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COMPUTER AIDED NONLINEAR ELECTRICAL NETWORKS ANALYSIS

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16. Abstract

This report describes techniques that are used in simulating an electrical circuit with nonlinear elements for use in computer-aided circuit analysis programs. Elements of the circuit include capacitors, resistors, inductors, transistors, diodes and voltage and current sources (constant or time varying). Simulation features are discussed for DC, AC, and/or transient circuit analysis. Calculations are based on the model approach of formulating the circuit equations. A particular solution of transient analysis for nonlinear storage elements is described.

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COMPUTER AIDED NONLINEAR ELECTRICAL
NETWORKS ANALYSIS

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C O N T E N T S

1. INTRODUCTION

Modeling of the electrical circuits which contain passive elements (R, L, C) and active nonlinear elements (transistors, diodes), allows for only very simple circuits to be solved using analytical procedures. To solve an electrical network Kirchhoff's 1st or 2nd law (node analysis or mesh analysis, respectively) and Ohm's law are used. More than three nodes or meshes in the linear case requires a complex analytical solution since it leads to a system of algebraic equations with three or more unknowns. This fact restricts the use of the analytical method. Further, the nonlinear elements cannot be included in the network, because even the most simple circuit with nonlinear elements - a series connection between the resistor and the diode - cannot be solved by the analytical procedure. Therefore, computers using numerical analysis procedures are needed. This acceptance of the computer as a tool allows for an extreme increase in both the possibilities and demands. The size of the network may be up to or exceeding 500 nodes or meshes, and the nonlinear elements are included with all their nonlinear characteristics.

Assessment begins with a description of the network, an understanding of which elements it may contain, and the selection of the proper mathematical procedure to use. Elements which the network can contain are resistors, inductors, capacitors, transistors, (BJT, JFET, MOSFET), diodes and voltage and current sources (constant, time varying and those which depend upon some other branch voltage or current).

Thus, both linear and nonlinear elements are included. Linear elements are the constans, and the nonlinear are described as I = f(V) of V = f(I). An example of a nonlinear element is the diode, where the diode current is equal to

$$I_D = I_S \left(e^{\frac{2V_D}{kT}} - 1 \right)$$

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The transistor cannot be described in such a simple way (with one equation). This in done with a number of equations which lead to a transistor model.

The nonlinear elements are therefore described by their model. For example, the bipolar transistor can be interpreted with Ebers-Moll's model as shown in Fig 1.
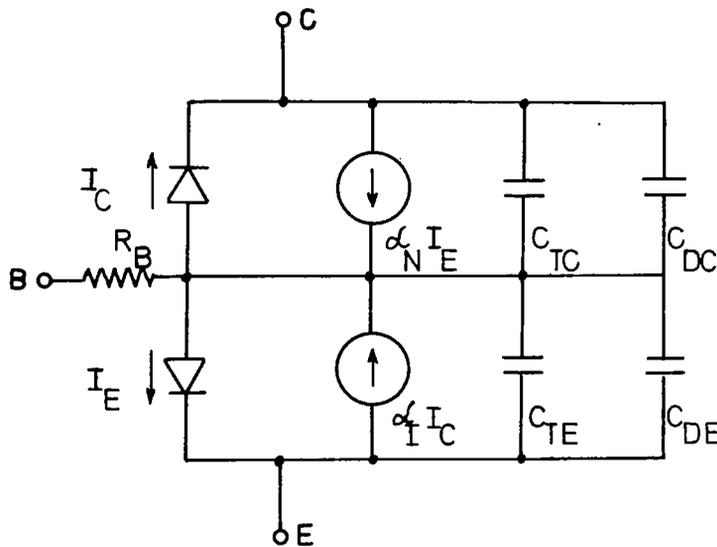
$$I_c = I_{cs} \left(e^{\frac{qV_{BC}}{kT}} - 1 \right)$$

$$I_\varepsilon = I_{ES} \left(e^{\frac{qV_{BE}}{kT}} - 1 \right)$$

$$C_{DE} = \tau_N \alpha_N I_E \frac{q}{kT}$$

$$C_{DC} = \tau_I \alpha_I I_c \frac{q}{kT}$$

$$C_{TC} = \frac{C_{TCO}}{\sqrt{1 - \frac{V_{DC}}{V_\phi}}}$$

$$C_{TE} = \frac{C_{TEO}}{\sqrt{1 - \frac{V_{BE}}{V_\phi}}}$$

Fig. 1. Ebers-Moll's model of the
bipolar transistor.

After the network elements have been described the voltage and current sources are given. The sources can be fixed, time varying, and dependent. For example, in transistors, a collector current generator is depedent upon the emitter current so,

$$I_C = \alpha_N I_E \;,$$

and the generator I_C is a dependent current source.

After the electrical network is described, a choice of the

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necessary calculation is made. The following types of analysis can be done :

1. An operating point of the circuit, so called DC analysis - analysis of the steady state conditions.

2. A response to the small alternating signal, so called AC analysis which is executed so that first the DC solution of the network (we calculate the operating point for all nonlinear elements of the network) is found i.e. we linearize every nonlinear element about its operating point.

The linearization of a nonlinear element is shown using a diode as an example (**Fig. 2**).
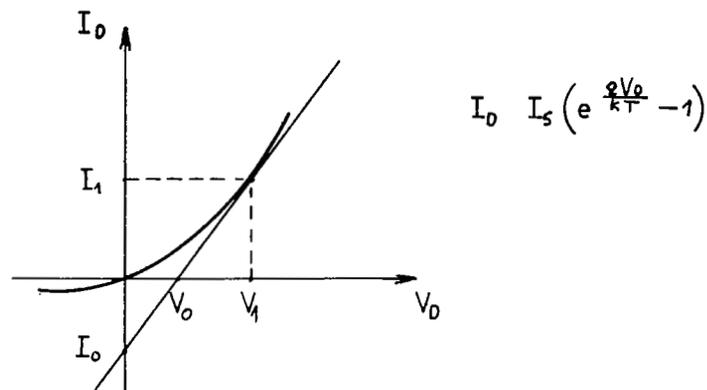
$$I_D \quad I_S \left(e^{\frac{q V_D}{kT}} - 1 \right)$$

Fig. 2. Diode characteristics.

Let V_1, I_1 be the diode operating point. The linearization is obtained so that, instead of the nonlinear characteristics

$$I_D = I_S \left(e^{\frac{q V_D}{kT}} - 1 \right)$$

the tangent in the point V_1, I_1 is used :

$$\left. \frac{d I_D}{d V_D} \right|_{V = V_1} \qquad \text{which gives the conductance}$$

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$$G = \frac{d\ I_D}{d\ V_D} = \frac{q}{kT}\ I_S\ e^q\ \frac{q}{kT} \simeq \frac{q}{kT}\ I_D \bigg|_{V = V_1} ,$$

so, in an electrical network the diode is presented only with the conductance G.

After linearizing the network, the small alternating signal is applied and the response at a given frequency f is calculated. In this type of analysis the response inside the frequency bandwidth from f_1 to f_2 is of interest. In this case, the network calculation is repeated several times, and the distance of the frequencies between f_1, f_2 is arbitrarily divided. Usualy, the geometrical division is taken, i.e.

$$f = q^m\ f_1\ (f_1,\ qf_1,\ q^2f_1\ \ldots\ q^nf_1 = f_2).$$

For example, to calculate the response of the video amplifier between 1 Hz and 1,05 MHZ with q=2 gives :

$$f = 1,\ 2,\ 4,\ 8,\ \ldots\ ,\ 524288,\ 1,048576.$$

In the given example, the calculation will be repeated 20 times.

As a result, the frequency and phase amplifier characteristics are obtained.

3. The third type of analyisis is the time domain analysis or transient analysis. In this type of analysis the DC network state is calculated and then the analysis is begun. This type is the most complex and involves the use of elaborate mathemetical and programming procedures.

Nonlinear elements are presented with their linear models as in Fig 3.
After the nonlinear elements are linearized, the network, which now represents a number of resistors and fixed voltage and current sources, is solved. When the DC condition is determined, due to

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the reactive elements C, L and the changeable signal source, the state of the network must be forseen at some time $t = t + \Delta t$. After this is accomplished (this will be covered later), the network is linearized again, but now with the new conditions, since voltages and currents through the nonlinear elements have changed. This procedure is repeated step by step until time $t = t_{stop}$ is reached, at which time the analysis ends.
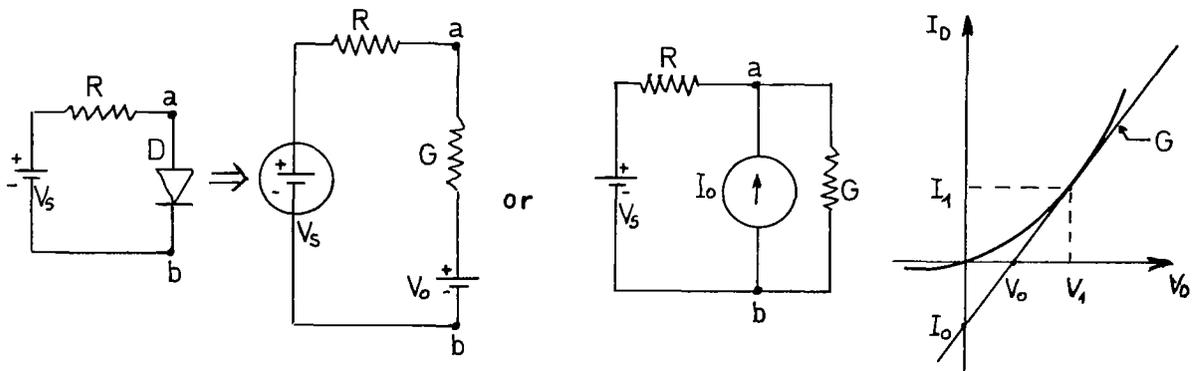

Fig. 3. Linearized diode model.

2. NETWORK ANALYSIS NODAL APPROACH

Network analysis can be performed using either Kirchhoff's 1st or 2nd Law (nodal analysis or mesh analysis, respectively). The nodal analysis is more convenient and will be applied below using a simple resistive network (Fig. 4).
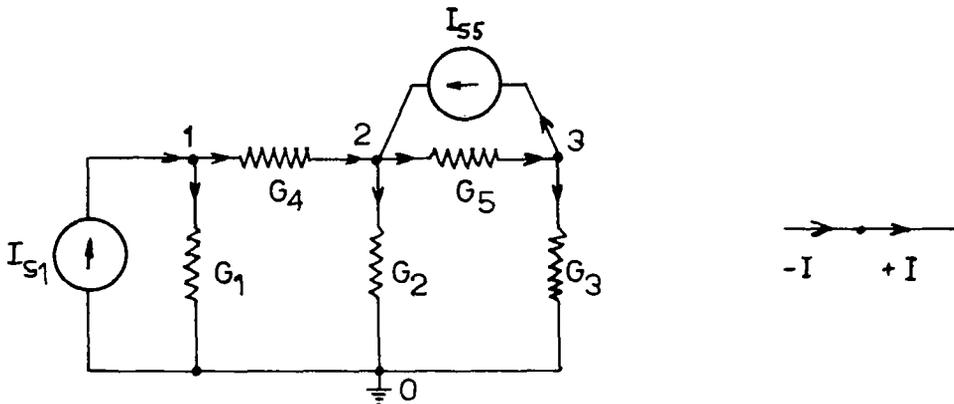
Fig. 4. Oriented linear network.

The network description must be done in a simple and definite way. Nodes and branches of the network are assigned symbols. One node is called the reference node 0. In the given example the network is described as in Fig. 5.

R1	1	0	1 k
R2	2	0	2 k
R3	3	0	5 k
R4	1	2	1 k
R5	2	3	1 k
IS1	0	1	1 A
IS5	3	2	.1 A
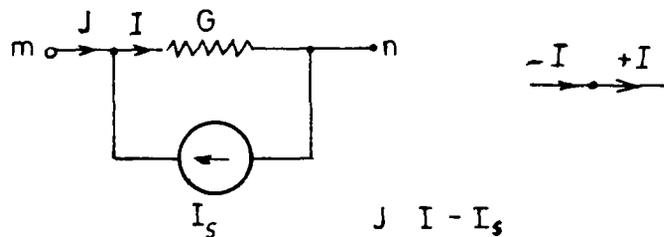
Fig. 5. Network description and network branch definition.

The given description in Fig. 5 completely defines the network so that the network may be reconstructed and solved. Solution of the various equations is done by computer analysis. Obviously, such a program is large and has 2000 - 10 000 instructions.

2.1. Conductance Matrix

After the network is described all the relevant equations are written. All mathematical equations should be in the matrix form, because this form is very convenient for mathematical manipulation and programming.

For the network in Fig. 4, the following nodal equations are written :

$$\text{node 1} \quad I_1 + 0 + 0 + I_4 + 0 = I_{s1}$$

$$\text{node 2} \quad 0 + I_2 + 0 - I_4 + I_5 = I_{s5} \qquad (1)$$

$$\text{node 3} \quad 0 + 0 + I_3 + 0 - I_5 = -I_{s5}$$

and further

$$\text{node 1} \quad G_1 V_1 + G_4 (V_1 - V_2) = I_{s1}$$

$$\text{node 2} \quad -G_4 (V_1 - V_2) + G_2 V_2 + G_5 (V_2 - V_3) = I_{s5}$$

$$\text{node 3} \quad -G_5 (V_2 - V_3) + G_3 V_3 = -I_{s5}$$

and rearranged

$$\text{node 1} \quad (G_1 + G_4)V_1 - G_4 V_2 + 0 = I_{s1}$$

$$\text{node 2} \quad -G_4 V_1 + (G_2 + G_4 + G_5) V_2 - G_5 V_3 = I_{s5} \qquad (2)$$

$$\text{node 3} \quad 0 - G_5 V_2 + (G_3 + G_5) V_3 = -I_{s5}$$

In the matrix form, term (2) is given as

$$
\begin{array}{c}
\text{node 1} \\
\text{node 2} \\
\text{node 3}
\end{array}
\left|
\begin{array}{ccc}
G_1+G_4 & -G_4 & 0 \\
-G_4 & G_2+G_4+G_5 & -G_5 \\
0 & -G_5 & G_3+G_5
\end{array}
\right|
\left|
\begin{array}{c}
V_1 \\
V_2 \\
V_3
\end{array}
\right|
=
\left|
\begin{array}{c}
I_{s1} \\
I_{s5} \\
-I_{s5}
\end{array}
\right|
\quad (3)
$$

$$\text{node 1} \quad \text{node 2} \quad \text{node 3}$$

or, in a shortened form : $\underset{\sim}{G}\, \underline{V}_n = \underline{I}_{sn}$

where $\underset{\sim}{G}$ is a conductance matrix from (3), vector $\underline{V}_n$ is the voltage vector of the nodes towards the reference node (node $\emptyset$), and $\underline{I}_{sn}$ on the right side is the current vector. Matrix $\underset{\sim}{G}$ is made from the elemental network (Fig. 4) as follows :

- a diagonal element ii is the sum of all the conductances that are connected with node i,
- other elements in the row are on the places ij, so that on the place ij comes the conductance which connects node i with node j, but with a negative symbol,
- conductances, that are from node i connected with a referent node, are only in the diagonal.

The current sources vector element represents the sum of all current sources which enter that node. This procedure is very convenient for arranging the matrices from the topological network descriptopn which is described in Fig 5. Further, the relation $\underset{\sim}{G}\, \underline{V}_n = \underline{I}_{sn}$ may be derived using Kirchhoff's nodal analysis, Ohm's Law, and the definition of a voltage drop on a resistor as demonstrated below.

1st Kirchhoff's Law. A standard network branch is defined in Fig. 6. Between nodes m and n is a conductance G ;

parallel to it is a current source, I_s.
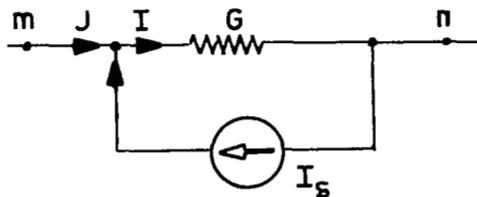
$$J = I - I_s$$

Fig. 6. Standard network branch.

The total branch current is $J = I - I_s$. A network branch can contain only the condudtance G, the current source I_s, or both at the same time. Considering this definition, term (1) can be written as

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{vmatrix} \begin{vmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \end{vmatrix} = 0$$

or as

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{vmatrix} \begin{vmatrix} I_1 - I_{s1} \\ I_2 - \emptyset \\ I_3 - \emptyset \\ I_4 - \emptyset \\ I_5 - I_{s5} \end{vmatrix} = \emptyset$$

or, in the matrix form

$$\underset{\sim}{A}\ \underline{J} = \emptyset$$

$$\underset{\sim}{A}\,(\underline{I} - \underline{I}_s) = 0$$

$$\underset{\sim}{A}\,\underline{I} = A\,\underline{I}_s\,,\qquad\qquad 1^{st}\ \text{K.L.}\qquad(4)$$

Matrix $\underset{\sim}{A}$ iz called the incidence matrix and can be constructed from a topological network description (Fig. 4). Progressing from branch 1 to branch 5, a value of 1 is written in place of the node where the branch current exits, and -1 where the current enters.

Matrix $\underset{\sim}{A}_a$ is the complete incidence matrix, and also contains the reference node:

branch	1	2	3	4	5	
node 1	1	0	0	1	0	
node 2	0	1	0	-1	1	$= \underset{\sim}{A}_a$
node 3	0	0	1	0	-1	
node 4=∅	-1	-1	-1	0	0	

The fourth row of matrix $\underset{\sim}{A}_a$ may be reconstructed from the first three and is therefore unnecessary. For example, in the first column (branch 1) we find only +1. This means that -1 is in the fourth row, signifying that the first branch is connected between node 1 and the referente node ∅.

Fig. 4 indicates that the network has four nodes. Since the zero value may be associated to any single node, the real number of unknowns is three.

