(NASA-CR-149528) INFLUENCE OF TRANSVERSE SHEAR ON AN AXIAL CRACK IN A CYLINDRICAL SHELL (Lehigh Univ.) 42 p HC A03/MF A01 CSCL 13M

N77-17507

Unclas
G3/39 14894

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July 1976

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ABSTRACT

An axial crack in a cylindrical shell is investigated by use of a 10th order shell theory, which accounts for transverse shear deformations as well as a special kind of orthotropy. The symmetric problem is formulated in terms of two coupled singular integral equations, which are solved numerically. The asymptotic membrane and bending stress fields ahead of the crack are found to be self similar. Stress intensity factors are given as a function of the shell parameter for various values of the ratio crack length to shell thickness. Considerable differences from 8th order shell theory results are found for the bending stresses, while the membrane stresses of the 8th order theory seems to be a lower limit reached for very thin shells.

*Part of the present work was carried out during a visit to Lehigh University supported by NSF under the Grant ENG 73-045053 A01 and NASA-Langley under the Grant NGR 30[illegible]-007-011.

1. INTRODUCTION

Plate and shell theories based on Kirchhoff's assumption only enables the satisfaction of two plate-type boundary conditions. Broadly speaking the importance of this restriction depends on the variations of the field quantities compared to the plate or shell thickness. Thus serious shortcomings can be expected in problems where steep gradients are encountered, e.g., crack problems. A number of investigations [1-4] concerned with a crack in an infinite plate in bending have revealed considerable differences between the 4th order bending theory solution and solutions obtained by use of 6th order Reissner-type bending theory [5]. The differences concern both the magnitudes of the stresses and their distribution around the crack. In the light of these results it seems to be of considerable interest to supplement existing results for cracks in shallow shells described by 8th order shell theory with calculations, which explicitly incorporate the effect of transverse shear.

In the present paper a brief derivation of shallow shell field equations including transverse shear is given. The procedure follows closely that of Naghdi [6], but a certain type of orthotropy is included here. It is demonstrated, how these orthotropic equations can be obtained from the isotropic equations merely by use of suitable variable transformations. This also holds for the 8th order shell theory used in previous investigations [7-10], and these solutions can therefore be given a more general interpretation.

Although the shallow shell equations are fairly general, the rest of the paper is confined to an axial crack in a cylindrical shell. The method of solution is that of singular integral equations obtained by the complex Fourier transform. An investigation of the asymptotic stress field around the crack tips is included, and extensive numerical results are given.

2. *FUNDAMENTAL EQUATIONS*

In this section a brief derivation is given of the fundamental equations for a shallow elastic shell. A special kind of orthotropy is accounted for through the parameter δ. In the isotropic case the equations are those given by Naghdi [6]. The middle surface of the shell is described in the cartesian coordinate system of Fig. 1 as $Z=Z(X_\alpha)$, $\alpha=1,2$. The stress resultants are given in terms of their components in the $\{X_\alpha, Z\}$ system. A different approach has been used by Sih and Hagendorf [11] for an isotropic spherical shell.

When only a vertical load $q(X_\alpha)$ is included, the equilibrium equations are

$$N_{\alpha\beta,\beta} = 0 \tag{2.1}$$

$$V_{\alpha,\alpha} + (Z_{,\alpha}N_{\alpha\beta})_{,\beta} + q = 0 \tag{2.2}$$

$$M_{\alpha\beta,\beta} - V_\alpha = 0 \tag{2.3}$$

$N_{\alpha\beta}$, $M_{\alpha\beta}$ and V_α are the membrane forces the moments and the transverse shear forces.

For a shallow shell the strains $\varepsilon_{\alpha\beta}$ are defined by

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(U_{\alpha,\beta} + U_{\beta,\alpha} + Z_{,\alpha}W_{,\beta} + Z_{,\beta}W_{,\alpha}\right) \tag{2.4}$$

where U_α and W are the displacement components in the $\{X_\alpha, Z\}$ system. The normals to the shell in its original configuration change directions by the angles β_α. The slope of the middle surface changes by $W_{,\alpha}$, and thus the effect of the transverse shear is expressed by

$$\theta_\alpha = W_{,\alpha} + \beta_\alpha \tag{2.5}$$

From (2.4) a compatibility equation is extracted in the form

$$e_{\alpha\gamma}e_{\beta\delta}(\varepsilon_{\alpha\beta,\gamma\delta} + Z_{,\alpha\beta}W_{,\gamma\delta}) = 0 \tag{2.6}$$

where $e_{\alpha\gamma}$ is the permutation symbol.

When elastic shells are considered, Hooke's law yields

$$h\varepsilon_{\alpha\beta} = a_{\alpha\beta\gamma\delta}N_{\gamma\delta} \tag{2.7}$$

where h is the thickness of the shell. By use of the stress function $F(X_\alpha)$ defined by

$$N_{\alpha\beta} = e_{\alpha\gamma}e_{\beta\delta}F_{,\gamma\delta} \tag{2.8}$$

the equilibrium equation (2.1) is satisfied, while (2.2) takes the form

$$M_{\alpha\beta,\alpha\beta} + Z_{,\alpha\beta}e_{\alpha\gamma}e_{\beta\delta}F_{,\gamma\delta} + q = 0 \tag{2.9}$$

The compatibility equation (2.6) becomes

$$e_{\alpha\kappa}e_{\beta\lambda}e_{\gamma\mu}e_{\delta\nu}a_{\alpha\beta\gamma\delta}F_{,\kappa\lambda\mu\nu} + hZ_{,\alpha\beta}e_{\alpha\gamma}e_{\beta\delta}W_{,\gamma\delta} = 0 \qquad (2.10)$$

In general solutions of the system of differential equations (2.3), (2.9) and (2.10) will be quite complicated. However, considerable simplifications arise, if the differential operators can be factorized. Below only orthotropic materials with $\{X_\alpha\}$ as principal axes are considered. This implies the following form of (2.7),

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{h}\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_1}{E_1} & 0 \\ -\frac{\nu_2}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix} \qquad (2.11)$$

where $\nu_2E_1 = \nu_1E_2$. In this case the factorization property amounts to

$$\left(-\frac{\nu_1}{E_1} + \frac{1}{2G_{12}}\right) = \frac{1}{\sqrt{E_1E_2}} \qquad (2.12)$$

Now introduce the geometrical mean values

$$E = \sqrt{E_1E_2} \quad , \quad \nu = \sqrt{\nu_1\nu_2} \qquad (2.13)$$

and the orthotropy parameter δ defined by

$$\delta^4 = \frac{E_1}{E_2} = \frac{\nu_1}{\nu_2} \qquad (2.14)$$

In terms of these three parameters (2.11) takes the form

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{hE} \begin{bmatrix} \delta^{-2} & -\nu & 0 \\ -\nu & \delta^2 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix} \tag{2.15}$$

Following from an assumption of linear variation of the stress components $\sigma_{\alpha\beta}$ over the thickness the moments are then given by

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \frac{h}{12} \frac{E}{1-\nu^2} \begin{bmatrix} \delta^2 & \nu & 0 \\ \nu & \delta^{-2} & \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \beta_{1,1} \\ \beta_{2,2} \\ \beta_{1,2}+\beta_{2,1} \end{bmatrix} \tag{2.16}$$

A linear relation between the angles θ_α and the shear forces V_α is assumed. In order to make elimination of the moments from (2.9) possible this relation must be of the form

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \frac{1}{hB} \begin{bmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{2.17}$$

B is the effective transverse shear modulus, which in the isotropic case may be taken as $\frac{5}{6}$ G [5].

The system of differential equations can now be simplified by eliminating the moments. When the notation

$$\nabla_\delta^2 = \delta \frac{\partial^2}{\partial X_1^2} + \delta^{-1} \frac{\partial^2}{\partial X_2^2} \tag{2.18}$$

is adopted, the result is

$$\nabla_\delta^4 F + hE\left(\frac{\partial^2 Z}{\partial X_1^2}\frac{\partial^2}{\partial X_2^2} - 2\frac{\partial^2 Z}{\partial X_1 \partial X_2}\frac{\partial^2}{\partial X_1 \partial X_2} + \frac{\partial^2 Z}{\partial X_2^2}\frac{\partial^2}{\partial X_1^2}\right)W = 0 \tag{2.19}$$

and

$$\frac{h^3}{12}\frac{E}{1-\nu^2}\nabla_\delta^4 W - \left(1 - \frac{h^2}{12(1-\nu^2)}\frac{E}{B}\nabla_\delta^2\right)\left(\frac{\partial^2 Z}{\partial X_1^2}\frac{\partial^2}{\partial X_2^2} - 2\frac{\partial^2 Z}{\partial X_1 \partial X_2}\frac{\partial^2}{\partial X_1 \partial X_2}\right.$$

$$\left. + \frac{\partial^2 Z}{\partial X_2^2}\frac{\partial^2}{\partial X_1^2}\right)F = \left(1 - \frac{h^2}{12(1-\nu^2)}\frac{E}{B}\nabla_\delta^2\right)q \tag{2.20}$$

with the extra conditions

$$\beta_1 + \frac{\partial W}{\partial X_1} = \frac{h^2}{12(1-\nu^2)}\frac{E}{B}\left[\nabla_\delta^2 \beta_1 + \frac{1+\nu}{2\delta}\frac{\partial}{\partial X_2}\left(\frac{\partial \beta_2}{\partial X_1} - \frac{\partial \beta_1}{\partial X_2}\right)\right] \tag{2.21}$$

$$\beta_2 + \frac{\partial W}{\partial X_2} = \frac{h^2}{12(1-\nu^2)}\frac{E}{B}\left[\nabla_\delta^2 \beta_2 + \delta\frac{1+\nu}{2}\frac{\partial}{\partial X_1}\left(\frac{\partial \beta_1}{\partial X_2} - \frac{\partial \beta_2}{\partial X_1}\right)\right] \tag{2.22}$$

