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ABSTRACT

The plane elastostatic problem of internal and edge cracks in an infinite orthotropic strip is considered. The problems for the material types I and II are formulated in terms of singular integral equations. For the symmetric case the stress intensity factors are calculated and are compared with the isotropic results. The results show that because of the dependence of the Fredholm kernels on the elastic constants, unlike the crack problem for an infinite plane, in the strip the stress intensity factors are dependent on the elastic constants and are generally different than the corresponding isotropic results.

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1. INTRODUCTION

In plane elastostatic problems for an infinite orthotropic medium containing a line crack [1-3] or a series of collinear cracks [4] it was shown that the stress intensity factor is identical to that found for isotropic materials. However, if the geometry of the medium is bounded, it is expected that in orthotropic solids the material constants would influence the stress intensity factors. A bounded specimen geometry which is sufficiently simple for the purpose of analysis and at the same time is of sufficient practical interest is that of a long strip containing internal or edge cracks. The main objective of the present paper is by considering this problem to give some idea about the degree of influence of the material orthotropy on the stress intensity factors. The equivalent isotropic case is one of the more widely studied crack problems in technical literature (see, for example, [5-13]).

2. FORMULATION OF THE PROBLEM

Consider the plane problem for the orthotropic strip shown in Figure 1. Referring to, for example [14], the equilibrium equations for an orthotropic plane may be expressed as

$$\begin{aligned} &\beta_1 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \beta_3 \frac{\partial^2 v}{\partial x \partial y} = 0 \ , \\ &\frac{\partial^2 v}{\partial x^2} + \beta_2 \frac{\partial^2 v}{\partial y^2} + \beta_3 \frac{\partial^2 u}{\partial x \partial y} = 0 \ , \end{aligned} \tag{1}$$

where u, v are the x,y-components of the displacement vector and

$$\beta_1 = \frac{E_{11}}{(1-\nu_{12}\nu_{21})G_{12}} \quad , \quad \beta_2 = \frac{E_{22}}{E_{11}} \quad , \quad \beta_3 = 1+\nu_{21}\beta_1 \tag{2}$$

for generalized plane stress, and

$$\beta_1 = \frac{b_{11}}{G_{12}} \quad , \quad \beta_2 = \frac{b_{22}}{G_{12}} \quad , \quad \beta_3 = 1 + \frac{b_{12}}{G_{12}} \tag{3}$$

for plane strain. Here, E_{ij}, ν_{ij}, G_{ij} are the engineering elastic constants, $(i,j)=(1,2,3)$, the indexes $(1,2,3)$ corresponding to the directions (x,y,z), and the matrix (b_{ij}) is given by

$$\begin{aligned} &(b_{ij}) = B = A^{-1} \quad , \quad A = (a_{ij}) \quad , \quad (i,j)=(1,2,3) \quad , \\ &a_{ii} = 1/E_{ii} \quad ; \quad a_{ij} = -\nu_{ij}/E_{ii} = a_{ji} \quad , \quad (i\neq j) \end{aligned} \tag{4}$$

The solution of the problem shown in Figure 1 may be obtained by using the standard superposition technique. Thus, for the purpose of evaluating the stress intensity factors and obtaining information relevant to the fracture of the solid, it is sufficient to consider the problem in which statically self-equilibrating crack surface tractions are the only external loads.

To solve the differential equations (1) let

$$\begin{aligned} u(x,y) &= \frac{2}{\pi}\int_0^\infty [f_1(\alpha,x)\cos\alpha y + g_1(\alpha,y)\sin\alpha x]d\alpha \quad , \\ v(x,y) &= \frac{2}{\pi}\int_0^\infty [f_2(\alpha,x)\sin\alpha y + g_2(\alpha,y)\cos\alpha x]d\alpha \quad . \end{aligned} \tag{5}$$

Substituting from (5) into (1) the functions f_i and g_i, $(i=1,2)$ are obtained as follows:

$$f_1(\alpha,x) = \sum_1^4 A_j(\alpha)e^{s_j\alpha x} \quad , \quad f_2(\alpha,x) = \sum_1^4 c_jA_j(\alpha)e^{s_j\alpha x} \quad ,$$

$$g_1(\alpha,y) = \sum_1^4 B_j(\alpha)e^{s_j\alpha y/\beta_5} \quad , \quad g_2(\alpha,y) = \sum_1^4 d_jB_j(\alpha)e^{s_j\alpha y/\beta_5} \quad , \tag{6}$$

where $s_1,\ldots,s_4$ are the roots of the following characteristic equation:

$$s^4 + \beta_4 s^2 + \beta_5^2 = 0 \quad , \quad s_3 = -s_1 \quad , \quad s_4 = -s_2 \quad , \tag{7}$$

and the constants β_4, β_5, c_j, and d_j, $(j=1,\ldots,4)$ are given by

$$\beta_4 = (\beta_3^2 - \beta_1\beta_2 - 1)/\beta_1 \quad , \quad \beta_5^2 = \beta_2/\beta_1 \quad ,$$

$$c_1 = -c_3 = (1 - \beta_1 s_1^2)/\beta_3 s_1 \quad , \quad c_2 = -c_4 = (1 - \beta_1 s_2^2)/\beta_3 s_2 \quad , \tag{8}$$

$$d_1 = -d_3 = (s_1^2 - \beta_1\beta_5^2)/\beta_3 s_1 \beta_5 \quad , \quad d_2 = -d_4(s_2 - \beta_1\beta_5^2)/\beta_3 s_2 \beta_5 \quad .$$

