REAL EIGENVALUE ANALYSIS IN NASTRAN

BY THE

TRIDIAGONAL REDUCTION (FEER) METHOD

Malcolm Newman and Paul F. Flanagan
Analytical Mechanics Associates, Inc.

and

James L. Rogers, Jr.
NASA Langley Research Center

SUMMARY

Implementation of the Tridiagonal Reduction method in Level 16 of NASTRAN for real eigenvalue extraction in structural vibration and buckling problems is described. The basic concepts underlying the method are summarized and special features, such as the computation of error bounds and default modes of operation are discussed. In addition, the new user information and error messages and optional diagnostic output relating to the Tridiagonal Reduction method are presented.

Some numerical results and initial experiences relating to usage in the NASTRAN environment are provided, including comparisons with other existing NASTRAN eigenvalue methods.

INTRODUCTION

The Tridiagonal Reduction or FEER method is based on an efficient algorithm which extracts eigensolutions in the neighborhood of a specified shift point in the eigenspectrum from a tridiagonal eigenvalue problem of highly reduced order. In essence, the size of the reduced problem is of the same order of magnitude as the number of desired roots, even when the fully discretized system possesses many thousand degrees of freedom. The computational scheme employed is basically a truncated version of the Lanczos Algorithm (ref. 1) as first proposed by Crandall (ref. 2), but its present level of performance derives from a series of improvements and numerical refinement which were started back in the late 1960's by Ojalvo and Newman (ref. 3) and subsequently developed into large-scale computational programs of general utility (refs. 4, 5, and 6).

The Tridiagonal Reduction method employs only a single, initial shift of eigenvalues and hence usually requires only one matrix decomposition. It consequently tends to be much more efficient than the inverse power method when many eigensolutions are required.

The version of the Tridiagonal Reduction method reported on here is the one implemented in Level 16 of NASTRAN for real eigenvalue extraction in structural vibration and buckling problems. The basic concepts underlying the method are summarized only briefly, since a more thorough exposition of the theoretical aspects are available in reference 7. The major emphasis of this paper is directed toward describing the features of this new capability in NASTRAN, outlining user procedures, and reporting on initial experiences relating to its usage in the NASTRAN environment.

PRELIMINARY OPERATIONS

The problem is to find a specified number of real eigenvalues, λ_a, and corresponding eigenvectors, $\{\phi\}$, for

$$[K - \lambda_a M]\{\phi\} = 0 \tag{1}$$

It is further required that these eigensolutions constitute a set lying closest to a specified point, λ_o, in the eigenspectrum.

The definitions of the eigenvalue, the matrices [K] and [M], and their mathematical properties, depend on the type of problem being solved. For real analysis, only two separate problem types need be considered: structural vibration and buckling problems. The matrix definitions and mathematical distinctions for these two cases are summarized in table I.

The essential differences between the two types of problems center around the properties of the [M] matrix, which is nonnegative for vibration mode problems, but indefinite for buckling problems, thereby permitting the existence of both positive and negative eigenvalues in the latter case. In addition, the stiffness matrix may be singular for vibration problems while it is always positive definite in buckling applications, which implies that the buckling analysis is performed on a kinematically stable structure.

Further, if the user requests vibration modes in the neighborhood of a specified frequency, ω_o, equation (1) can be written as

$$[\bar{K}]\{\phi\} = \lambda' [M_{aa}]\{\phi\} \tag{2}$$

where

$$[\bar{K}] = [K_{aa} - \omega_o^2 M_{aa}] \tag{3a}$$

and

$$\lambda' = \omega^2 - \omega_o^2 \tag{3b}$$

The resulting effective stiffness matrix, $[\bar{K}]$, is indefinite in this case, since it possesses both positive and negative eigenvalues. This requires that

a non-square root decomposition scheme be used in subsequent operations. However, $\omega_o = 0$ is taken as a default value, or it may be specified by the user. In this case, a specified number of natural frequencies starting with the lowest will be computed. In order to utilize a more efficient Cholesky decomposition of $[\bar{K}]$ under these conditions, a small negative shift $\lambda_o = -\alpha^2$ is used yielding

$$[\bar{K}] = [K_{aa} + \alpha^2 M_{aa}] \tag{4a}$$

and

$$\lambda' = \omega^2 + \alpha^2 \tag{4b}$$

The resulting effective stiffness matrix $[\bar{K}]$ is positive definite thereby allowing a square-root decomposition to be performed when the roots are computed in the neighborhood of zero. Since no shifting is performed in buckling problems, the effective stiffness matrix is $[\bar{K}] = [K_{aa}]$, which is always positive definite, again permitting the use of a Cholesky decomposition.