3. *DIMENSIONLESS PARAMETERS*

The following curvature measures are introduced

$$\frac{1}{R_1} = -\frac{\partial^2 Z}{\partial X_1^2}, \quad \frac{1}{R_2} = -\frac{\partial^2 Z}{\partial X_2^2}, \quad \frac{1}{R_{12}} = -\frac{\partial^2 Z}{\partial X_1 \partial X_2} \tag{2.23}$$

When dimensionless variables and parameters are defined as shown in Table 1, the differential equations (2.19)-(2.22) reduce to

$$\nabla^4\phi - \frac{1}{\lambda^2}\left(\lambda_1^2\frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2\frac{\partial^2}{\partial x \partial y} + \lambda_2^2\frac{\partial^2}{\partial x^2}\right)w = 0 \tag{3.2}$$

$$\nabla^4 w + \lambda^2(1-\kappa\nabla^2)\left(\lambda_1^2\frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2\frac{\partial^2}{\partial x \partial y} + \lambda_2^2\frac{\partial^2}{\partial x^2}\right)\phi = \lambda^4(1-\kappa\nabla^2)\frac{a}{h}q \tag{3.3}$$

and

$$(1-\kappa\nabla^2)\beta_x + \frac{\partial w}{\partial x} = \kappa \frac{1+\nu}{2} \frac{\partial}{\partial y} \left(\frac{\partial \beta_y}{\partial x} - \frac{\partial \beta_x}{\partial y}\right) \tag{3.4}$$

$$(1-\kappa\nabla^2)\beta_y + \frac{\partial w}{\partial y} = \kappa \frac{1+\nu}{2} \frac{\partial}{\partial x} \left(\frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x}\right) \tag{3.5}$$

The usual static and geometric boundary conditions can also be formulated without explicit use of the orthotropy parameter δ. As a consequence the following presentation refers to an isotropic shell. The corresponding specially orthotropic solutions are easily found by use of Table 1. It should be pointed out that the shell parameter λ_2 is δ^2 times the shell parameter used by Yuceoglu and Erdogan [12].

4. *THE CYLINDER - INTEGRAL REPRESENTATION*

When considering the cylinder shown in Fig. 2, $\lambda_1=\lambda_{12}=0$. The homogeneous equations corresponding to (3.2) and (3.3) then take the form

$$\nabla^4\Phi - \left(\frac{\lambda_2}{\lambda}\right)^2 \frac{\partial^2 w}{\partial x^2} = 0 \tag{4.1}$$

$$\nabla^4 w + (\lambda\lambda_2)^2 (1-\kappa\nabla^2) \frac{\partial^2 \Phi}{\partial x^2} = 0 \tag{4.2}$$

Elimination of either w or Φ from (4.1) and (4.2) leads to the same 8th order differential equation

$$\nabla^8 w + \lambda_2^4 (1-\kappa\nabla^2) \frac{\partial^4 w}{\partial x^4} = 0 \tag{4.3}$$

Although the introduction of a finite transverse shear stiffness represented by $1/\kappa$ does not increase the order of the differential equations (4.1) and (4.2), the order of (3.4) and (3.5) is increased thereby enabling the satisfaction of five boundary conditions as compared to four, when $\kappa=0$.

The procedure now is to give integral representations for $\Phi(x,y)$, $w(x,y)$, $\beta_x(x,y)$ and $\beta_y(x,y)$. When a crack is present along part of the x-axis as shown in Fig. 2, different expressions must be given for the half-planes $y>0$ and $y<0$. Introduce the representations

$$\Phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\Phi}(\xi,y)e^{-i\xi x}d\xi \tag{4.4}$$

$$w(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{w}(\xi,y)e^{-i\xi x}d\xi \tag{4.5}$$

where $\tilde{\Phi}(\xi,y)$ and $\tilde{w}(\xi,y)$ are given by

$$\tilde{\Phi}(\xi,y) = \int_{-\infty}^{\infty}\Phi(x,y)e^{i\xi x}dx \tag{4.6}$$

$$w(\xi,y) = \int_{-\infty}^{\infty}w(x,y)e^{i\xi x}dx \tag{4.7}$$

Due to (4.3) and the corresponding equation for $\Phi(x,y)$ both $\tilde{w}(\xi,y)$ and $\tilde{\Phi}(\xi,y)$ can be expressed as linear combinations of $\exp(m_j y)$, where $m_j=m_j(\xi)$ are the roots of the characteristic equation

$$(m^2-\xi^2)^4 + (\lambda_2\xi)^4[1 - \kappa(m^2-\xi^2)] = 0 \tag{4.8}$$

The notation

$$p = m^2 - \xi^2 \tag{4.9}$$

is now introduced leading to the quartic equation

$$p^4 + (\lambda_2\xi)^4(1-\kappa p) = 0 \tag{4.10}$$

The solutions $p_j(\xi)$, $j=1,..,4$ to (4.10) are given in Appendix A. The Solutions $m_j(\xi)$ to (4.8) are selected such that

$$\mathrm{Re}[m_j]<0 \quad , \quad m_{j+4} = -m_j \quad , \quad j=1,..,4 \tag{4.11}$$

When the displacement transform function $\tilde{w}(\xi,y)$ is given in the form

$$\tilde{w}(\xi,y) = \begin{cases} \sum_{j=1}^{4} R_j(\xi)e^{m_j y} & , \quad y>0 \\ \sum_{j=5}^{8} R_j(\xi)e^{m_j y} & , \quad y<0 \end{cases} \tag{4.12}$$

the transform function $\tilde{\Phi}(\xi,y)$ is found by substitution of (4.12) into (4.1).

$$\tilde{\Phi}(\xi,y) = \begin{cases} -\left(\frac{\lambda_2}{\lambda}\right)^2 \sum_{j=1}^{4} \left(\frac{\xi}{p_j}\right)^2 R_j(\xi)e^{m_j y} & , \quad y>0 \\ -\left(\frac{\lambda_2}{\lambda}\right)^2 \sum_{j=5}^{8} \left(\frac{\xi}{p_j}\right)^2 R_j(\xi)e^{m_j y} & , \quad y<0 \end{cases} \tag{4.13}$$

Introduce the function $\Psi(x,y)$ as

$$\Psi(x,y) = \frac{\partial\beta_x}{\partial y} - \frac{\partial\beta_y}{\partial x} \tag{4.14}$$

The equations (3.4) and (3.5) give the equation

$$(1- \kappa \frac{1-\nu}{2} \nabla^2)\Psi = 0 \tag{4.15}$$

from which the transform function $\tilde{\Psi}(\xi,y)$ is found to be

$$\tilde{\Psi}(\xi,y) = \begin{cases} A_1(\xi)e^{r_1 y} , & y>0 \\ A_2(\xi)e^{r_2 y} , & y<0 \end{cases} \tag{4.16}$$

where

$$r_{1,2} = \pm\sqrt{\xi^2 + \frac{2}{\kappa(1-\nu)}} \tag{4.17}$$

Now $\Psi(x,y)$ is assumed to be independent of β_x and β_y. The equations (3.4) and (3.5) are then solved, and the solution verified by substitution into the original equations. When a function $\chi(x,y)$ with the transform

$$\tilde{\chi}(\xi,y) = \begin{cases} \sum_{j=1}^{4} \left(\frac{\xi\lambda_2}{p_j}\right)^4 R_j(\xi)e^{m_j y} , & y>0 \\ \sum_{j=5}^{8} \left(\frac{\xi\lambda_2}{p_j}\right)^4 R_j(\xi)e^{m_j y} , & y<0 \end{cases} \tag{4.18}$$

is introduced, the solution is found to be

$$\beta_x = \frac{\partial\chi}{\partial x} + \kappa\,\frac{1-\nu}{2}\,\frac{\partial\Psi}{\partial y} \tag{4.19}$$

$$\beta_y = \frac{\partial\chi}{\partial x} - \kappa\,\frac{1-\nu}{2}\,\frac{\partial\Psi}{\partial x} \tag{4.20}$$

The solution to the field equations has now been represented in terms of ten unknown functions $A_j(\xi)$, $j=1,2$ and $R_j(\xi)$, $j=1,..8$, which must be determined from the boundary conditions of the problem.