Assuming that x and y are axes of symmetry for loading as well as geometry, the unknown functions $A_j(\alpha)$ and $B_j(\alpha)$, $(j=1,..,4)$ are determined from the following conditions:

$$u(x,y)\to 0 \quad , \quad v(x,y)\to 0 \quad \text{for } y\to\infty \quad , \tag{9}$$

$$\sigma_{xx}(h,y) = 0 \quad , \quad \sigma_{xy}(h,y) = 0 \quad , \quad 0\le y<\infty \quad , \tag{10}$$

$$u(0,y) = 0 \quad , \quad \sigma_{xy}(0,y) = 0 \quad , \quad 0\le y<\infty \quad , \tag{11}$$

$$\sigma_{xy}(x,0) = 0 \quad , \quad 0\le y\le h \quad , \tag{12}$$

$$\left.\begin{aligned} &\sigma_{yy}(x,+0) = -p(x) \quad , \quad a<|x|<b \quad , \\ &v(x,0) = 0 \quad , \quad 0\le|x|<a \quad , \quad b<|x|<h \quad , \end{aligned}\right\} \tag{13}$$

where the crack surface traction $p(x)$ is a known function. The seven homogeneous conditions (9-12) may be used to eliminate seven of the unknowns and the mixed boundary conditions (13) would give a system of dual integral equations to determine the eighth. In this paper, defining a new unknown function

$$\phi(x) = \frac{\partial}{\partial x} v(x,0) \quad , \quad a<|x|<b \quad , \tag{14}$$

the problem is reduced to a singular integral equation by using the first equation of (13). From the second equation of (13) it is seen that

$$\phi(x,0) = 0 \quad , \quad 0\leq|x|<a \quad , \quad b<|x|<h \quad ,$$

$$\int_a^b \phi(x)dx = 0 \quad . \tag{15}$$

Examining the roots of (7) it may be observed that

(i) for $\beta_4<0$, $\beta_6 = \beta_4^2-4\beta_5^2>0$ there are four real roots, s_1, s_2 $s_3 = -s_1$, and $s_4 = -s_2$ ($s_1>0$, $s_2>0$); in this case the corresponding material will be classified as type I;

(ii) for $\beta_6<0$ the roots are complex, $s_1 = \omega_1 + i\omega_2 = -s_3$, $s_2 = \omega_1 - i\omega_2 = -s_4$ ($\omega_1>0$, $\omega_2>0$) and the related material will be classified as type II; and

(iii) for $\beta_4>0$, $\beta_6>0$ the roots would be pure imaginary $s_1 = i\omega_3 = -s_3$, $s_2 = i\omega_4 = -s_4$.

In practice generally β_4 is negative. Therefore the problems of interest are those relating to materials type I and II only. This classification seems to be necessary in order to pursue the formulation of the problem beyond equations (6) without introducing unnecessarily complicated complex algebra. Also, in this paper only the case of generalized plane stress will be considered. For plane strain it is sufficient to replace the quantities $E_x/(1-\nu_{xy}\nu_{yx})$, $E_y/(1-\nu_{xy}\nu_{yx})$, and $\nu_{yx}E_x/(1-\nu_{xy}\nu_{yx})$ by b_{11}, b_{22}, and b_{12}, respectively (see equation 4).

Because of symmetry considering only one quarter of the medium shown in Figure 1, and using the standard stress-displacement relations for plane stress, after somewhat lengthy but routine analysis, for material type I (i.e., for real s_1 and s_2) the problem may be reduced to the following integral equation:

$$\int_a^b \left[\frac{1}{t-x}+\frac{1}{t+x}+k_1(x,t)-k_1(x,-t)\right]\phi(t)dt$$

$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y m_{14}}\,p(x) \quad , \quad a<x<b \quad , \tag{16}$$

under the additional condition (15). Here the kernel is given by

$$k_1(x,t) = \frac{1}{m_{14}}\int_0^\infty \left[K_1(x,\alpha)e^{-(h-t)\alpha\beta_5/s_1} + K_2(x,\alpha)e^{-(h-t)\alpha\beta_5/s_2}\right]d\alpha \quad . \tag{17}$$

The expressions for K_1, K_2, and m_{14} are given in Appendix A.

