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IMPLICIT APPROXIMATE-FACTORIZATION SCHEMES FOR THE EFFICIENT SOLUTION OF STEADY TRANSONIC FLOW PROBLEMS

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16. Abstract

Implicit approximate-factorization algorithms (AF) are developed for the solution of steady-state transonic flow problems. The performance of the AF solution method is evaluated relative to that of the standard solution method for transonic flow problems, successive line over-relaxation (SLOR). Both methods are applied to the solution of the nonlinear, two-dimensional transonic small-disturbance equation. Results indicate that the AF method requires substantially less computer time than SLOR to solve the nonlinear finite-difference matrix equation for a transonic flow field. This increase in computational efficiency is achieved with no appreciable increase in computer storage or coding complexity.

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IMPLICIT APPROXIMATE-FACTORIZATION SCHEMES FOR THE EFFICIENT SOLUTION OF STEADY TRANSONIC FLOW PROBLEMS

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INTRODUCTION

The objective of this work was to determine the feasibility of using implicit approximate-factorization algorithms (AF), instead of successive line over-relaxation algorithms (SLOR), to solve steady-state transonic flow problems. In this investigation, two AF solution procedures are compared with SLOR in terms of efficiency, reliability, and flexibility.

Murman and Cole (ref. 1) developed the first computer-programmable SLOR algorithm for the solution of transonic flow problems. That solution procedure proved to be about an order of magnitude more efficient computationally than the first computer-programmed transonic flow algorithm, the explicit, time-accurate procedure of Magnus and Yoshihara (ref. 2). Since that time, the Murman-Cole procedure has been improved, extended, and applied to a variety of aerodynamic problems (reviews are given in refs. 3 and 4). SLOR algorithms have generally proved to be reliable, in the sense of convergence, and flexible, in the sense that they can be easily adapted to a wide range of applications.

A practical limitation on the class of problems that can be treated is the computer time required to obtain a converged solution. Hence a number of potentially more efficient iterative solution procedures have been proposed as alternatives to SLOR. These are semidirect methods using fast Poisson solvers, extrapolation, the multigrid approach, and implicit approximate factorizations.

In recent years fast direct methods have been developed for solving matrix equations for the discrete Laplacian (refs. 5-7). For the transonic, small-disturbance potential equation

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$$(k - \phi_x)\phi_{xx} + \phi_{yy} = 0 \qquad (1)$$

this suggests an iteration procedure of the form

$$\phi_{xx}^{n+1} + \phi_{yy}^{n+1} = (1 + \phi_x{}^n - k)\phi_{xx}^n \qquad (2)$$

such that a discrete Poisson equation is solved for each iteration n. Such a scheme has been proposed by Martin and Lomax (ref. 8) and by Martin (ref. 9). Schemes of this type produce substantial increases in computational efficiency, especially for cases with embedded supersonic regions that are small. However, their application has been limited because the most efficient Poisson solver algorithms require uniform grid spacing in all but one coordinate direction, and they require special, complex treatment of internal boundaries. A hybrid Poisson solver-SLOR scheme proposed by Jameson (ref. 10) avoids these difficulties for some special cases of two-dimensional flows by introducing a coordinate mapping in a simple domain.

Extrapolation has proved useful for the acceleration of SLOR convergence in cases in which a dominant eigenvalue exists and can be identified (refs. 11 and 12). It has also been used, in a somewhat different form, to accelerate convergence in the fast Poisson solver approach (refs. 8 and 9).

The multigrid method was first proposed by Federenko (ref. 13), developed by Brandt (ref. 14), and recently applied to the solution of transonic flow fields by South and Brandt (ref. 15). Preliminary results indicate a substantial increase in efficiency over SLOR for computations on uniform meshes. However, the method is sensitive to the aspect ratio of the mesh cells, leading to complications in the treatment of nonuniform grids.

Here we test approximate-factorization schemes to determine if they are capable of providing accelerated convergence and whether there are any restrictions limiting their application to transonic flow calculations.

APPROXIMATE FACTORIZATIONS

General Requirements

We seek the solution to the difference equation

$$L\phi = (A\delta_{xx} + \delta_{yy})\phi = 0 \qquad (3)$$

where δ_{xx} and δ_{yy} are second central difference operators, ϕ is the solution vector, and A is a constant. An iterative solution procedure to solve (3) can be written

$$NC^n + \sigma L\phi^n = 0 \tag{4}$$

where C^n is the correction vector $\phi^{n+1} - \phi^n$ computed for cycle $n+1$. We will call $R \equiv L\phi^n$ the residual at the nth iteration, and σ is a parameter to be determined.

Let $e^n = \phi^n - \phi$ be the error after the nth cycle. Then $C^n = e^{n+1} - e^n$, and equations (3) and (4) can be combined to give

$$\left.\begin{aligned} e^{n+1} &= M\, e^n = M^2\, e^{n-1} = \ldots\, M^{n+1}\, e^o \\ M &= I - \sigma N^{-1} L \end{aligned}\right\} \tag{5}$$

For the iteration procedure to converge, the modulus of every eigenvalue of M must be less than unity.

The type of an iterative solution procedure is determined by the choice of the operator N. To ensure convergence, N should be chosen to resemble the operator L as closely as possible and, furthermore, $N^{-1}L\phi^n$ must be easily computable. If N is identical to L, for example, then $M = I - \sigma N^{-1}L$ vanishes (for $\sigma = 1$), and an exact solution to $L\phi = 0$ is obtained in a single cycle. In the special case of the discrete Laplacian in a rectangular, uniform, Cartesian grid, this can be accomplished by the use of one of the fast Poisson algorithms. It is usually impractical to invert L, however, and one must use an iteration scheme in which N is an approximation to L. For example, for SLOR

$$NC^n_{i,j} = \left[\delta_{yy} - \frac{2A}{\Delta x^2} + \frac{\sigma A}{\Delta x^2} E_x^{-1}\right] C^n_{i,j} = -\sigma L\phi^n_{i,j} \tag{6}$$

where i,j are grid point indices in the x,y directions, $E_x^{-1}C_{i,j} = C_{i-1,j}$, and σ is the relaxation parameter. Note that N contains the δ_{yy} operator in L but contains only the lower diagonal part of the δ_{xx} operator. Hence, SLOR will require more than one cycle to converge. Actually, the required number of cycles will increase as the number of x grid points increases, because a grid point is only influenced by a single grid point to the right of it in the x direction during one cycle.

The underlying idea of the approximate factorization methods is to construct N as a product of two or more factors

$$N = N_1 N_2 \ldots N_k$$

each of which is restricted to a form leading to equations that can be easily solved. For example, each factor may have a block tridiagonal structure. On the other hand, the added flexibility resulting from the use of several factors allows L to be more closely approximated by N. Moreover, algorithms can be constructed along these lines in such a way that the final N operator is fully implicit, so that every grid point is influenced by every other grid point during each cycle.

AF Scheme 1

Considering equation (3) in the case when $A > 0$ so that the equation is elliptic, an approximate factorization can be realized as

$$(\alpha - \delta_{yy})(\alpha - A\delta_{xx})C^n = \sigma\alpha L\phi^n \tag{7}$$

where α is a parameter to be chosen, and σ is an over-relaxation factor. Multiplying out the two factors on the left, the operator N, which should now be an approximation to αL, has the form

$$N = \alpha L - \alpha^2 I - \delta_{yy}A\delta_{xx}$$

The success of the method will depend on αL dominating the error terms $-\alpha^2 I - \delta_{yy}A\delta_{xx}$. The required equations can easily be solved by letting $f^n = (\alpha - A\delta_{xx})C^n$ and solving directly for f the tridiagonal matrix equation

$$(\alpha - \delta_{yy})f^n = \sigma\alpha L\phi^n \tag{8}$$

followed by a direct solution for C^n of the tridiagonal matrix equation

$$(\alpha - A\delta_{xx})C^n = f^n \tag{9}$$

To estimate the rate of convergence for this scheme (which is a reformulation of the Peaceman-Rachford scheme) note that equation (7) can be written in the form

$$(\alpha - \delta_{yy})(\alpha - A\delta_{xx})e^{n+1} = \sigma\alpha L\, e^n \tag{10}$$

Now, assuming periodic boundary conditions and a uniform Cartesian grid, let

$$e^n(x,y) = \sum_{p,q=1}^{\infty} G^n(p,q)e^{ipx}\, e^{iqy} \tag{11}$$

Because equation (10) is linear, we need consider only a single arbitrary Fourier component. Substituting in (10) and rearranging gives, in the case when $\sigma = 2$,

$$\beta = \frac{G^{n+1}}{G^n} = \frac{\alpha - \frac{2}{\Delta y^2}(1 - \cos\eta)}{\alpha + \frac{2}{\Delta y^2}(1 - \cos\eta)} \cdot \frac{\alpha - \frac{2A}{\Delta x^2}(1 - \cos\xi)}{\alpha + \frac{2A}{\Delta x^2}(1 - \cos\xi)} \tag{12}$$

where $\eta = q\Delta y$ and $\xi = p\Delta x$. For stability, that is, for the error to approach zero in the iteration procedure,

$$|\beta| < 1 \tag{13}$$

and this condition is satisfied for all ξ and η eigenvalues in (12).