5. *BOUNDARY CONDITIONS - SYMMETRIC LOADING*

For the axial crack shown in Fig. 2 the following five static quantities are prescribed at the crack surface,

$$N_{yy} = \frac{\partial^2\phi}{\partial x^2} \quad , \quad N_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

$$M_{yy} = \frac{a}{h}\frac{1}{\lambda^4}\left(\frac{\partial\beta_y}{\partial y} + \nu\frac{\partial\beta_x}{\partial x}\right)$$

$$M_{xy} = \frac{a}{h}\frac{1}{\lambda^4}\frac{1-\nu}{2}\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right) \tag{5.1}$$

$$\frac{\partial V_y}{\partial x} = \frac{\partial\beta_y}{\partial x} + \frac{\partial^2 w}{\partial x\partial y}$$

The last expression is used in differentiated form in order to secure dimensional consistency. By substitution of the integral representations of the previous section the following expressions are found for the half-plane $y>0$. The corresponding expressions for $y<0$ follow trivially from changes of the indices.

$$N_{yy}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\xi^4\left(\frac{\lambda_2}{\lambda}\right)^2\sum_{j=1}^{4}p_j^{-2}R_j(\xi)e^{m_jy}\}e^{-i\xi x}d\xi \tag{5.2}$$

$$N_{xy}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{-i\xi^3\left(\frac{\lambda_2}{\lambda}\right)^2\sum_{j=1}^{4}m_jp_j^{-2}R_j(\xi)e^{m_jy}\}e^{-i\xi x}d\xi \tag{5.3}$$

$$M_{yy}(x,y) = \frac{a}{h}\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\xi^4\left(\frac{\lambda_2}{\lambda}\right)^4\sum_{j=1}^{4}[p_j+(1-\nu)\xi^2]p_j^{-4}R_j(\xi)e^{m_jy}$$

$$+ \frac{\kappa(1-\nu)^2}{2\lambda^4}\,i\xi r_1A_1(\xi)e^{r_1y}\}e^{-i\xi x}d\xi \tag{5.4}$$

$$M_{xy}(x,y) = \frac{a}{h}\frac{1-\nu}{2\pi}\int_{-\infty}^{\infty}\{-i\xi^5\left(\frac{\lambda_2}{\lambda}\right)^4\sum_{j=1}^{4}m_jp_j^{-4}R_j(\xi)^{m_jy}$$

$$+ \frac{1}{2\lambda^4}[1+\kappa(1-\nu)\xi^2]A_1(\xi)e^{r_1y}\}e^{-i\xi x}d\xi \tag{5.5}$$

$$\frac{\partial}{\partial x} V_y(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{-i\xi^5 \kappa \lambda_2^4 \sum_{j=1}^{4} m_j p_j^{-3} R_j(\xi) e^{m_j y}$$

$$+ \kappa \frac{1-\nu}{2} \xi^2 A_1(\xi) e^{r_1 y}\} e^{-i\xi x} d\xi \tag{5.6}$$

The problem of static boundary conditions on the crack in Fig. 2 could in principle be formulated directly in terms of the unknown functions $A_j(\xi)$ and $R_j(\xi)$. This would lead to a system of coupled dual integral equations, which is not easily solved even by numerical methods. An alternative and numerically more suitable technique consists in the following. The integral representations given in Section 4 are used to express geometric quantities, which are in a sense complementary to the static quantities (5.1). The most direct choice is the generalized displacements corresponding to the generalized forces (5.1).

For the sake of brevity we shall limit our attention to the case of symmetric self-equilibrating loading.

$$N_{yy}(x,y) = N_{yy}(x,-y) \quad , \quad N_{xy}(x,y) = -N_{xy}(x,-y)$$

$$M_{yy}(x,y) = M_{yy}(x,-y) \quad , \quad M_{xy}(x,y) = -M_{xy}(x,-y) \tag{5.7}$$

$$V_y(x,y) = -V_y(x,-y)$$

In this case

$$R_{j+4}(\xi) = R_j(\xi) \quad , \quad A_2(\xi) = -A_1(\xi) \tag{5.8}$$

whereby the number of unknown functions reduces to five.

Due to (5.7) the boundary conditions for N_{xy}, M_{xy} and V_y are homogeneous. We therefore only need to introduce generalized displacements corresponding to N_{yy} and M_{yy}. By use of Hooke's law (2.15) and the equilibrium equations (2.1) we find

$$\lim_{y\to 0} \frac{\partial^3 \phi}{\partial y^3} = -\lim_{y\to 0} \frac{\partial^2 v}{\partial x^2} \tag{5.9}$$

It is convenient to use the following two functions as unknowns,

$$g(x) = \lim_{y\to 0+} \frac{\partial}{\partial x} v(x,y) \tag{5.10}$$

$$f(x) = \lim_{y\to 0+} \frac{\partial}{\partial x} \beta_y(x,y) \tag{5.11}$$

Their integral representations are

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{i\xi(\frac{\lambda_2}{\lambda})^2 \sum_{j=1}^{4} m_j^3 p_j^{-2} R_j(\xi)\} e^{-ix\xi} d\xi \tag{5.12}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{-i\xi^5 \lambda_2^4 \sum_{j=1}^{4} m_j p_j^{-4} R_j(\xi) + \kappa \frac{1-\nu}{2} \xi^2 A_1(\xi)\} e^{-ix\xi} d\xi \tag{5.13}$$

The five unknown functions $A_1(\xi)$ and $R_j(\xi)$, $j=1,..4$ are now eliminated by use of the three homogeneous boundary conditions and inversion of (5.12) and (5.13). It is noted that $g(x)=f(x)=0$ for $x\notin[-1,1]$. After a few reductions we find

$$A_1(\xi) = -2 \int_{-1}^{1} f(t) e^{it\xi} dt \tag{5.14}$$

and the four equations

$$\sum_{j=1}^{4} \xi m_j R_j(\xi) = -i \int_{-1}^{1} f(t) e^{it\xi} dt \tag{5.15}$$

$$\sum_{j=1}^{4} p_j^{-1} \xi m_j R_j(\xi) = -i \left(\frac{\lambda}{\lambda_2}\right)^2 \int_{-1}^{1} g(t) e^{it\xi} dt \tag{5.16}$$

$$\sum_{j=1}^{4} p_j^{-2} \xi m_j R_j(\xi) = 0 \tag{5.17}$$

$$\sum_{j=1}^{4} p_j^{-3} \xi m_j R_j(\xi) = i \frac{1-\nu}{\xi^2 \lambda_2^4} \int_{-1}^{1} f(t) e^{it\xi} dt \tag{5.18}$$

Due to the systematic nature of these equations the solution is straightforward. When use is made of the characteristic equation (4.10), we get

$$R_1(\xi) = \frac{i}{\xi m_j (p_1 - p_2)(p_1 - p_3)(p_1 - p_4)} \{-p_1^2 [p_1 + (1-\nu)\xi^2] \int_{-1}^{1} f(t) e^{it\xi} dt$$

$$+ (\lambda\lambda_2)^2 \xi^4 (1 - \kappa p_1) \int_{-1}^{1} g(t) e^{it\xi} dt\} \tag{5.19}$$

The expressions for $R_2(\xi)$, $R_3(\xi)$ and $R_4(\xi)$ are found by interchanging the indices.

It should be noted that $A_1(\xi)=0$ does not follow from the limit process $\kappa \to 0$. Special care must therefore be taken, when relating results of the present theory to results from 8th order shallow shell theory.

6. *SINGULAR INTEGRAL EQUATIONS*

When $A_1(\xi)$ and $R_j(\xi)$ from (5.14) and (5.19) are substituted into (5.2) and (5.4), integral expressions for $N_{yy}(x,y)$ and $M_{yy}(x,y)$ are obtained in terms of the two unknown functions f(t) and g(t). For y=0 the integrals are defined by the limit $y \to 0+$. Contributions to the integrals, which are non-integrable for y=0 are extracted by use of the equations (5.14)-(5.18).

The identity (5.17) is multiplied by $\xi|\xi|(\lambda_2/\lambda)^2\exp(-|\xi y|)$ and added to the integrand in (5.2).

$$N_{yy}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\frac{\lambda_2}{\lambda})^2\{\sum_{j=1}^{4}|\xi|m_j(\frac{\xi}{p_j})^2R_j(\xi)$$

$$[1+\frac{|\xi|}{m_j}e^{(m_j+|\xi|)|y|}]\}e^{-|\xi y|}e^{-ix\xi}d\xi \tag{6.1}$$

Asmymptotic expansion for large values of ξ yields

$$1+\frac{|\xi|}{m_j}e^{(m_j+|\xi|)|y|} \sim 1-(1-\frac{1}{2}\frac{p_j}{\xi^2}+\frac{3}{8}\frac{p_j^2}{\xi^4}+\cdots)e^{(m_j+|\xi|)|y|} \tag{6.2}$$

Substitution of (6.2) into (6.1) and use of the identity (5.16) lead to a non-integrable term of the form

$$N_{yy}^{s}(x,0) = \lim_{y\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\frac{\lambda_2}{\lambda})^2\{-\sum_{j=1}^{4}\frac{|\xi|m_j}{2p_j}R_j(\xi)\}e^{-|\xi y|}e^{-ix\xi}d\xi$$

$$= \lim_{y\ 0}\frac{-i}{4\pi}\int_{-1}^{1}g(t)\int_{\infty}^{\infty}\mathrm{sgn}(\xi)e^{-|\xi y|}e^{i\xi(t-x)}d\xi dt$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\frac{g(t)}{t-x}dt \tag{6.3}$$

The last integral must be evaluated as the Cauchy principal value.