Ohm's Law. For every network branch the connection between the voltage and the current may be written as follows :

$$
\begin{aligned}
G_1 V_{b1} &= I_1 \\
G_2 V_{b2} &= I_2 \\
G_3 V_{b3} &= I_3 \implies \\
G_4 V_{b4} &= I_4 \\
G_5 V_{b5} &= I_5
\end{aligned}
\qquad
\begin{vmatrix}
G_1 & & & & \\
& G_2 & & 0 & \\
& & G_3 & & \\
& 0 & & G_4 & \\
& & & & G_5
\end{vmatrix}
\begin{vmatrix}
V_{b1} \\ V_{b2} \\ V_{b3} \\ V_{b4} \\ V_{b5}
\end{vmatrix}
=
\begin{vmatrix}
I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5
\end{vmatrix}
, \quad \text{or}
$$

$$\underset{\sim}{G}_D \; \underline{V}_b = \underline{I} \; , \quad \text{Ohm's law,}$$

where $\underset{\sim}{G}_D$ is the diagonal conductance matrix, vector $\underline{V}_b$ is the branch voltage vector, and $\underline{I}$ is the branch current vector.

Branch Voltage Drop Definition. The voltage drop across a resistor is obtained by calculating the branch voltage from the difference between the potentials of the branch ends:

$$
\begin{aligned}
V_1 &= V_{b1} \\
V_2 &= V_{b2} \implies \\
V_3 &= V_{b3} \\
V_1 - V_2 &= V_{b4} \\
V_2 - V_3 &= V_{b5}
\end{aligned}
\qquad
\begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & -1 & 0 \\
0 & 1 & -1
\end{vmatrix}
\begin{vmatrix}
V_1 \\ V_2 \\ V_3 \\ \\ \\
\end{vmatrix}
=
\begin{vmatrix}
V_{b1} \\ V_{b2} \\ V_{b3} \\ V_{b4} \\ V_{b5}
\end{vmatrix}
\quad \text{or}
$$

$$\underset{\sim}{A}^t \; \underline{V}_n = \underline{V}_b \; . \tag{6}$$

A matrix which connects node and branch voltages is the transposed incidence matrix $\underset{\sim}{A}^t$.

Now we connect the terms (4), (5), and (6)

$$\underset{\sim}{A}\ \underline{I} = \underset{\sim}{A}\ \underline{I}_s \ ,$$

$$\underset{\sim}{G}_D\ \underline{V}_b = \underline{I} \ , \tag{7}$$

$$\underset{\sim}{A}^t\ \underline{V}_n = \underline{V}_b \ .$$

A simple insertion gives:

$$\underset{\sim}{A}\ \underset{\sim}{G}_D\ \underline{V}_b = \underset{\sim}{A}\ \underline{I}_s \ ,$$

$$\underset{\sim}{A}\ \underset{\sim}{G}_D\ \underset{\sim}{A}^t\ \underline{V}_n = \underset{\sim}{A}\ \underline{I}_s \ , \tag{8}$$

$$\underset{\sim}{G}\ \underline{V}_n = \underset{\sim}{A}\ \underline{I}_s \ ,$$

where $\underset{\sim}{G} = \underset{\sim}{A}\ \underset{\sim}{G}_D\ \underset{\sim}{A}^t$. The conductance matrix is obtained using the incidence matrix and the diagonal conductance matrix.

From term (8), node voltages are calculated using :

$$\underline{V}_n = \underset{\sim}{G}^{-1}\ \underset{\sim}{A}\ \underline{I}_s \ . \tag{9}$$

All other parameters can be calculated from the known node voltages using term (7) :

$$\text{branch voltages } \underline{V}_b = \underset{\sim}{A}^t\ \underline{V}_n \ ,$$

$$\text{branch currents } \underline{I} = \underset{\sim}{G}_D\ \underline{V}_b = \underset{\sim}{G}_D\ \underset{\sim}{A}^t\ \underline{V}_n \ .$$

The matrix $\underset{\sim}{G} = \underset{\sim}{A}\ \underset{\sim}{G}_D\ \underset{\sim}{A}^t$ for the network in Fig. 4 is calculated as:

$$\underset{\sim}{A}\ \underset{\sim}{G}\ \underset{\sim}{A}^t =
\begin{vmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{vmatrix}
\begin{vmatrix} G_1 & & & & \\ & G_2 & & 0 & \\ & & G_3 & & \\ & 0 & & G_4 & \\ & & & & G_5 \end{vmatrix}
\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} =$$

$$\begin{vmatrix} G_1 & 0 & 0 & G_4 & 0 \\ 0 & G_2 & 0 & -G_4 & G_5 \\ 0 & 0 & G_3 & 0 & -G_5 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} (G_1+G_4) & -G_4 & 0 \\ -G_4 & (G_2+G_4+G_5) & -G_5 \\ 0 & & (G_3+G_5) \end{vmatrix} = \underset{\sim}{G} \ .$$

2.2 Dependent Current Sources

Any network branch can contain a current source. Current sources
may be either dependent or independent. Independent current
sources are placed on the right side vector as already shown.
Dependent current sources are treated as follows : the source
is described using the independent (control) branch voltage
and is placed on the right side vector. Next, the terms are
switched from the right to the left side (Fig. 7).

The independent branch current is given as

$$I_1 = G_1 \ (V_{m1} - V_{m2}) \ ,$$

and the dependent current source is

$$I_2 = \beta \ I_1 = \beta \ G_1 \ (V_{m1} - V_{m2}).$$

This kind of dependent current source is called a current
controlled dependent current source. If the source dependens
upon the voltage, it is called a voltage controlled dependent
current source and is expressed as

$$I_2 = G_m \ (V_{m1} - V_{m2}) \ .$$
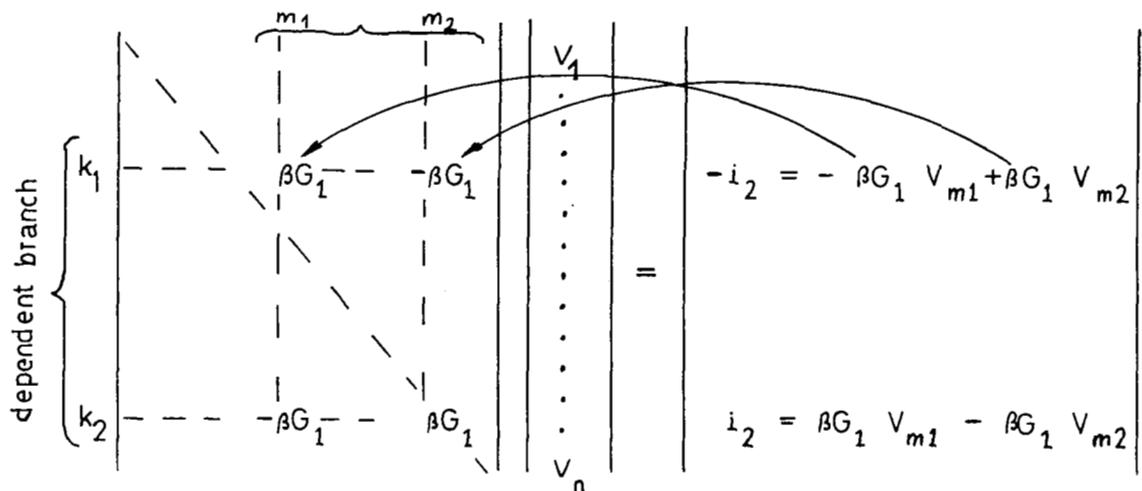
Fig. 7. Inserting of the dependent current
source into matrix $\underset{\sim}{G}$.

The dependent current source is determined by four terms in the
matrix $\underset{\sim}{G}$ which are not symmetrical with regard to a diagonal. Thus,
the matrix $\underset{\sim}{G}$ is generally symmetrical; however if there are some
dependent current sources, matrix $\underset{\sim}{G}$ will have some nonsymmetrical
terms. In this case, dependent current sources represented by
following matrix $\underset{\sim}{G}$ terms:

$$\text{dependent branch} \rightarrow \begin{vmatrix} \beta G_1 & - \beta G_1 \\ - \beta G_1 & \beta G_1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} G_m & - G_m \\ -G_m & G_m \end{vmatrix}$$

$$\text{control branch} \qquad\qquad \text{control branch}$$
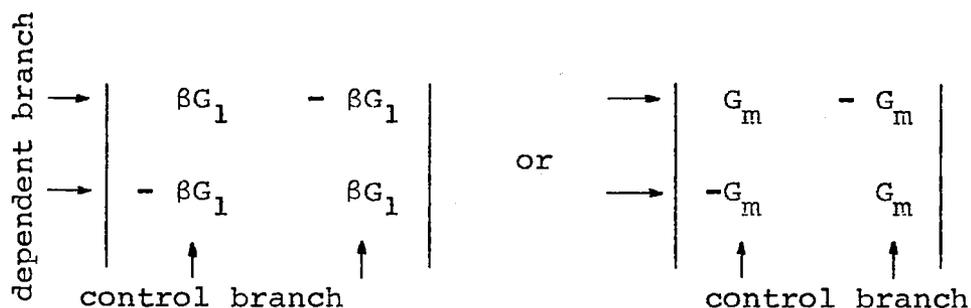
2.3. Voltage Sources in the Network

A voltage source which is alone in the branch, cannot be directly included in the network which is analysed by the nodal method. This is not possible because, by Ohm's law, term (5), makes it impossible to calculate the current of the branch.

Observing a network which has a voltage source, it is obvious that the number of the unknown node voltages is decreased by one. The known voltage source V_s, if attached between nodes V_{k1} and V_{k2}, gives $V_{k1} = V_{k2} + V_s$, or $V_{k2} = V_{k1} - V_s$. This requires a smaller number of unknowns in the case of the network which is being solved by the nodal method. Voltage sources in the network are of two basic types: sources connected with one pole to the referent node and sources connected between any two nodes.

a) Voltage sources connected with one pole to the referent node

Most voltage sources that are applied in the network are grounded with one pole. These are supply voltages and input signal voltages. First described is the insertion of these sources in the matrix term (8). We will start from the example in Fig. 8. The network is described below assuming that the voltage source current (I_x) is known :
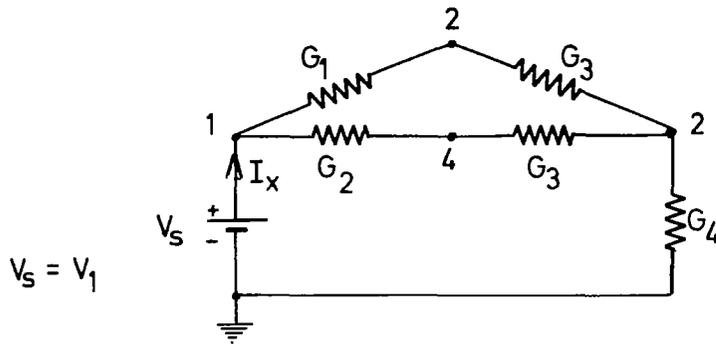
Fig. 8. A network with a voltage source with one pole grounded.

$$
\begin{vmatrix}
G_1+G_2 & -G_1 & 0 & -G_2 \\
-G_1 & G_1+G_3 & -G_3 & 0 \\
0 & -G_3 & G_3+G_4+G_5 & -G_5 \\
-G_2 & 0 & -G_5 & G_2+G_5
\end{vmatrix}
\begin{vmatrix}
V_s \\
V_2 \\
V_3 \\
V_4
\end{vmatrix}
=
\begin{vmatrix}
I_x \\
0 \\
0 \\
0
\end{vmatrix}
\qquad (10)
$$

where $V_1 = V_s$.

Switching the first column to the right the arrangement of the unknowns V_n is changed from V_s, V_2, V_3, V_4 to V_2, V_3, V_4, V_s, i.e. the known voltage V_s is moved to the bottom of the vector $\underline{V}_n$. After this operation, the matrix $\underline{G}$ is nonsymmetrical and the current I_x has remained in the same place. The second operation switches the entire first equation to the last place, so that the arrangement of the current vector $\underline{I}$ is changed: I_x, 0, 0, 0 becomes 0, 0, 0, I_x. Aftr this, the matrix has become symmetrical again and the unknown current I_x is placed at the bottom of the current vector. When both operations are carried out, term (10) becomes:

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$$\left|\begin{array}{ccc:c} G_1+G_3 & -G_3 & 0 & -G_1 \\ -G_3 & G_3+G_4+G_5 & -G_5 & 0 \\ 0 & -G_5 & G_2+G_5 & -G_2 \\ \hdashline -G_1 & 0 & G_2 & G_1+G_2 \end{array}\right| \left|\begin{array}{c} V_2 \\ V_3 \\ V_4 \\ \hline V_s \end{array}\right| = \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline I_x \end{array}\right| \qquad (11)$$

This term can be written in the condensed form as

$$\left|\begin{array}{c:c} \underset{\sim}{G}_{nn} & \underset{\sim}{G}_{ns} \\ \hdashline \underset{\sim}{G}_{sn} & \underset{\sim}{G}_{ss} \end{array}\right| \left|\begin{array}{c} \underline{V}_n \\ \hline \underline{V}_s \end{array}\right| = \left|\begin{array}{c} \underline{I}_s \\ \hline \underline{I}_x \end{array}\right| \qquad (12)$$

From term (12) we get:

$$\underset{\sim}{G}_{nn}\,\underline{V}_n + \underset{\sim}{G}_{ns}\,\underline{V}_s = \underline{I}_s \qquad (13)$$
$$\underset{\sim}{G}_{sn}\,\underline{V}_n + \underset{\sim}{G}_{ss}\,\underline{V}_s = \underline{I}_x \;.$$

The first equation from (13) gives:

$$\underset{\sim}{G}_{nn}\,\underline{V}_n = (\underline{I}_s - \underset{\sim}{G}_{ns}\,\underline{V}_s), \qquad (14)$$

and further

$$\underline{V}_n = \underset{\sim}{G}_{nn}^{-1}\,(\underline{I}_s - \underset{\sim}{G}_{ns}\,\underline{V}_s)\;.$$

Thus the described procedure eliminates a number of the unknowns in the system and includes voltage sources. The second equation from (13), after calculating the node voltages $\underline{V}_n$ from (14), enables us to calculate currents $\underline{I}_x$ which flow through the voltage sources.

By rearrangement of term (11) using (13) and (14) we obtain:

$$
\begin{vmatrix}
G_1+G_3 & -G_3 & 0 \\
-G_3 & G_3+G_4+G_5 & -G_5 \\
0 & -G_5 & G_2+G_5
\end{vmatrix}
\begin{vmatrix}
V_2 \\
V_3 \\
V_4
\end{vmatrix}
=
\begin{vmatrix}
G_1 V_s \\
0 \\
G_2 V_s
\end{vmatrix}
\quad \text{and}
$$

$$
I_x = -G_1 V_2 + G_2 V_4 + (G_1+G_2)\, V_s \; .
$$

If the network contains other voltage sources with one grounded pole, they are eliminated by switching all columns which contain the voltages V_s and the rows which contain the relevant currents I_x to the right side, i.e. to the bottom of the matrix term.

b) Voltage sources connected between any
   two nodes

In this case, the procedure is more complicated and is rarely applied in the electrical network nodal analysis. Because of this, the entire algebraic procedure will not be given. The basic principle involved is covered briefly. For a voltage source such as that in Fig. 9, the network, where the voltage source V_s is placed, can be described as if it consisted of two separate voltage sources V_s. After this, there are two voltage source branches and node V_4 is eliminated. The series connection between the voltage and the conductance is changed regarding Norton's theorem with a parallel connection between the current source and the conductance as shown in Fig. 9.
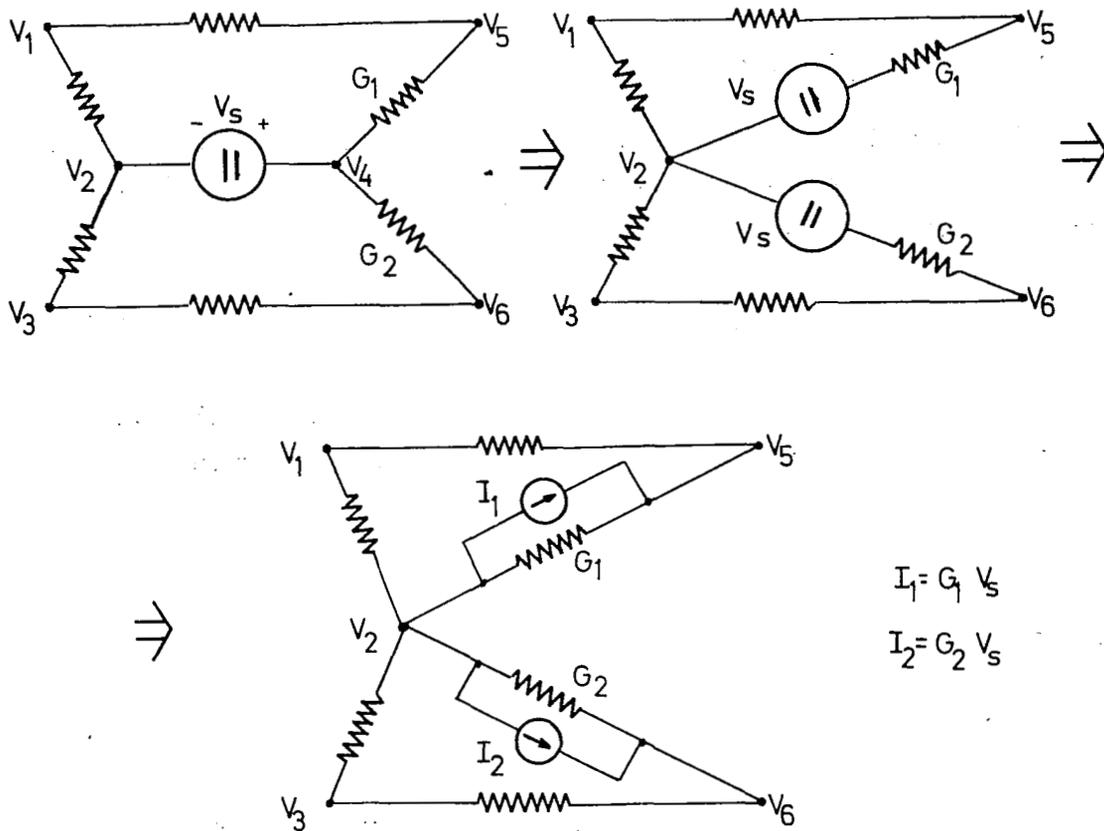
Fig. 9. A voltage source which is connected
between two nodes.