The number of iterations required to reduce the residual to a predetermined amount depends on $|\beta|$; that is, fewer iterations are required to achieve a specified degree of convergence as $|\beta|$ decreases. The convergence rate, therefore, depends strongly on the choice of α. The choice

$$\alpha = \frac{2}{\Delta y^2}(1 - \cos\eta) \tag{14}$$

gives $\beta = 0$ for the particular eigenvalue η. Hence, for a problem with k grid points in the y-direction, corresponding to k eigenvalues, the solution process will converge to zero error after k cycles.

Precise estimation of the eigenvalues is generally not practical. Instead, a repeating sequence of α's can be used with each element of the sequence chosen to maintain small values of $|\beta|$ in a given range of eigenvalues (see ref. 16). Maximum and minimum values of α are estimated and then used to form a geometric sequence to cover the entire eigenvalue spectrum. The lowest eigenvalue is given approximately by $\eta \sim \Delta y$, which, from equation (14), gives $\alpha_\ell \sim 1$. For the high frequency error components, $\eta \sim \pi$ corresponding to $\alpha_h \sim 4/\Delta y^2$. The sequence

$$\alpha_k = \alpha_\ell \left(\frac{\alpha_h}{\alpha_\ell}\right)^{(2k-1)/2M} \qquad k = 1,2, \ldots, M \tag{15}$$

is then used repetitively during the course of the computation. The sequence is adjusted in the following section to optimize convergence for particular transonic flow field computations.

In solving equation (1) for transonic flow computations, scheme (7) is suitable for the treatment of the subsonic part of the flow, where $A = (k - \phi_x) > 0$. To extend the scheme to treat flows with embedded supersonic zones, the factorization is modified at supersonic points, where $A = (k - \phi_x) < 0$, to the form

$$(-A\overleftarrow{\delta}_{xx} - \delta_{yy})\alpha C^n = \sigma\alpha L\phi^n \tag{16}$$

where $\overleftarrow{\delta}_{xx}$ is the upwind second difference operator. Here, because of the use of upwind differencing, it is feasible to shift the term $A\overleftarrow{\delta}_{xx}$ to the first factor while retaining an easily invertible form, and the second factor is essentially eliminated. On setting $\sigma = 1$, the result is a transition to the fully implicit marching scheme of Murman and Cole (ref. 1) in the supersonic zone. Transonic operators are used at the sonic and shock points to maintain conservation form (ref. 17). A term $\beta\overleftarrow{\delta}_x$ ($\overleftarrow{\delta}_x$ is a first-order, first upwind difference operator) is added to the first factor as a stabilizer with $\beta \sim \Delta x$. At the shock point, $D\overleftarrow{\delta}_x$ is added to the second factor for the first few cycles to prevent instabilities associated with shock motion (e.g., see ref. 3 or 18).

AF Scheme 2

Implicit approximate-factorization schemes for the low-frequency transonic equation

$$K_1\phi_{xt} = (k - \phi_x)\phi_{xx} + \phi_{yy} \tag{17}$$

where K_1 is a constant, were investigated by Ballhaus and Steger (ref. 18). For the conservative schemes tested, an instability appeared whenever the following two conditions were satisfied: (1) the differencing was shifted from upwind to central across a shock and (2) the shock propagated at a rate greater than one spatial grid point per time step. However, a nonconservative-in-time (but conservative in the steady state) scheme was found to be much less susceptible to this instability. For this scheme it is not necessary to add $D\overleftarrow{\delta}_x$ at the shock to maintain stability as for AF scheme 1. This is an attractive feature, since the appropriate value of D is difficult to determine. The scheme, a factorization of (17) which we will call AF scheme 2 (ref. 18), is

$$[\alpha - (1 - \varepsilon_j)A_j\overrightarrow{\delta}_x - \varepsilon_{j-1}A_{j-1}\overleftarrow{\delta}_x][\alpha\overleftarrow{\delta}_x - \delta_{yy}]C^n$$

$$= \left\{\alpha\delta_{yy} + \alpha\overleftarrow{\delta}_x[(1 - \varepsilon_j)A_j\overrightarrow{\delta}_x + \varepsilon_{j-1}A_{j-1}\overleftarrow{\delta}_x]\right\}\phi^n \tag{18}$$

where $\varepsilon_j = \begin{bmatrix}0\\1\end{bmatrix}$ for $A_j \begin{bmatrix}>\\<\end{bmatrix}$ zero, $A_j = k - (\phi^n_{j+1} - \phi^n_{j-1})/2\Delta x$, $\alpha = (\Delta t)^{-1}$, and $\overrightarrow{\delta}_x$, $\overleftarrow{\delta}_x$ are first-order, first forward, and backward difference operators. Note that the algorithm can be coded so that the ϕ array need be stored for only a single time level (ref. 18). Note also that for subsonic flow ($A_j > 0$), AF scheme 2 can be derived from AF scheme 1, equation (7), by replacing α by $\alpha\overleftarrow{\delta}_x$ and then removing $\overleftarrow{\delta}_x$ from the second factor and the right-hand side.

Substitution of form (11) into (18) leads to an error growth factor β that is not real. Choosing α to minimize β is therefore more difficult, and, in fact, there is no choice of α that can cause β to be zero for a given eigenvalue. Approximate values of α_ℓ and α_h for equation (15) are 1 and Δy^{-1}. Of course, there is no guarantee that (15) is the optimal functional form for the acceleration parameter sequence.

Thus, it appears at first glance that AF scheme 2 cannot be so highly tuned for rapid convergence as AF scheme 1, but it shows potential for greater convergence reliability because of its treatment of shock waves.

OPTIMIZED ACCELERATION PARAMETERS

Determination of the optimum relaxation parameter for SLOR, or the optimum acceleration parameter sequence for approximate factorizations, for a practical computation is a formidable task to achieve analytically. However, comparisons of both optimum and non-optimum SLOR and approximate factorizations would be useful in assessing the relative merits of these solution procedures. Therefore, we have selected particular representative transonic flow problems for which "optimum" acceleration parameters can be determined by numerical optimization.

The numerical optimization problem is formulated in the following way. First, we select as the *objective,* that is, the quantity to be minimized, a combination of parameters that represents the computational efficiency associated with a given set of decision variables:

$$OBJ = (R_n/R_1)^{1/N} \tag{19}$$

OBJ is related to the average decrease in the residual per iteration, R_n is the maximum residual at the nth iteration, and N is the number of iterations required to reduce the maximum residual to some specified value.

Next, we define the decision variables, which are parameters that affect the convergence rate. For SLOR the decision variables are the subsonic relaxation parameters, one for each of the multiple meshes used to converge the solution. The supersonic relaxation parameter is always unity. For both AF schemes, the accleration parameter sequence is

$$\alpha_k = \alpha_\ell (1/\gamma)^{2k-1} \qquad k = 1,2,3,\ . \ . \ . \ , M \tag{20}$$

and the decision variable used in the optimization is γ. The initial guess is $\gamma = (\alpha_\ell/\alpha_h)^{1/2M}$, where α_ℓ and α_h are given in the previous section. For coarse, medium, and fine grids, values of 4, 6, and 8, respectively, were used for M.