By a similar, but slightly more complicated procedure, the following non-integrable contribution to the moment $M_{yy}(x,0)$ is found

$$M^{s}_{yy}(x,0) = \frac{a}{h}\,\frac{1-\nu^2}{\lambda^4}\,\frac{1}{2\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt \tag{6.4}$$

After extraction of the non-integrable parts of the integrals the boundary conditions take the form of two singular integral equations.

$$\int_{-1}^{1}\left[\frac{1}{t-x} + K_{11}(t-x)\right]g(t)dt + \int_{-1}^{1}K_{12}(t-x)f(t)dt = 2\pi N_{yy}(x,0)$$

$$-1<x<1 \tag{6.5}$$

$$\int_{-1}^{1}K_{21}(t-x)g(t)dt + \int_{-1}^{1}\left[\frac{1-\nu^2}{\lambda^4}\,\frac{1}{t-x} + K_{22}(t-x)\right]f(t)dt = 2\pi\,\frac{h}{a}\,M_{yy}(x,0)$$

$$-1<x<1 \tag{6.6}$$

When the symmetry properties $m_j(\xi)=m_j(-\xi)$ and $p_j(\xi)=p_j(-\xi)$ are noted, the functions $K_{jk}(t-x)$ are easily brought in the following form

$$K_{11}(t-x) = \int_{0}^{\infty}\{2\sum_{1}^{4}\frac{\xi^3}{m_1}\,\frac{p_1^2}{(p_1-p_2)(p_1-p_3)(p_1-p_4)} - 1\}\,\sin\xi(t-x)d\xi \tag{6.7}$$

$$K_{12}(t-x) = K_{21}(t-x) = \int_{0}^{\infty}\{2\left(\frac{\lambda_2}{\lambda}\right)^2\sum_{1}^{4}\frac{\xi^3}{m_1}\,\frac{p_1 + (1-\nu)\xi^2}{(p_1-p_2)(p_1-p_3)(p_1-p_4)}\}$$

$$\sin\xi(t-x)d\xi \tag{6.8}$$

$$K_{22}(t-x) = \int_{0}^{\infty}\{2\left(\frac{\lambda_2}{\lambda}\right)^4\sum_{1}^{4}\frac{\xi^3}{m_1 p_1^2}\,\frac{[p_1 + (1-\nu)\xi^2]^2}{(p_1-p_2)(p_1-p_3)(p_1-p_4)}$$

$$+ 2\kappa \frac{(1-\nu)^2}{\lambda^4} \xi r_1 - \frac{1-\nu^2}{\lambda^4} \} \sin\xi(t-x)d\xi \tag{6.9}$$

The summations imply interchange of the indices.

Due to the presence of terms of order ξ^2 the expression (6.9) is not suitable for numerical integration. The problem is solved by constructing the following identity from (5.15), (5.18) and (5.19).

$$2(\frac{\lambda_2}{\lambda})^4 \sum_i^4 \xi^2 \frac{[p_1 + (1-\nu)\xi^2]^2}{p_1^2(p_1-p_2)(p_1-p_3)(p_1-p_4)} + 2\kappa \frac{(1-\nu)^2}{\lambda^4} \xi^2 + 4 \frac{1-\nu}{\lambda^4} = 0 \tag{6.10}$$

$K_{22}(t-x)$ can then be evaluated by the more suitable expression

$$K_{22}(t-x) = \int_0^\infty \{2(\frac{\lambda_2}{\lambda}) \sum_i^4 \frac{\xi^2}{m_1 p_1^2} \frac{[p_1 + (1-\nu)\xi^2]^2}{(p_1-p_2)(p_1-p_3)(p_1-p_4)} (\xi + m_1)$$

$$+ 2\kappa \frac{(1-\nu)^2}{\lambda^4} \xi(\xi + r_1) + \frac{(1-\nu)(3-\nu)}{\lambda^4} \} \sin\xi(t-x)d\xi \tag{6.11}$$

In order to get continuity of the displacements outside the crack the solution must also satisfy the two conditions

$$\int_{-1}^{1} g(t)dt = 0 \quad , \quad \int_{-1}^{1} f(t)dt = 0 \tag{6.12}$$

The integral equations given above only apply when $\kappa>0$. In order to obtain the corresponding equations for the 8th order theory, where $\kappa=0$, the function $A_1(\xi)$ must explicitly be set equal to zero. Only M_{yy}'s dependence on $f(t)$ includes $A_1(\xi)$, and the changes are therefore restricted to the last integral in (6.6). The coefficient $(1-\nu^2)/\lambda^4$ must be replaced by $(1-\nu)(3+\nu)/\lambda^4$, and the kernel $K_{22}(t-x)$ is now

$$K_{22}^{0}(t-x) = \int_0^\infty \{2(\frac{\lambda_2}{\lambda}) \sum_1^4 \frac{\xi^2}{m_1 p_1^2} \frac{[p_1 + (1-\nu)\xi^2]^2}{(p_1-p_2)(p_1-p_3)(p_1-p_4)} (\xi+m_1)$$

$$+ \frac{(1-\nu)^2}{\lambda^4} \} \sin\xi(t-x)d\xi \qquad (6.13)$$

7. THE ASYMPTOTIC STRESS FIELD

The solution to the two singular integral equations (6.5) and (6.6) is of the form [13]

$$g(t) = (1-t^2)^{-\frac{1}{2}}G(t)$$
$$f(t) = (1-t^2)^{-\frac{1}{2}}F(t) \qquad (7.1)$$

In order to obtain the asymptotic stress field in a neighborhood of the crack tip x=1 the following formula, which is derived in Appendix B, is used

$$\int_{-1}^{1} g(t)e^{i\xi t}dt \sim \sqrt{\frac{\pi}{2|\xi|}} \{G(1)\exp[i(\xi-\mathrm{sgn}(\xi)\ \frac{\pi}{4})]$$

$$+ G(-1)\exp[-i(\xi-\mathrm{sgn}(\xi)\ \frac{\pi}{4})]\} \qquad (7.2)$$

Substitution of (7.2) and the similar formula for f(t) into (5.14)-(5.18) makes possible an asymptotic analysis similar to the one which lead to the extraction of the Cauchy integrals. Now, however, (6.2) must be expressed in the slightly different form

$$1 + \frac{|\xi|}{m_j} e^{(m_j+|\xi|)|y|} \sim \frac{1}{2}\frac{p_j}{\xi^2} (1+|\xi y|) + \sum_{j=2}^{\infty} O[(\frac{p_j}{\xi^2})^j](1+c_j|\xi y|^j) \qquad (7.3)$$

c_j, $j=1,2,\ldots$ are constants, which do not influence the result.

The asymptotic expressions are obtained by application of the following formula [14; 3.944]

$$\int_0^\infty \xi^{\mu-1} e^{-\beta\xi} \begin{pmatrix}\sin\\ \cos\end{pmatrix}(\delta\xi)d\xi = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\mu/2}} \begin{pmatrix}\sin\\ \cos\end{pmatrix} (\mu \text{ arctg } \frac{\delta}{\beta}) \qquad (7.4)$$

It is seen that a factor $(\xi\beta)^j$ under the integral sign does not change the order of the result for $\beta,\delta\to 0$. The order of the terms in (7.3) can then be evaluated as if $|\xi y|$ were a constant. It turns out that only the first term of (7.2) contributes to the singular stress field around $x=1$.

Substitution of (7.2) and (7.3) into (6.1) yields

$$N_{yy}(x,y) \sim \frac{G(1)}{2\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\xi}} (1+\xi|y|)e^{-\xi|y|}\sin[\xi(1-x) - \frac{\pi}{4}]d\xi \qquad (7.5)$$

In the same way we find

$$N_{xx}(x,y) \sim \frac{G(1)}{2\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\xi}} (1-\xi|y|) e^{-\xi|y|}\sin[\xi(1-x) - \frac{\pi}{4}]d\xi \qquad (7.6)$$

$$N_{xy}(x,y) \sim \frac{G(1)}{2\sqrt{2\pi}} \int_0^\infty \sqrt{\xi}\, y e^{-\xi|y|}\cos[\xi(1-x) - \frac{\pi}{4}]d\xi \qquad (7.7)$$

The asymptotic expressions (7.5)-(7.7) are also valid for $\kappa=0$. For $\kappa>0$ the asymptotic expressions for the moments are

$$M_{yy}(x,y) \sim \frac{h}{12a} \frac{F(1)}{2\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{\xi}} (1+\xi|y|)e^{-\xi|y|}\sin[\xi(1-x) - \frac{\pi}{4}]d\xi \qquad (7.8)$$

$$M_{xx}(x,y) \sim \frac{h}{12a}\frac{F(1)}{2\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{\xi}}\,(1-\xi|y|)e^{-\xi|y|}\sin[\xi(1-x)-\frac{\pi}{4}]d\xi \qquad (7.9)$$

$$M_{xy}(x,y) \sim \frac{h}{12a}\frac{F(1)}{2\sqrt{2\pi}}\int_0^\infty \sqrt{\xi}\;ye^{-\xi|y|}\cos[\xi(1-x)-\frac{\pi}{4}]d\xi \qquad (7.10)$$

It follows from (7.5)-(7.10) that the asymptotic membrane and bending stress fields are self similar for $\kappa>0$. This is in agreement with results reported in [1-4] and [11]. For $\kappa=0$ the asymptotic bending stress field is found to be different and depend on Poisson's ratio ν [7,10].