For material type II the roots of the characteristic equation (7) are complex. Defining

$$s_1 = \omega_1 + i\omega_2 = -s_3 \quad , \quad s_2 = \omega_1 - i\omega_2 = -s_4 \quad , \tag{18}$$

and, assuming that $\omega_1>0$, in this case the integral equation becomes

$$\int_a^b [\frac{1}{t-x}+\frac{1}{t+x}+k_2(x,t)-k_2(x,-t)]\phi(t)dt$$

$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y r_{14}}\,p(x) \quad , \quad a<x<b \quad , \tag{19}$$

again subject to condition (15). The kernel k_2 is given by

$$k_2(x,t) = \int_0^\infty K_3(x,t,\alpha)e^{-\omega_1\alpha(h-t)}d\alpha \quad , \tag{20}$$

where the function K_3 and the related constants r_i are defined in the Appendix B.

One may note that in the special case of single internal crack, (i.e., for a=0, b<h) the integral equations (16) and (19) may be written as

$$\int_{-b}^b [\frac{1}{t-x}+k_i(x,t)]\phi(t)dt = -\frac{\pi}{M_i}\,p(x) \quad , \quad -b<x<b \quad ,$$

$$(i=1,2) \quad , \quad M_1 = \frac{2E_y m_{14}}{1-\nu_{xy}\nu_{yx}} \quad , \quad M_2 = \frac{2E_y r_{14}}{1-\nu_{xy}\nu_{yx}} \tag{21}$$

where i=1 and 2 correspond to materials type I and II, respectively. In this case the single-valuedness condition (15) becomes

$$\int_{-b}^b \phi(x)dx = 0 \quad . \tag{22}$$

3. STRESS INTENSITY FACTORS

The standard definition of the stress intensity factors at the crack tips a and b is

$$k(a) = \lim_{x\to a} \sqrt{2(a-x)}\,\sigma_{yy}(x,0) \quad ,$$

$$k(b) = \lim_{x\to b} \sqrt{2(x-b)}\,\sigma_{yy}(x,0) \quad . \tag{23}$$

To determine the asymptotic behavior of the cleavage stress σ_{yy} around the crack tips, first it may be observed that the index of the singular integral equations (16) and (19) is +1. Therefore, the solution is of the following form [15]:

$$\phi(t) = f(t)[(t-a)(b-t)]^{-\frac{1}{2}} \quad . \tag{24}$$

Next, it should be pointed out that the left hand side of (16) and (19) gives $\sigma_{yy}(x,0)$ for x outside the interval (a,b) as well as within. Thus, making a change in variable

$$x = \frac{b-a}{2}\,s + \frac{b+a}{2} \quad , \quad t = \frac{b-a}{2}\,r + \frac{b+a}{2} \tag{25}$$

for example, (16) may be expressed as

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{r-s} + k(s,r)\right]\psi(r)dr = \frac{q(s)}{M_1} \quad , \quad \frac{b+a}{b-a} \le s \le \frac{2h-b-a}{b-a} \tag{26}$$

where

$$q(s) = \sigma_{yy}(x,0) \quad , \quad \psi(r) = \phi(t) = F(r)(1-r^2)^{-\frac{1}{2}} \tag{27}$$

and k(s,r) and F(r) are bounded functions. The objective is then to determine the asymptotic behavior of q(s) around $s=\mp 1$, ($|s|>1$) in terms of the unknown function F(r) which is obtained by solving the integral equation (26) in $-1<s<1$ where $q(s)=-p(x)$ is known. To do this let us assume that the bounded function F(r) can be represented by the following infinite series in Tchebyshev polynomials:

$$F(r) = \sum_{0}^{\infty} A_n T_n(r) \quad . \tag{28}$$

Substituting from (28) into (26) one obtains

$$\sum_{0}^{\infty} A_n[G_n(s) + H_n(s)] = \frac{q(s)}{M_1} \tag{29}$$

$$G_n(s) = \frac{1}{\pi}\int_{-1}^{1} \frac{T_n(r)dr}{(r-s)\sqrt{1-r^2}} \quad , \quad H_n(s) = \frac{1}{\pi}\int_{-1}^{1} \frac{k(s,r)T_n(r)}{\sqrt{1-r^2}} dr \quad . \tag{30}$$

Here, $H_n(s)$ is a bounded function. To determine G_n one may use the expression

$$\frac{1}{\pi i}\int_{-1}^{1} \frac{T_n(r)dr}{(r-z)\sqrt{r^2-1}} = \frac{(z-\sqrt{z^2-1})^n}{\sqrt{z^2-1}} \tag{31}$$

where z is the complex variable in the plane cut along (-1,1). Observing that on the real line $(z^2-1)^{\frac{1}{2}}$ is an odd function, from (30) and (31) it follows that

$$G_n(s) = -\frac{[s-\mathrm{sgn}(s)\sqrt{s^2-1}]^n}{\mathrm{sgn}(s)\sqrt{s^2-1}} \quad . \tag{32}$$

As $s\to\mp 1$ (32) yields

$$s \to +1,\ (s>1):\quad G_n(s) = -\frac{1}{\sqrt{s^2-1}} + R_1(s)\ ,$$

$$s \to -1,\ (s<-1):\quad G_n(s) = \frac{(-1)^n}{\sqrt{s^2-1}} + R_2(s)\ , \tag{33}$$

where the functions R_1 and R_2 are bounded at $s=\mp 1$.