3. NUMERICAL SOLUTION OF ALGEBRAIC EQUATIONS

3.1. Gaussian Elimination

Chapter 2 shows how matrix $\underset{\sim}{G}$ and vector $\underline{I}_s$ are obtained, thus the network is described by $\underset{\sim}{G}\, \underline{V}_n = \underline{I}_{sn}$. The node voltages solution is symbolically written as

$$\underline{V}_n = \underset{\sim}{G}^{-1}\, \underline{I}_{sn} \tag{15}$$

where $\underset{\sim}{G}^{-1}$ is an inverse matrix of $\underset{\sim}{G}$. The solution will be explained as it is the custom in mathematics in the example

$$\underset{\sim}{A}\, \underline{x} = \underline{b} \ ,$$

or, rewritten as

$$\underline{x} = \underset{\sim}{A}^{-1}\, \underline{b} \ . \tag{16}$$

The solution by Cramer's rule requires n! operations and cannot be used for larger matrices. $(10! = 3,628,800$, and $20! = 2.432 \cdot 10^{18})$.

Because of this, the system is solved by the Gaussian elimination method, or the modified Gaussian method which is sometimes called Crout's reduction. This procedure requires about $n^3/3$ operations where n is the number of unknowns. The solution of the Gaussian method involves eliminating the first unknown from the first equation and inserting it into all (n-1) remaining equations. The procedure will be described in an example of three equations with three unknowns using the modified Gaussian method. In the original Gaussian elimination, the x_1 is eliminated from the first row, and according to the modified method term $a_{11} x_1$ is eliminated. So, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix} ,$$

which can be rewritten as

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3 \qquad .$$

Division of the terms of the first row with a_{11} yields:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \dfrac{a_{21}}{a_{11}} & a_{22} & a_{23} \\ \dfrac{a_{31}}{a_{11}} & a_{32} & a_{33} \end{vmatrix} \Longrightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}^1 & a_{22} & a_{23} \\ a_{31}^1 & a_{32} & a_{33} \end{vmatrix} . \qquad (17)$$

Elimination of $a_{11}x_1$ from the first equation gives :

$$a_{11} x_1 = b_1 - a_{12} x_2 - a_{13} x_3 \qquad .$$

By insertion that into the second and the third equation we have:

$$\frac{a_{21}}{a_{11}} (b_1 - a_{12} x_2 - a_{13} x_3) + a_{22} x_2 + a_{23} x_3 = b_2$$

$$\frac{a_{31}}{a_{11}} (b_1 - a_{12} x_2 - a_{13} x_3) + a_{32} x_2 + a_{33} x_3 = b_3 ,$$

and, after reduction,

$$(a_{22} - \frac{a_{21}}{a_{11}} a_{12})x_2 + (a_{23} - \frac{a_{21}}{a_{11}} a_{13}) x_3 = b_2 - \frac{a_{21}}{a_{11}} b_1$$

$$\text{(18)}$$

$$(a_{32} - \frac{a_{31}}{a_{11}} a_{12})x_2 + (a_{33} - \frac{a_{31}}{a_{11}} a_{13}) x_3 = b_3 - \frac{a_{31}}{a_{11}} b_1 \quad ,$$

or, in a shorter form

$$a_{22}^1 x_2 + a_{23}^1 x_3 = b_2^1$$

$$a_{32}^1 x_2 + a_{33}^1 x_3 = b_3^1 .$$

The described procedure eliminates the unknown x_1, i.e. the order of the system is reduced from n to $(n-1)$. Continuing this procedure by eliminating x_2 gives:

$$(a_{33}^1 - \frac{a_{32}^1}{a_{22}^1} a_{23}^1)x_3 = b_3^1 - \frac{a_{32}^1}{a_{22}^1} b_2^1 \quad ,$$

or

$$a_{33}^2 x_3 = b_3^2 .$$

After the elimination has ended, the result is:

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \qquad \text{(19.a)}$$

$$0 \quad + a_{22}^1 x_2 + a_{23}^1 x_3 = b_2^1 \qquad \text{(19.b)}$$

$$0 \quad + \quad 0 \quad + a_{33}^2 x_3 = b_3^2 . \qquad \text{(19.c)}$$

The unknowns may be easily calculated from term (19). From

equation (19.c):

$$x_3 = b_3^2/a_{33}^2 \; .$$

Inserting x_3 into equation (19.b) gives :

$$x_2 = (b_2^1 - a_{23}^1 \, x_3)/a_{22}^1 \; .$$

Next, x_2 and x_3 are inserted into equation (19.a) and x_1 is calculated.

Note that the elimination is done in the following way :

$$
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
\dfrac{a_{21}}{a_{11}} & a_{22} & a_{23} \\
\dfrac{a_{31}}{a_{11}} & a_{32} & a_{33}
\end{vmatrix}
\begin{vmatrix}
x_1 \\ x_2 \\ x_3
\end{vmatrix}
=
\begin{vmatrix}
b_1 \\ b_2 \\ b_3
\end{vmatrix}
\tag{20}
$$

which gives:

$$
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22}^1 & a_{23}^1 \\
0 & a_{32}^1 & a_{33}^1
\end{vmatrix}
\begin{vmatrix}
x_1 \\ x_2 \\ x_3
\end{vmatrix}
=
\begin{vmatrix}
b_1 \\ b_2^1 \\ b_3^1
\end{vmatrix}
$$

The elimination is repeated until the last unknown is reached.

The important information is the number of operations required for the Gaussian elimination.

According to the described procedure the following is needed (D : division, M : multiplication, and S : subtraction):

ELIMINATION	FIRST COLUMN	REDUCED MATRIX		VECTOR B		SOLUTION X		
	D	M	S	M	S	M	S	D
x_1	$(n-1)$	$(n-2)^2$	$(n-1)^2$	$(n-1)$	$(n-1)$	$(n-1)$	$(n-1)$	1
x_2	$(n-2)$	$(n-2)^2$	$(n-2)^2$	$(n-2)^2$	$(n-1)$	$(n-2)$	$(n-2)$	1
$\vdots$ x_{n-1}	$\vdots$ 1	$\vdots$ 1	$\vdots$ 1	$\vdots$ 1	$\vdots$ 1	$\vdots$ 1	$\vdots$ 1	$\vdots$ 1
x_n	–	–	–	–	–	–	–	1
	$\sum_{i=1}^{n-1} i^2$	$\sum_{i=1}^{n-1} i^2$	$\sum_{i=1}^{n-1} i^2$	$\sum_{i=1}^{n-1} i$	$\sum_{i=1}^{n-1} i$	$\sum_{i=1}^{n-1} i$	$\sum_{i=1}^{n-1} i$	n
APPROXIMATELY	$n^2/2$	$n^3/3$	$n^3/3$	$n^2/2$	$n^2/2$	$n^2/2$	$n^2/2$	0

The number of operations is defined with two types of sums of real numbers , $\sum i$ and $\sum i^2$:

$$\sum_{i=1}^{n} i = \frac{(n+1)n}{2}$$ because the given sum is the real number arithmetic series which gives $\frac{n^2}{2} + \frac{n}{2}$, and is approximated with $\frac{n^2}{2}$.

$$\sum_{i=1}^{n} i^2$$ is the sum of the real numbers' squares, and is equal to

$$\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

which is approximated with $\frac{n^3}{2}$. [1]

Thus reduction of matrix $\underset{\sim}{A}$ requires

$$\frac{n^3}{3} M + \frac{n^2}{2} D + \frac{n^3}{3} S \ ,$$

where only the first term $\frac{n^3}{3} M$ is kept, because it needs the most operations and the longest execution time. Vector $\underline{b}$ rearrangement requires

$$\frac{n^2}{2} M + \frac{n^2}{2} S$$

operations of which only the first term, $\frac{n^2}{2} M$, is kept. To obtain the solution approximately $\frac{n^2}{2} M$ operations are needed.

For example, a system of 30 unknowns will need :

$$\frac{n^3}{3} + n^2 = 9000 + 900 = 9900$$

operations. It is obvious that $\frac{n^3}{3}$ multiplications require most of the time and effort in calculations.

A numerical example for the Gaussian elimination is

$$\begin{vmatrix} 2 & 2 & 3 \\ 4 & 5 & 8 \\ 6 & 2 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 4 \end{vmatrix}$$

---

[1]

An approximate term for $\sum\limits_{i=1}^{n} i^2$ can be obtained from the integral

$$\int_{1}^{n} i^2 \, di = \frac{i^3}{3} \Big|_{1}^{n} = \frac{n^3}{3} - \frac{1}{3} \ .$$

$$\begin{vmatrix} 2 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & -4 & -7 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & -4 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$$

The solution is :

$$x_3 = 1,$$

$$x_2 = 0 - 2 \cdot 1 = -2,$$

$$x_1 = (1 - 3 \cdot 1 - 2(-2))/2 = 1.$$

Checking is done by inserting the solutions into the given equations :

$$2 - 4 + 3 = 1$$

$$4 - 10 + 8 = 2$$

$$6 - 4 + 2 = 4.$$

Regarding term (16), $x = A^{-1} b$, if vector b changes, the matrix A^{-1} remains unaltered and the solution x is obtained by a simple procedure of multiplying matrix A^{-1} with the new vector b. When applying Gaussian elimination, instead of the inverse matrix A^{-1}, the unknown vector x is calculated from term (19). Though it appears at first that the vector b change requires the repeating of the whole Gaussian elimination, i.e. approximately $n^3/3$ operations, this is not necessary ; and the number of operations is only n^2 (if only vector b is changed). It is obvious that the coefficients a_{ij} do not have to be calculated again in term (19) but only vector b (b_1, b_2^1, b_3^2, ... b_n^{n-1}); and for vector b

processing the terms in the columns which are placed under the diagonal are used. These terms disappear during the elimination and zeros appear in their places.

However, if we keep these terms, the calculation, because of the vector $\underline{b}$ change, requires $\frac{n^2}{2}$ for solving vector $\underline{b}$, and $n^2/2$ operations for getting the solution, totally n^2 operations.

This Gaussian elimination supplement, by which all terms under the main diagonal are kept, leads te the so called LU decomposition. Let the matrix of term (19) be the upper triangular matrix $\underset{\sim}{U}$:

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & 0 & a_{33}^2 \end{vmatrix} = \begin{vmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{vmatrix} = \underset{\sim}{U}
$$

The terms which appear under the diagonal during the elimination are called the lower triangular matrix $\underset{\sim}{L}$:

$$
\begin{vmatrix} 1 & 0 & 0 \\ a_{21}^1 & 1 & 0 \\ a_{31}^1 & a_{32}^2 & 1 \end{vmatrix} = \begin{vmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{vmatrix} = \underset{\sim}{L} \quad ,
$$

where diagonal terms which are all equal 1 are added to the terms under the diagonal.

It can be shown that :

$$\underset{\sim}{L}\, \underset{\sim}{U} = \underset{\sim}{A} \; .$$

Substituting matrix $\underset{\sim}{A}$ with $\underset{\sim}{L}\,\underset{\sim}{U}$ we proceed:

$$\underset{\sim}{A}\, \underline{x} = \underline{b} \; , \tag{21.a}$$

$$\underset{\sim}{L} \underset{\sim}{U} \underline{x} = \underline{b} ,$$
(21.b)

$$\underset{\sim}{U} \underline{x} = \underline{y} ,$$
(21.c)

$$\underset{\sim}{L} \underline{y} = \underline{b} ,$$

which gives

$$\underline{y} = \underset{\sim}{L}^{-1} \underline{b} .$$
(21.d)

Now, by inserting $\underline{y}$ into (21.6) we obtain

$$\underline{x} = \underset{\sim}{U}^{-1} \underline{y} .$$
(21.e)

Further examination of terms (21) reveals the following :
term (21.a), i.e. matrices $\underset{\sim}{L}$ and $\underset{\sim}{U}$, are obtained by the normal
Gaussian elimination procedure; term (21.d), i.e. vector $\underline{y}$, is
the symbolic representation of the vector $\underline{b}$ elimination procedure,
term (21.e) is a symbolic representation of solution $\underline{x}$ from the
reduced matrix $\underset{\sim}{A}$.

3.2. Improved Precision of the Solution

To increase accuracy, the obtained system $\underset{\sim}{A} x = \underline{b}$ solution is
called $\bar{x}_1 + \bar{x}_n$. Inserting $\bar{x}_1 + \bar{x}_n$ into the original equations
gives:

$$a_{11} \bar{x}_1 + a_{12} \bar{x}_2 + \dots + a_{1n} \bar{x}_n = \bar{b}_1 \neq b_1$$

$$a_{21} \bar{x}_1 + a_{22} \bar{x}_2 + \dots + a_{2n} \bar{x}_n = \bar{b}_2 \neq b_2$$

$$\vdots$$

$$a_{n1} \bar{x}_1 + a_{n2} \bar{x}_2 + \dots + a_{nn} \bar{x}_n = \bar{b}_n \neq b_n$$

By this procedure the calculation accuracy is estimated. If all
$\bar{x}_1$ to $\bar{x}_n$ are correctly calculated, then all b's must equal b.
From the difference $(b_1 - \bar{b}_1)$ to $(b_n - \bar{b}_n)$, the calculation

28

precision is estimated. This simple control procedure allows improvement of the solution accuracy by subtracting the calculated equation from the original one:

$$a_{11}(x_1 - \bar{x}_1) + a_{12}(x_2 - \bar{x}_2) + \ldots + a_{1n}(x_n - \bar{x}_n) = b_1 - \bar{b}_1$$
$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$
$$a_{n1}(x_1 - \bar{x}_1) + a_{n2}(x_2 - \bar{x}_2) + \ldots + a_{nn}(x_n - \bar{x}_n) = b_n - \bar{b}_n$$

or

$$a_{11}\,\Delta x_1 + a_{12}\,\Delta x_2 + \ldots + a_{1n}\,\Delta x_n = \Delta b_1$$
$$\vdots \qquad\qquad\qquad\qquad\qquad \vdots$$
$$a_{n1}\,\Delta x_1 + a_{n2}\,\Delta x_2 + \ldots + a_{nn}\,\Delta x_n = \Delta b_n.$$

Since matrix $\underset{\sim}{A}$ has remained the same, and only term $b_1 \div b_n$ has changed (into $b_1 - b_n$) the calculation is repeated with only n^2 operations to solve for the unknowns $\Delta x_1 \div \Delta x_n$. New values of x are:

$$x_1 = \bar{x}_1 + \Delta x_1$$
$$\vdots$$
$$x_n = \bar{x}_n + \Delta x_n$$

This procedure can be repeated several times to obtain the best accuracy.

3.3. The Inverse Matrix

Using Cramer's Rule to obtain the inverse matrix is a very complicated procedure. The Gaussian elimination method requires approximately n^3 operations and is much more efficient.

It is done as follows :

the system $\underset{\sim}{A} \underline{x} = \underline{b}$ is given; the symbolic solution is $\underline{x} = \underset{\sim}{A}^{-1} \underline{b}$.
The inverse matrix $\underset{\sim}{A}^{-1}$ elements will be called αij (to differ
from the matrix $\underset{\sim}{A}$ elements a_{ij}).

Supposing that matrix $\underset{\sim}{A}^{-1}$ is known, then :

$$
\begin{aligned}
x_1 &= \alpha_{11} b_1 + \alpha_{12} b_2 + \ldots + \alpha_{1n} b_n \\
x_2 &= \alpha_{21} b_1 + \alpha_{22} b_2 + \ldots + \alpha_{2n} b_n \\
&\ \vdots \qquad\qquad\qquad\qquad \vdots \\
x_n &= \alpha_{n1} b_1 + \alpha_{n2} b_2 + \ldots + \alpha_{nn} b_n
\end{aligned}
\tag{22}
$$

If all vector $\underline{b}$ elements are zeros, except the first one which
is $b_1 = 1$, then the upper system is reduced to:

$$
\begin{aligned}
x_1 &= \alpha_{11} \\
x_2 &= \alpha_{21} \\
&\ \vdots \\
x_n &= \alpha_{n1}
\end{aligned}
\tag{23}
$$

If all $b_i = \emptyset$ except $b_2 = 1$, then :

$$
\begin{aligned}
x_1 &= \alpha_{21} \\
x_2 &= \alpha_{22} \\
&\ \vdots \\
x_n &= \alpha_{n2}
\end{aligned}
$$

Now, Gaussian elimination will be applied to the above system
$\underset{\sim}{A} \underline{x} = \underline{b}$, where $\underline{b}$ is :

$$\underline{b}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix}, \quad \underline{b}_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix}, \quad \underline{b}_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{vmatrix}, \ldots \underline{b}_n = \begin{vmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{vmatrix}, \quad (25)$$

which can be written as :

$$\begin{vmatrix} a_{11} \cdots\cdots a_{1n} \\ \cdot \qquad\quad \cdot \\ \cdot \qquad\quad \cdot \\ \cdot \qquad\quad \cdot \\ \cdot \qquad\quad \cdot \\ a_{n1} \cdots\cdots a_{nn} \end{vmatrix} \begin{vmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{vmatrix}, \ldots \begin{vmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{vmatrix}. \quad (26)$$

The first solution values, $x_1 \div x_n$, have to be indentical with equations (23), the second series values have to be identical with equations (24), etc. The obtained solutions ($x_1 \div x_n$) therefore represent the inverse matrix coefficients. The number of needed operations is :

- matrix $\underline{A}$ solution requires $n^3/3$ operations
- solving the right-side column, $\underline{b}$, $n^2/2$ operations
- calculating $x_1 \div x_n$, $n^2/2$ operations

Now the matrix $\underline{A}$ solution is not necessary any more. Continuing from the second column :

- solving the second right side column $\underline{b}$, but starting from $b_2 = 1$, because $b_1 = 0$, $(n-1)^2/2$ operations
- calculating $x_1 \div x_n$, $n^2/2$ operations
- solving the third right side column $\underline{b}$, starting from $b_3 = 1$, $(n-2)^2/2$ operations
- calculating $x_1 \div x_n$, $n^2/2$ operations, etc.