When the coordinates r and θ are defined by

$$x-1 = r\cos\theta \quad , \quad y = r\sin\theta \qquad (7.11)$$

application of (7.4) and use of trigonometric formulae lead to the result

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} \sim \frac{G(1)+zF(1)}{2\sqrt{2r}}\begin{bmatrix}\frac{3}{4}\cos(\theta/2)+\frac{1}{4}\cos(5\theta/2)\\ \frac{5}{4}\cos(\theta/2)-\frac{1}{4}\cos(5\theta/2)\\ -\frac{1}{4}\sin(\theta/2)+\frac{1}{4}\sin(5\theta/2)\end{bmatrix} \qquad (7.12)$$

For specially orthotropic materials r and θ are not polar coordinates. With the usual definition of the stress intensity factor

$$K_1 = \lim_{X_1\to a}\sqrt{2(X_1-a)}\;\sigma_{22}(X_1,0) \qquad (7.13)$$

we get

$$K_1 = -\frac{1}{2} E \sqrt{a} \; [G(1) + zF(1)] \tag{7.14}$$

By inspection of the integral equations (6.5) and (6.6) it is seen that the stress intensity factor at the surfaces $Z=\mp h/2$ depends only on the parameters ν, λ_2 and κ. Thus the thickness effect and the influence of the effective transverse shear modulus are combined in the parameter $\kappa = E/(B\lambda^4)$.

8. *NUMERICAL SOLUTION*

The singular integral equations (6.5) and (6.6) are solved by use of a quadrature formula of closed type developed in [15]. The integral equations are replaced by the algebraic equations

$$\sum_{i=1}^{n} W_{n,i}\{[\frac{1}{t_i - x_k} + K_{11}(t_i - x_k)]G(t_i) + K_{12}(t_i - x_k)F(t_i)\} = 2N_{yy}(x_k, 0) \tag{8.1}$$

$$\sum_{i=1}^{n} W_{n,k}\{K_{21}(t_i - x_k)G(t_i) + [\frac{1-\nu^2}{\lambda^4}\frac{1}{t_i - x_k} + K_{22}(t_i - x_k)]F(t_i)\} = \frac{2h}{a} M_{yy}(x_k, 0) \tag{8.2}$$

where

$$t_i = \cos\left(\frac{i-1}{n-1}\pi\right) \quad , \quad i=1,2,\ldots,n \tag{8.3}$$

$$x_k = \cos\left(\frac{2k-1}{2n-2}\pi\right) \quad , \quad k=1,2,\ldots,n-1 \tag{8.4}$$

and

$$W_{n,1} = W_{n,n} = \frac{1}{2(n-1)}$$

$$W_{n,i} = \frac{1}{n-1} \quad , \quad i=2,3,\ldots,n-1 \tag{8.5}$$

The extra conditions (6.12) are replaced by

$$\sum_{i=1}^{n} W_{n,i}G(t_i) = 0 \quad , \quad \sum_{i=1}^{n} W_{n,i}F(t_i) = 0 \tag{8.6}$$

The convergence of the method is estimated by evaluation of the coefficients b_j in the following expansion

$$G(t) \simeq \frac{1+t}{2} G(1) + \frac{1-t}{2} G(-1) + (1-t^2) \sum_{j=0}^{n-3} b_j U_j(t) \tag{8.7}$$

where $U_j(t)$ is the Chebyshev polynomial of the second kind of degree j. The coefficients b_j can be expressed explicitly in terms of the solution to the algebraic equations [15].

$$b_j = \frac{2}{n-1} \sum_{i=2}^{n-1} [G(t_i) - \frac{1+t_i}{2} G(1) - \frac{1-t_i}{2} G(-1)]U_j(t_i) \tag{8.8}$$

The necessary values of the bounded kernels $K_{11}(t_i-x_k)$, $K_{12}(t_i-x_k)$ and $K_{22}(t_i-x_k)$ were calculated by use of Filon's integration formula.

9. *RESULTS*

Numerical results are given in the form of stress intensity factors for two loading situations, constant membrane load and constant bending moment. For each loading situation two stress intensity factors are given, one for the average stress and one for the surface stress from bending. Introduce the following normalization of the stress intensity factors. For compressive membrane load σ_m

$$k_{mm} = K_1(0)(\sigma_m \sqrt{a})^{-1} \tag{9.1}$$

$$k_{bm} = [K_1(h/2) - K_1(0)](\sigma_m \sqrt{a})^{-1} \tag{9.2}$$

For a constant bending moment with maximum surface stress σ_b

$$k_{bb} = [K_1(h/2) - K_1(0)](\sigma_b \sqrt{a})^{-1} \tag{9.3}$$

$$k_{mb} = K_1(0)(\sigma_b \sqrt{a})^{-1} \tag{9.4}$$

The results are given as functions of the shell parameter λ_2 for various values of a/h in Tables 2-5 and Figs. 3-6. The effective transverse shear modulus $\frac{5}{12}\frac{E}{1+\nu}$ has been used together with $\nu=0.3$.

The first column of the tables and the dashed curve in the figures correspond to 8th order shell theory ($\kappa=0$). As expected from the difference in the asymptotic moment fields the bending stresses show some differences. The membrane stresses from the 8th order theory, however, are found to be representative for very thin shells h/a<10. It is important to note that in general 8th order shell theory gives non-conservative estimates of the membrane stresses.

The extrapolated values of k_{bb} for $\lambda_2=0$ are in good agreement with the results obtained in [2-4] for a plate.

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APPENDIX A

Solution of the equation $p^4 + (\lambda_2\xi)^4(1-\kappa p) = 0$

Following the method in [16] the solution of the quartic equation

$$p^4 + (\lambda_2\xi)^4(1-\kappa p) = 0 \tag{A.1}$$

requires the real root of the cubic equation

$$x^3 - 4(\lambda_2\xi)^4 x - \kappa^2(\lambda_2\xi)^8 = 0 \tag{A.2}$$

Now introduce the parameter

$$\eta = 3\sqrt{3}\ \left(\frac{\kappa\lambda_2\xi}{4}\right)^2 \tag{A.3}$$

The real solution to (A.2) is determined as the sum of β_1 and β_2 given by

$$\beta_{1,2}^3 = \left(\frac{3}{\sqrt{3}}\right)^3(\lambda_2\xi)^6\left[\eta \pm \sqrt{\eta^2-1}\right] \tag{A.4}$$

This relation is rewritten in the form

$$\beta_{1,2}^3 = \left(\frac{2}{\sqrt{3}}\right)^3(\lambda_2\xi)^6 \begin{cases} \exp\,(\pm\, i\ \mathrm{Arccos}\,\eta) , & 0 \leq \eta < 1 \\ \exp\,(\pm\ \mathrm{Arccosh}\,\eta) , & 1 \leq \eta \end{cases} \tag{A.5}$$

The real root of (A.2) is then given by

$$x = \beta_1 + \beta_2 = \frac{4}{\sqrt{3}}\,(\lambda_2\xi)^2 \begin{cases} \cos\,(\frac{1}{3}\,\mathrm{Arccos}\,\eta) , & 0 \leq \eta < 1 \\ \cosh\,(\frac{1}{3}\,\mathrm{Arccosh}\,\eta) , & 1 \leq \eta \end{cases} \tag{A.6}$$

The roots p of (A.1) are found from

$$p^2 \mp \sqrt{x}\, p + \frac{1}{2} x \mp \frac{\kappa}{2\sqrt{x}} (\lambda_2 \xi)^4 = 0 \qquad \text{(A.7)}$$

leading to

$$p = \lambda_2 \xi \sqrt{\frac{\cos(\theta/3)}{\sqrt{3}}} \left[\pm 1 (\pm) i \sqrt{1 \mp \sqrt{\eta/\cos^3(\theta/3)}}\right]$$

$$0 \leq \eta < 1 \qquad \eta = \cos\theta \qquad \text{(A.8)}$$

For $1 \leq \eta$ cos( ) is replaced by cosh ( ).

APPENDIX B

Asymptotic expansion of $I(\xi) = \int_{-1}^{1} \frac{\phi(t)}{\sqrt{1-t^2}} e^{i\xi t} dt$

Rewrite the integral in the form

$$I(\xi) = \int_{-1}^{1} \frac{\phi(t)}{\sqrt{1-t^2}} e^{i\xi t} dt$$

$$= \int_{0}^{1} \{[\phi(t)+\phi(-t)]\cos(\xi t)+i[\phi(t)-\phi(-t)]\sin(\xi t)\} \frac{dt}{\sqrt{1-t^2}} \tag{B.1}$$

Introduce the following series expansion of $\phi(t)$

$$\phi(t) = \sum_{j=0}^{\infty} c_j T_j(t) \tag{B.2}$$

and use the results [14; 7.355]

$$\int_{0}^{1} T_{2n+1}(t)\sin(\xi t) \frac{dt}{\sqrt{1-t^2}} = (-1)^n \frac{\pi}{2} J_{2n+1}(|\xi|)\mathrm{sgn}(\xi)$$

$$\int_{0}^{1} T_{2n}(t)\cos(\xi t) \frac{dt}{\sqrt{1-t^2}} = (-1)^n \frac{\pi}{2} J_{2n}(|\xi|) \tag{B.3}$$

$T_j(t)$ is the Chebyshev polynomial of the first kind of degree j, and $J_n(\xi)$ is the Bessel function of the first kind of order n. The integral then is

$$I(\xi) = \pi \sum_{j=0}^{\infty} (-1)^j \; c_{2j}J_{2j}(|\xi|) + i \; \mathrm{sgn}(\xi)c_{2j+1}J_{2j+1}(|\xi|)] \tag{B.4}$$

From the asymptotic formula [14]

$$J_n(|\xi|) = \sqrt{\frac{2}{\pi|\xi|}} \cos(|\xi| - \frac{n\pi}{2} - \frac{\pi}{4})[1+O(\xi^{-1})] \tag{B.5}$$

we get

$$I(\xi) \sim \sqrt{\frac{2\pi}{|\xi|}} \sum_{j=0}^{\infty} [c_{2j}\cos(|\xi| - \frac{\pi}{4}) + i \ \mathrm{sgn}(\xi) c_{2j+1} \sin(|\xi| - \frac{\pi}{4})] \tag{B.6}$$

By use of $T_{2n}(\pm 1)=1$ and $T_{2n+1}(\pm 1)=\pm 1$ the leading term of the asymptotic expansion of $I(\xi)$ is found to be

$$I(\xi) \sim \sqrt{\frac{\pi}{2|\xi|}} \{\phi(1)\exp[i(\xi - \mathrm{sgn}(\xi) \frac{\pi}{4})]$$

$$+ \phi(-1)\exp[-i(\xi - \mathrm{sgn}(\xi) \frac{\pi}{4})]\} \tag{B.7}$$

TABLE 1

Dimensionless variables.