Now, observing that $H_n(\mp 1) = \text{finite}$, $T_n(1) = 1$, $T_n(-1) = (-1)^n$, from (29) and (33) the asymptotic behavior of $q(s)$ is found to be

$$s \to +1,\ (s>1):\quad \frac{q(s)}{M_1} = -\frac{F(1)}{\sqrt{s^2-1}} + R_3(s)\ ,$$

$$s \to -1,\ (s<-1):\quad \frac{q(s)}{M_1} = \frac{F(1)}{\sqrt{s^2-1}} + R_4(s)\ . \tag{34}$$

where again the functions R_3 and R_4 are bounded at $s=\mp 1$. Going back to the original quantities by using (25) and (27), (34) becomes

$$x \to b,\ (x>b):\quad \frac{\sigma_{yy}(x,0)}{M_1} = -\frac{F(1)\sqrt{\ell}}{\sqrt{2(x-b)}} + R_5(x)\ ,$$

$$x \to a,\ (x<a):\quad \frac{\sigma_{yy}(x,0)}{M_1} = \frac{F(-1)\sqrt{\ell}}{\sqrt{2(a-x)}} + R_6(x)\ , \tag{35}$$

where the functions R_5 and R_6 are also bounded at $x=b$ and $x=a$. Thus, from (23) and (35) the stress intensity factors are found to be

$$k(b) = -M_1 F(1)\sqrt{\ell}\ ,\quad k(a) = M_1 F(-1)\sqrt{\ell}\ . \tag{36}$$

In the case of fully imbedded cracks the integral equations (16), (19) or (21) can always be reduced to the normalized form (26) and can be solved by using the technique described in [16].

4. EDGE CRACKS

In equations (16) and (19) the kernels $k_1(x,t)-k_1(x,-t)$ and $k_2(x,t)-k_2(x,-t)$ are bounded provided $b<h$ (see Figure 1). For $b=h$, that is in the case of edge cracks, the integral equations are still valid but these kernels do not remain bounded as x and t go to the end point $b=h$ and, consequently, the singular behavior of the solution at $x=b=h$ may no longer be described by (24). Expressing the kernels in (16) and (19) as

$$k_i(x,t) = k_{if}(x,t) + k_{is}(x,t) \quad , \quad (i=1,2)$$

where k_{if} is bounded in the closed interval $[a,h]$, the unbounded parts k_{is}, $(i=1,2)$ may be obtained from (17) and (20) by examining the asymptotic behavior of the integrals for large values of α. Thus, after some routine analysis we find

$$k_{1s}(x,t) = \frac{1}{m_{14}m_{15}}\left[\frac{m_{16}}{s_1(h-x)+(h-t)\beta_5/s_1} + \frac{m_{17}}{s_2(h-x)+(h-t)\beta_5/s_1}\right.$$

$$\left. + \frac{m_{18}}{s_1(h-x)+(h-t)\beta_5/s_2} + \frac{m_{19}}{s_2(h-x)+(h-t)\beta_5/s_2}\right] \quad , \qquad (37)$$

$$k_{2s}(x,t) = \frac{1}{r_{14}r_{19}}\left[\frac{r_{25}\omega_2(t-x)}{\omega_2^2(t-x)^2+\omega_1^2(2h-t-x)^2}\right.$$

$$+ \frac{r_{26}\omega_2(2h-t-x)}{\omega_2^2(2h-t-x)^2+\omega_1^2(2h-t-x)^2}$$

$$+ \frac{r_{27}\omega_1(2h-t-x)}{\omega_2^2(t-x)^2+\omega_1^2(2h-t-x)^2}$$

$$+ \frac{r_{28}\omega_1(2h-t-x)}{\omega_1^2(2h-t-x)^2+\omega_2^2(2h-t-x)^2}\Big] \tag{38}$$

where the constants m_i and r_i are given in the appendices. Thus, for example, the integral equation (16) may be expressed as

$$\int_a^h \left[\frac{1}{t-x}+\frac{1}{t+x}+k_{1s}(x,t) + k_{1f}(x,t) - k_1(x,t)\right]\phi(t)dt$$

$$= -\frac{\pi}{M}\, p(x) \quad , \quad a<x<h \quad . \tag{39}$$

In (39) for the purpose of asymptotic analysis transferring the terms involving the bounded kernels to the righthand side one may write

$$\int_a^h \left[\frac{1}{t-x}+k_{1s}(x,t)\right]\phi(t)dt = P_1(x) \quad , \quad a<x<h \tag{40}$$

where P_1 is a bounded function in [a,h]. Letting now

$$\phi(t) = \frac{f_1(t)}{(h-t)^{\alpha}(t-a)^{\beta}} \quad , \quad 0<Re(\alpha,\beta)<1 \quad , \tag{41}$$

where f_1 is H-continuous in [a,h], and following the procedure outlined in [15], the characteristic equations for α and β are found to be

$$\cot\pi\beta = 0 \quad , \quad \beta = 1/2 \quad , \tag{42}$$

$$-\cos\pi\alpha + \frac{1}{m_{14}m_{15}}\left[m_{16}(\beta_5/s_1^2)^{\alpha}(s_1/\beta_5) + m_{17}(s_1/\beta_5)\right.$$

$$\left. + m_{18}(s_2/\beta_5) + m_{19}(\beta_5/s_2^2)^{\alpha}(s_2/\beta_5)\right] = 0 \quad . \tag{43}$$