The finite difference AF and SLOR codes were coupled with an executive optimization code, CONMIN (ref. 19). CONMIN is designed to minimize an objective, for a set of decision variables, subject to specified constraints.

Here the only constraint imposed was a limit of 2 on the relaxation parameter for SLOR.

The present optimization formulation was designed to be as simple and economical as possible. Only single parameter optimizations are performed on each grid for both the AF and SLOR schemes. There is certainly no guarantee that either the functional form (20) or the arbitrarily chosen values of M used are optimum.

COMPUTED RESULTS

The convergence characteristics of the SLOR and AF schemes are investigated here for a simple test problem. The problem is that of computing the flow field about a nonlifting, parabolic-arc airfoil with a thickness-to-chord ratio of 10 percent. Three free-stream Mach numbers are considered: (1) $M_\infty = 0.7$, a subcritical case; (2) $M_\infty = 0.84$, a supercritical case; and (3) $M_\infty = 0.9$, a highly supercritical case. Computed surface pressure coefficients for these three cases are shown in figure 1. For all of the computations reported here, the solution process was considered completed when the absolute value of the maximum flow field residual dropped below 10^{-9}. Here the residual is defined as $\Delta x^2 L\phi$.

The optimum convergence history of the AF and SLOR schemes for the $M_\infty = 0.7$ case is shown in figure 2(a). This computation was performed using a single, fine grid with (128×32) (x,y) grid points. SLOR required 350 iterations as opposed to 34 and 45 for AF schemes 1 and 2. The peaks in the AF schemes' convergence histories correspond to the smaller values of α(i.e., larger values of Δt) in the eight element sequence.

SLOR convergence can usually be accelerated by the use of coarse-to-fine mesh interpolations. For example, converged solutions on a (64×16) grid were interpolated onto the fine (128×32) grid to provide a good starting solution. The resulting fine grid convergence history is illustrated in figure 2(b). Interpolation reduced the number of SLOR fine-mesh iterations from 350 to 141. Its effect on the AF schemes was small. Similar results are shown in figure 3 for the $M_\infty = 0.84$ case. The use of multiple meshes accelerated SLOR convergence here as in the $M_\infty = 0.7$ case. It also improved the AF-2 scheme convergence somewhat, but perhaps not enough to justify the additional complexity and computational work required in generating coarser grid solutions. However, the use of multiple meshes seems to be extremely important for AF-1. Good initial guesses approximately locate the shock wave and thus help prevent the "moving shock instability" mentioned in the preceding section. Difficulties with this type of instability were obtained in the "fine grid only" case, and hence results for that case are not reported in figure 3(a). For $M_\infty = 0.9$, the supersonic bubble extends vertically from the airfoil surface to a point close to the far-field grid boundary. The optimum convergence history for this case is illustrated in figure 4. Instabilities were encountered with AF-1 for both grid systems. In the fine grid only case, AF-2 required an order of magnitude fewer iterations for convergence than SLOR. Coarse-to-fine mesh interpolation cut the required number of SLOR iterations in half but had very little effect on the AF-2 scheme convergence. The local peaks in the SLOR convergence history, figure 4(b), correspond to changes in the location of the maximum residual.

Up to this point we have compared the convergence performance of optimum SLOR and optimum AF, where here "optimum" refers to values of acceleration and relaxation parameters obtained from the single parameter numerical optimization procedure outline in the preceding section. However, in engineering work optimum parameters are almost never used. A good solution procedure must therefore provide fast, reliable convergence for a reasonable range of nonoptimum parameters. To test the nonoptimum behavior of the SLOR and AF schemes, we applied the optimum values obtained for the subsonic case, $M_\infty = 0.70$, to the two supercritical cases, $M_\infty = 0.84$, 0.90; results are shown in figures 5 and 6. Table 1 summarizes all of the results presented here in terms of number of iterations required to achieve convergence. Table 2 gives optimum values of the acceleration and relaxation parameters. Results for coarse and medium grids are also included.

It is clear from Table 2 that the optimum acceleration parameter sequence is much more sensitive to the grid than to M_∞. For example, optimum values of γ for AF-2 range from 1.363 to 1.455 for the fine grid over all three Mach numbers and both mesh systems. Hence, there is only slight deterioration in convergence performance in the nonoptimum cases, and the effect of mesh system is very small. The nonoptimum SLOR results appear to be dominated by the difference between the optimum relaxation parameters for the fine grid only ($\alpha_{OPT} = 1.965$) and for the fine grid with interpolation ($\alpha_{OPT} = 1.919$). The $M_\infty = 0.70$ optimum acceleration parameter for the fine grid only case is nearly equal to that for $M_\infty = 0.84$. However, for the interpolated case, the optimum values for the two Mach numbers are substantially different. Hence, the fine grid only case converges faster, although less smoothly. Results for the AF-2 scheme at $M_\infty = 0.90$ show the same trends as for $M_\infty = 0.84$, that is, the convergence performance is (1) nearly equal to that for the optimum case and (2) insensitive to mesh interpolation. In the fine grid case, SLOR diverges because of the excessively large relaxation parameter.

CONCLUDING REMARKS

The present study indicates that implicit, approximate-factorization schemes can provide rapid and reliable convergence in finite-difference transonic flow computations. These schemes can be easily coded, require about the same storage as, and only about 50 percent more computational work per iteration than, present successive line over-relaxation algorithms. They can be used to obtain solutions on nonuniformly-spaced grids, a feature not presently available in some of the other convergence acceleration schemes.

Of the two AF schemes investigated here, AF-1 converged the most rapidly. However, it was found to be very susceptible to the "moving shock wave instability" discussed in references 3 and 18. AF-2 was found to be insensitive to this instability for the cases computed here, and its convergence was found to be at least as reliable as that of SLOR and substantially faster.

There should be no fundamental problems in applying the AF schemes to lifting cases. However, the application of these schemes to practical three-dimensional cases, or to the solution of the full potential equation, should be more difficult. The problem is to devise a stable, easily solvable, implicit factorization for the governing equations, which are complicated by the presence of cross derivatives.

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TABLE 1.- NUMBER OF ITERATIONS REQUIRED FOR CONVERGENCE

M_∞	Grid system	Parameters	AF-1	AF-2	SLOR
0.7	Fine only	Opt	34	45	350
	Interpolated	Opt	29	44	141
0.84	Fine only	Opt	--	84	352
		Non Opt	--	99	373
	Interpolated	Opt	51	66	253
		Non Opt	58	101	508
0.90	Fine only	Opt	--	82	819
		Non Opt	--	91	--
	Interpolated	Opt	--	72	411
		Non Opt	--	93	688

TABLE 2.- OPTIMUM ACCELERATION AND RELAXATION PARAMETERS

M_∞	Grid system	γ(AF-1)	γ(AF-2)	SLOR
0.70	Fine only	1.813	1.386	1.965
	Interpolated	(1) 2.276	(1) 1.576	(1) 1.830
		(2) 1.979	(2) 1.428	(2) 1.903
		(3) 1.843	(3) 1.363	(3) 1.919
0.84	Fine only	---	1.400	1.961
	Interpolated	(1) 2.319	(1) 1.730	(1) 1.821
		(2) 2.087	(2) 1.531	(2) 1.911
		(3) 1.895	(3) 1.455	(3) 1.960
0.90	Fine only	---	1.414	1.926
	Interpolated	(1) ---	(1) 1.765	(1) 1.858
		(2) ---	(2) 1.417	(2) 2.01[a]
		(3) ---	(3) 1.404	(3) 1.976

Note: Numbers in parentheses correspond to mesh systems listed below:

Mesh	No. x points	No. y points
(1)	32	8
(2)	64	16
(3)	128	32

[a]Stability was maintained in some cases for relaxation parameters slightly greater than 2.