Coordinates

$x = \frac{1}{\sqrt{\delta}} \frac{X_1}{a}$	$y = \sqrt{\delta}\, \frac{X_2}{a}$	$\zeta = \frac{Z}{a}$

Displacements

$u = \sqrt{\delta}\, \frac{U_1}{a}$	$v = \frac{1}{\sqrt{\delta}} \frac{U_2}{a}$	$w = \frac{W}{a}$
$\beta_x = \sqrt{\delta}\, \beta_1$	$\beta_y = \frac{1}{\sqrt{\delta}} \beta_2$	

Stresses and stress resultants

$\sigma_{xx} = \frac{\sigma_{11}}{\delta E}$	$\sigma_{yy} = \frac{\delta\sigma_{22}}{E}$	$\sigma_{xy} = \frac{\sigma_{12}}{E}$
$N_{xx} = \frac{N_{11}}{\delta hE}$	$N_{yy} = \frac{\delta N_{22}}{hE}$	$N_{xy} = \frac{N_{12}}{hE}$
$M_{xx} = \frac{M_{11}}{\delta h^2 E}$	$M_{yy} = \frac{\delta M_{22}}{h^2 E}$	$M_{xy} = \frac{M_{12}}{h^2 E}$
$V_x = \frac{V_1}{\sqrt{\delta} hB}$	$V_y = \frac{\sqrt{\delta} V_2}{hB}$	$\Phi = \frac{F}{a^2 hE}$

Parameters

$$\lambda_1^4 = 12(1-\nu^2)\,\frac{\delta^2 a^4}{h^2 R_1^2} \qquad \lambda_2^4 = 12(1-\nu^2)\,\frac{a^4}{\delta^2 h^2 R_2^2} \qquad \lambda_{12}^4 = 12(1-\nu^2)\,\frac{a^4}{h^2 R_{12}^2}$$

$$\lambda^4 = 12(1-\nu^2)\,\frac{a^2}{h^2} \qquad \kappa = \frac{1}{\lambda^4}\frac{E}{B}$$

Table 2. Stress intensity factors k_{mm}

λ_2	$\kappa = 0$	a/h=1	a/h=2	a/h=5	a/h=10
0.01	1.000	1.000	1.000	1.000	1.000
0.25	1.014	1.015	1.015	1.015	1.015
0.50	1.056	1.061	1.058	1.057	1.057
0.75	1.119	1.135	1.123	1.120	1.119
1.00	1.198	1.233	1.208	1.200	1.199
1.5	1.391	1.485	1.420	1.398	1.394
2.0	1.613	1.788	1.668	1.625	1.618
3.0	2.095	2.478	2.220	2.122	2.105
4.0	2.588	3.254	2.808	2.634	2.603
5.0	3.075	4.100	3.414	3.146	3.096
6.0	3.552	4.944	4.069	3.656	3.580
7.0	4.021	-	4.723	4.154	4.054
8.0	4.484	-	-	4.649	4.515
10.0	5.376	-	-	-	5.422
12.0	6.297	-	-	-	-

Table 3. Stress intensity factors k_{bm}

λ_2	$\kappa = 0$	a/h=1	a/h=2	a/h=5	a/h=10
0.01	0.0001	0.0001	0.0001	0.0001	0.0001
0.25	0.0328	0.0235	0.0221	0.0212	0.0208
0.50	0.0866	0.0602	0.0571	0.0551	0.0544
0.75	0.142	0.0951	0.0912	0.0890	0.0891
1.00	0.194	0.1243	0.1206	0.1193	0.1200
1.5	0.279	0.1622	0.1604	0.1636	0.1674
2.0	0.336	0.1757	0.1748	0.1851	0.1942
3.0	0.371	0.1507	0.1397	0.1661	0.1887
4.0	0.313	0.0801	0.0395	0.0762	0.1156
5.0	0.176	-0.0266	-0.1089	-0.0698	-0.0140
6.0	-0.025	-0.1510	-0.2965	-0.2605	-0.1865
7.0	-0.279	-	-0.4991	-0.4826	-0.3952
8.0	-0.579	-	-	-0.7369	-0.6343
10.0	-1.306	-	-	-	-1.1829
12.0	-2.186	-	-	-	-

Table 4. Stress intensity factors k_{bb}

λ_2	$\kappa = 0$	a/h=1	a/h=2	a/h=5	a/h=10
0.01	1.000	0.747	0.699	0.662	0.644
0.25	0.998	0.745	0.698	0.660	0.643
0.50	0.992	0.738	0.692	0.655	0.639
0.75	0.983	0.728	0.684	0.649	0.637
1.00	0.973	0.716	0.674	0.641	0.628
1.5	0.950	0.693	0.653	0.623	0.612
2.0	0.927	0.671	0.632	0.605	0.597
3.0	0.881	0.633	0.594	0.572	0.564
4.0	0.838	0.603	0.561	0.540	0.535
5.0	0.801	0.580	0.534	0.513	0.506
6.0	0.767	0.562	0.511	0.489	0.483
7.0	0.737	-	0.492	0.468	0.463
8.0	0.709	-	-	0.450	0.446
10.0	0.661	-	-	-	0.413
12.0	0.621	-	-	-	-

Table 5. Stress intensity factors k_{mb}

λ_2	$\kappa = 0$	a/h=1	a/h=2	a/h=5	a/h=10
0.01	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.0043	0.0069	0.0060	0.0051	0.0047
0.50	0.0115	0.0184	0.0158	0.0136	0.0126
0.75	0.0191	0.0302	0.0261	0.0227	0.0212
1.00	0.0267	0.0414	0.0363	0.0315	0.0294
1.5	0.0407	0.0607	0.0544	0.0478	0.0447
2.0	0.0530	0.0761	0.0698	0.0619	0.0582
3.0	0.0731	0.0977	0.0933	0.0846	0.0799
4.0	0.0882	0.1121	0.1096	0.1012	0.0961
5.0	0.0998	0.1223	0.1209	0.1136	0.1081
6.0	0.1088	0.1282	0.1303	0.1231	0.1175
7.0	0.1161	-	0.1370	0.1303	0.1249
8.0	0.1221	-	-	0.1359	0.1308
10.0	0.1309	-	-	-	0.1399
12.0	0.1388	-	-	-	-

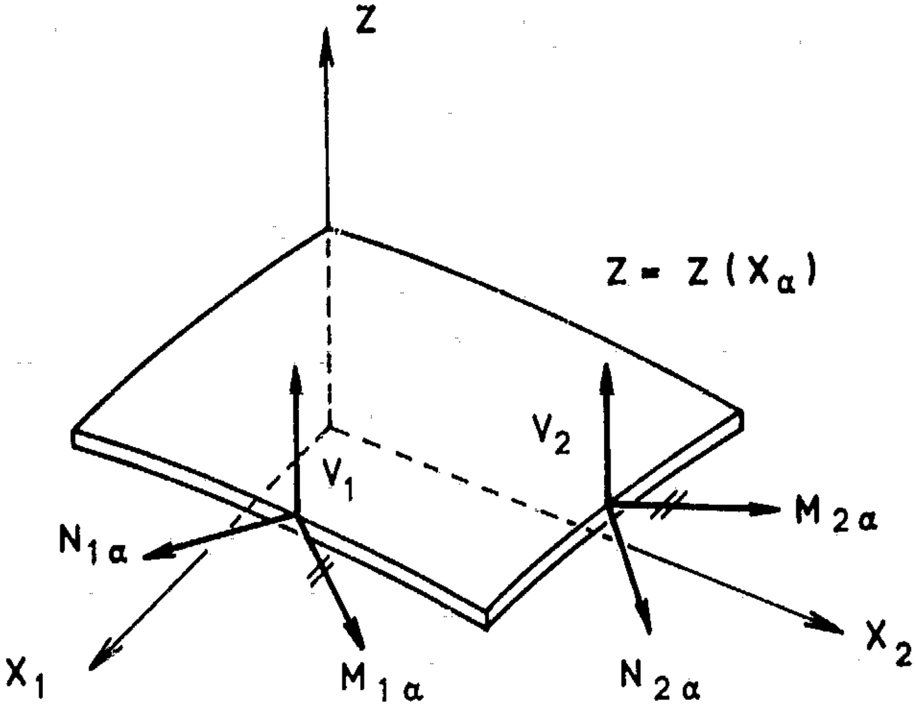

Fig. 1. Stress resultants.