Similarly, for material type II described by (19), assuming the solution again as given by (41), the characteristic equations become

$$\cot\pi\beta = 0 \quad , \quad \beta = 1/2 \tag{44}$$

$$\cos\pi\alpha + \frac{1}{r_{14}r_{19}}\left[r_{29} + \frac{\omega_2 r_{25} - \omega_1 r_{27}}{\omega_1^2 + \omega_2^2}\cos(2\alpha\tan^{-1}\frac{\omega_2}{\omega_1}) - \frac{\omega_1 r_{25} + \omega_2 r_{27}}{\omega_1^2 + \omega_2^2}\sin(2\alpha\tan^{-1}\frac{\omega_2}{\omega_1})\right] . \tag{45}$$

At the imbedded crack tip x=a it is seen that the singularity has the expected 1/2 power. On the other hand, as in the isotropic case, (43) and (45) have no root for which $0<\mathrm{Re}(\alpha)<1$, meaning that at x=h there is no power singularity. One may also proceed and investigate the possibility of a logarithmic singularity for the solution. Thus, letting $\alpha=0$ in (41) and defining the sectionally holomorphic function

$$F_1(z) = \frac{1}{\pi}\int_a^h \frac{\phi(t)}{t-z}\,dt \tag{46}$$

we find [15]

$$F_1(z) = \frac{f_1(a)e^{\pi i\beta}}{\sin\pi\beta}\,\frac{1}{(z-a)^\beta} + \frac{f_1(h)}{\pi(h-a)^\beta}\log(z-h) + P_2(z) \quad ,$$

$$\frac{1}{\pi}\int_a^b \frac{\phi(t)}{t-x}\,dt = \frac{f_1(a)\cot\pi\beta}{(x-a)^\beta} + \frac{f_1(h)}{\pi(h-a)^\beta}\log(h-x) + P_3(x) \quad ,$$

$$\frac{1}{\pi}\int_a^b \frac{\phi(t)dt}{t-(2h-x)} = F_1(2h-x) \quad , \ldots \tag{47}$$

where P_2 and P_3 are bounded at x=h and have at most a singularity of lower order than β at x=a. Substituting from (47) into (40), multiplying through by $(x-a)^{\beta}$ and letting $x \to a$, it is found that $\cot\pi\beta=0$, giving again $\beta=1/2$. On the other hand, in the neighborhood of the end point x=h one obtains

$$[1-\frac{1}{m_{14}m_{15}\beta_5}(s_1m_{16}+s_1m_{17}+s_2m_{18}+s_2m_{19})]\log(h-x)+P_4(x)=P_1(x) \quad (48)$$

where P_4 contains all the bounded terms around x=h on the left hand side of (40). Similarly, for the material type II one finds

$$[1+\frac{1}{(\omega_1^2+\omega_2^2)r_{14}r_{19}}(\omega_2r_{25}-\omega_2r_{26}-\omega_1r_{27}-\omega_1r_{28})]\log(h-x)+Q_4(x)=Q_1(x) \quad (49)$$

It turns out that, as in the case of isotropic materials [10], the coefficient of the logarithmic term in (48) and (49) is identically zero, meaning that the solution may not have logarithmic singularity at x=h. In the edge crack problem the integral equation (39) and the similar equation for the material type II are solved by defining

$$\phi(x)=\frac{f(x)}{\sqrt{x-a}}=\frac{F(s)}{\sqrt{s+1}} \quad (50)$$

and by using the numerical technique described in [10]. In this case the stress intensity factor at x=a becomes

$$k(a)=M_1F(-1)\sqrt{2\ell}\ , \quad \ell=(h-a)/2\ . \quad (51)$$

5. RESULTS AND DISCUSSION

As an example the following two orthotropic materials will be considered:

$$\text{Type I:} \quad \begin{aligned} E_x &= 8 \times 10^6 \text{ psi } (55.16 \times 10^9 \text{ N/m}^2) \quad , \\ E_y &= 24.75 \times 10^6 \text{ psi } (170.65 \times 10^9 \text{ N/m}^2) \quad , \\ G_{xy} &= 0.7 \times 10^6 \text{ psi } (4.83 \times 10^9 \text{ N/m}^2) \quad , \\ \nu_{xy} &= 0.036 \quad , \end{aligned}$$

$$\text{Type II:} \quad \begin{aligned} E_x &= 3.1 \times 10^6 \text{ psi } (21.37 \times 10^9 \text{ N/m}^2) \quad , \\ E_y &= 9.7 \times 10^6 \text{ psi } (66.88 \times 10^9 \text{ N/m}^2) \quad , \\ G_{xy} &= 2.6 \times 10^6 \text{ psi } (17.93 \times 10^9 \text{ N/m}^2) \quad , \\ \nu_{xy} &= 0.2 \quad . \end{aligned}$$

Tables 1-3 show some of the calculated results for the stress intensity factors. In all the calculations it was assumed that the crack surface traction was constant, i.e.,

$$\sigma_{yy}(x,0) = -p(x) = -p_0 \tag{52}$$

which corresponds to uniform tension of the strip away from the crack region. Table 1 shows the results for an internal crack of length 2b (see Figure 1, a=0) which was found by solving (21). The stress intensity factors used in the tables are defined by (23) and are calculated by using (36) for internal and (51) for edge cracks. The stress intensity factors for symmetrically located two collinear cracks (Figure 1) are given by Table 2. Table 3 gives the results for symmetric edge cracks.

