} S^\sigma \tag{D38}$$

or

$$\left(\frac{DJ^\alpha}{dt}\right)_{(y)} = \frac{\partial J^\alpha}{\partial y^\sigma} W^\sigma + J^\phi \begin{Bmatrix} \alpha \\ \phi \quad \sigma \end{Bmatrix} S^\sigma$$

$$= \frac{\partial J^\alpha}{\partial y^\sigma} S^\sigma + J^\phi \begin{Bmatrix} \alpha \\ \phi \quad \sigma \end{Bmatrix} S^\sigma \tag{D39}$$

by equations (D11). The term $\dfrac{dJ^\alpha}{dt_{(y)}}$ in equation (D38) is the rate of change of J^α at a y grid point, or

$$\frac{dJ^{\alpha}}{dt_{(y)}} = \left(\frac{\partial J^{\alpha}_{(y)}}{\partial t}\right)_{y^{\phi}} \tag{D40}$$

Hence, by equations (D38) to (D40),

$$\left(\frac{\partial J^{\alpha}_{(y)}}{\partial t}\right)_{y^{\phi}} = \frac{\partial J^{\alpha}}{\partial y^{\sigma}} S^{\sigma} \tag{D41}$$

Rate of Change of Gradient of Gravitational Potential

Since $\bar{x}^3$ is vertical, the gradient of the gravitational potential is

$$\frac{\partial \theta}{\partial \bar{x}^i} = \frac{\partial \theta}{\partial \bar{x}^3} \delta_i^3 \tag{D42}$$

Thus,

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial y^{\beta}}\right)\right]_{y^{\sigma}} = \lambda \frac{\partial}{\partial y^4}\left(\frac{\partial \theta}{\partial y^{\beta}}\right) = \lambda \frac{\partial}{\partial y^{\beta}}\left(\frac{\partial \theta}{\partial y^4}\right)$$

$$= \lambda \frac{\partial}{\partial y^{\beta}}\left(\frac{\partial \theta}{\partial \bar{x}^3} \frac{\partial \bar{x}^3}{\partial y^4}\right)$$

$$= \frac{\partial}{\partial y^{\beta}}\left[\frac{\partial \theta}{\partial \bar{x}^3} \left(\frac{\partial \bar{x}^3}{\partial t}\right)_{y^{\sigma}}\right]$$

or, by equations (D32) and (D42) and the vanishing of $J^3_{(\bar{x})}$,

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial y^{\beta}}\right)\right]_{y^{\sigma}} = \frac{\partial}{\partial y^{\beta}}\left[\frac{\partial \theta}{\partial \bar{x}^3} \left(W^3_{(\bar{x})} - J^3_{(\bar{x})}\right)\right]$$

$$= \frac{\partial}{\partial y^{\beta}}\left(\frac{\partial \theta}{\partial \bar{x}^i} W^i_{(\bar{x})}\right)$$

$$= \frac{\partial}{\partial y^{\beta}}\left(\frac{\partial \theta}{\partial y^k} W^k_{(y)}\right)$$

Hence, by equations (D11),

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial y^{\beta}}\right)\right]_{y^{\sigma}} = \frac{\partial}{\partial y^{\beta}}\left(\frac{\partial \theta}{\partial y^{\phi}} S^{\phi}_{(y)}\right) \tag{D43}$$

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