In any event, a decomposition or factoring of $[\bar{K}]$ is next performed:

$$[\bar{K}] = [L][d][L]^T \text{ (shifted vibration mode problems)} \tag{5a}$$

or

$$[\bar{K}] = [C][C]^T \begin{array}{l}\text{(buckling problems}\\ \text{or vibration modes}\\ \text{in the neighborhood}\\ \text{of zero desired)}\end{array} \tag{5b}$$

where $[L]$ and $[C]$ are lower triangular factors and $[d]$ is a diagonal matrix.

To facilitate computation of eigenvalues closest to the point of interest within the eigenspectrum, inverse forms of the eigenvalue problems are employed. The general form of the inverse problem may be written as

$$[B][X] = \Lambda[D][X] \tag{6}$$

where the above terms are defined in table II.

THE REDUCTION ALGORITHM

A reduction of the order of the above eigenvalue problem is effected through the transformation

$$\underset{nx1}{\{\bar{X}\}} = \underset{nxm}{[V]} \underset{mx1}{\{y\}} \tag{7}$$

where $\{\bar{X}\}$ is an approximation of $\{X\}$, n is the order of the unreduced problem, and $m \leq n$. The transformation matrix is taken to be orthonormal to [D] so that

$$[V]^T [D][V] = [I] \tag{8}$$

From equations (6), (7), and (8) it is seen that

$$[A] \{y\} = \bar{\Lambda}\{y\} \tag{9}$$

where

$$\underset{mxm}{[A]} = [V]^T [B][V] \tag{10}$$

and $\bar{\Lambda}$ is an approximation of the eigenvalue, Λ.

Thus, equation (9) is an $m + n$ order eigenvalue problem where $m \leq n$. The essence of the reduction scheme lies in the choice of the transformation matrix, [V]. In the present case, the Lanczos algorithm is used to build up the [V] matrix, vector by vector, that is,

$$\underset{nxm}{[V]} = [\{v_1\}\{v_2\} \ . \ . \ .\{v_m\}] \tag{11}$$

such that the reduced mxm matrix [A] is tridiagonal and its eigenvalues accurately approximate the roots of the physical model closest to the specified point of interest in the eigenspectrum.

The theoretical recurrence formulas for generation of the columns of the [V] matrix are

$$\left.\begin{aligned} a_{i,i} &= \{v_i\}^T [B]\{v_i\} \\ \{\bar{v}_{i+1}\} &= [D]^{-1} [B]\{v_i\} - a_{i,i}\{v_i\} - d_i\{v_{i-1}\} \\ d_{i+1} &= [\{\bar{v}_{i+1}\}^T [D]\{\bar{v}_{i+1}\}]^{1/2} \end{aligned}\right\} \ i = 1,m \tag{12a}$$

$$\{v_{i+1}\} = \frac{1}{d_{i+1}} \{\bar{v}_{i+1}\}; \qquad i = 1, \ m - 1 \tag{12b}$$

where the sequence is initialized by choosing a random starting vector for $\{v_i\}$ and setting $d_i = 0$, $\{v_o\} = \{0\}$. In order to prevent numerical drift in the computations, each vector, $\{v_{i+1}\}$, is reorthogonalized to all previously computed $\{v\}$ - vectors before reentering equation (28a).

Again, the reader is referred to reference 7 for a more complete discussion of the underlying theoretical and numerical details.