The tables also contain the stress intensity factors for the isotropic strip which are included for comparison. A close examination of the integral equations (16), (19), or (21) would indicate that in orthotropic materials since the Fredholm kernel k_1 or k_2 is heavily dependent on the material constants, the solution must also depend on the constants. On the other hand, in isotropic materials even though the structure of the integral equation is identical to that of (16) or (19) (see, for example, [10]), the kernel of the integral equation is independent of the elastic constants and the constants appear in the equation as a multiplying factor (in the form of $(1+\kappa)/4\mu$) only. The stress intensity factors given in the tables indicate that the results for the orthotropic strip are indeed different than the isotropic results. The tables also show that for approximately the same modulus ratio E_y/E_x (in the example approximately 3/1), depending on the remaining constants, the materials may not only be of different type (I or II), also the stress intensity factors may be greater (in this case, in material type II) or smaller (in material type I) than the isotropic values. In orthotropic materials there are three independent material parameters, namely, G_{xy}/E_y, E_x/E_y, and ν_{xy}. Therefore, it does not seem to be feasible to make a systematic study and demonstrate the effect of the material orthotropy on the stress intensity factors. However, it appears that there exists a difference between isotropic and orthotropic results and in highly orthotropic materials it may be significant.

In solving this problem, the numerical analysis produced a somewhat unexpected result. First, it should be pointed out that the results given in the tables are accurate to roughly three significant digits, the remaining digits may not be accurate. On the other hand, after rotating the material 90 degrees (i.e., taking the strip parallel to the less stiff axis and the crack along the stiffer axis) and fully expecting to obtain a different set of results, the print out for the stress intensity factors came out to be identical - in all eight digits - to the original values obtained for the strip which was parallel to the stiff axis. Furthermore, the ratio of the function F(r) defined by (27) at all points in $-1<r<1$ for the 0 and 90 degree orientations was found to be constant, indicating that the crack surface displacements for the two cases are related by (see (14), (27), and (36))

$$v_0(x,0)M_0 = v_{90}(x,0)M_{90} \tag{53}$$

where the constant M is defined by (21). This simply shows that the kernels k_1 and k_2 which appear in the integral equations (16), (19), and (21) remain invariant under a 90 degree rotation for a given orthotropic strip.

Table 1. The stress intensity factor $k(b)/p_0\sqrt{b}$ for an internal crack of length 2b in isotropic and orthotropic strips.

b/h	Isotropic	Orthotropic	
		Type I	Type II
→0	→1.0	→1.0	→1.0
0.1	1.0060	1.0044	1.0064
0.2	1.0246	1.0182	1.0261
0.3	1.0578	1.0428	1.0611
0.4	1.1094	1.0811	1.1155
0.5	1.1869	1.1387	1.1966
0.6	1.3033	1.2264	1.3183
0.7	1.4888	1.3674	1.5099
0.8	1.8160	1.6241	1.8471
0.9	2.5809	2.2487	2.6278

Table 2. The stress intensity factors k(a) and k(b) for symmetric collinear internal cracks in a strip.

a/h	b/h	$k(a)/p_0\sqrt{\ell}$			$k(b)/p_0\sqrt{\ell}$		
		Isotr.	Type I	Type II	Isotr.	Type I	Type II
→0	0.4	(→∞)	(→∞)	(→∞)	→1.569	→1.530	→1.575
0.1	0.5	1.176	1.160	1.179	1.115	1.100	1.117
0.2	0.6	1.109	1.095	1.111	1.094	1.080	1.096
0.4	0.8	1.097	1.081	1.099	1.122	1.098	1.127
0.5	0.9	1.127	1.104	1.132	1.221	1.170	1.231
0.6	→1	→1.600	→1.531	→1.613	(→∞)	(→∞)	(→∞)
0.1	0.9	1.678	1.595	1.689	1.694	1.607	1.705
0.5	0.95	1.194	1.160	1.200	1.445	1.351	1.461
0.5	0.98	1.268	1.226	1.275	1.875	1.721	1.883
0.5	→1	→1.640	→1.600	→1.661	(→∞)	(→∞)	(→∞)

Table 3. The stress intensity factor $k(a)p_0\sqrt{\ell}$ for the case of symmetric edge cracks, $\ell=(h-a)/2$.