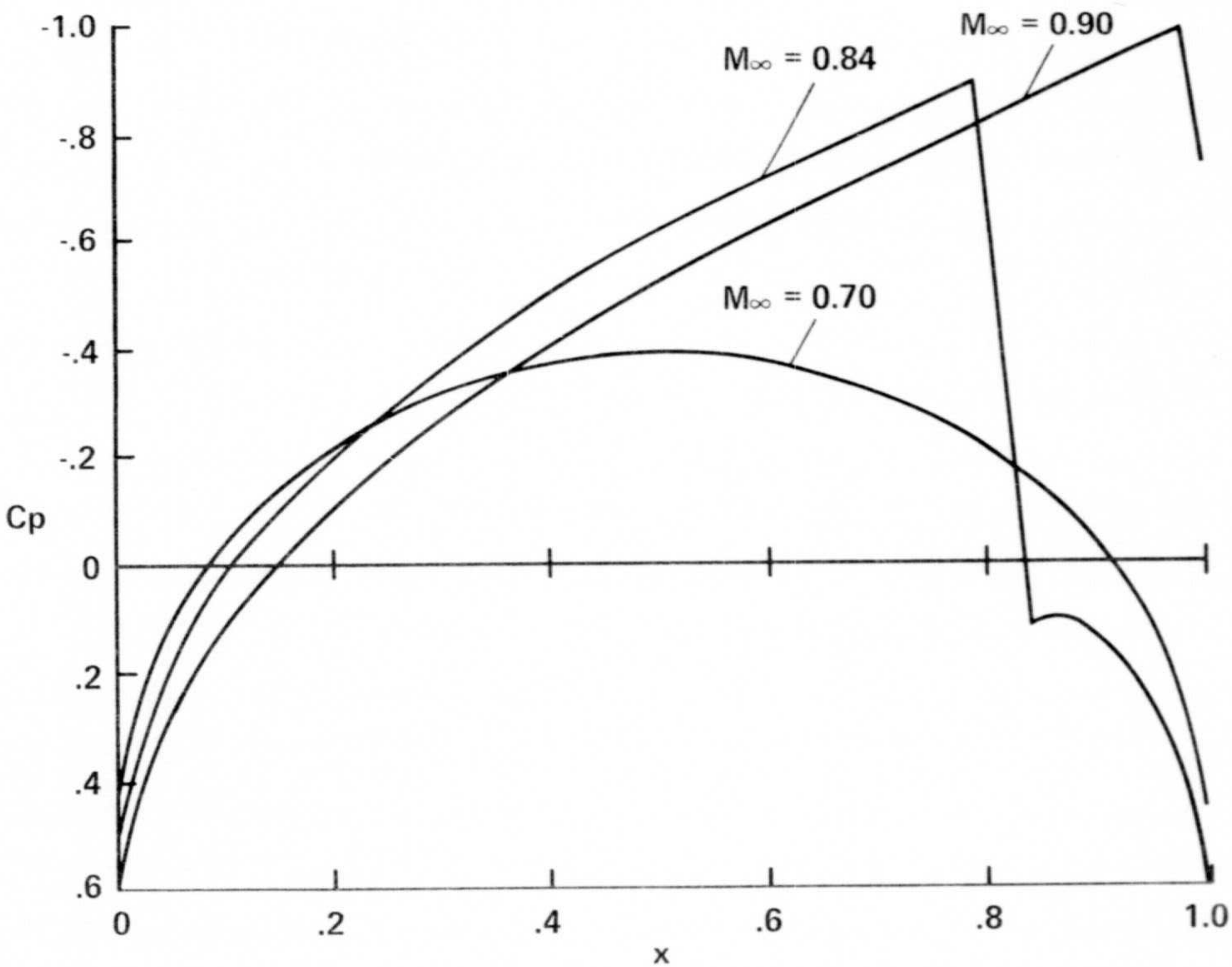
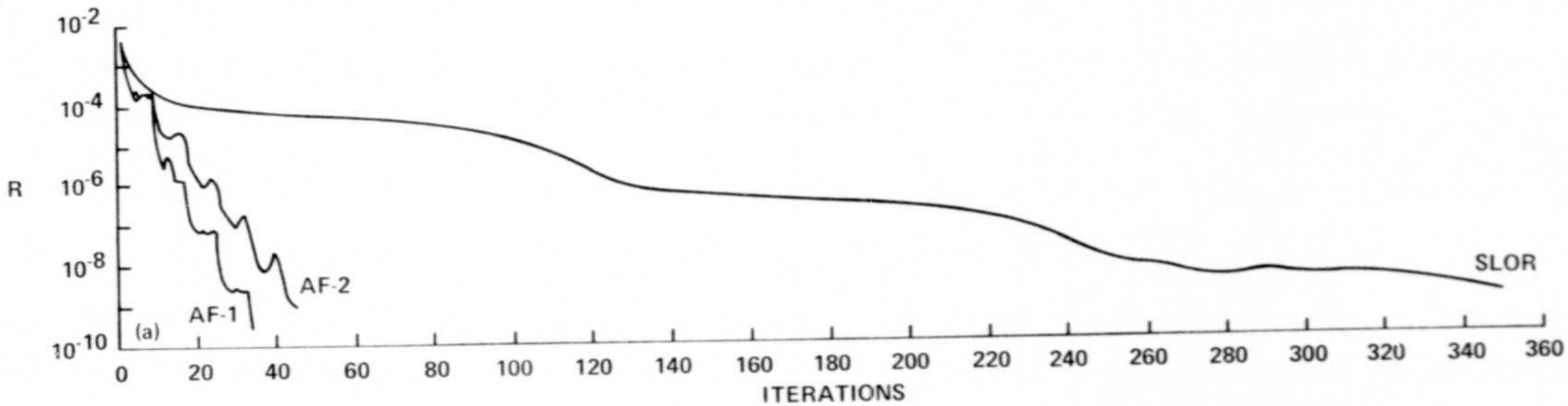
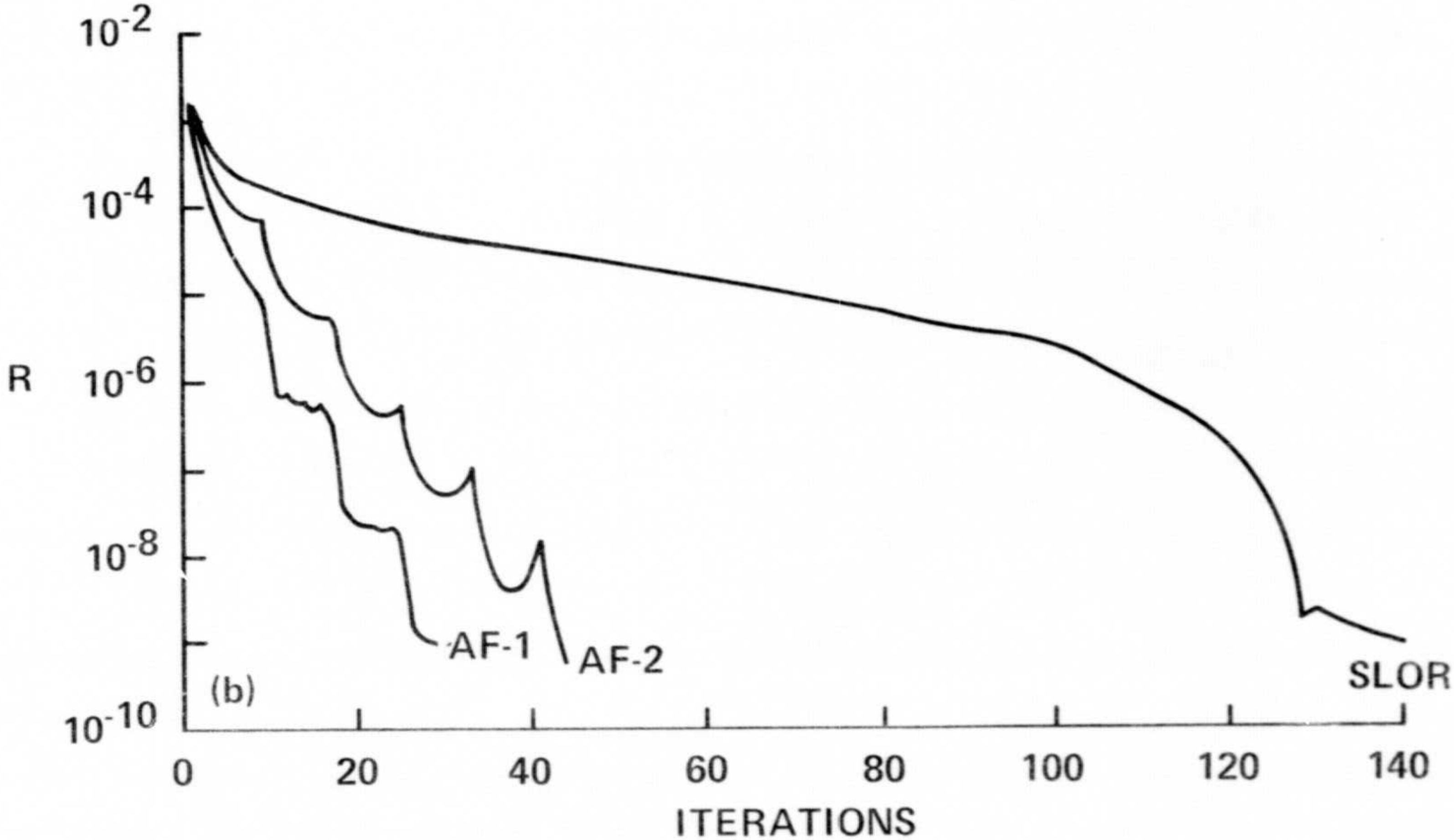