The eigenvalues, $\bar{\Lambda}$, and eigenvectors, $\{y\}$ of equation (9) are extracted using a Q - R algorithm and eigenvector computational procedure similar to that employed in the Givens method. They are then converted to physical form as follows:

$$\bar{\lambda}_1 = -\frac{1}{\bar{\Lambda}_i} \quad \text{(buckling problems)} \tag{13a}$$

$$\bar{\omega}_i^2 = \frac{1}{\bar{\Lambda}_i} - \alpha^2 \quad \text{(unshifted vibration mode problems)} \tag{13b}$$

$$\bar{\omega}_i^2 = \frac{1}{\bar{\Lambda}_i} + \omega_o^2 \quad \text{(shifted vibration mode problems)} \tag{13c}$$

$$\{\bar{\phi}_i\} = [C^{-1}]^T [V]\{y_i\} \quad \text{(buckling or unshifted vibration mode problems)} \tag{13d}$$

$$\{\bar{\phi}_i\} = [V]\{y_i\} \quad \text{(shifted vibration mode problems)} \tag{13e}$$

SIZE CRITERIA FOR THE REDUCED EIGENPROBLEM

The total number of eigensolutions is equal to the rank, r, of the [M] matrix. Thus, the size of the reduced problem, m, cannot be greater than r. If, in addition, f eigensolutions have previously been computed by NASTRAN (modes generated prior to a restart plus rigid body modes generated by using a SUPØRT card in the bulk data deck), these must be swept out of the problem by making all the $\{v\}$ vectors orthogonal to the previously computed eigenvectors. This implies that the maximum size of the reduced problem is further reduced to

$$\bar{r} = r - f \tag{14}$$

As a result of numerical experiments, it has been found that in cases where $m << \bar{r}$, a first grouping of more than $m/2$ eigenvalues closest to the shift point are in accurate agreement with the corresponding number of exact eigenvalues. The remaining reduced-system roots are spread across the remaining exact eigenspectrum.

In view of the above considerations, the order of the reduced problem solved by NASTRAN is

$$m = \min [(2\bar{q} + 10), \bar{r}] \tag{15a}$$

where

$$\bar{q} = q - f \tag{15b}$$

and q is the total number of eigenvalues requested by the user. The value "10" appearing in equation (15a) is somewhat conservative and was adopted to improve the accuracy of the user-requested eigenvectors, which tend to deteriorate more rapidly than the eigenvalues. However, if the user is not too concerned with the eigenvectors farthest removed from the shift point, he can reduce the problem size and decrease the run time by requesting a smaller value of q on the EIGR or EIGB bulk data card, with the assurance that in almost all cases, at least $q + 5$ accurate eigenvalues will be computed by the Tridiagonal Reduction method.

ERROR BOUNDS ON THE COMPUTED EIGENVALUES

The maximum absolute relative errors in the computed physical eigenvalues (see ref. 7) are obtained from

$$\left| 1 - \frac{\bar{\lambda}_{ai}}{\lambda_{ai}} \right| \leq \frac{\left| d_{m+1} \cdot y_{mi} \right|}{\left| \bar{\Lambda}_i \left(1 + \lambda_o \bar{\Lambda}_i \right) \right|}; \quad i = 1, m \tag{16}$$

where λ_{ai} is an exact system root, y_{mi} is the last element of the reduced-system eigenvector corresponding to $\bar{\lambda}_{ai}$ and λ_o is the shift point. Thus, it is seen that the eigenvalue errors are all proportional to d_{m+1}, which is the next off-diagonal term that would be generated, had the reduced tridiagonal matrix, [A], been increased from order m to order $m + 1$.

If the physical eigenvalue, $1/\bar{\Lambda}_i + \lambda_o$, corresponds to a rigid body mode, the above computation is invalid and therefore bypassed. A rigid body mode is assumed to occur whenever

$$\left| \frac{1}{\bar{\Lambda}_i} + \lambda_o \right| \leq 10^{-t/3} \tag{17}$$

where t is the number of decimal digits carried by the computer. In this case, the relative error is set to a flat zero.

The eigenvalues are processed in order of increasing distance from the center of range of interest, λ_o, to determine whether their associated error values meet an acceptable tolerance set by the user on the EIGR or EIGB bulk data card (the default value is 0.001/n percent. The first eigenvalue not meeting this tolerance test, as well as all subsequent eigenvalues farther removed from the center of interest, are considered to lack sufficient accuracy and are therefore rejected. Finally, acceptable eigenvalues obtained in the above manner are reordered in terms of increasing physical value for subsequent processing by NASTRAN.

NASTRAN USER'S INSTRUCTIONS

Figures 1 and 2 show modifications of the EIGR and EIGB cards in the NASTRAN bulk data deck which accommodate user implementation of the Tridiagonal Reduction method for real eigenvalue analysis. The modifications are constituted of additions to the standard user instructions and are underscored for ease in identification.