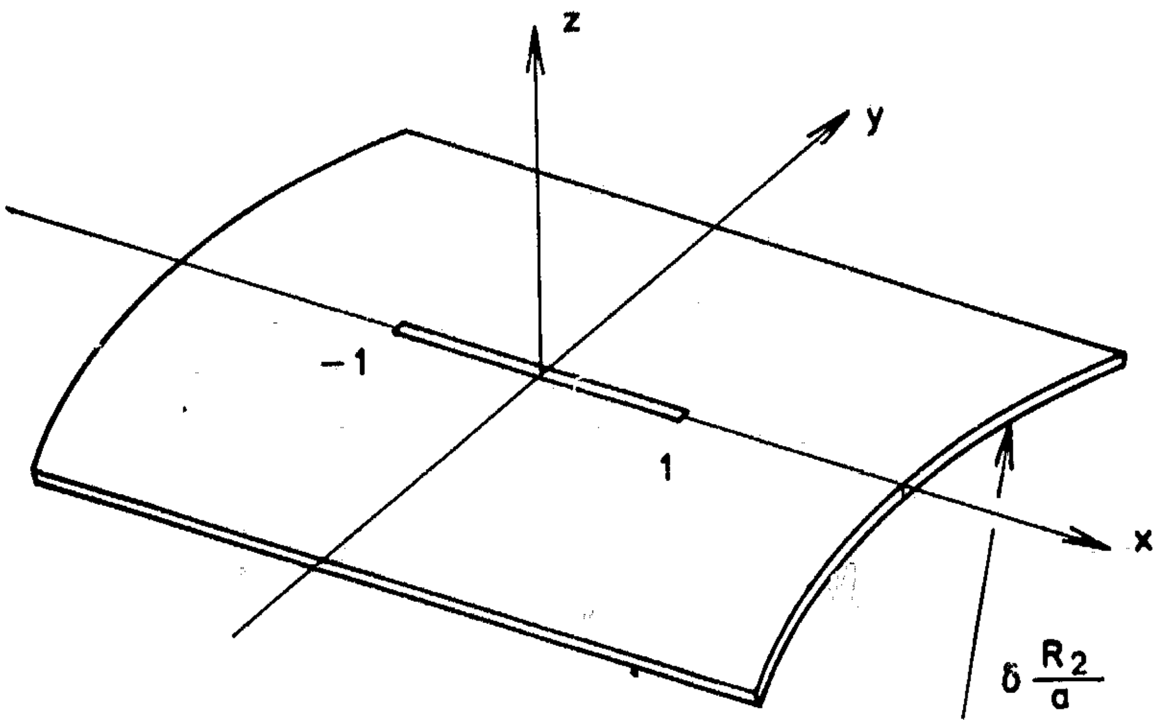

Fig. 2. Cylindrical shell with axial crack.