Figure 1.- Surface pressure coefficients on a 10 percent thick parabolic-arc airfoil for the three test case Mach numbers.



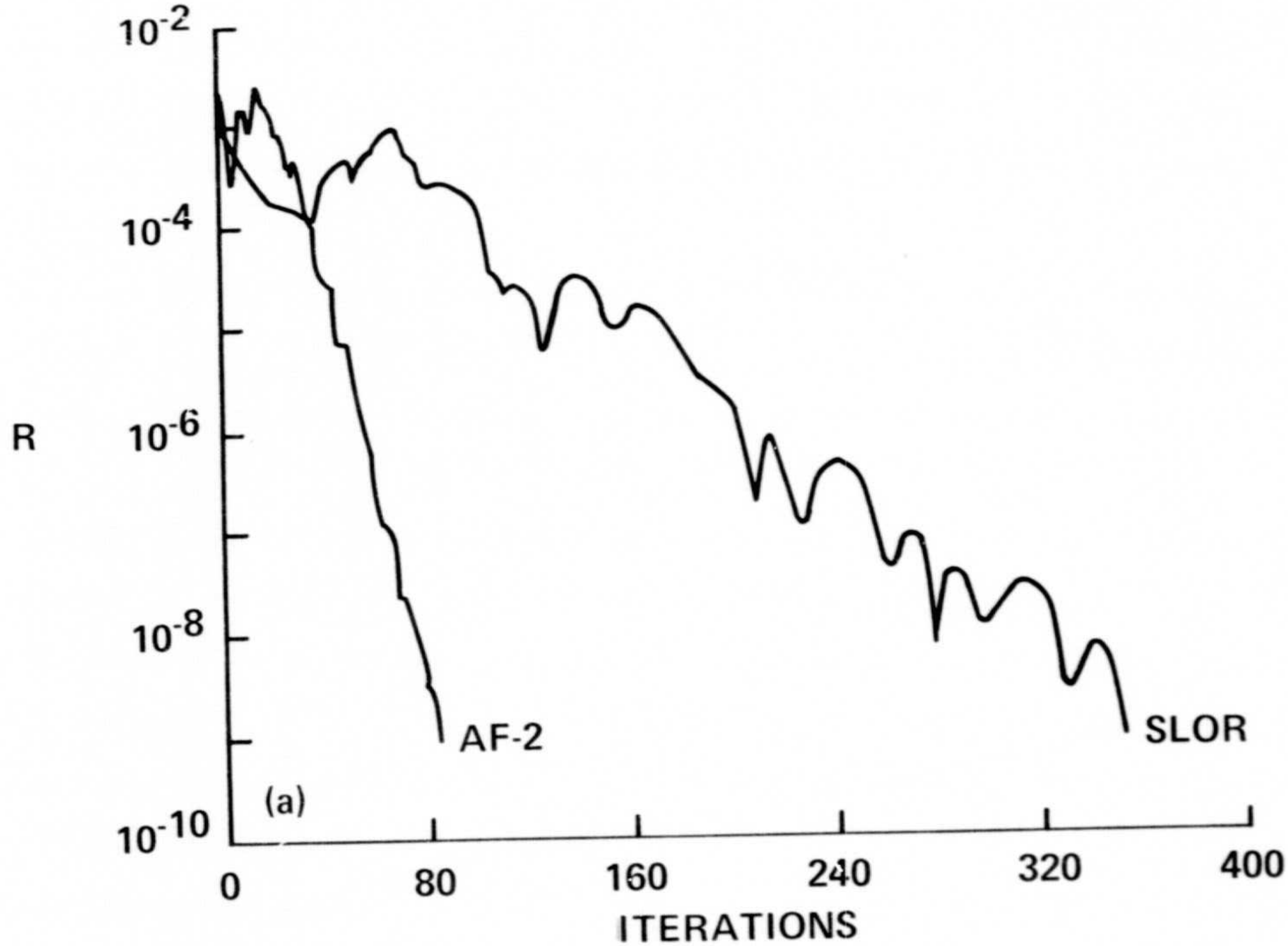
(a) Fine grid only.

Figure 2.- Optimum convergence histories for M_∞ = 0.7.



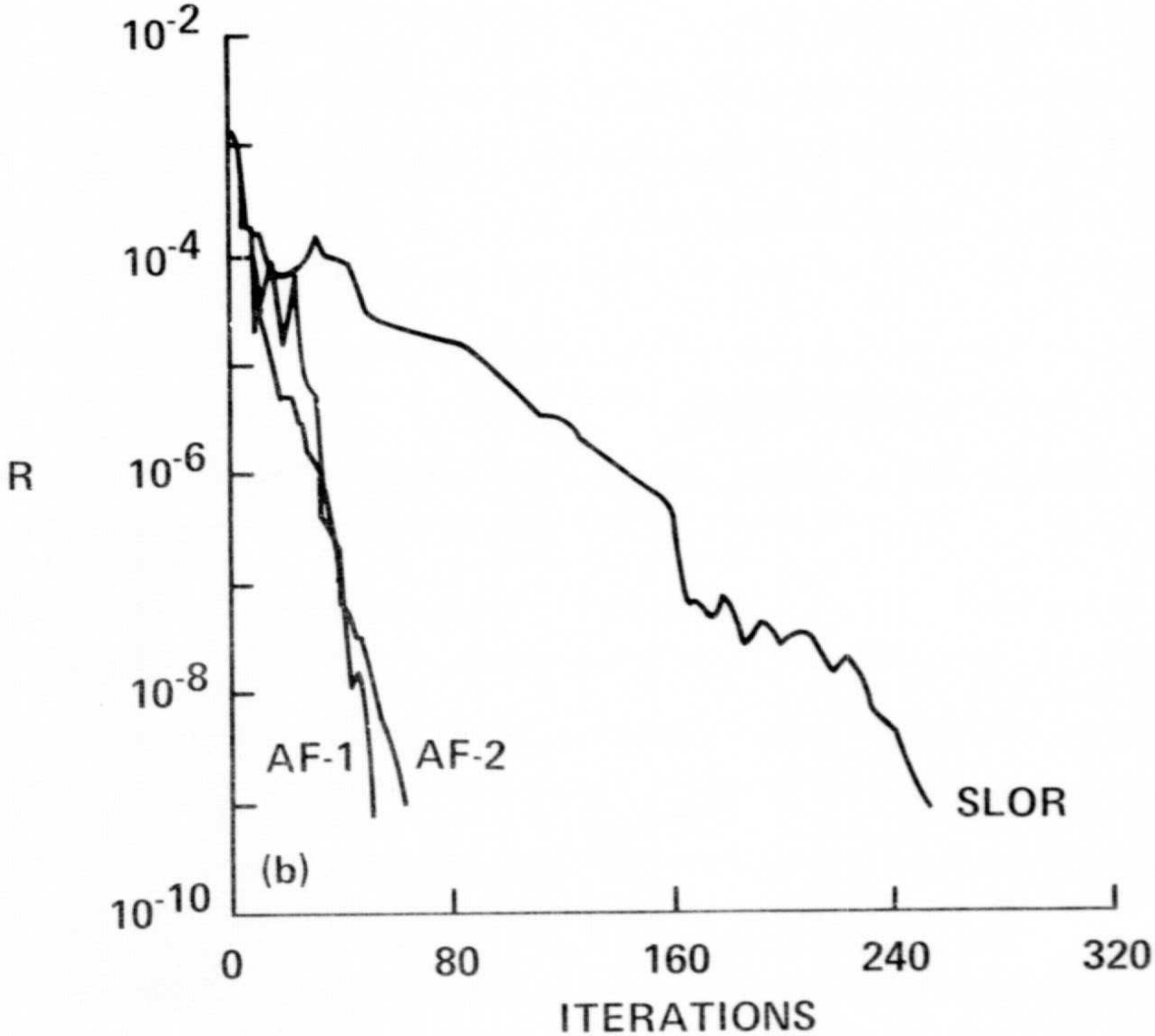
(b) Fine grid with mesh interpolation.