When the Tridiagonal Reduction method is invoked, the F2 or L2 parameter on these cards represents the maximum allowable value of the computed relative error in a physical eigenvalue. If this value is exceeded the associated eigensolution is not accepted for further processing by NASTRAN. A detailed list of the maximum relative errors in the computed eigenvalues can be obtained by requesting DIAG 16 in the NASTRAN Executive Control Deck.

USER MESSAGES AND OPTIONAL DIAGNOSTICS

Functional Module User Messages

The following is a description of the NASTRAN user messages which may be generated by NASTRAN during the execution of the Tridiagonal Reduction method and which are unique to this method. Explanatory information is provided following the text of each message and, in the case of a fatal message, corrective action is indicated. Refer to the NASTRAN Users' Manual, Section 6, for a complete listing of other system and user messages.

Fatal messages cause the termination of the execution following the printing of the message text. These messages will always appear at the end of the NASTRAN output. Warning and information messages will appear at various places in the output stream. Such messages convey only warnings or information to the user. Consequently, the execution continues in a normal manner following the printing of the message text.

Message List

2385*** USER WARNING MESSAGE 2385, DESIRED NUMBER ØF EIGENVALUES EXCEED THE EXISTING NUMBER, ALL EIGENSØLUTIØNS WILL BE SØUGHT.

The desired number of eigenvalues specified on the EIGR card (NEP) or the EIGR card (ND) exceeds the rank of the $[K^d_{aa}]$ or $[M_{aa}]$ matrix, which is the maximum number of existing eigenvalues.

2386*** USER FATAL MESSAGE 2386, STIFFNESS MATRIX SINGULARITY CANNØT BE REMØVED BY SHIFTING.

Check the specification of masses on CØNM1, CØNM2, CMASSi, material definition and element property cards to insure that the degrees of freedom in the analysis set are not all massless.

2387*** USER WARNING MESSAGE 2387, PRØBLEM SIZE REDUCED TØ **** DUE TØ ØRTHØGØNALITY DRIFT ØR NULL TRIAL VECTOR. ALL EXISTING MODES HAVE BEEN ØBTAINED. USE DIAG 16 TØ DETERMINE ERRØR BØUNDS.

The Tridiagonal Reduction method cannot generate a reduced problem size of the order prescribed in Section 10.6.2.3 of the Theoretical Manual. However, the desired number of accurate eigenvalues specified on the EIGB card (NEP) or the EIGR card (ND) may have been obtained. A detailed list of the computed error bounds can be obtained by requesting DIAG 16 in the EXECUTIVE CØNTRØL DECK.

2388*** USER WARNING MESSAGE 2388, USER SPECIFIED RANGE NØT USED FØR FEER BUCKLING, THE RØØTS ØF LØWEST MAGNITUDE ARE ØBTAINED.

The value of L1 specified on the EIGB card is ignored for buckling analysis by the Tridiagonal Reduction (FEER) method.

2389*** USER WARNING MESSAGE 2389, PRØBLEM SIZE REDUCED. NØ MØRE TRIAL VECTØRS CAN BE ØBTAINED.

The desired number of eigenvalues specified on the EIGB card (NEP) or the EIGR card (ND) exceeds the number that can be calculated by the Tridiagonal Reduction (FEER) method. Check whether the requested number of eigenvalues exceeds the rank of the $[K^d_{aa}]$ or $[M_{aa}]$ matrix, which equals the number of existing eigenvalues.

2390*** USER WARNING MESSAGE 2390, **** FEWER ACCURATE EIGENSØLUTIØNS THAN THE **** REQUESTED HAVE BEEN FOUND. USE DIAG 16 TØ DETERMINE ERRØR BØUNDS.

The number of eigenvalues passing the eigenvalue relative-error test is less than the number requested on the EIGB or EIGR card. The maximum allowable error is specified in field 5 on the above cards. A detailed list of the computed error bounds can be obtained by requesting DIAG 16 in the EXECUTIVE CØNTROL DECK. A checkpoint and restart should be employed to obtain additional accurate eigensolutions.

2391*** USER FATAL MESSAGE 2391, PRØGRAM LØGIC ERROR IN FEER.