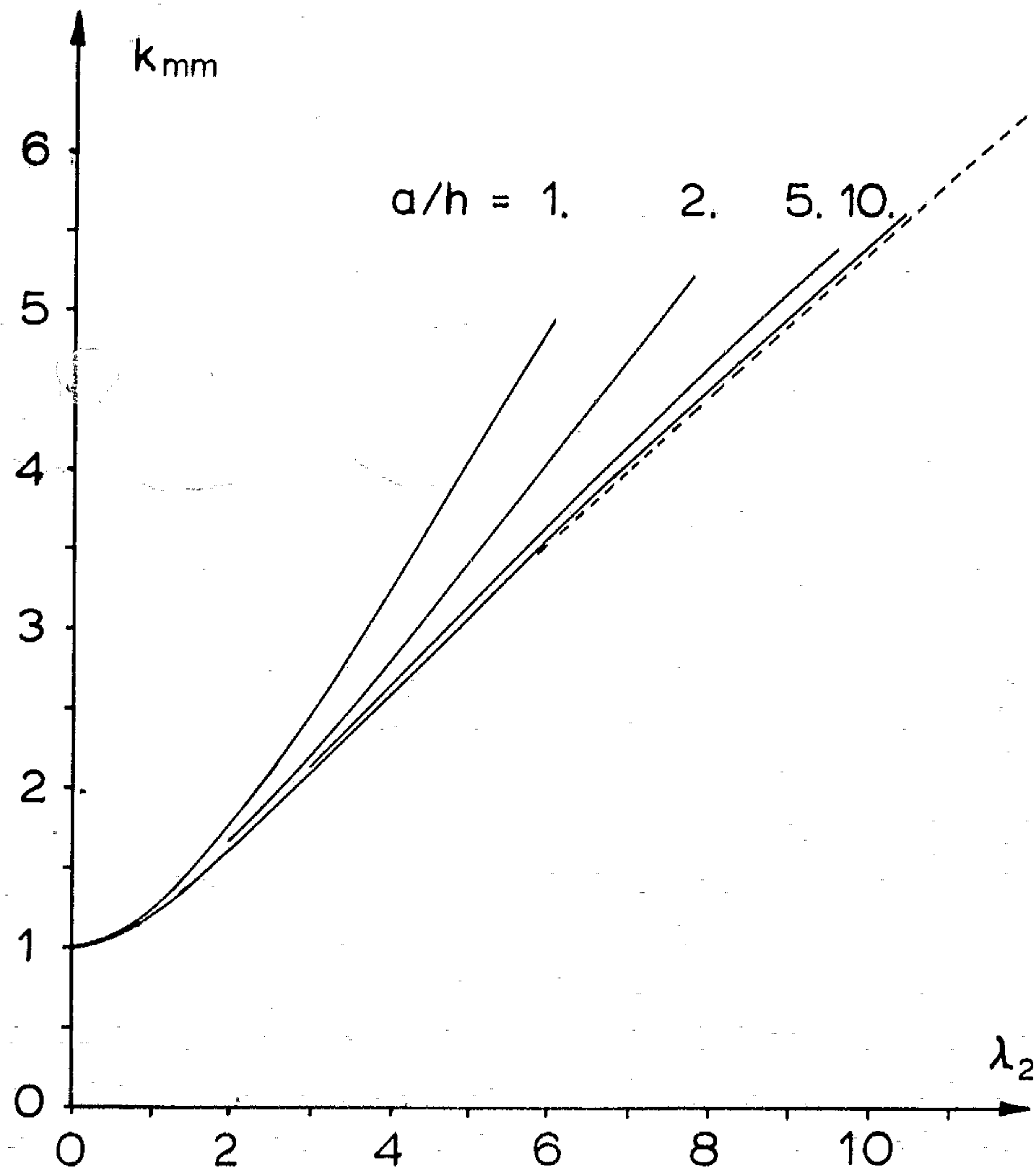

Fig. 3. Stress intensity factor k_{mm}

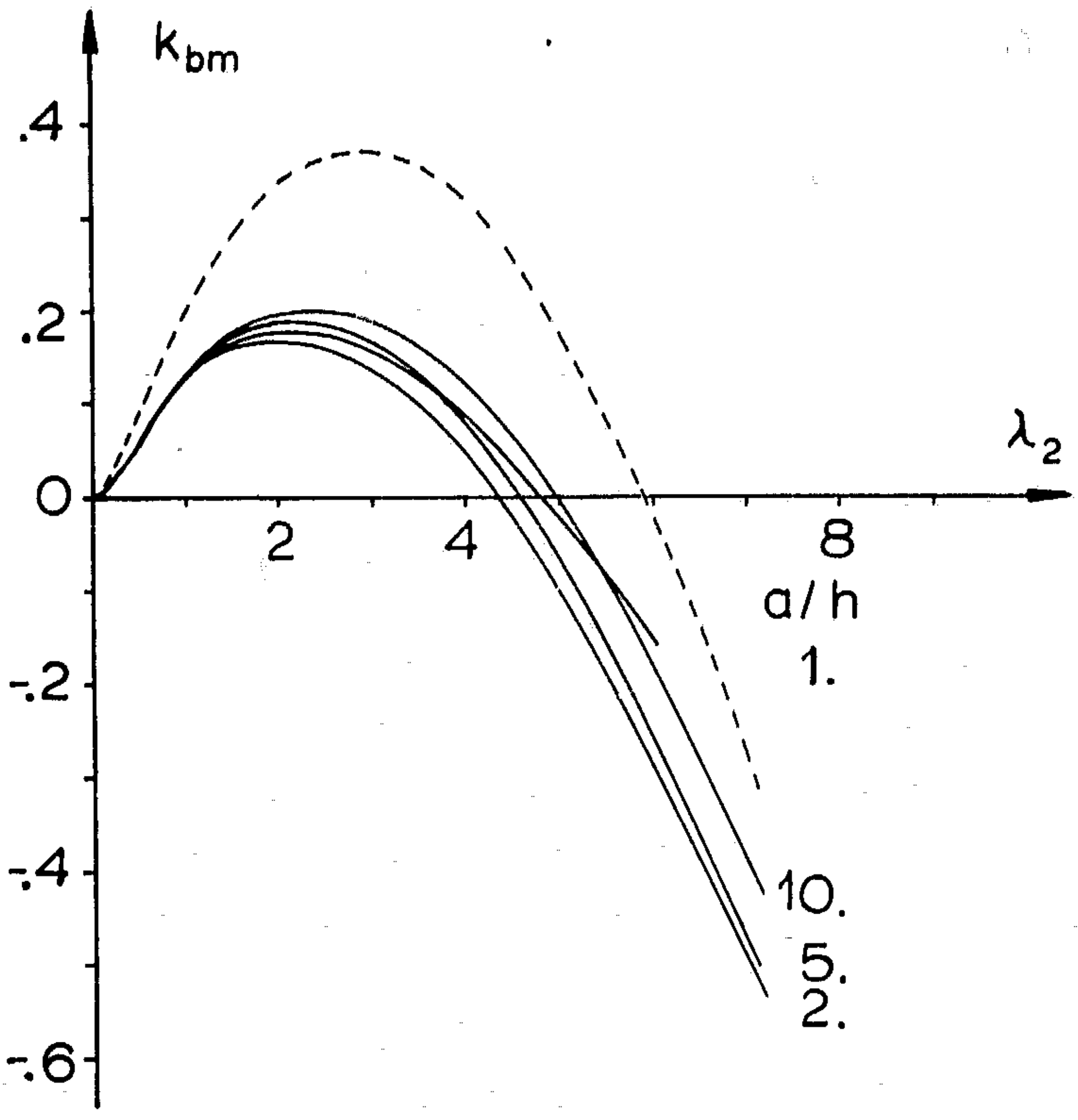

Fig. 4. Stress intensity factor k_{bm}

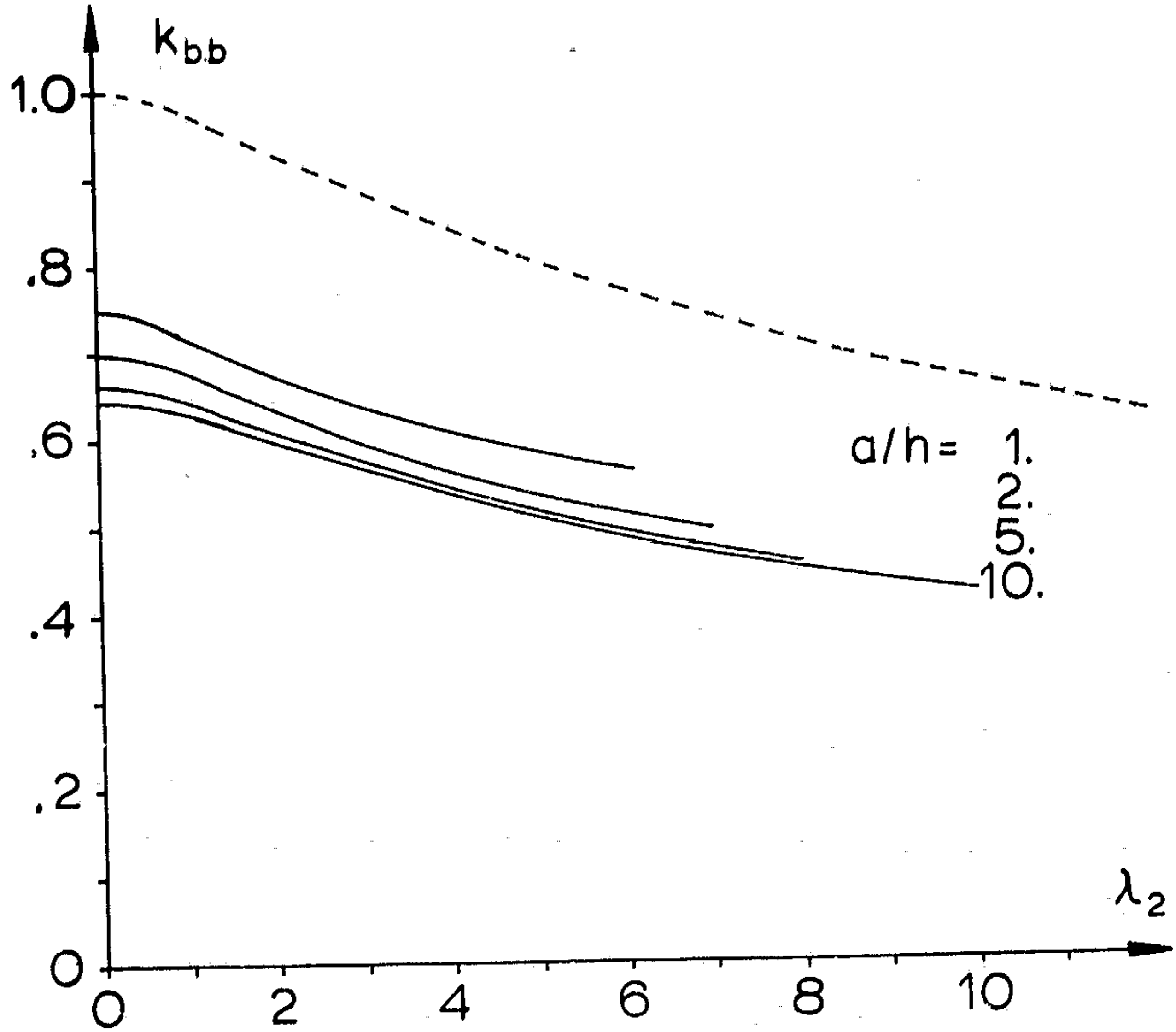

Fig. 5. Stress intensity factor k_{bb}

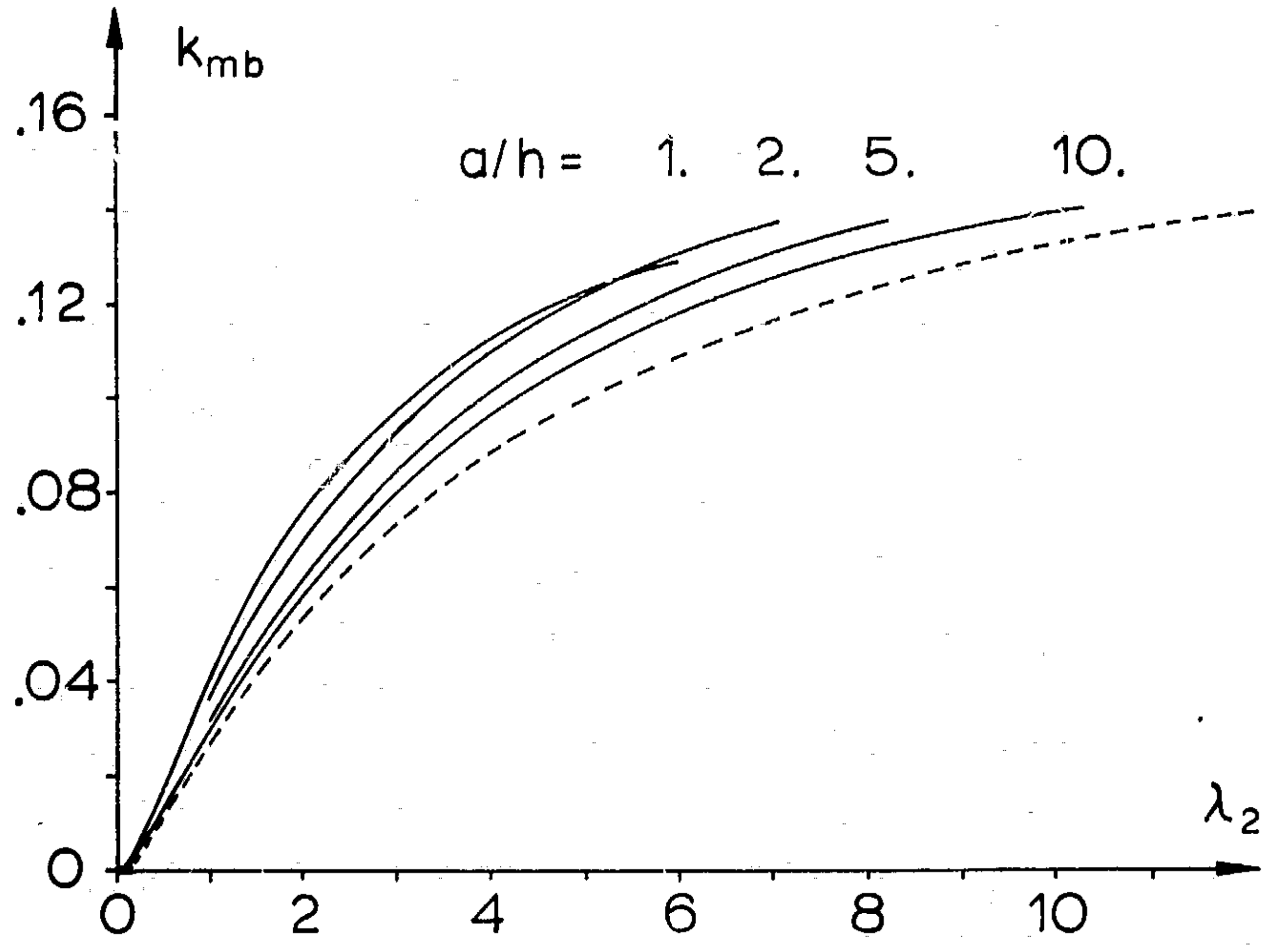

Fig. 6. Stress intensity factor k_{mb}