Totally, we need :

$$\frac{n^3}{3} + \left(\frac{n^2}{2} + \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \ldots \frac{2^2}{2} + \frac{1}{2} \right) + n\frac{n^2}{2} .$$

The above middle term is equal to :

$$\frac{1}{2} \sum_{i=1}^{n} i^2 \doteq \frac{1}{2} \frac{n^3}{3} \doteq \frac{n^3}{6} .$$

The total number of operations is therefore:

$$\frac{n^3}{3} + \frac{n^3}{6} + \frac{n^3}{2} = n^3 .$$

This requires only three times more calculations than is needed for the system $\underset{\sim}{A}\,\underline{x} = \underline{b}$ solution.

3.4. Gaussian Elinination Example

To illustrate the simplicity of the Gaussian elimination the following FORTRAN program for solving the system of unknowns n is given:

```
aFOR, IS
C
C      RESERVING FIELD FOR THE MATRIX  A   COEFFICIENTS,
C      VECTOR   B   AND SOLUTIONS   X
C
       DIMENSION A(100,100),B(100),X(100)
C
C      READING MATRIX A COEFICIENTS AND VECTOR  B
C
       READ(5,100) N
       READ(5,101) ((A(I,J),I=1,N),J=1,N),(B(I),I=1,N)
   100 FORMAT (I5)
   101 FORMAT (10F8.1)
       DO 15 J=1,N
       K=J+1
C
C      SETTLING MATRIX  A   FIRST COLUMN AND VECTOR   B
C
       DO 16 I=K,N
       A(I,J)=A(I,J)/A(J,J)
   16  B(I)=B(I)-A(I,J)*B(J)
C
C      SETTLING REDUCED (N-1)*(N-1) MATRIX  A   PART
C
       DO 17 L=K,N
       DO 18 M=K,N
   18  A(L,M)=A(L,M)-A(L,J)*(J,M)
   17  CONTINUE
   15  CONTINUE
C
C      CALCULATING UNKNOWN VECTOR   X
C
       C=0.
       DO 19 J=1,N
       L=N-J+1
       K=N-J
       X(L)=(B(L)-C)/A(L,L)
       C=0.
       IF(K-1) 1,2,2
   2   DO 20 M=L,N
   20  C=C+A(K,M)*X(M)
   19  CONTINUE
C
C      WRITING SOLUTION, VECTOR  X   VALUES
C
   1   WRITE(6,102) (X(J),J=1,N)
   102 FORMAT(1X,10F12.3,/)
       STOP
       END
```

3.5. Sparse Matrices

Matrix G, which describes an electrical network has a great percentage of matrix coefficients equal to zero. This large number of zeros should be eliminated so that the operation is done only with coefficients other than zero. The small number of non-zeros (NZ) is the result of the electrical network nature. Matrix G is constructed as follows :

$$5 - - - \begin{array}{|ccccccc|} 0 & -G_1 & 0 & -G_2 & (G_1+G_2+G_3+G_4) & 0 & -G_3 \end{array}$$
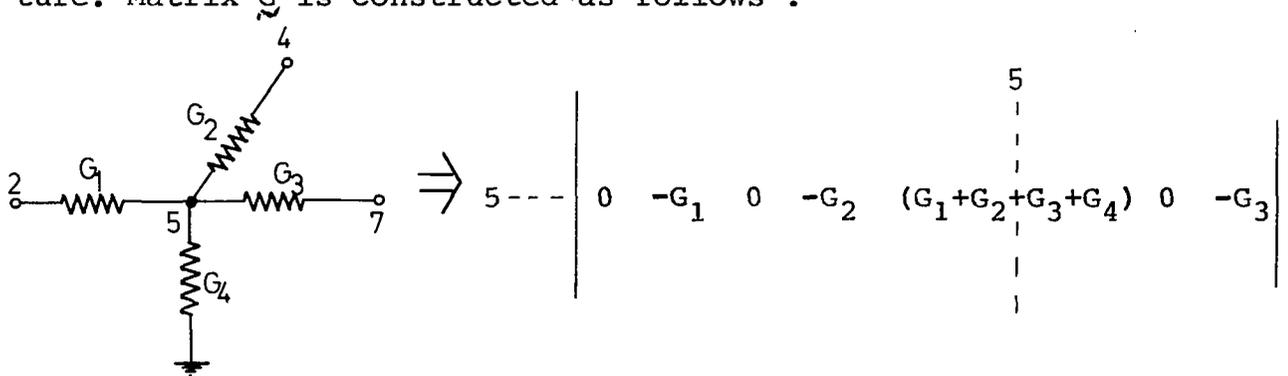
In one row there is a diagonal term and as many terms as the number of other nodes with which the observed node is connected (we do not count a connection to the referent node).

In the given example, the fifth row of matrix G has $1 + 3 = 4$ elements. Electrical networks generally have 2 to 6 branches connected to one node. Suppose that the average is 5 elements per row and 50 nodes. Then, matrix G will have $50 \times 5 = 250$ NZ elements and 2250 Z(zero) elements, i.e. of 2500 elements there will be 10% NZ and 90% Z. It is clear that such a great number of Z elements should be eliminated to obtain a more efficient calculation.

The second property is that matrix G is symmetrical regarding the main diagonal. Since matrix G does not contain any dependent current source, it is symmetrical. However, if the network contains dependent current sources, then there are some non-symmetrical terms too. This may be demonstrated by the Ebers-

Moll's transistor model (Fig. 10).
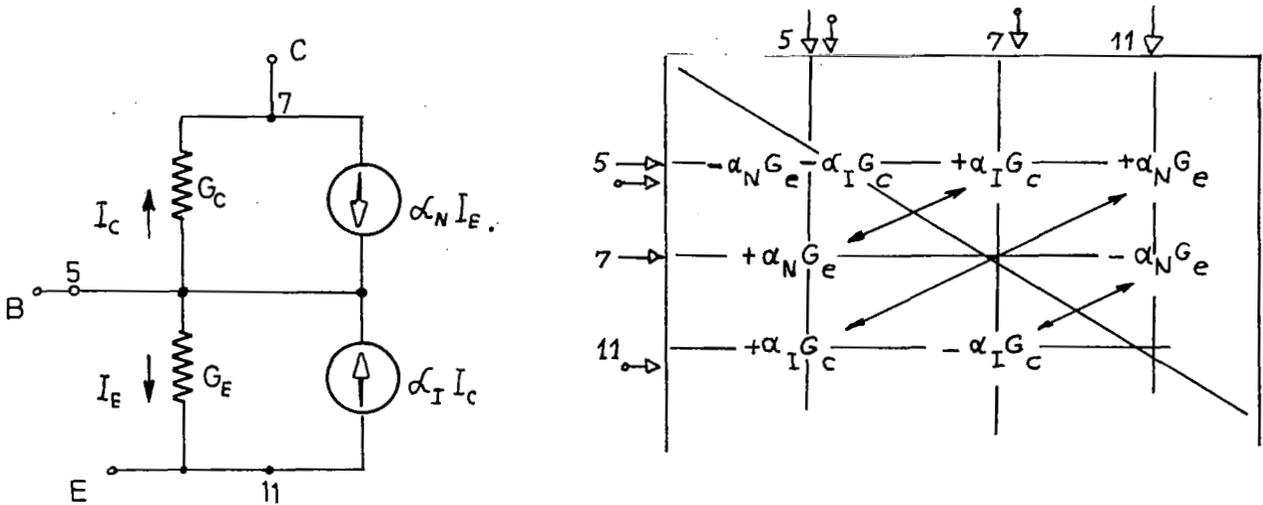
Fig. 10. Ebers-Moll's transistor model gives a symmetrical structure to the $\underset{\sim}{G}$ matrix.

In the given example every dependent source gives three non-symmetrical terms. It is fortunate that branch 5,7 and branch 5,11 are interdependent. In such cases, where there is a reciprocal dependence of two branches, the matrix is symmetrical; however, symmetrical coefficients do not have the same values. Thus when considering sparse matrices their symmetrical structure will be taken into account.

Thus there are over 90% zero elements, matrix $\underset{\sim}{G}$ has symmetrical structure, and the number of non zeros is different per given row. Gaussian elimination should be performed in a way such that it does not occupy too much of the memory, and that it does not require too many operations. Furthermore note the following : the matrix $\underset{\sim}{G}$ coefficient g_{ij} will be processed only when frontal coefficients in the row i, and column j, differ from zero, regardless of whether the coefficient $g_{ij} \neq 0$, or $g_{ij}=0$. In case $g_{ij}=0$ after processing, the element becomes $g_{ij} \neq 0$, so a new non-zero element is generated. The fact that Gaussian elimination

generates new non-zeros leads to the growing number of elements in matrix $\underset{\sim}{G}$ during the elimination. The purpose is therefore to arrange rows and columns so that a minimum of non-zeros is generated during the elimination. This will occupy the least amount of words in memory, and will require the least number of operations.

Since the Gaussian elimination procedure has been covered, storing and arranging of the matrix $\underset{\sim}{G}$ elements will be demonstrated.

    a) Matrix $\underset{\sim}{G}$ element storing

Non-zero elements are read by rows as follows :

row	column	value
1	1	x
	3	x
	7	x
2	2	x
	3	x
	7	x
	9	x
3	5	x
.	.	.
.	.	.
.	.	.

$R(1) = 1$ $C(1) = 1$ $A(1) = x$
$R(2) = 4$ $C(2) = 3$ $A(2) = x$
$R(3) = 8$ $C(3) = 7$
              $C(4) = 2$
              $C(5) = 3$
              $C(6) = 7$
              $C(7) = 9$
              $C(8) = 5$ $A(8) = x$

If matrix $\underset{\sim}{G}$ is n x n, and NZ is the number of non-zeros, then

      n words are needed for the row index, R
      NZ words are needed for the column index, C
      NZ words are needed for the element value, A

i.e. totaly n + 2NZ words. According to the given example with

n = 50 and 10% NZ elements, this gives 50 + 250 + 250 = 550 words (instead of 25oo)

b) Matrix $\underset{\sim}{G}$ arrangement

In general the matrix $\underset{\sim}{G}$ is arranged so that the row and column with the least elements is in the first row. The row and column with the most elements go in the last row. Rows and columns are used because in the matrix with symmetrical structure, row j and column j have the same number of elements.

Therefore, switching the row means that the column must also be switched to keep matrix $\underset{\sim}{G}$ structurally symmetrical.

After this first arrangement is done, there are several methods on how to proceed with the $\underset{\sim}{G}$ matrix rearrangement, one of which is given in reference (5). Let us see one of the possible methods whose principle is given in reference (6).

The top of the matrix $\underset{\sim}{G}$ has row with the least elements, say it has one off-diagonal element. This row does not generate new non-zeros. Regarding Fig. 11, after eliminating the first row and column, row k will lose 1 non-zero and will become a row with 1 off-diagonal element too. Thus row k, too, should be switched to the top of the matrix. In other words, after the first unknown is eliminated we treat the rest of the matrix $\underset{\sim}{G}$, i.e. (n-1)(n-1) matrix as follows: we scan (n-1) rows left and the row with the least elements is put on the top, and the second unknown is eliminated. If some new elements according to the elimination nature are generated they should be added into the proper rows. The same procedure is repeated for (n-2)(n-2) matrix and so on. At the end a row elimination order is obtained which has to be stored in a separate file. This file is used when the Gaussian elimination of the same matrix $\underset{\sim}{G}$ is to be repeated. This procedure gives very good results with respect to minimum number of operations and minimum number

of new generated elements (so called "fill-in"). The very minimum is not obtained and as far as is known such method has not yet been proposed at present.
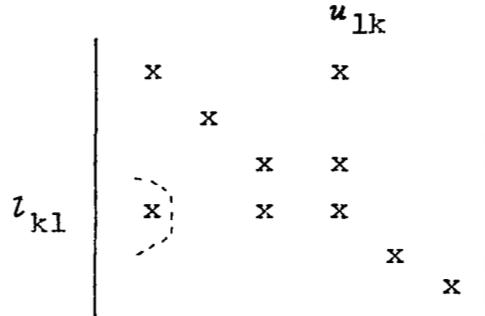

Fig. 11. *An example showing that row and column with two elements do not generate a new element.*

An example for matrix with 24 unknowns and an average of 4 elements per row (including diagonal term) is given. Matrix is generated randomly using the Monte Carlo Method (Fig. 12.a). Gaussian elimination result executed on the originally given matrix is shown on Fig. 12.b and the number of multiplications (and subtractions) is 452, number of division is 86 and number of fill-ins is 100. When the reordering scheme described above was applied, Gaussian elimination result is shown in Fig 12.c and the number of multiplications (and subtractions) is 131, number of divisions is 51 and number of fill-ins is 30. On the left column of Fig. 12.c the row sequence elimination is given.

a)

```
  123456789012345678901234
 1X.............X.........
 2.X........X.............
 3..X.X......X.......X...
 4...X..............X....
 5..X.XX....X...X..XX...X.
 6...XX......X........X...
 7.....X.........X......
 8.....X....X......X....
 9.......XX...X.X.......
10.......XX.....X...X....
11.X..XX....X....X.......
12.X........X...X.......
13......X....X.X........
14X......X...XX...X...X.
15...X......XXX.........
16.......XXX...X........
17..........X...XX.XXX..
18...X.X.........XXX..X..
19...XX.X......X...XX.X...
20.......X.....X..X....
21..XX..........X.X.X...
22..............XX...X..
23...X.......X.......XX
24................XX
```

b)

```
  123455789012345678901234
 1X..........X...........
 2.X.......X.............
 3..X.X......X.......X...
 4...X...............X...
 5.X.XX....XX..X..XX.X.X.
 6....XX...XX..X..XX.X.X.
 7.....X...........X...
 8.......X...X.....X...
 9.......XX...X.X.......
10.......XX..X.X...X....
11X..XX...XX..XX.XX.X.X.
12..X.XX...XX..XXXX.X.X.
13....X...X.X...X.....
14X.....XX...XX..XX..X.
15...XX...XXXXX XXXXX.X.
16.....X XXX.XX XXXXXX.X.
17.......X..XXXXX XXXX.
18...XXX..XX..X XXXX XXX.
19...XX.X.XXXXX XXXXX XX.
20......X...XX XXXX XXX.
21.XXXX...XX..XXXXX XXX.
22...........XXXXXX XX.
23...XX...XX.XX XXXX XXXX.
24.....................XX
```

c)

```
  124748230233654681799015
 1X...........X.........
 2.X..............X....X.
 4..X..............X.....
 7..X...............X....
24....X.....X............
 8....X.X.............X..
12.....X....X.......X...
13....X.X....X......X...
20......X........X..X..
22.....X.......X.X....
23...X....X...X.........X
 3.....X...X....XX....X
 6..........X.......X.XX
15......X...XX.....X..X
14X.......X..XX...XX..X
16.................X..X.XX.
18..X....X.....X.X.X..X
21..X........X....XX.X..X
17.....X.XX.X.....XXX.XX.X
 9...........XX...XXXXX
19.....X.X...XXX.XXXXXXXX
10........X.....X..XXXXX
11X..........X..X...XXXX
 5.......XXXX.XXXXXXXX
```

Fig. 12. a) Randomly generated 24x24 matrix with an average of 4 nonzero elements per row,

b) Gaussian elimination result executed on the given matrix,

c) Gaussian elimination result after reordering.

4. SENSITIVITY COEFFICIENTS

After the node voltages are defined, it is often necessary to calculate the sensitivity of an individual voltage to the network parameter changes. A sensitivity analysis is usually done for only one node, i.e. the output voltage. Here, the described procedure concerns the **linear** networks and is designed for use of the total differential. Let $V_o = f(x_1, x_2, \ldots x_n)$ be an output voltage; x_i supplies the network parameters - resistors and current sources. The total differential from V_o is

$$dV_o = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n .$$

The solution requires finding the differential dV_o, but from the matrix term for the network.

The network is described by the term

$$\underset{\sim}{A} \underset{\sim}{G}_D \underset{\sim}{A}^t \underline{V}_n = \underset{\sim}{A} \underline{I}_s$$

and the total differential is equal to

$$\underset{\sim}{A} dG_D \underset{\sim}{A}^t \underline{V}_n + \underset{\sim}{A} \underset{\sim}{G}_D \underset{\sim}{A}^t d\underline{V}_n = \underset{\sim}{A} d\underline{I}_s$$

which, with $\underset{\sim}{G} = \underset{\sim}{A} \underset{\sim}{G}_D \underset{\sim}{A}^t$, follows for $d\underline{V}_n$

$$d\underline{V}_n = \underset{\sim}{G}^{-1} \underset{\sim}{A} d\underline{I}_s - \underset{\sim}{G}^{-1} \underset{\sim}{A} dG_D \underset{\sim}{A}^t \underline{V}_n ,$$

$$d\underline{V}_n = \left| G^{-1} A \right| \left| d\underline{I}_s - dG_D A^t \underline{V}_n \right| .$$

Let the matrix $\underset{\sim}{G}^{-1} \underset{\sim}{A} = \underset{\sim}{H}$ and $\underset{\sim}{A}^t \underline{V}_n = \underline{V}_b$. The upper term is written in the extended form (n is the number of nodes, and b is the number of branches) as:

$$\begin{vmatrix} dV_1 \\ \cdot \\ \cdot \\ \cdot \\ dV_r \\ \cdot \\ dV_n \end{vmatrix} = \begin{vmatrix} h_{11} \cdots h_{1j} \cdots h_{1b} \\ \cdot \\ \cdot \\ h_{r1} \cdots h_{r1} \cdots h_{rb} \\ \cdot \\ h_{n1} \cdots h_{nj} \cdots h_{nb} \end{vmatrix} \begin{vmatrix} dI_{s1} - V_{b1}\, dG_1 \\ dI_{sj} - V_{bj}\, dG_j \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ dI_{sb} - V_{bb}\, dG_b \end{vmatrix}$$

Matrix $\underset{\sim}{H}$ represents the sensitivity coefficients matrix, because

$$S_{I_{rj}} = \frac{dV_r}{dI_{sj}} = h_{rj} \; .$$

$$S_{G_{rj}} = \frac{dV_r}{dG_j} = -h_{rj}\, V_{bj} \; .$$

$S_{I_{rj}}$ is the voltage sensitivity of the the r-th node in relation to a current source in the branch j. $S_{G_{rj}}$ is the voltage sensitivity of the r-th node in the relation to the branch j conductivity.