Figure 2.- Concluded.



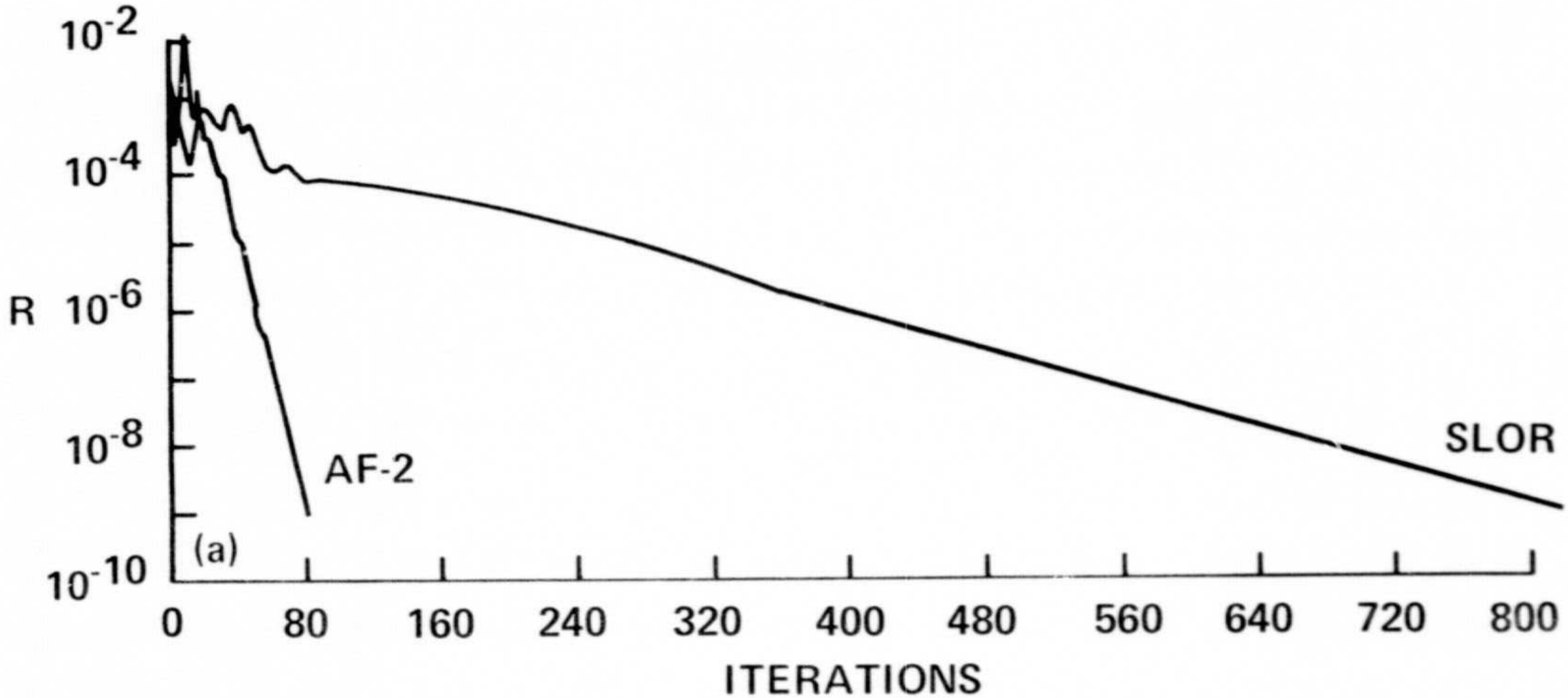
(a) Fine grid only.

Figure 3.- Optimum convergence histories for $M_\infty = 0.84$.



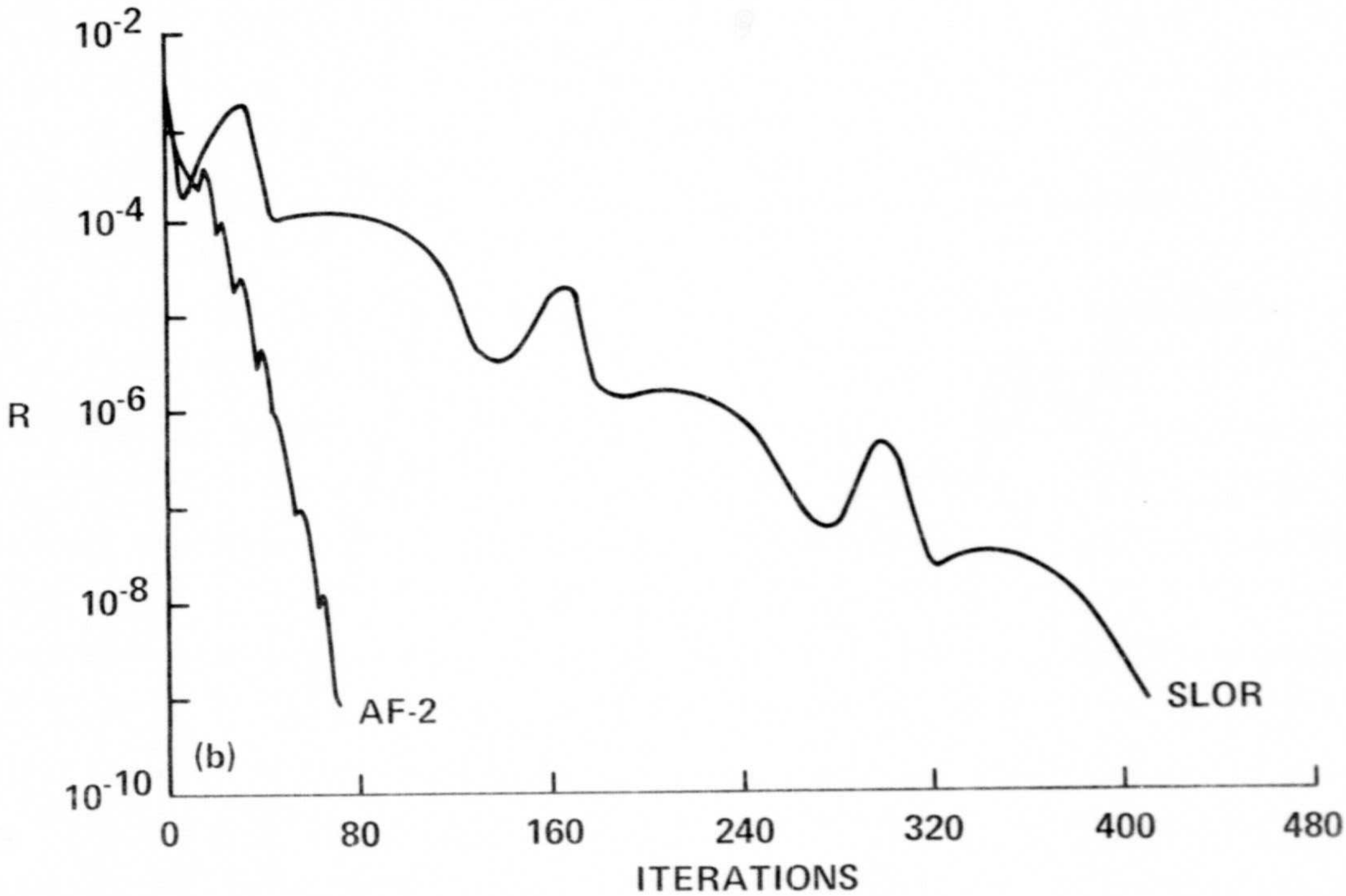
(b) Fine grid with mesh interpolation.