a/h	$k(a)/p_0\sqrt{\ell}$		
	Isotropic	Type I	Type II
0.1	2.980	2.978	2.982
0.2	2.218	2.208	2.220
0.3	1.907	1.887	1.912
0.4	1.742	1.710	1.750
0.5	1.640	1.600	1.661
0.6	1.600	1.531	1.613
0.7	1.574	1.486	1.590
0.8	1.567	1.462	1.587
0.9	1.576	1.458	1.593
0.98	1.582	1.467	1.598

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APPENDIX A

Expressions for the functions K_1 and K_2 and the constants m_i (see equation 17) (material type I):

$$K_1(x,\alpha) = \frac{1}{2s_1 m_{13} P(\alpha)} [m_7 \frac{\cosh(s_1\alpha x)}{\cosh(s_1\alpha h)} (m_1 m_{10}\beta_5 \tanh(s_2\alpha h)$$

$$+ m_4 m_{11}\beta_5) - m_8 \frac{\cosh(s_2\alpha x)}{\cosh(s_2\alpha h)} (m_3 m_{11}\beta_5)$$

$$+ m_1 m_9 \beta_5 \tanh(s_1\alpha h))] \quad ,$$

$$K_2(x,\alpha) = \frac{1}{2s_2 m_{13} P(\alpha)} [m_7 \frac{\cosh(s_1\alpha x)}{\cosh(s_1\alpha h)} (-m_4 m_{11}\beta_5$$

$$- \frac{m_{11}}{m_{12}} m_2 m_{10}\beta_5 \tanh(s_2\alpha h)) + m_8 \frac{\cosh(s_2\alpha x)}{\cosh(s_2\alpha h)} \cdot$$

$$\cdot (\frac{m_{11}}{m_{12}} m_2 m_9 \beta_5 \tanh(s_1\alpha h) + m_3 m_{11}\beta_5)] \quad ,$$

$$P(\alpha) = m_3 m_{10} \tanh(s_2\alpha h) - m_4 m_9 \tanh(s_1\alpha h) \quad ,$$

$$m_1 = 1+\nu_{yx}s_1 d_1/\beta_5 \quad , \quad m_2 = 1+\nu_{yx}s_2 d_2/\beta_5 \quad , \quad m_3 = s_1+\nu_{yx}c_1 \quad ,$$

$$m_4 = s_2+\nu_{yx}c_2 \quad , \quad m_5 = \nu_{xy}+d_1 s_1/\beta_5 \quad , \quad m_6 = \nu_{xy}+d_2 s_2/\beta_5 \quad ,$$

$$m_7 = \nu_{xy}s_1+c_1 \quad , \quad m_8 = \nu_{xy}s_2+c_2 \quad , \quad m_9 = c_1 s_1 - 1$$

$$m_{10} = c_2 s_2 - 1 \quad , \quad m_{11} = d_1 - s_1/\beta_5 \quad , \quad m_{12} = d_2 - s_2/\beta_5$$

$$m_{13} = d_1 - d_2 m_{11}/m_{12} \quad , \quad m_{14} = (m_5 - m_6 m_{11}/m_{12})/2m_{13} \quad ,$$

$$m_{15} = m_3m_{10}-m_4m_9 \quad , \quad m_{16} = m_7\beta_5(m_1m_{10}+m_4m_{11})/2s_1m_{13} \quad ,$$

$$m_{17} = -m_8\beta_5(m_3m_{11}+m_1m_9)/2s_1m_{13} \quad ,$$

$$m_{18} = -m_7m_{11}\beta_5(m_4+m_2m_{10}/m_{12})/2m_{13}s_2 \quad ,$$

$$m_{19} = m_8m_{11}\beta_5(m_3+m_9m_2/m_{12})/2s_2m_{13} \quad .$$

APPENDIX B

Expressions for the function K_3 and the constants r_i (see equation 20) (material type II):

$$K_3(x,t,\alpha) = \frac{2}{r_{14}Q(\alpha)} \{[-r_6 \sin(\omega_2\alpha x)\sinh(\omega_1\alpha x)$$

$$+ r_5 \cos(\omega_2\alpha x)\cosh(\omega_1\alpha x)] \cdot [r_{16} \sin[\omega_2\alpha(h-t)] \cdot$$

$$\cdot (r_9 \sin(\omega_2\alpha h)\cosh(\omega_1\alpha h) + r_{10} \cos(\omega_2\alpha h) \sinh(\omega_1\alpha h))$$

$$- r_{18} (\frac{\omega_2}{\omega_1} \cos[\omega_2\alpha(h-t)] - \sin[\omega_2\alpha(h-t)]) \cdot$$

$$\cdot (r_1 \sin(\omega_2\alpha h)\sinh(\omega_1\alpha h) + r_2 \cos(\omega_2\alpha h) \cosh(\omega_1\alpha h))]$$

$$+ [r_5 \sin(\omega_2\alpha x)\sinh(\omega_1\alpha x) + r_6 \cos(\omega_2\alpha x) \cosh(\omega_1\alpha x)] \cdot$$

$$\cdot [-r_{16} \sin[\omega_2\alpha(h-t)] \cdot (r_9\cos(\omega_2\alpha h)\sinh(\omega_1\alpha h)$$