To find the matrix $\underset{\sim}{H} = \underset{\sim}{G}^{-1} \underset{\sim}{A}$, since matrices $\underset{\sim}{G}$ and $\underset{\sim}{A}$ have been done, the only problem is the matrix $\underset{\sim}{G}$ inversion which requires approximately n^3 operations. The multiplying of matrix $\underset{\sim}{G}^{-1}$ by $\underset{\sim}{A}$ and other operations is not a big burden for the computation. To determine the sensitivity coefficients of one output voltage (for example node V_r voltage), there is no need to calculate the entire coefficient matrix $\underset{\sim}{H}$, but only its row r, because:

$$dV_r = h_{r1}\,(dI_{s1} - V_{b1}\, dG_1) + h_{r2}\,(dI_{s2} - V_{b2}\, dG_2) + \ldots + h_{rb}\,(dI_{sb} - V_{bb}\, dG_b) \; .$$

Before the above mentioned $\underset{\sim}{H}$ matrix r-th row is obtained, the r-th row of the matrix $\underset{\sim}{G}^{-1}$ must be done. The previously described matrix inversion procedure, using Gaussian elimination (see page 29), gives one column per one inversion step. To calculate the

sensitivity coefficients of one row, the matrix $\underset{\sim}{G}$ inversion should be executed by rows. This is possible and is done with the help of the following relation:

$$(G^t)^{-1} = (G^{-1})^t .$$

The purpose of this relation is to invert the transposed matrix $\underset{\sim}{G}$ by the usual procedure, i.e. columns. After this, the inversed matrix $\underset{\sim}{G}^t$ column is equal to the inversed matrix $\underset{\sim}{G}$ row. Thus the sensitivity coefficients of the desired output node voltage are obtained by $\frac{n^3}{3}$ operations.

The sensitivity application and its connection with the relative error of a very simple example is as follows (Fig. 13):

$V_s = 10$ V,

$R_1 = 10$ Ω $G_1 = 0.1$ 1/Ω

$R_2 = 5$ Ω $G_2 = 0.2$ 1/Ω
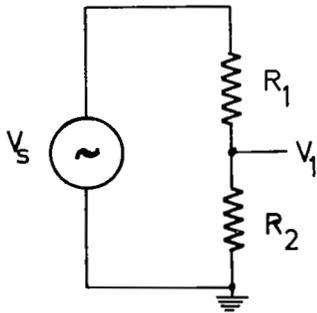
resistor tolerance $\pm$ 10%.

Fig. 13. Sensitivity computation example.

Voltage V_1 sensitivity to the parameter G_1 is

$$S_{G_{11}} = \frac{dV_1}{dG_1} ,$$

$$V_1 = V_o \frac{G_1}{G_1+G_2} = 10 \frac{0.1}{0.3} = 3{,}33 \text{ V} ,$$

$$S_{G_{11}} = \frac{dV_1}{dG_1} = V_o \frac{G_2}{(G_1+G_2)^2} = 10 \frac{0.2}{(0.3)^2} = 22{,}2 .$$

If the resistor R_1 changes for + 10%, then G_1 changes from 0.1

to 0.091 so $G_1 = 0.009 \; 1/\Omega$, and the output voltage V_1 difference is

$$\Delta V_1 = S_{G_{11}} \; \Delta G_1 = 22.2 \; 0.009 = 0.202 \text{ V}$$

Working only with the relative errors (tolerances), the term is arranged like this :

$$\frac{\Delta V_1}{V_1} = \frac{S_G}{V_1} \; \Delta G_1 \; \frac{G_1}{G_1} = \frac{\Delta G_1}{G_1} \left| S_{G_1} \; \frac{G_1}{V_1} \right|$$

where

$$\frac{\Delta V_1}{V_1} = P_{V1}, \text{ and } \frac{\Delta G_1}{G1} = P_{G1}$$

$$P_{V1} = \left| S_{G1} \; \frac{G_1}{V_1} \right| P_{G1} ,$$

which gives

$$P_{V1} = \left| 22,2 \; \frac{0.1}{3,33} \right| P_{G1} = 0.667 \; P_{G1} .$$

This means that if R_1 changes by 10%, the voltage V_1 will change by 6.06%, and in total this amount gives $V_1 P_{V1} = 3.33 \cdot 0.0606 = 0.202$ V.

The observed change of value V_1 and the relative change P_{V1} are obtained from the same sensitivity coefficient, the only difference is in the form in which the solution is given.

5. NONLINEAR DC NETWORKS

The nonlinear network, other than the linear elements, contains nonlinear elements such as diodes and transistors. A nonlinear element cannot be directly presented in matrix $\underset{\sim}{G}$ with a constant term. So, in the nonlinear network the operating point of some nonlinear element is defined by Newton-Raphson's method. The method consists of the following: to find an intersection of function $y = f(x)$ (Fig. 14) with the abcissa, an x_1 is chosen and $y_1 = f(x_1)$ is defined (point A).
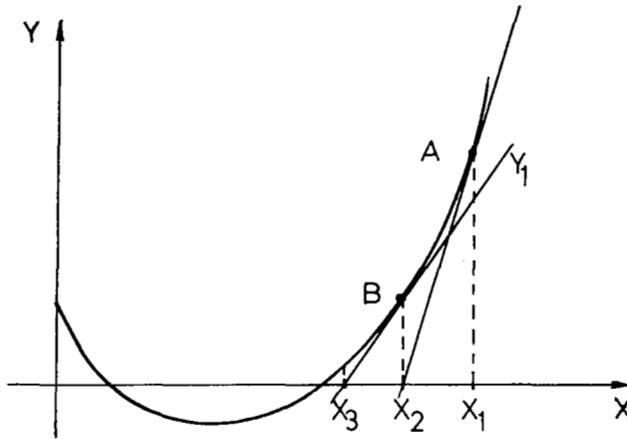

Fig. 14. Example of Newton-Raphson's method.

A tangent $y' = f'(x_1)$, is drawn through point A. The intersection of the tangent and abcissa is x_2. The procedure is repeated until either y or the absolute difference $(x_2 - x_1) = \Delta x$ becomes as small as is required.

The procedure is now adjusted for analyzing nonlinear electrical networks with a larger number of elements. It is understandable that in a complicated network, gradually approaching the solution cannot be illustrated graphically. The entire procedure will be

described by the example of a series connection between a diode and resistor (Fig. 15) and then applied to any network.

$$I_S \left(e^{\frac{V_D}{V_r}} - 1 \right)$$
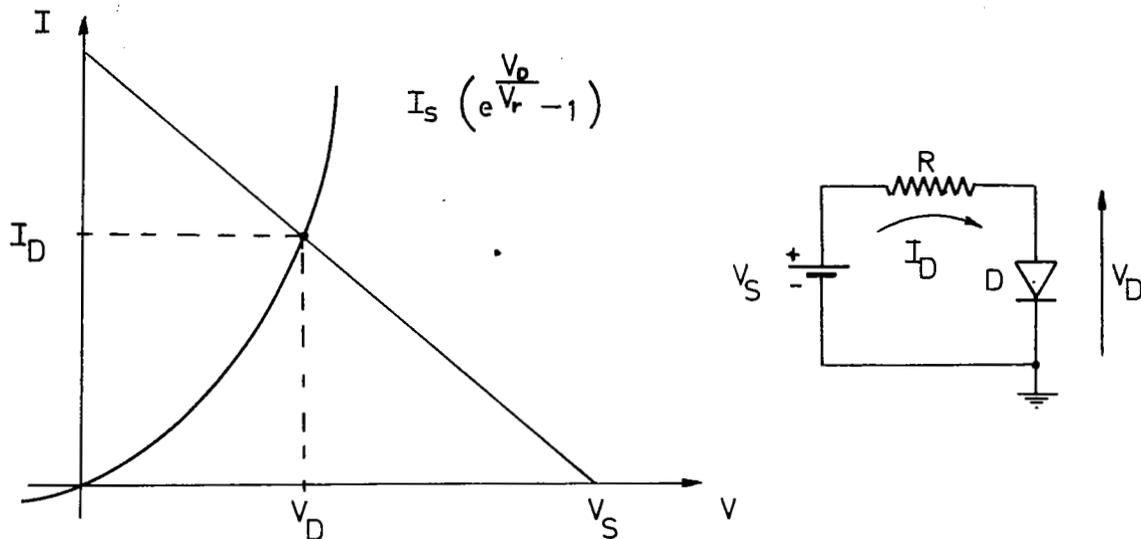
Fig. 15. *A simple nonlinear network.*

The given network can be calculated by the algebraic procedure only if it consists of linear elements. Thus the diode must be linearized (Fig. 16).
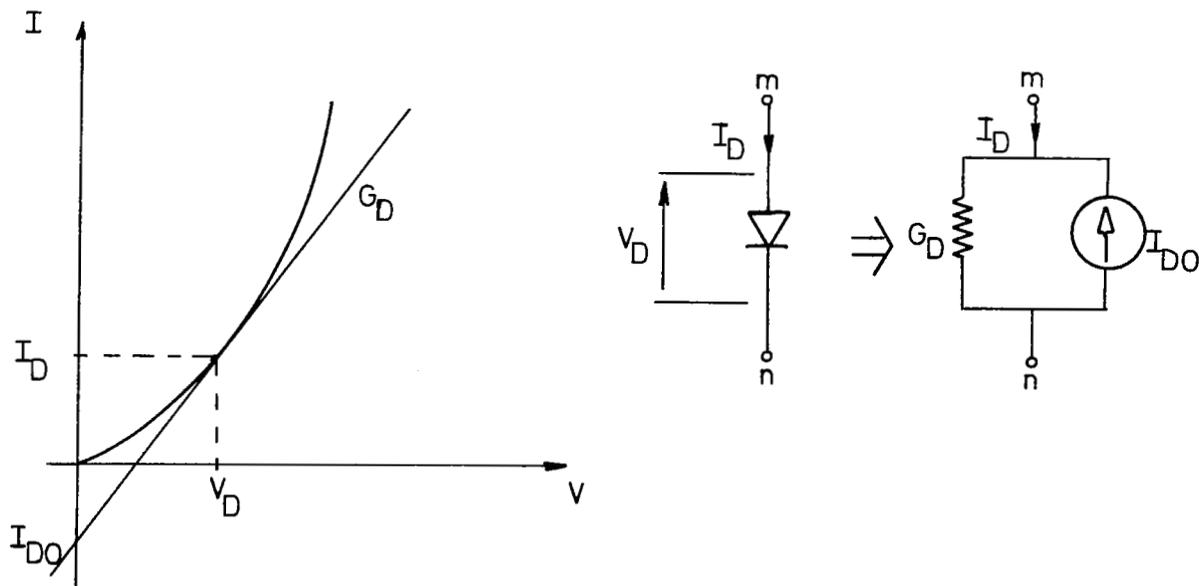

Fig. 16. *Diode presentation using the linear model.*

With a known diode voltage,

$$I_D = I_S \ (e^{\frac{V_D}{V_T}} - 1) \ , \quad V_T = \frac{kT}{q} \simeq 26 \text{ mV}$$

$$G_D = \frac{dI_D}{dV_D} = (I_S/V_T) \ e^{\frac{V_D}{V_T}} = (I_D + I_S)/V_T \ .$$

Through the point I_D, V_D a line with the slope G_D is drawn:

$$(I - I_D) = G_D \ (V - V_D) \ ,$$

$$I = G_D \ (V - V_D) + I_D, \quad \text{if} \quad V=0 \quad \text{we have}$$

$$I_{DO} = I_D - G_D \ V_D \quad \text{or}$$

$$I_{DO} = G_D * (V_T - V_D) - I_S = G_D \ V_T \ (1 - \ln \frac{I_D + I_S}{I_S}) - I_S \ .$$

Between nodes m and n, instead of the diode, the conductance G_D and the current source I_{DO} are connected.

In the example of Fig. 16, the diode initial values are I_D, V_D, (Fig. 17.a).

The line which corresponds to conductance G_D intersects the conductance line, G, at voltage V_D'. The current is calculated by $I_D' = f(V_D')$, a tangent is drawn through the new point on the diode's characteristic, and the intersection with line G is found.

The procedure is repeated until voltage $\Delta V = (V_D' - V_D)$ is within the selected error limits. Voltage ΔV is the tolerance (usually 50 µV) because it is not possible to calculate the voltage V_D exactly.
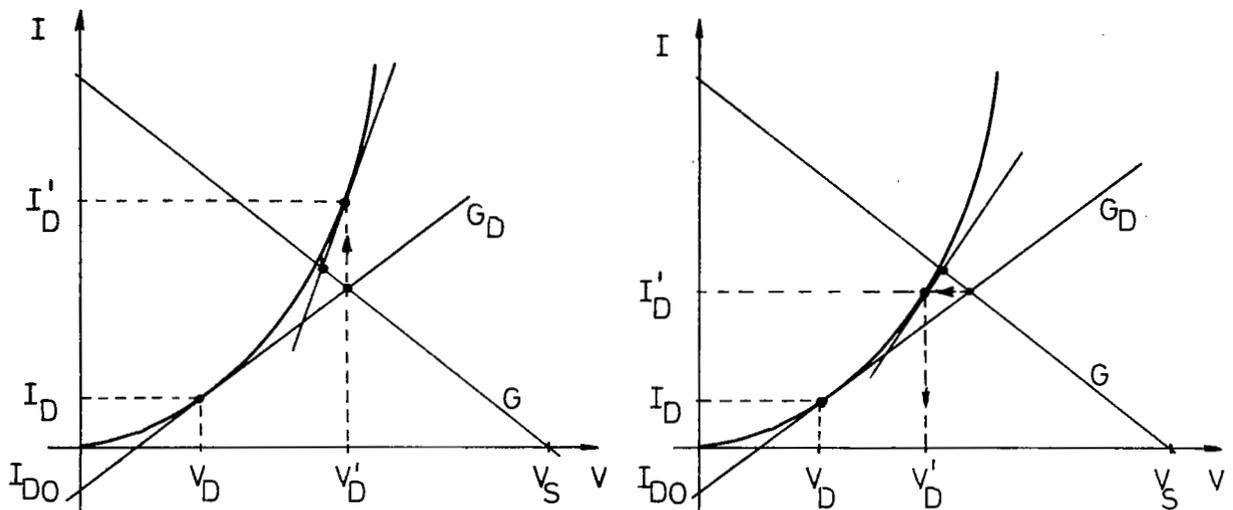
$$\Delta V = |V_D^{'} - V_D| \qquad\qquad \Delta V = |V_D^{'} - V_D|$$

$$G_D = \frac{I_S}{V_T} e^{\frac{V_D}{V_T}} \qquad\qquad G_D = (I_D + I_S)/V_T$$

$$I_{DO} = G_D (V_T - V_D) - I_S \qquad I_{DO} = G_D V_T (1 - \ln \frac{I_D + I_S}{I_S}) - I_S$$

a)   b)

Fig. 17. Newton-Raphson's method application
example.

In a complex network, in place of each nonlinear element, the conductance G_D is placed parallel to the current source I_{DO}. The system is solved using the node voltages $V_1 \div V_n$. From the node voltages the nonlinear element's (branch) voltages are calculated using $\underline{V}_b = \underline{\underline{A}}^t \underline{V}_n$. After the value V_b, i.e. V_D, for every single nonlinear element is obtained, tolerance voltage $\Delta V = (V_D^{'} - V_D)$ is calculated and, if it is within the 50 μV tolerance the procedure is completed. If not, voltage V_D for each nonlinear element whose difference voltage ΔV is not whitin the given tollerance is set **equal** to voltage $V_D^{'}$ and new values G_D and I_{DO} are calculated

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and inserted into the network. The procedure is repeated until the difference voltage ΔV for every single nonlinear element is less than 50 μV.

Electrical networks nonlinear elements are diodes and transistors that are modeled using diodes. Because of this we have to notify some exponential function e^{V_D/V_T} characteristics. During the elimination, some difficulties can arise in the calculations if $V_D^{'}$ and V_D (new and old diode voltage values) differ by more than $2V_T$, i.e. $(V_D^{'} - V_D) > 2V_T$. This is explained in the following example:

$$I_D = I_S \, e^{\frac{V_D}{V_T}} \, , \qquad I_D^{'} = I_S \, e^{\frac{V_D^{'}}{V_T}} = I_S \, e^{\frac{V_D + 2V_T}{V_T}} \, ,$$

Then
$$I_D^{'}/I_D = e^2 = 7,8 \; .$$

Due to the exponential dependency, a new current, $I_D^{'}$, is eight times larger than the current I_D. Such a large difference is undesirable and therefore the growth of the diode voltage is restricted to $2VT = 52$ mV, i.e. $V_D^{'} = V_D + 2V_T$ (instead of the calculated voltage $V_D^{'}$)

A modification of Newton-Raphson's method provides better answers to the pecularities of the diode characteristics. This is shown in Fig. 17.b.