Figure 3.- Concluded.



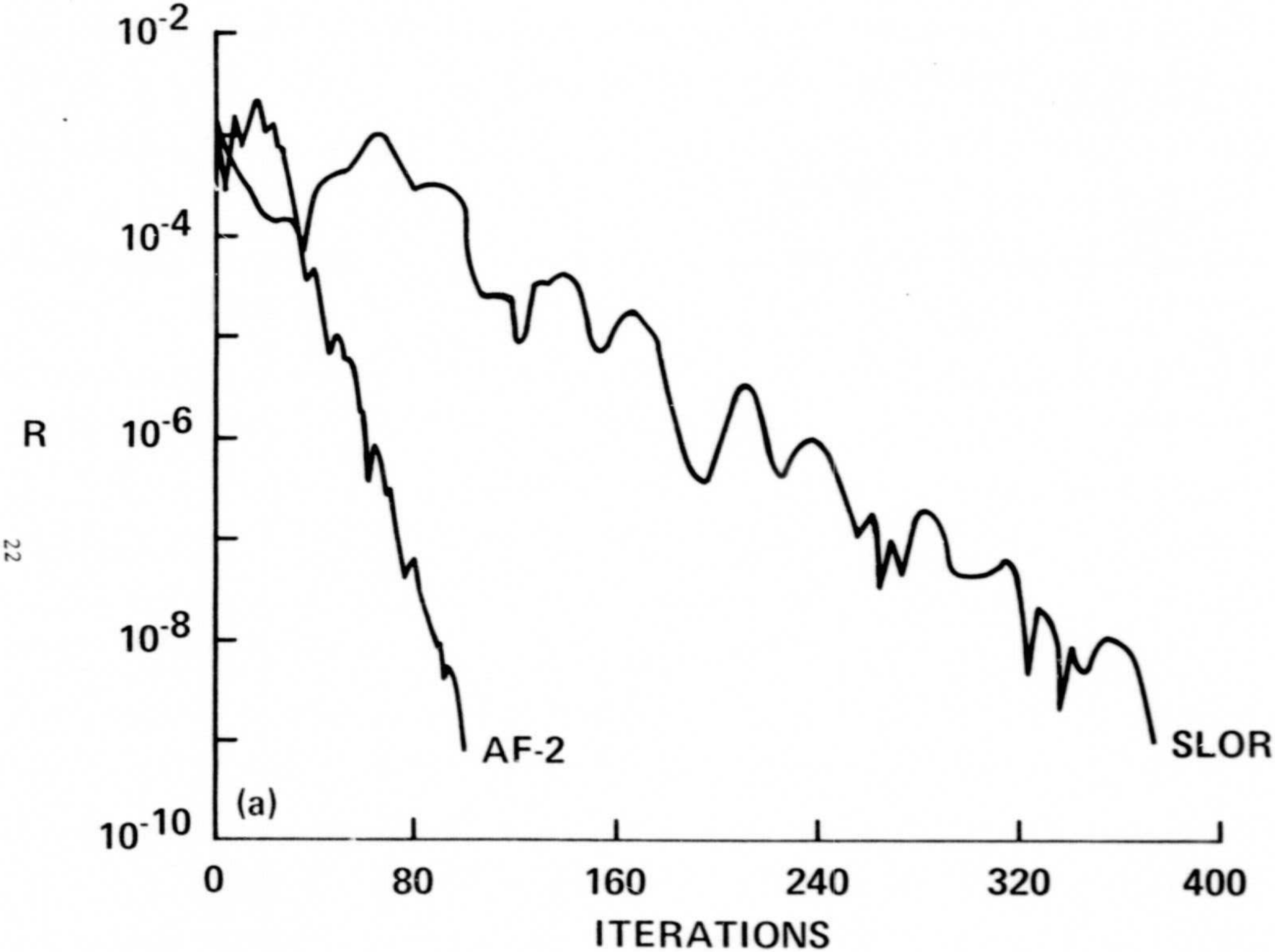
(a) Fine grid only.

Figure 4.- Optimum convergence histories for $M_\infty = 0.90$.



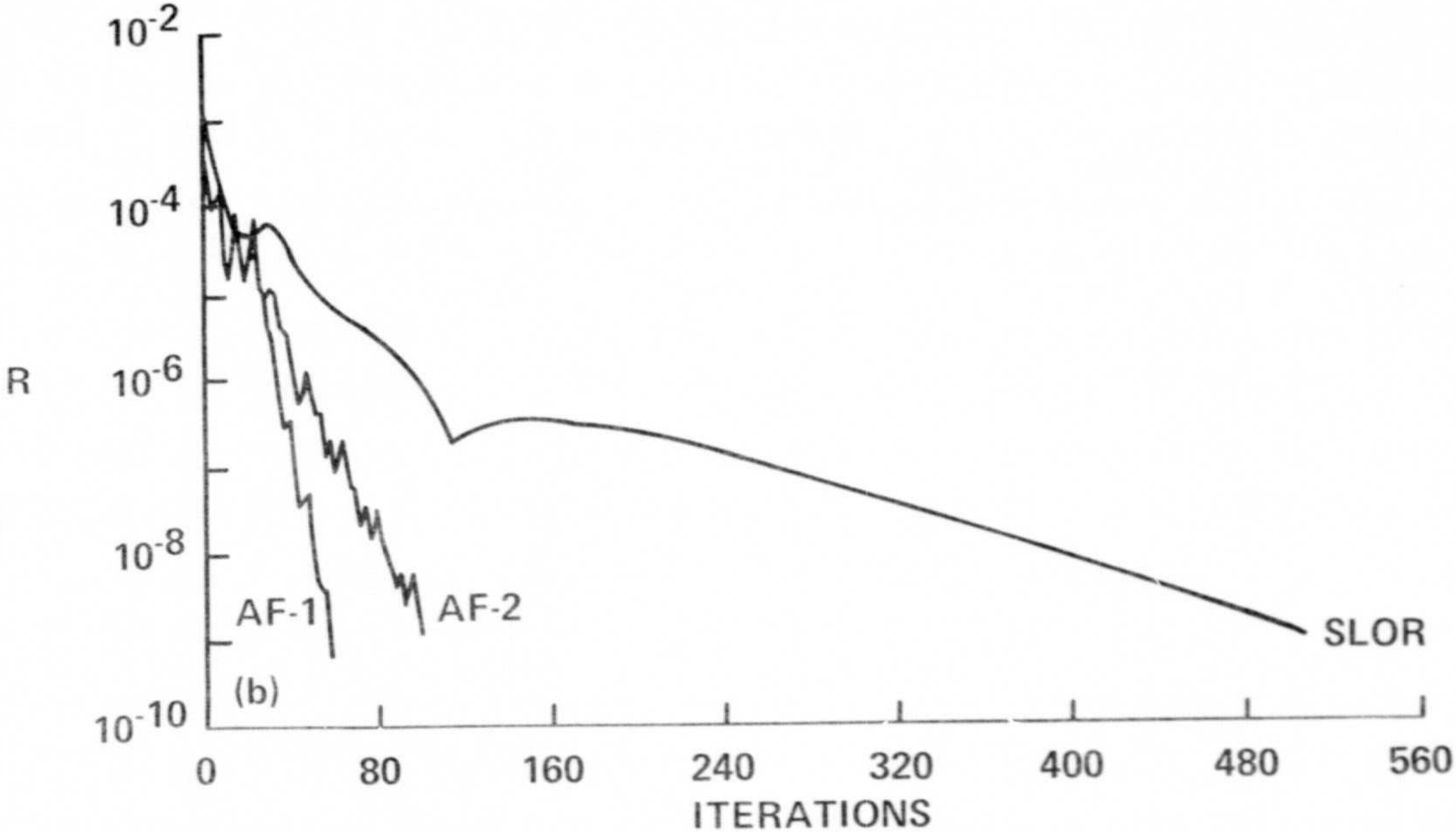
(b) Fine grid with mesh interpolation.