An unexpected EØF or word count has been encountered.

2392*** USER INFØRMATIØN MESSAGE 2392, **** MØRE ACCURATE EIGENSØLUTIØNS THAN THE **** REQUESTED HAVE BEEN FOUND. USE DIAG 16 TØ DETERMINE ERRØR BØUNDS.

The number of eigenvalues passing the eigenvalue relative-error test is greater than the number requested on the EIGB or EIGR card. The maximum allowable error is specified in field 5 on

the above cards. A detailed list of the computed error bounds can be obtained by requesting DIAG 16 in the EXECUTIVE CØNTRØL DECK.

2393*** USER WARNING MESSAGE 2393, THE REDUCED-SYSTEM EIGENVECTØR CØRRESPØNDING TØ EIGENVALUE **** DØES NØT MEET CØNVERGENCE CRITERIØN. ABSØLUTE RELATIVE ERRØR BETWEEN SUCCESSIVE ITERATES IS ****.

The accuracy of the corresponding physical eigenvector is in doubt. Refer to the Eigenvalue Summary Table for the largest error in the generalized mass matrix.

The Eigenvalue Summary Table

The following summary of the eigenvalue analysis performed is automatically printed when rigid formats using the Tridiagonal Reduction (FEER) method are invoked:

1. The number of eigenvalues extracted.

2. Number of starting points used.

   This corresponds to the total number of random starting and restart vectors used by the FEER process.

3. Number of starting point moves.

   Not used in FEER (set equal to zero).

4. Number of triangular decompositions.

   Always equal to one, except for unshifted vibration problems (roots starting from the lowest requested). In this case, a maximum of three shifts and three decompositions are employed to remove possible stiffness matrix singularities.

5. Total number of vector iterations.

   The total number of reorthogonalizations of all the trial vectors employed.

6. Reason for termination.

   (0) Normal termination.

   (1) Fewer than the requested number of eigenvalues and eigenvectors have been extracted.

(3) The problem size has been reduced. However, the desired number of accurate eigensolutions specified on the EIGB or EIGR card may have been obtained. A detailed list of the computed error bounds can be obtained by requesting DIAG 16 in the EXECUTIVE CØNTRØL DECK.

7. Largest off-diagonal modal mass term and the number failing the mass orthogonality criterion.

Optional Diagnostic Output

The user can obtain special detailed information relating to the generation of the reduced problem size, the elements of the reduced tridiagonal matrix, computed error bounds, and other numerical tests by requesting DIAG 16 in the NASTRAN Executive Control Deck.

The meaning of this information is explained below in the order in which it appears in the DIAG 16 output.

ØRDER - The order of the unreduced problem (size of the $[K_{aa}]$ matrix)

MAX RANK - The maximum number of existing finite eigensolutions as initially detected by FEER

RED ØRDER - The order of the reduced eigenproblem which will be solved to obtain the number of accurate solutions requested by the user

ØRTH VCT - The number of previously computed accurate eigenvectors on the eigenvector file which were generated prior to a restart or by the NASTRAN rigid body mode generator

USER SHIFT - Used only in frequency problems. The user specified shift after conversion from cycles to radians - squared

INTERNAL SHIFT - Used only in frequency problems. A small positive value automatically computed to remove singularities if the user has specified a zero shift. Otherwise, the negative of the user shift

SINGULARITY CHECK - PASS: the shifted stiffness matrix is nonsingular
****: the number of internal shifts needed to remove stiffness matrix singularities

TRIDIAGØNAL ELEMENTS RØW j, **, ***, **** - Lists the computed tridiagonal elements of the reduced eigenmatrix:

j - Matrix row
** - Diagonal element
*** - Off-diagonal element
**** - First estimate of off-diagonal element in the next row

ØRTH ITER - The number of times a reorthogonalization of a trial vector has been performed

MAX PRØJ - The maximum projection of the above trial vector on the previously computed accurate trial vectors (prior to the current reorthogonalization)

NØRMAL FACT - The normalization factor for the reorthogonalized trial vector

ØPEN CØRE NØT USED *** FEER 3 - Open core not used by Subroutine FEER 3, in single-precision words

FEER QRW ELEMENT *, ITER **, ***, RATIO ****, PRØJ *****:

* - The internal eigenvalue number in the order of its extraction by FEER

** - The number of inverse power iterations performed to extract the associated eigenvector of the reduced system (this is not a physical eigenvector)

*** - If a multiple root has been detected, the number of times that the previous multiple-root, reduced-system eigenvectors have been projected out of the current multiple-root eigenvector before repeating the inverse power iterations

**** - The absolute ratio of maximum, reduced-system eigenvector elements for successive inverse power iterations

***** - The maximum projection of a current multiple-root eigenvector on previously computed eigenvectors for the same root

PHYSICAL EIGENVALUE *, **, THEØR ERRØR *** PERCENT, PASS OR FAIL:

* - The internal eigenvalue number in the order of its extraction by FEER

** - The associated physical eigenvalue (λ for buckling problems, ω^2 for frequency problems)

*** - Theoretical upper bound on the relative eigenvalue error, in percent

PASS - The computed error is less than or equal to the allowable specified on the EIGB or EIGR bulk data card (default is 0.001/n percent, where n is the total number of unconstrained degrees of freedom)

FAIL - The computed error is greater than the allowable and this mode is not accepted for further processing

ØPEN CØRE NØT USED *** FEER 4 - Open core not used by Subroutine FEER 4, in single precision words

FEER CØMPLETE *, **, ***, ****

*	- The remaining CPU time available following decomposition of the shifted stiffness matrix, in seconds (the total time is specified on the TIME card in the Executive Control Deck)
**	- The remaining CPU time, in seconds after completing Subroutine FEER 3
***	- The remaining CPU time, in seconds after completing Subroutine FEER 4
****	- The total operation count for FEER after decomposition of the shifted stiffness matrix. One operation is considered to be a multiplication or division followed by an addition

COMPARISON OF FEER WITH EXISTING NASTRAN EIGENVALUE METHODS

The AH1G helicopter three-dimensional built-up dynamic model was chosen for the comparison because it represented a moderately large, "real-world" problem. The order of the stiffness matrix was 1706, while its maximum rank was 1277. In addition, the average number of active columns (CAVG) per pivot row was 81. During the original analysis of the problem, the GIVENS method was used and 30 eigenvectors were found. Therefore, for comparison purposes, 30 eigenvalues were requested in runs 2-6 as shown in table III. The problem size for the GIVENS run, however, was reduced to 241 by using OMIT BULK DATA cards. A series of eight test cases were run. Each case was executed on a CDC 6600 computer in $164K_8$ core (except run 6 which used $230K_8$). The only changes made to the input deck were to the EIGR card. The first six eigenvalues found in each run were the rigid body modes. The timing results for these tests are shown in table III.

In runs 1-3 where OMIT cards were used, the obvious choice is the GIVENS method, since the other two methods require twice as much CPU time and get fewer eigenvalues. FEER reduced the problem order to 58, while the other two methods worked with an order of 241. The 30 eigenvalues found agreed exactly for all three methods, while the 30 eigenvectors from each method agreed to the fourth significant digit. The eigenvectors were all normalized to MAX by the EIGR card.

In runs 4-5 no OMIT cards were used and, therefore, the problem size (1706 degrees of freedom) was too large to use the GIVENS method. FEER showed a significant speed advantage over the INVERSE POWER method. In fact, the INVERSE POWER problem terminated after finding only 22 eigenvalues because of insufficient time to calculate another root and no eigenvectors were obtained. It is therefore reasonable to estimate that FEER (run 4) was at least twice as fast as the INVERSE POWER method (run 5). In each of these runs, since no OMIT

cards were used, no time was spent in modules SMP1 and SMP2. This reduced the total run time. FEER again reduced the problem order to 58, while INVERSE POWER worked on a problem size of 1706. The first 22 eigenvalues found in these two runs agreed to the fifth or sixth significant digit. No eigenvectors were printed for run 5 (INVERSE POWER), thus no comparison was made. The following list shows how the eigenvalues found in run 4 differ from the eigenvalues found in run 1.