After obtaining the solution the nonlinear element current is calculated using

$$\underline{I} = \underset{\sim}{G}_D \, \underline{V}_b = \underset{\sim}{G}_D \, \underset{\sim}{A}^t \, \underline{V}_n, \quad \text{i.e. } \underline{I}_D^{'} \text{ and, using these currents,}$$

the new values G_D and I_{DO} are calculated. By this procedure, the solution with fewer iterations is obtained.

6. LINEAR AC NETWORKS

Before beginning linear AC network analysis explanation of how
any nonlinear network may be linearized must be given. Generally,
networks are nonlinear because, in addition to R, L, C, elements
they also contain nonlinear and active elements; diodes and tran-
sistors. In this type of analysis, nonlinear elements must be
linearized about the operating point and the values inserted
into the network. In the diode linearization example, the tangent
of the operating point according to the known term is calculated
as shown:

$$G_D = \frac{dI_D}{dV_D} = \frac{I_S}{V_T} e^{\frac{V_D}{V_T}} = (I_D + I_S)/V_T \ .$$
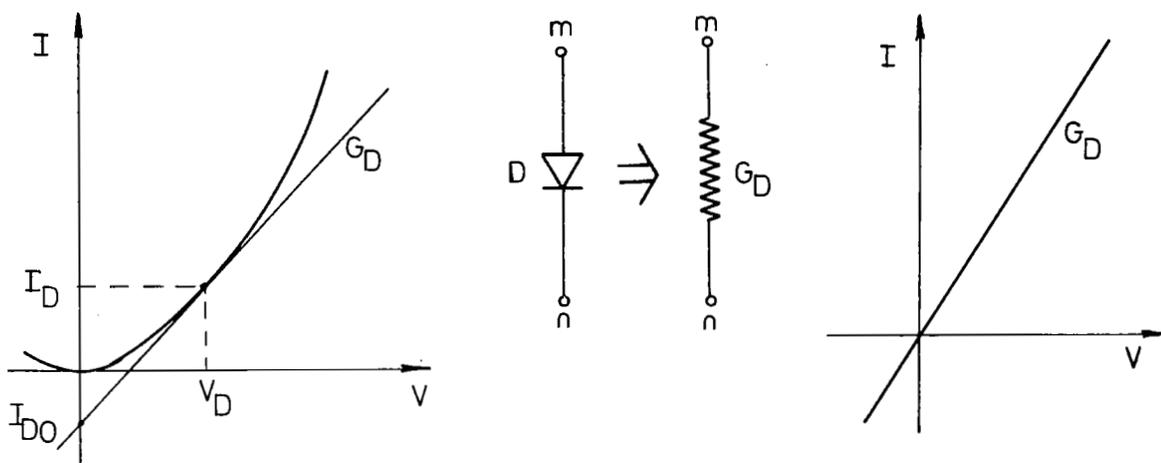
*Fig. 18. Diode linearization for AC
analysis.*

DC analysis of a nonlinear network requires the diode model
according to Fig. 16, while AC analysis needs only the conductance,
G_D (Fig. 18). Once we have performed DC analysis, all models of
nonlinear elements needed for AC analysis are existing.

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After the DC state is defined, all nonlinear elements are introduced into matrix $\underset{\sim}{G}$ with adequate conductances, G_D, and current sources, I_{DO}. Conductances, G_D, are kept in the network while current sources I_{DO} are removed. After this procedure, linear AC analysis may begin.

Generally the matrix terms of DC analysis are used. This provides the admittance matrix $\underset{\sim}{Y}$, instead of the conductance matrix $\underset{\sim}{G}$, and voltage $\underline{V}_n$ and current sources $\underline{I}_s$ vectors are complex. So, we write :

$$\underset{\sim}{Y} \, \underline{V}_n^* = \underline{I}_{sn}^* \, . \quad ^{1)} \tag{27}$$

where

$$\underset{\sim}{Y} = |G_{ij} + j\, B_{ij}| \; , \quad \underline{V}_n^* = \underline{V}_n + j\, \underline{\overline{V}}_n \; ,$$

$$\underline{I}_{sn}^* = \underline{I}_{sn} + j\, \underline{\overline{I}}_{sn} \; .$$

The upper term could be used as it is, in which case the first row of term (27) will be :

$$(G_{11}+j\, B_{11})(V_1+j\, \overline{V}_1) + (G_{12}+j\, B_{12})(V_2+j\, \overline{V}_2)+ \; \ldots \; = I_{sn1}+j\, \overline{I}_{sn1} \, . \tag{28}$$

This form is inconvenient for the work on the computer, so it must be rearrenged. First, the standard branch for AC networks is defined (Fig. 19)

---

1)

Mark V* represents the voltage having the real component V and the imaginary component $\overline{V}$.
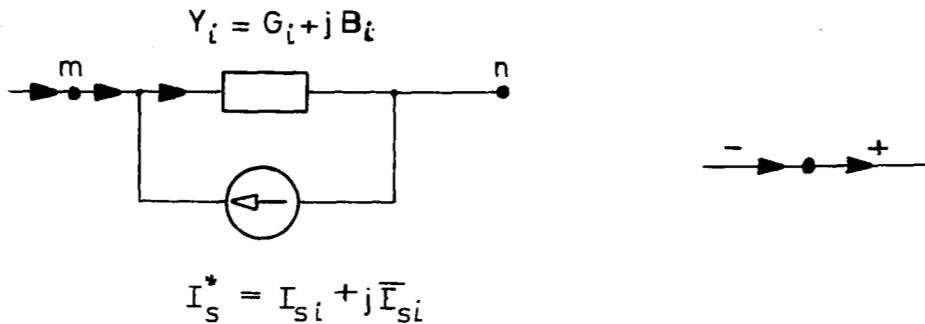
Fig. 19. A standard branch for AC networks.

A diagonal admittance matrix is

$$\underset{\sim}{Y}_D = \left| G_{ii} + j\, B_{ii} \right|$$

and can be presented as the sum of matrices $\underset{\sim}{G}_D$ and $\underset{\sim}{B}_D$ as follows:

$$\underset{\sim}{Y}_D = \underset{\sim}{G}_D + j\, \underset{\sim}{B}_D \; .$$

Now, the procedure for obtaining the admittance matrix can be applied usnig the diagonal admittance matrix, i.e.

$$\underset{\sim}{Y} = \underset{\sim}{A}\, \underset{\sim}{Y}_D\, \underset{\sim}{A}^t = \underset{\sim}{A}\, (\underset{\sim}{G}_D + j\, \underset{\sim}{B}_D)\, \underset{\sim}{A}^t =$$

$$= \underset{\sim}{A}\, \underset{\sim}{G}_D\, \underset{\sim}{A}^t + j\underset{\sim}{A}\underset{\sim}{B}_D\, \underset{\sim}{A}^t = \underset{\sim}{G} + j\, \underset{\sim}{B} \; .$$

(29)

This term implies that the admittance matrix can be constructed from two matrices, $\underset{\sim}{G}$ and $\underset{\sim}{B}$, where each one is calculated independently using the incidence matrix $\underset{\sim}{A}$. Thus, matrix $\underset{\sim}{G}$ is constructed using the previously established rule (see page 8) without regard to reactive branches of the network. Matrix $\underset{\sim}{B}$ is constructed using the same rule without regard to resistive branches of the network. Using this procedure, term (27) can be written as:

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$$(\underset{\sim}{G} + j \underset{\sim}{B})(\underline{V}_n + j \underline{V}_n) = (\underline{I}_{sn} + j \overline{\underline{I}}_{sn}), \tag{30}$$

or

$$\underset{\sim}{G} \underline{V}_n + j \underset{\sim}{B} \underline{V}_n + j \underset{\sim}{G} \underline{V}_n - \underset{\sim}{B} \underline{V}_n = \underline{I}_{sn} + j \overline{\underline{I}}_{sn} . \tag{31}$$

The real part is equalized with the real one and the imaginary with the imaginary one:

$$\underset{\sim}{G} \underline{V}_n - \underset{\sim}{B} \overline{\underline{V}}_n = \underline{I}_{sn}$$

$$\underset{\sim}{B} \underline{V}_n + \underset{\sim}{G} \overline{\underline{V}}_n = \overline{\underline{I}}_{sn}$$

Which gives the matrix term:

$$
\begin{vmatrix} \underset{\sim}{G} & -\underset{\sim}{B} \\ \\ \underset{\sim}{B} & \underset{\sim}{G} \end{vmatrix}
\begin{vmatrix} \underline{V}_n \\ \\ \overline{\underline{V}}_n \end{vmatrix}
=
\begin{vmatrix} \underline{I}_{sn} \\ \\ \overline{\underline{I}}_{sn} \end{vmatrix}
\tag{32}
$$

The upper term is used directly for solving the linear AC networks. The first row, written more extensively, reads:

$$G_{11} V_1 + G_{12} V_2 + \ldots + G_{1n} V_n - B_{11} \overline{V}_1 - B_{12} \overline{V}_2 \ldots - B_{1n} \overline{V}_n = I_{sn1}$$

$$\tag{32.a}$$

This form is much more suitable than that of term (28).

In term (32) note that the number of unknowns is two times larger than the number of nodes because of the fact that all node voltages have real and imaginary components. Matrix $\underset{\sim}{Y}$ is four times larger than matrix $\underset{\sim}{G}$ because instead of (n x n) elements it contains (2n x 2n) elements. Thus, the calculation is also larger.

Applying the described procedure on the network in Fig. 20. matrices $\underset{\sim}{G}$ and $\underset{\sim}{B}$ are written as:

$$G = \begin{vmatrix} G_1 & -G_1 & 0 \\ -G_1 & G_1+G_2 & 0 \\ 0 & 0 & G_4 \end{vmatrix} \qquad B = \begin{vmatrix} 0 & 0 & 0 \\ 0 & B_2+B_3 & -B_3 \\ 0 & -B_3 & B_3 \end{vmatrix} ,$$
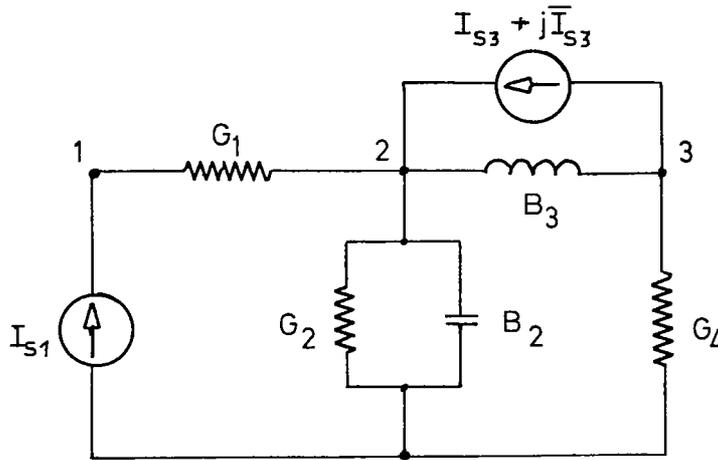

Fig. 20. The linear AC network example.

where $B_2 = \omega C_2$, $B_3 = \dfrac{-1}{\omega L_3}$,

$$I_1 = I_{10} \sin \omega t , \qquad I_3 = I_{30} \sin (\omega t + \phi).$$

The entire matrix term which describes the network is:

$$\begin{vmatrix} G_1 & -G_1 & 0 & 0 & 0 & 0 \\ -G_1 & G_1+G_2 & 0 & 0 & -B_2-B_3 & B_3 \\ 0 & 0 & G_4 & 0 & B_3 & -B_3 \\ \hline 0 & 0 & 0 & G_1 & -G_1 & 0 \\ 0 & B_2+B_3 & -B_3 & -G_1 & G_1+G_2 & 0 \\ 0 & -B_3 & B_3 & 0 & 0 & G_4 \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \\ V_3 \\ \hline \overline{V}_1 \\ \overline{V}_2 \\ \overline{V}_3 \end{vmatrix} = \begin{vmatrix} I_{s1} \\ I_{s3} \\ -I_{s3} \\ \hline 0 \\ \overline{I}_{s3} \\ -\overline{I}_{s3} \end{vmatrix} ,$$

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or, for the first two rows:

$$G_1V_1 - G_1V_2 \qquad\qquad = I_{s1}$$

$$= G_1V_1 + (G_1+G_2)V_2 - (B_2+B_3)\,\bar{V}_2 + B_2\,\bar{V}_3 = I_{s3} \quad , \text{ etc.}$$

Here, the analysis is performed at the chosen frequency. If we change the frequency, the matrix G remains the same, while the matrix B changes. The terms of matrix B are of the form ωC, $\frac{1}{\omega L}$ or their sum and they change with the frequency.

This means that for every frequency value, matrix B elements must be recalculated and inserted into term (32), and new values for node voltages $V_n = \underline{V}_n + j\,\bar{\underline{V}}_n$ are calculated. Regarding the fact that the observed frequency area is usually large, the frequencies are shown in logarithmical scale. The frequencies change as $f_n = 10^{nx}\, f_o$, where x is calculated from the number of observed frequencies per decade. For example, if 4 frequencies per decade in the same logarithmical increments are desired, then $4x = 1$, $x = 0.25$ so

$$f_o = 10^0\, f_o \qquad = 1.000 \quad f_o$$

$$f_1 = 10^{0.25}\, f_o = 1.780 \quad f_o$$

$$f_2 = 10^{0.50}\, f_o = 3.165 \quad f_o$$

$$f_3 = 10^{0.75}\, f_o = 5.630 \quad f_o$$

$$f_4 = 10^{1.} \quad f_o = 10.000 \quad f_o \quad , \text{ etc.}$$

Input and output voltages and currents, $(V+j\bar{V})$ and $(I+j\bar{I})$, can in

some programs be changed to $\sqrt{v^2 + \bar{v}^2}\ \underline{/\Phi_v}$ and $\sqrt{I^2 + I^2}\ \underline{/\Phi_i}$.
Thus by changing the input signal frequency as a result of AC
analysis, the amplitude and phase network characteristics in
any node are obtained.

For dependent current sources, and voltage sources, the principle
described for the DC analysis is used, except that consideration
should be given to their complex nature.

7. TRANSIENT ANALYSIS OF NONLINEAR NETWORKS

In this chapter a description is given of the procedure for analyzing the nonlinear electrical networks which, in addition to the resistance R, contain capacitance C, and inductance L. All three kinds of elements, R, L, C can be nonlinear. The input signal is the time function. Thus the transient analysis of nonlinear networks will be discussed.

Up to this point, a static analysis of nonlinear networks has been examined. This means that only the resistor, linear or nonlinear, was included. Next, the networks with constant capacitances C and inductances L were included, as well as linearized nonlinear elements and the input signal was sinusoidal.

First, the transient analysis of linear networks is described, the solution stability relating to that is examined, and the networks which contain the nonlinear C and L elements are analyzed. Finally, the nonlinear resistances are added to the network.

7.1. Transient Analysis of Linear Networks

Suppose that Fig. 21 represents one part of a given network.
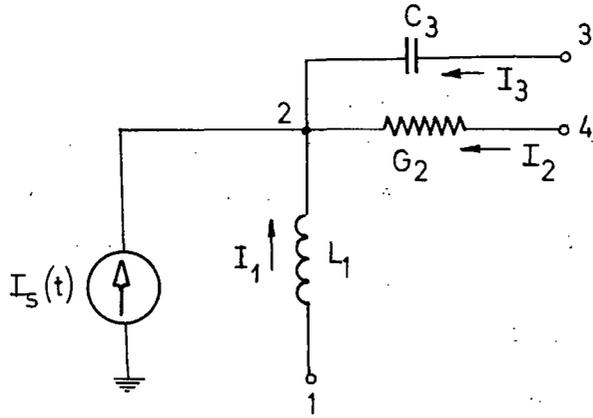
Fig. 21. A part of a linear network.

For node 2, Kirchhoff's nodal analysis implies:

$$- I_1 - I_2 - I_3 = I_s(t).$$

In analyzing the network by the nodal method, as earlier stated, all unknown currents are presented by node voltages:

$$I_L = \frac{1}{L} \int_0^t U_L \, dt \rightarrow I_1 = \frac{1}{L_1} \int_0^t (V_1 - V_2) \, dt \quad ,$$

$$I_C = G \, V_G \qquad \rightarrow I_2 = G_2 (V_4 - V_2) \quad , \tag{33}$$

$$I_C = C \frac{d \, U_C}{dt} \qquad \rightarrow I_3 = C_3 \frac{d(V_3 - V_2)}{dt} \quad ,$$

which, when inserted into the equation of the current balance in node 2, yields:

$$- \frac{1}{L_1} \int_0^t (V_1 - V_2) dt - G_2(V_4 - V_1) - C_3 \frac{d(V_3 - V_2)}{dt} = I_s(t) \ .$$

This simple example shows that the old system of algebraic equations, $\underset{\sim}{G} \ \underline{V}_n = \underline{I}_s$, will change to a system of integral - differential equations. Using the computer the analytical solution is unobtainable; this means that the integral-differential equation must be changed into an algebraic one and numerical method should be applied. The mathematical approach that was used for solving the resistance networks is also employed for solving dynamic (linear and nonlinear) electrical networks. Solving the differential equations is called integration. There are many numerical integration methods. Three of the most common **are** described here. When choosing the integration method, the one which is simple enough for the application and which gives a stable solution should be chosen. The first of the three methods is Euler's method, which is often unstable but it is the least complex. This method is used to describe the principle of the differential equation's numerical integration.

For an example of Euler's method, refer to Fig. 22.
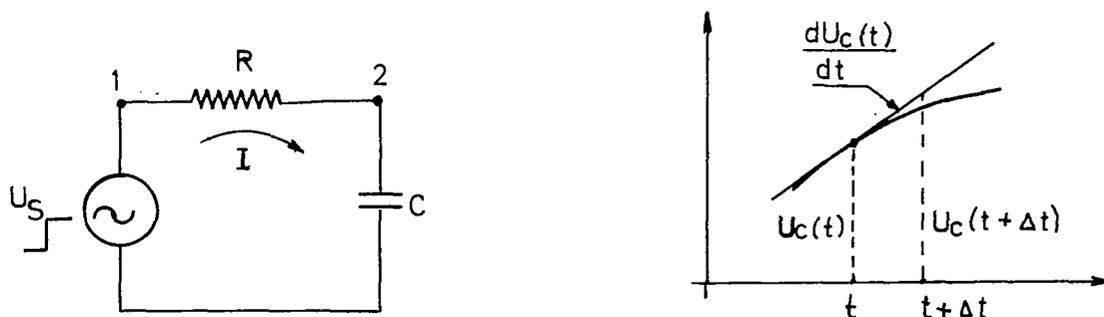
Fig. 22. Simple RC circuit.