Figure 4.- Concluded.



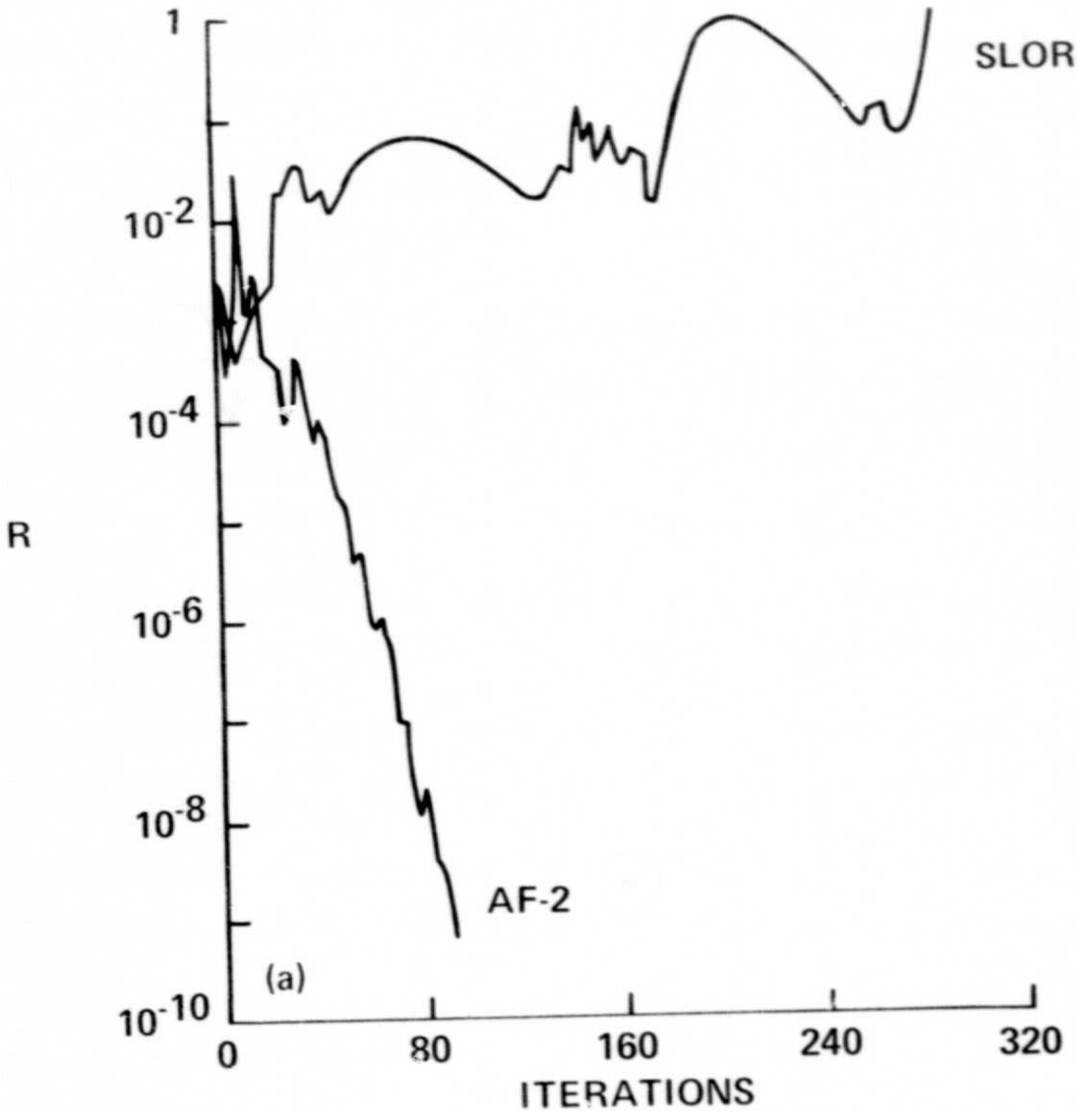
(a) Fine grid only.

Figure 5.- Nonoptimum convergence histories for $M_\infty = 0.84$.



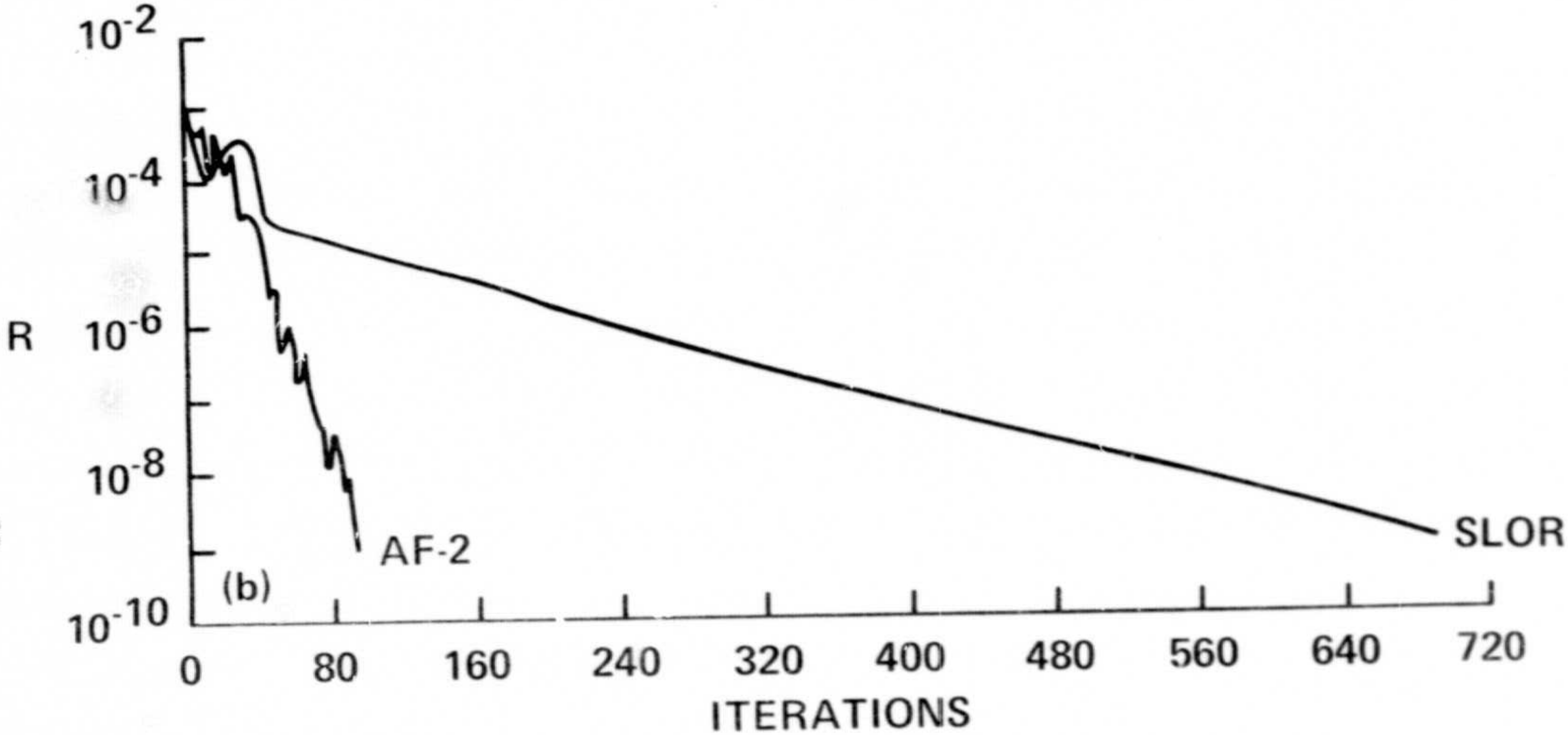
(b) Fine grid with mesh interpolation.

Figure 5.- Concluded.



(a) Fine grid only.

Figure 6.- Nonoptimum convergence histories for M_∞ = 0.90.



(b) Fine grid with mesh interpolation

Figure 6.- Concluded.