Mode No.	Eigenvalues from FEER with no OMITS	Eigenvalues from GIVENS with OMITS
1	0.0	0.0
2	0.0	0.0
3	0.0	0.0
4	0.0	0.0
5	0.0	0.0
6	0.0	0.0
7	3.607787E+02	3.607830E+02
8	4.359254E+02	4.359756E+02
9	1.936684E+03	1.936968E+03
10	2.469448E+03	2.469903E+03
11	7.174892E+03	8.380196E+03
12	8.765565E+03	8.954930E+03
13	1.020949E+04	1.056439E+04
14	1.066561E+04	1.174620E+04
15	1.173722E+04	1.273280E+04
16	1.497434E+04	1.512035E+04
17	1.511484E+04	1.553434E+04
18	1.632569E+04	1.646955E+04
19	1.960200E+04	2.166074E+04
20	2.235550E+04	2.253426E+04
21	2.367533E+04	2.371600E+04
22	2.391382E+04	2.445306E+04
23	2.441746E+04	2.556377E+04
24	2.627131E+04	2.869502E+04
25	2.864655E+04	3.245894E+04
26	3.612023E+04	4.061178E+04
27	4.052112E+04	4.249005E+04
28	4.220211E+04	4.612667E+04
29	4.488473E+04	5.430758E+04
30	4.600289E+04	5.673561E+04

Significant differences can be seen between the two sets of eigenvalues beyond the fourth nonrigid body mode. Similar differences occurred for the eigenvectors. These results indicate that the engineer must seriously consider whether the savings in CPU time achieved by using the Guyan Reduction method (OMITS) are worth the price paid in loss of accuracy or, at best, doubtful accuracy in the results. Run 6 was an additional run using FEER with no OMIT cards, but with the storage increased from $164K_8$ to $230K_8$, to see if the increased core would decrease the run time for a problem of this size. This was not the case. Runs 2, 4, and 6 show that no matter which core size was

used or even whether or not OMIT cards were used, the CPU time in the READ module remains more or less constant.

A final pair of runs (7 and 8) were made to determine the effect of extracting only half (15) of the eigenvalues of the previous runs. For this case, FEER is 65 percent faster than INVERSE POWER and 32 percent faster than the total time for GIVENS with omits in run 1. In run 7, FEER used a reduced order of 30. INVERSE POWER method, however, missed six lower eigenvalues in the frequency range selected (0 to 500 cycles). Comparison of the eigenvectors to those in runs 4 and 5 reveal that the eigenvectors found by INVERSE POWER in run 8 agreed with the eigenvectors of both run 4 and run 5. The first 10 eigenvectors (including the six rigid body modes) found by FEER in run 7 agreed to significant digits with the first 10 eigenvectors found in run 4, after which the eigenvectors in run 7 deteoriated quickly.

CONCLUDING REMARKS

The GIVENS method, using the OMIT feature in NASTRAN, demonstrated the shortest run time for the problem considered but, as shown by the results, the approximations inherent in the Guyan Reduction scheme led to reduced accuracy in the eigensolutions. In fact, only four out of the 24 nonrigid body, eigenvalue modes computed in this manner were of good accuracy. Thus, if the problem contains more than just a few hundred mass degrees of freedom (thereby precluding the use of the GIVENS method without a Guyan Reduction), and the user does not wish to depend on a "judicious" choice of which of these degrees of freedom to omit, then the most reliable course is to omit none of them. In this case, FEER is the obvious choice over the INVERSE POWER method. This conclusion should be tempered somewhat by the fact that the FEER eigenvectors tend to deteriorate in accuracy as the computed modes become more remote from the shift point (center of the desired frequency range). However, it is not expected that this would seriously affect the accuracy of a dynamic response analysis in which there is limited frequency range for the dominant forcing functions.

It should be recognized that the above conclusions and run-time comparisons are only preliminary as of the time this report was written. As indicated by the operation count studies in reference 5, further numerical comparisons should show progressively increasing efficiency of FEER over the INVERSE POWER method as the problem size and matrix bandwidth increase.

REFERENCES

1. Lanczos, C.: An Iteration Method for the Solution of the Eigenvalue Problem of Linear Differential and Integral Operators. J. Res. Nat. Bur. Stand., Vol. 45, 1950, pp. 225-282.

2. Crandall, S. H.: Engineering Analysis. McGraw-Hill, New York, 1956, pp. 106-109.

3. Ojalvo, I. U.; and Newman, M.: Vibration Modes of Large Structures by an Automatic Matrix-Reduction Method. AIAA J., Vol. 8, No. 7, 1970, pp. 1234-1239.