The mesh equation is:

$$U_s = I\,R + \frac{1}{C} \int_0^t I\,dt \; , \qquad \text{or}$$

$$U_s = RC\,\frac{dU_C}{dt} + U_C \quad . \tag{34.a}$$

Equation (34.a) is the differential equation which connects the variable U_C and its derivative dU_C/dt at any time, t. If the signal voltage U_s and the voltage on the capacitor U_C are known, the voltage derivative U_C can be calculated from equation (34.a) as follows:

$$\frac{dU_C}{dt} = \frac{U_s}{RC} - \frac{U_C}{RC} \quad . \tag{34.b}$$

Now, the variable U_C value in time $t+\Delta t$ is calculated using Euler's method:

$$U_C(t+\Delta t) = U_C(t) + \Delta t\,\frac{dU_C(t)}{dt} \quad . \tag{35}$$

The voltage U_C curve is approximated by the tangent. The approximation becomes more accurate as the time interval Δt decreases. After the calculation, the new voltage value U_C is inserted in term (34.b) and the new derivative value is calculated. This continues until the end of the given time interval is reached. The error which appears according to term (35) can be calculated by using Taylor's series:

$$U_C(t+\Delta t) = U_C(t) + \Delta t\,U_C'(t) + \frac{\Delta t^2}{2!}\,U_C''(t) + \frac{\Delta t^3}{3!}\,U_C'''(t) + \ldots \tag{36}$$

The error is, as is shown in the comparison of terms (35) and (36), equal to:

$$\frac{\Delta t^2}{2!} U_C''(t) + \frac{\Delta t^3}{3!} U_C'''(t) + \dots\dots$$

Thus it is proportional to $\frac{\Delta t^2}{2!} U_C''(t)$. To avoid forming differential equations, and to keep the mathematical procedure that was used for resistive networks, equation (35) is written as:

$$U_{n+1} = U_n + \Delta t \frac{dU_n}{dt} ,$$

$$U_{n+1} = U_n + \frac{\Delta t}{C} I_n . \qquad (37)$$

An equivalent circuit for the capacitance C following from equation (37), is shown in Fig. 23 which is a voltage source.
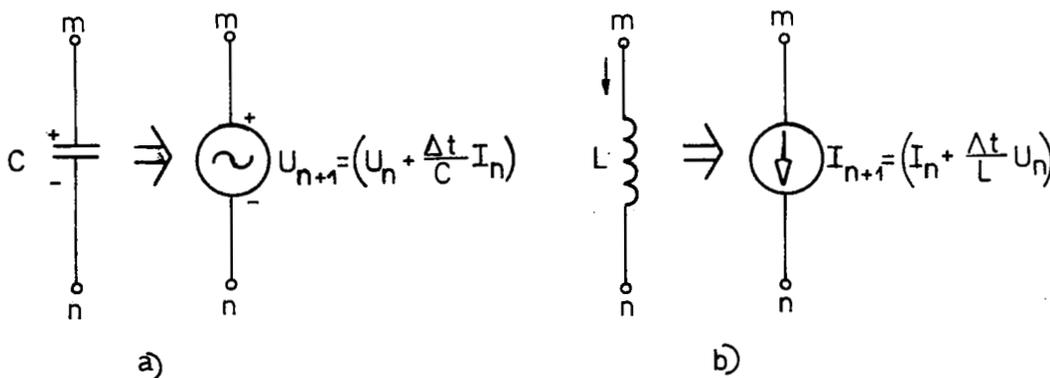

Fig. 23. a) An equivalent circuit for the capacitance C,
b) An equivalent circuit for the inductance L,
according to Euler's formula.

A calculation of the example in Fig. 22 (using the principle shown in Fig. 23) is given in Fig. 24.

$$I_n = \frac{U_s - U_n}{R}$$

$$\Delta t = 1 \mu s$$

$$U_{n+1} = U_n + \frac{\Delta t}{C} I_n$$
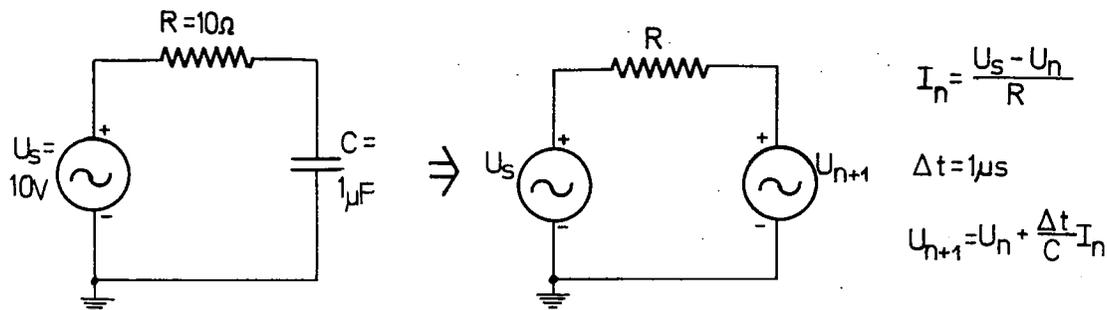
n Δt	U_n (V)	I_n (A)	(U_{n+1}) (V)
0	0.0	1.0	1.0
1	1.0	0.9	1.9
2	1.9	0.81	2.71
3	2.71	0.729	3.50
4	3.50	.	.

Fig. 24. RC network equivalent circuit with the numerical results.

Note that the network containing capacitance C is now included in the network that contains only voltage sources and resistors. Mathematically, this means that the differential equation is now an algebraic one.

For the inductance L, the fundamental relation between the voltage and current, $U_L = L \, dI_L/dt$. Euler's integration will give

$$I_{n+1} = I_n + \Delta t \frac{dI_n}{dt} \, , \qquad \text{respectively}$$

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$$I_{n+1} = I_n + \frac{\Delta t}{L} U_n \quad . \tag{38}$$

An equivalent circuit for inductance L is, using term (38), a current source and is shown in Fig. 23.b). Because of the capacitance C, the described procedure is not convenient for the nodal analysis application since it includes the voltage sources in the network.

The second integration method is a modified Euler's method, called Backward Euler (B. E.), which is more convenient and gives a stable solution of the differential equation. Equation (35), using this method, is written as:

$$U_C (t+\Delta t) = U_C(t) + \Delta t \frac{dU_C(t+\Delta t)}{dt} , \qquad \text{respectively}$$

$$U_{n+1} = U_n + \Delta t\, U'_{n+1}$$

and, after substituting $U'_{n+1} = I_{n+1}/C$, gives

$$U_{n+1} = U_n + \frac{\Delta t}{C} I_{n+1} \quad . \tag{39}$$

This procedure is justified because the relation between voltages U_{n+1} and U_n can be a derivative in time t_n, just as it can be a derivative in time t_{n+1}. From term (39):

$$I_{n+1} = \frac{C}{\Delta t} U_{n+1} - \frac{C}{\Delta t} U_n \tag{40}$$

which leads to the equivalent circuit as in Fig. 25.a. The current I_{n+1} is equal to the sum of currents through the conductance G_C and the current I_C, which satisfies equation (40).
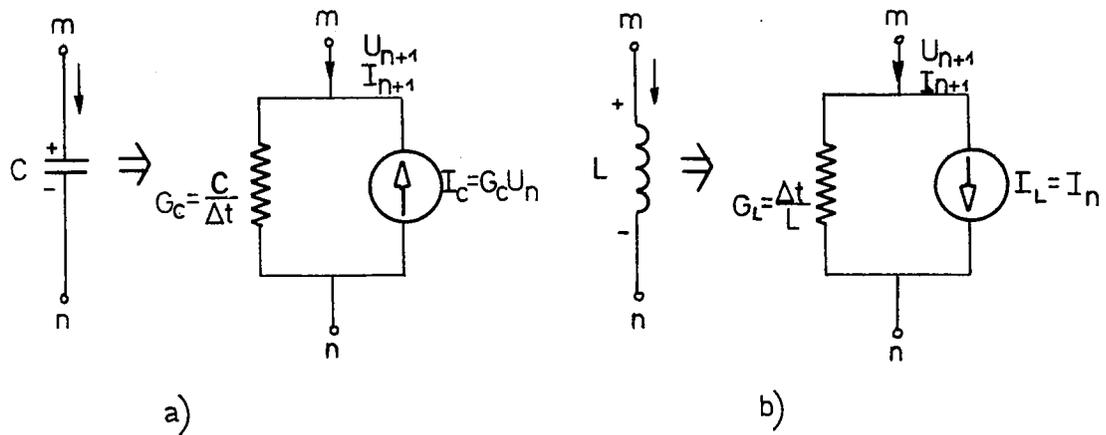
Fig. 25.a) An equivalent circuit of the capacitance C,
b) An equivalent circuit of the inductance L,
according to the Backward Euler method.

Similarly, with inductance L:

$$I_{n+1} = I_n + \Delta t \, I'_{n+1} \quad ,$$

$$I_{n+1} = I_n + \frac{\Delta t}{L} U_{n+1} \quad , \tag{41}$$

which gives an equivalent circuit as in Fig. 25.b. The Backward
Euler method is much more suitable because both C and L are
presented with the same kind of equivalent circuit and they
satisfy demands of the network calculation by the nodal method,
i.e. there are no voltage sources. Because of this, the Backward
Euler method is used in many programs for dynamical electrical
network calculations. The error generated by this method is the
same as that of the original Euler's method, i.e. it is
proportional with

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$$\frac{\Delta t^2}{2!} U_C^{\prime\prime} (t), \quad \text{respectively} \quad \frac{\Delta t^2}{2!} I_L^{\prime\prime}(t).$$

The third method is the trapezoidal integration method which can be explained in two ways:

a) The voltage of the capacitance C in time $t+\Delta t$ is calculated by

$$U_C (t+\Delta t) = U_C(t) + \frac{\Delta t}{2} (\frac{dU_C(t)}{dt} + \frac{dU_C(t+\Delta t)}{dt}),$$

respectively $\quad U_{n+1} = U_n + \frac{\Delta t}{2} (U_n^{\prime} + U_{n+1}^{\prime})$, $\hfill$ (42.a)

or, by substituting $\quad U_n^{\prime} = \frac{I_n}{C}$, $U_{n+1}^{\prime} = \frac{I_{n+1}}{C}$

$$U_{n+1} = U_n + \frac{\Delta t}{2C} (I_n + I_{n+1}) . \hfill (42.b)$$

This procedure uses the derivative mean arithmetical value in time t and $t+\Delta t$. From term (42.b)

$$I_{n+1} = \frac{2C}{\Delta t} U_{n+1} - (\frac{2C}{\Delta t} U_n + I_n) \hfill (43)$$

which leads to the equivalent circuit in Fig. 26.a. The current I_{n+1} is equal to the sum of the currents through the conductance G_C and the current I_C, which satisfies equation (43).

Similarly, by manipulating the inductance L:
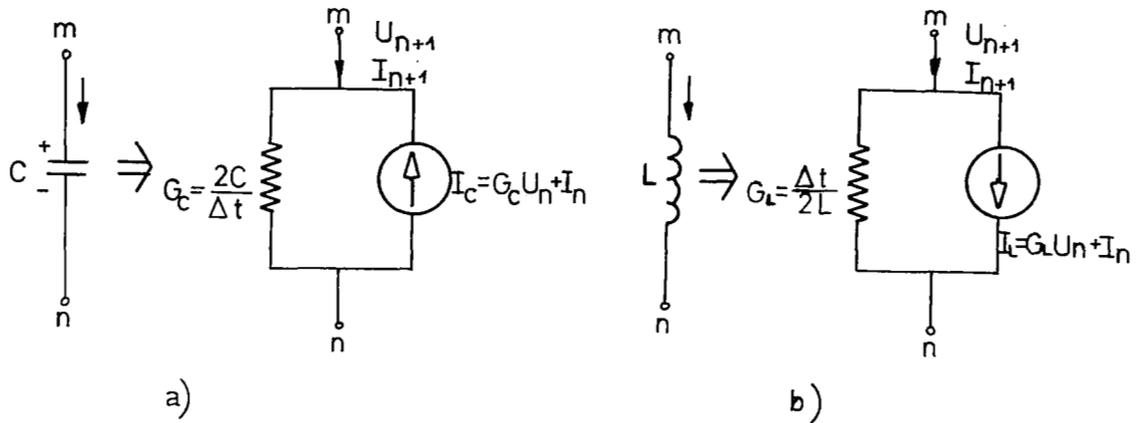
Fig. 26.a) The capacitance C equivalent circuit,
b) The inductance L equivalent circuit
according to the trapezoidal forumula.

$$I_{n+1} = I_n + \frac{\Delta t}{2} (I_n' + I_{n+1}') \quad ,$$

or substituting $\quad I_n' = \frac{U_n}{L} \quad , \quad I_{n+1}' = \frac{U_{n+1}}{L}$

$$I_{n+1} = I_n + \frac{\Delta t}{2L} (U_n + U_{n+1}) \; . \tag{44}$$

After arranging the right side terms;

$$I_{n+1} = \frac{\Delta t}{2L} U_{n+1} + (\frac{\Delta t}{2L} U_n + I_n) \; , \tag{45}$$

which leads to the the equivalent circuit in Fig. 26.b. The current I_{n+1} is equal to the sum of the currents thorough the conductance G_L and the current I_L, which satisfies equation (45).

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The error which appears in the described integration procedure
(formula 42.a) is calculated by comparison using Taylor's series:

$$U_{n+1} = U_n + \frac{\Delta t}{2} (U_n' + U_{n+1}') \quad ,$$

$$U_{n+1} = U_n + \Delta t \, U_n' + \frac{\Delta t}{2} (U_{n+1}' - U_n') \quad . \qquad (46.a)$$

Now, using Euler's method the term for the first derivative in
time t+Δt is given as

$$U_{n+1}' = U_n' + \Delta t \, U_n'' \quad ,$$

from which

$$U_{n+1}' - U_n' = \Delta t \, U_n''$$

inserting this into the
term (46.a) gives

$$U_{n+1} = U_n + \Delta t \, U_n' + \frac{\Delta t^2}{2} U_n''. \qquad (46.b)$$

This integration method is more accurate than Euler's methods,
because the function is approximated with the first three terms
of Taylor's series, i.e. it includes the square term, so that the
error is proportional with $\Delta t^3/3! \, U_n'''$.

  b) The second approach to the trapezoidal integration gives
     the same formula.

The voltage of the capacitance C, at ime t+Δt, is equal to

$$U(t+\Delta t) = U(t) + \frac{1}{C} \int_{t}^{t+\Delta t} I \, dt =$$

$$= U(t) + \frac{1}{C} (I(t+\Delta t) + I(t)) \frac{\Delta t}{2},$$

or, simplified,

$$U_{n+1} = U_n + \frac{\Delta t}{2C} (I_n + I_{n+1}). \qquad (46.c)$$

This is the same as term (42.b). In Eq. (46.c) the voltage growth on the capacitor is calculated by integrating the capacitor current by the trapezoidal rule (Fig. 27).
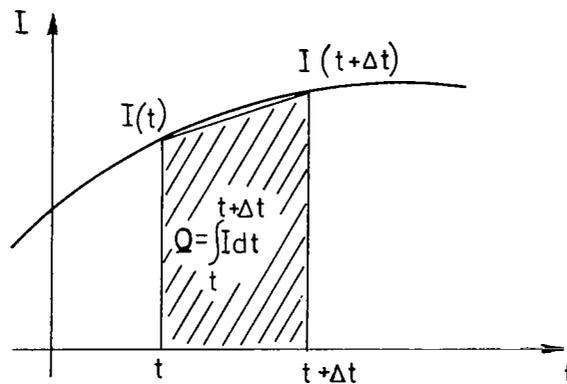
Fig. 27. Trapezoidal integration.

7.2. Integration Method Precision and Stability

Precision and stability of the three described methods will now be explained by a simple example of the RC circuit (Fig. 28.)

$$E = U + V$$

$$t = n\Delta t$$
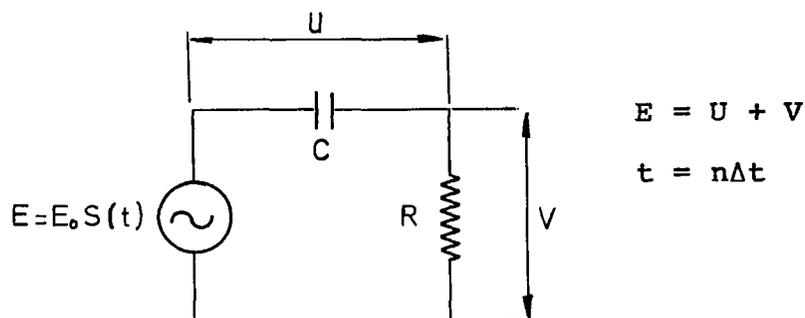
Fig. 28. Simple RC circuit

a) Euler's method (E)

At any moment, $t = n\Delta t$,

$$E = U_n + V_n \quad , \text{ or } U_n = E - V_n \qquad (47.a)$$

and at $t+\Delta t = (n+1)\Delta t$

$$E = U_{n+1} + V_{n+1} \quad , \text{ or } \quad U_{n+1} = E - V_{n+1}, \qquad (47.b)$$

and also

$$I_n = \frac{V_n}{R} \cdot \qquad (47.c)$$

The voltage of the capacitor is

$$U_{n+1} = U_n + \frac{\Delta t}{C} I_n \qquad (47.d)$$

Using the first three terms (47. a,b and c), the correspoding values for U_{n+1}, U_n and I_n are inserted into term (47.d) and give:

$$E - V_{n+1} = E - V_n + \frac{\Delta t}{RC} V_n \quad ,$$

$$V_{n+1} = V_n (1 - \frac{\Delta t}{RC}) \qquad (48)$$

If the voltage, V_n, is to be time dependent, then ($\tau = RC$):

$$V_1 = V_o (1 - \frac{\Delta t}{\tau}) = E_o (1 - \frac{\Delta t}{\tau}) \quad ,$$

because at $t = 0$, the starting voltage V_o on the resistor equals the source voltage E_o. Then

$$V_2 = V_1 (1 - \frac{\Delta t}{\tau}) = E_o (1 - \frac{\Delta t}{\tau})^2 \quad ,$$

and, applying this procedure n times gives

$$V_n = E_o (1 - \frac{\Delta t}{\tau})^n \quad . \qquad (49)$$

b) Backward Euler method (B.E.)