$$- r_{10} \sin(\omega_2\alpha h)\cosh(\omega_1\alpha h)) + r_{18} (\frac{\omega_2}{\omega_1} \cos[\omega_2\alpha(h-t)]$$

$$- \sin[\omega_2\alpha(h-t)]) \cdot (-r_2 \sin(\omega_2\alpha h)\sinh(\omega_1\alpha h)$$

$$+ r_1 \cos(\omega_2\alpha h) \cosh(\omega_1\alpha h))]\} ,$$

$$Q(\alpha) = r_{19} \sinh(\omega_1\alpha h)\cosh(\omega_1\alpha h) + r_{20} \sin(\omega_2\alpha h)\cos(\omega_2\alpha h) .$$

$$s_1 = \omega_1 + i\omega_2 , \quad s_2 = \omega_1 - i\omega_2 , \quad \omega_1 > 0 ,$$

$$c_1 = \beta_7 + i\beta_8 , \quad c_2 = \beta_7 - i\beta_8 , \quad d_1 = \beta_9 + i\beta_{10} , \quad d_2 = \beta_9 - i\beta_{10} ,$$

$$r_1 = \omega_1 + \nu_{yx}\beta_7 , \quad r_2 = \omega_2 + \nu_{yx}\beta_8 , \quad r_3 = 1 + \nu_{yx}(\omega_1\beta_9 - \omega_2\beta_{10})/\beta_5 ,$$

$$r_4 = \nu_{yx}(\omega_1\beta_{10} + \omega_2\beta_9)/\beta_5 , \quad r_5 = \omega_1\nu_{xy} + \beta_7 , \quad r_6 = \omega_2\nu_{xy} + \beta_8 ,$$

$$r_7 = \nu_{xy} + (\omega_1\beta_9 - \omega_2\beta_{10})/\beta_5 , \quad r_8 = r_4/\nu_{yx} , \quad r_9 = \omega_1\beta_7 - \omega_2\beta_8 - 1 ,$$

$$r_{10} = \omega_2\beta_7 + \omega_1\beta_8 , \quad r_{11} = -\beta_{10} + \omega_2/\beta_5 , \quad r_{12} = \beta_9 - \omega_1/\beta_5 ,$$

$$r_{13} = -\beta_9 - \beta_{10}r_{12}/r_{11} , \quad r_{14} = -(r_7 + r_8 r_{12}/r_{11})/2r_{13} ,$$

$$r_{15} = r_4 - r_3 r_{12}/r_{11} , \quad r_{16} = r_{15}\beta_5/4r_{13}\omega_1 , \quad r_{17} = -r_{11} - r_{12}^2/r_{11} ,$$

$$r_{18} = \omega_1 r_{17}/4r_{13} , \quad r_{19} = r_1 r_{10} - r_2 r_9 , \quad r_{20} = r_2 r_{10} + r_1 r_9 ,$$

$$r_{21} = -r_6 r_{16} r_9 - r_6 r_{18} r_1 + r_5 r_{10} r_{16} + r_2 r_5 r_{18} ,$$

$$r_{22} = (r_1 r_6 - r_2 r_5) r_{18}\omega_2/\omega_1 ,$$

$$r_{23} = -r_6 r_{10} r_{16} - r_2 r_6 r_{18} - r_5 r_9 r_{16} - r_1 r_5 r_{18} ,$$

$$r_{24} = (r_2 r_6 + r_1 r_5) r_{18}\omega_2/\omega_1 , \quad r_{25} = -r_{21} - r_{24} , \quad r_{26} = r_{21} - r_{24} ,$$

$$r_{27} = r_{22} - r_{23} , \quad r_{28} = r_{22} + r_{23} , \quad r_{29} = -(\omega_1 r_{28} + \omega_2 r_{26})/(\omega_1^2 + \omega_2^2) .$$

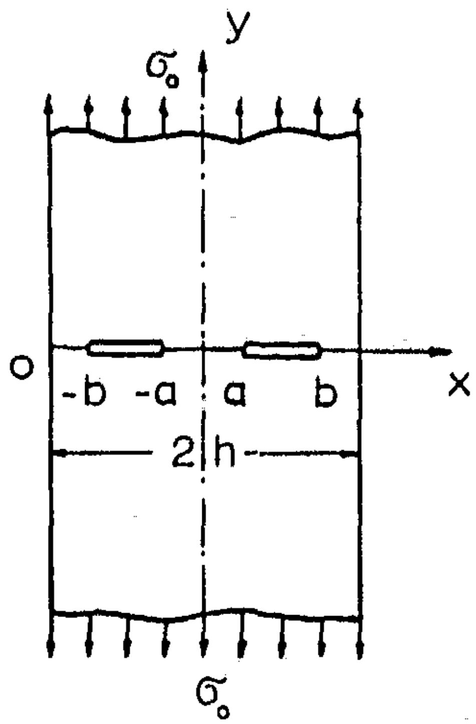

Figure 1. Infinite strip with two internal cracks.