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Equations (47.a, b and c) are the same, except that the voltage of the capacitor is:

$$U_{n+1} = U_n + \frac{\Delta t}{C} I_{n+1} \quad ,$$

which after the insertion gives,

$$E - V_{n+1} = E - V_n + \frac{\Delta t}{RC} V_{n+1} \quad ,$$

$$V_{n+1} = V_n \frac{1}{1 + \frac{\Delta t}{\tau}} \quad . \tag{50}$$

Thus the voltage at time $n\Delta t$ following the steps used in Euler's method is equal to

$$V_n = E_o \frac{1}{(1 + \frac{\Delta t}{\tau})^n} \quad . \tag{51}$$

c) Trapezoidal method (TR.)

The equations (47. a, b and c) are the same, except that the capacitor voltage is

$$U_{n+1} = U_n + \frac{\Delta t}{2C} (I_n + I_{n+1}) \quad ,$$

which, after the insertion, gives

$$E - V_{n+1} = E - V_n + \frac{\Delta t}{2RC} V_n + \frac{\Delta t}{2RC} V_{n+1} ,$$

$$V_{n+1} = V_n \frac{1 - \frac{\Delta t}{2\tau}}{1 + \frac{\Delta t}{2\tau}} . \tag{52}$$

Thus the voltage at time $n\Delta t$ following the steps used in Euler's method is equal to

$$V_n = E_o \left(\frac{1 - \frac{\Delta t}{2\tau}}{1 + \frac{\Delta t}{2\tau}} \right)^n . \tag{53}$$

The exact analytical solution for the observed RC circuit is:

$$V(t) = E_o e^{-t/\tau} \qquad \text{i.e. substituting } t = n\Delta t$$

$$V_n = E_o e^{-n\Delta t/\tau} . \tag{54}$$

Using equations from (48) to (54) an estimate of the accuracy and stability of all three described integration methods may be made. First, the precision which is obtained in the case when the time increases by Δt is evaluated. The results for $\Delta t = 0.1\tau$ and $\Delta t = \tau$ are given in the following table:

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PRECISION

	E. $(1 - \frac{\Delta t}{\tau})$	B.E. $\frac{1}{1 + \frac{\Delta t}{\tau}}$	TR. $\frac{1 - \Delta t/2\tau}{1 + \Delta t/2\tau}$	exact $e^{-\Delta t/\tau}$
$\Delta t = 0.1\tau$	0.90000	0.90909	0.90476	0.90483
$\Delta t = \tau$	0.0	0.5	0.333	0.368

The trapezoidal method best approximates the exact solution. The stability can be estimated by increasing the time step Δt and observing the result by odd and even n's. The three methods are compared in the following table:

STABILITY

	E. $(1 - \frac{\Delta t}{\tau})^n$	B.E. $\frac{1}{(1 + \frac{\Delta t}{\tau})^n}$	TR. $\left(\frac{1 - \frac{\Delta t}{2\tau}}{1 + \frac{\Delta t}{2\tau}}\right)^n$
stable for	$0 < \Delta t < \tau$	$0 < \Delta t$	$0 < \Delta t < 2\tau$
for very large Δt $\frac{\Delta t}{\tau} \gg 1$	$(-\frac{\Delta t}{\tau})^n$ divergent and osc.	$\frac{1}{(\frac{\Delta t}{\tau})^n}$ stable	$(-1)^n$ stable but. osc.

Euler's formula is stable when the time step is $\Delta t < \tau$. When Δt is very large, the solution oscillates and diverges more as n increases. This method is called an "explicit" integration method. It is generally stable only if the condition $\Delta t < \tau$ is satisfied. If the electrical network has more time constants, then time step Δt for the stable solution must be smaller than the circuit's smallest time constant, $\Delta t < \tau_{min}$.

The Backward Euler method is stable for any time step, Δt, and the solution's precision decreases as the coefficient $\Delta t / \tau$ increases. This method is an "implicit" integration method which is generally stable.

The trapezoidal formula is the most precise and is stable even if the numerical solution oscillates for $\Delta t \gg \tau$. This method is also an "implicit" integration method. The trapezoidal formula is preferrable to the Backward Euler method, although Backward Euler method is a stable one for any Δt. The trapezoidal method advantage is that the numerical oscillations appear at the same time as the loss in precision, yet the result does not diverge. This means that the appearance of the numerical oscillations provides a warning that the solution is incorrect, so that the calculation can be repeated with a smaller time step. In effect, if there are no numerical oscillations, then the precision is satisfactory . The Backward Euler does not give such a simple monitor of accuracy.

7.3. Equivalent Model of Nonlinear Storage Elements

An element whose charge does not change linearly with the voltage, i.e. $Q \neq CV$, but $Q = f(V)$, is a nonlinear storage element. The calculation of the charge increment ΔQ, which is a consequence of the voltage growth Δv in time Δt, is similar to Newton-Raphson's iteration procedure for nonlinear elements. But, instead of

Newton-Raphson's iteration which uses the tangent, "regula falsi" method is used which employs the secant. In Fig. 29 the function $Q = f(V)$ is illustrated graphically.
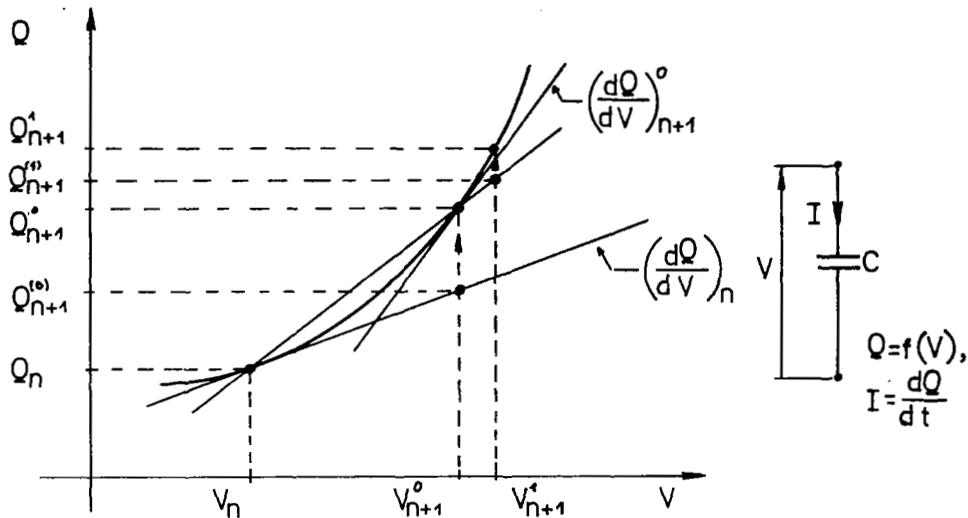
Fig. 29. The "regula falsi" method application on a nonlinear element $Q = f(V)$.

The nonlinear element, $Q = f(V)$, is linearized about the point Q_n, V_n by [1]

$$Q_{n+1}^{(o)} = Q_n + \left(\frac{dQ}{dV}\right)_n (V_{n+1}^o - V_n) \;. \tag{55}$$

The charge growth, $\Delta Q = Q_{n+1}^o - Q_n$, is equal to

$$Q_{n+1}^{(o)} - Q_n = (I_{n+1}^o + I_n)\Delta t/2 \;. \tag{56}$$

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[1] Charge $Q_{n+1}^{(o)}$ is given according to Eq. (55); Q_{n+1}^o used in Eq. (58) is obtained applying $Q_{n+1}^o = f(V_{n+1}^o)$. The same holds for $Q_{n+1}^{(1)}$, etc.

Inserting term (55) into (56) gives

$$(\frac{dQ}{dV})_n \ (V_{n+1}^O - V_n) = (I_{n+1}^O + I_n)\Delta t/2,$$

$$I_{n+1}^O = 2/\Delta t \ (\frac{dQ}{dV})_n \ V_{n+1}^O - \left[2/\Delta t \ (\frac{dQ}{dV})_n \ V_n + I_n\right]. \qquad (57)$$

Voltage V_{n+1}^O and current I_{n+1}^O are the values obtained in the first iteration. The second step gives the following term for $Q_{n+1}^{(1)}$:

$$Q_{n+1}^{(1)} = Q_n + \frac{Q_{n+1}^O - Q_n}{V_{n+1}^O - V_n} \ (V_{n+1}^1 - V_n) \quad \text{and}$$

$$Q_{n+1}^{(1)} - Q_n = (I_{n+1}^1 + I_n)\Delta t/2 \quad , \qquad \text{which gives}$$

$$I_{n+1}^1 = \frac{2}{\Delta t} \frac{Q_{n+1}^O - Q_n}{V_{n+1}^O - V_n} \ V_{n+1}^1 - \left[\frac{2}{\Delta t} \frac{Q_{n+1}^O - Q_n}{V_{n+1}^O - V_n} \ V_n + I_n\right] . \qquad (58)$$

The iterations are continued until the difference, $\Delta V = V_{n+1}^1 - V_{n+1}^O$, is within the error tolerance, which, as earlier mentioned is usually 50 µV. When calculating complex networks, the iterations are done together with Newton-Raphson's iterations for the nonlinear resistance elements. An equivalent circuit for the nonlinear storage element $Q = f(V)$ is given in Fig. 30 with the conductance

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$$G_C = \frac{2}{\Delta t} \frac{Q_{n+1}^O - Q_n}{v_{n+1}^O - V_n} \quad , \text{ and the current source, } I_C = G_C V_n + I_n.$$
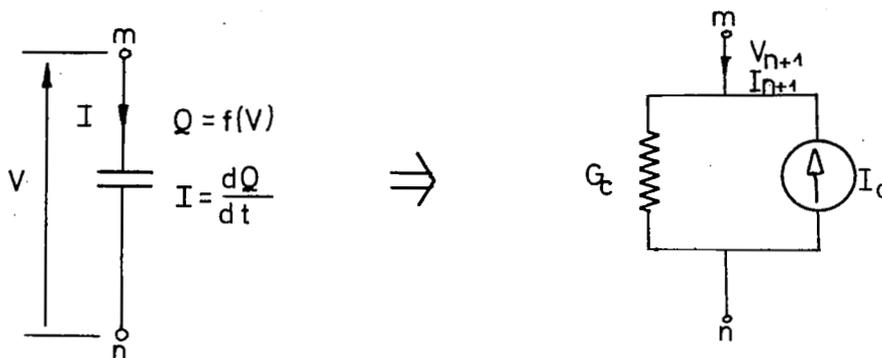
Fig. 30. *An equivalent circuit of the nonlinear storage element, Q = f(V).*

The secant slope, $\dfrac{Q_{n+1}^O - Q_n}{v_{n+1}^O - V_n}$, can be calculated with second

degree accuracy using the charge derivatives in places v_{n+1}^O and v_n:

$$\frac{Q_{n+1}^O - Q_n}{v_{n+1}^O - V_n} \doteq \frac{1}{2} \left[(\frac{dQ}{dV})_{n+1}^O + (\frac{dQ}{dV})_n \right] . \tag{59}$$

In the case of semiconductor devices, capacitances which are, by the charge change nature in semiconductors, may be defined as

$$C = \frac{dQ}{dV} \tag{60}$$

and the capacitance, C, is known in the form $C = f_c(V)$. As in the previous case, charge Q is obtained by defining the function

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$Q = f(V)$ using

$$Q = \int C \, dV = \int f_c(V) \, dV \, . \tag{61}$$

With the calculated charge Q, the method is continued as described in terms (57) and (58). In this case, when the capacitance is already given as the function of voltage, it is convenient to use term (59) for the secant slope. Then

$$\frac{Q^{\circ}_{n+1} - Q_n}{V^{\circ}_{n+1} - V_n} = \frac{1}{2} (C^{\circ}_{n+1} + C_n). \tag{62}$$

Thus the calculation is simplified and there is no need to calculate the integral from term (61). Terms (57) and (58) become:

$$I^0_{n+1} = \frac{2C_n}{\Delta t} V^{\circ}_{n+1} - \left[\frac{2C_n}{\Delta t} V_n + I_n \right] \quad \text{and} \tag{63}$$

$$I^1_{n+1} = \frac{C^{\circ}_{n+1} + C_n}{\Delta t} V^1_{n+1} - \left[\frac{C^{\circ}_{n+1} + C_n}{\Delta t} V_n + I_n \right]. \tag{64}$$

In the case of a nonlinear inductance element, instead of $\phi = LI$ we have $\phi = f(I)$ and the procedure is the same as that of nonlinear capacitance. According to this

$$\phi = f(I),$$

$$\phi^{(0)}_{n+1} = \phi_n + \left(\frac{d\phi}{dI} \right)_n (I^{\circ}_{n+1} - I_n) \quad .$$

Expanding $\Delta\phi$ gives

$$\phi_{n+1}^{(o)} - \phi_n = (V_{n+1}^o + V_n)\Delta t/2.$$

By eliminating the $\Delta\phi$ from upper terms we obtain:

$$(\frac{d\phi}{dI})_n (I_{n+1}^o - I_n) = (V_{n+1}^o + V_n)\Delta t/2 \quad , \text{ respectively}$$

$$I_{n+1}^o = \frac{\Delta t}{2(\frac{d\phi}{dI})_n} V_{n+1}^o + \left[\frac{\Delta t}{2(\frac{d\phi}{dI})_n} V_n + I_n\right] \tag{65}$$

The second iteration step, using $\phi_{n+1}^{(1)}$ yields:

$$\phi_{n+1}^{(1)} = \phi_n + \frac{\phi_{n+1}^o - \phi_n}{I_{n+1}^o - I_n} (I_{n+1}^1 - I_n), \text{ and}$$

$$\phi_{n+1}^{(1)} - \phi_n = (V_{n+1}^1 + V_n) \frac{\Delta t}{2} \quad , \text{ which gives}$$

$$I_{n+1}^1 = \frac{\Delta t}{2(\frac{\phi_{n+1}^o - \phi_n}{I_{n+1}^o - I_n})} V_{n+1}^1 + \left[\frac{\Delta t}{2(\frac{\phi_{n+1}^o - \phi_n}{I_{n+1}^o - I_n})} V_n + I_n\right]. \tag{66}$$

An equivalent circuit, according to term (66) consists of the conductance

$$G_L = \frac{\Delta t}{2\left(\dfrac{\phi_{n+1}^o - \phi_n}{I_{n+1}^o - I_n}\right)}$$ and the current source $I_L = G_L v_n + I_n$.

7.4. Transient nonlinear networks analysis procedure

This paragraph demonstrates the procedure for the numerical calculation of a network's transient analysis according to the flow diagram of Fig. 31. The network contains linear and nonlinear resistive elements, and linear conductances and inductances.

From Fig. 31 the following is apparent:

a) The network is defined and every nonlinear resistive element is replaced with the linear equivalent circuit (conductance G and current source I). Every capacitance C, and inductance L, is replaced with an equivalent circuit (conductance G and current source I).

b) To start a calculation, the initial capacitors' voltages, the initial currents through inductances, and the nonlinear elements assumed voltages are given.

c) The nonlinear elements voltage is calculated using Newton-Raphson's iterations since in step b) only the assumed voltages were given. The initial capacitors' voltages and the initial currents through inductances remain unchanged.

d) Next, the time is advanced by Δt. This is accomplished using numerical integration by calculating the voltage increment ΔV_c on capacitors and current increment ΔI_L through the inductances. After this is done we turn back to step c) and calculate new voltage values on the resistance elements.

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The procedure is continued by repeating steps c) and d) until the network's time response is calculated for the entire time interval.

```
┌─────────────────────────────────────────────────────────┐
│ Given network,                                           │
│ equivalent circuit for nonlinear elements,               │
│ equivalent circuit for C and L and                       │
│ independent voltage and current sources are set up.      │
└─────────────────────────────────────────────────────────┘
                          │
                          ▼
                  ┌───────────────┐
                  │    t = 0      │
                  └───────────────┘
                          │
                          ▼
┌─────────────────────────────────────────────────────────┐
│ Initial values $V_C$ on C and $I_L$ through L and        │
│ assumed values $V_{NL}$ and $I_{NL}$ of nonlinear        │
│ elements are given.                                      │
└─────────────────────────────────────────────────────────┘
                          │
                          ▼
┌─────────────────────────────────────────────────────────┐
│ Newton-Raphson's iterations for $V_{NL}$ and $I_{NL}$ of │
│ nonlinear elements are executed                          │
│ ($V_C$ and $I_L$ remain unchanged).                      │
└─────────────────────────────────────────────────────────┘
                          │
                          ▼
                  ┌───────────────┐
                  │ t = t + Δt    │
                  └───────────────┘
                          │
                          ▼
┌─────────────────────────────────────────────────────────┐
│ Integration (time advances for time step Δt):            │
│ $V_C = V_C + \Delta V_C$ and                             │
│ $I_L = I_L + \Delta I_L$ are calculated.                 │
└─────────────────────────────────────────────────────────┘
                          │
                          ▼
                  ┌───────────────┐
                  │  t = t stop   │
                  └───────────────┘
```

Fig. 31. Numerical calculation procedure of nonlinear electrical network transient analysis.

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