4. Newman, M.; and Pipano, A.: Vibration Modes via Program NEWLAS. Chapter C - Program FEER (Fast Eigenvalue Extraction Routine). IAI TR 4842/6277, June 1971, pp. 7-16.

5. Newman, M.; and Pipano, A.: Fast Modal Extraction in NASTRAN via the FEER Computer Program. NASA TM X-2893, Sept. 1973, pp. 485-506.

6. Ojalvo, I. U.: ALARM - A Highly Efficient Eigenvalue Extraction Routine for Very Large Matrices. Shock and Vibration Digest, Vol. 7, No. 12, Dec. 1975.

7. Newman, M.; and Flanagan, P. F.: Eigenvalue Extraction in NASTRAN by the Tridiagonal Reduction (FEER) Method. NASA CR-2731, Aug. 1976.

TABLE I. PROBLEM FORMULATIONS

Problem type	Quantity	Definition	NASTRAN notation	Most general properties
Structural vibration modes	[K]	Stiffness matrix - analysis set	$[K_{aa}]$	Symmetric, nonnegative, semidefinite matrix
	[M]	Mass matrix - analysis set	$[M_{aa}]$	Same
	λ_a	Square of a circular natural frequency	ω^2	Positive
Buckling	[K]	Stiffness matrix - analysis set	$[K_{aa}]$	Symmetric, positive-definite matrix
	[M]	Differential stiffness matrix - analysis set	$[K^d_{aa}]$	Symmetric, indefinite matrix
	λ_a	Buckling load parameter	$-\lambda$	Positive or negative

TABLE II. INVERSE EIGENPROBLEM DEFINITIONS

	Problem type	[B]	[D]	{X}	Λ
1.	Shifted vibration modes	$[M_{aa}][L^{-1}]^T[d]^{-1}[L^{-1}][M_{aa}]$	$[M_{aa}]$	$\{\phi\}$	$\frac{1}{\omega^2 - \omega_o^2}$
2.	Unshifted vibration modes (in the neghborhood of zero frequency	$[C^{-1}][M_{aa}][C^{-1}]^T$	[I] (identity matrix)	$[C]^T\{\phi\}$	$\frac{1}{\omega^2 + \alpha^2}$
3.	Buckling modes	$[C^{-1}][K^d_{aa}][C^{-1}]^T$	[I]	$[C]^T\{\phi\}$	$-\frac{1}{\lambda}$

TABLE III. COMPARISON OF EIGENVALUE METHODS

Run	Method	OMITS	Number of eigenvalues found	Time in READ module (CPU sec)	Time in SMP1 and SMP2 modules (CPU sec)	Total time in READ, SMP1, and SMP2 modules (CPU sec)
1	GIVENS	Yes	241	495	669	1164
2	FEER	Yes	30	1809	669	2478
3	INVERSE POWER	Yes	30	1994	669	2663
4	FEER	No	30	1837	0	1837
5	INVERSE POWER	No	22	3118	0	3118
6	FEER	No	30	1817	0	1817
7	FEER	No	15	879	0	879
8	INVERSE POWER	No	15	1451	0	1451

Input Data Card EIGR

Description: Defines data needed to perform real eigenvalue analysis.

Format and Example:

EIGR	SID	METHØD	F1	F2	NE	ND	NZ	E	+abc
EIGR	13	DET	1.9	15.6	10	12	0	1.-3	ABC

+abc	NØRM	G	C						
+BC	PØINT	32	4						

Field	Content
SID	Set identification number (Unique integer > 0)
METHØD	Method of eigenvalue extraction, one of the BCD values "INV", "DET", "GIV", "FEER", "UINV", or "UDET".
	INV Inverse power method, symmetric matrix operations.
	DET Determinant method, symmetric matrix operations.
	GIV Givens method of tridiagonalization.
	FEER Tridiagonal reduction method, symmetric matrix operations.
	UINV Inverse power method, unsymmetric matrix operations.
	UDET Determinant method, unsymmetric matrix operations.
F1, F2	Frequency range of interest (Required for METHØD = "DET", "INV", "UDET", or "UINV") (Real ≥ 0.0; F1 < F2). Frequency range over which eigenvectors are desired for METHØD = "GIV". The frequency range is ignored if ND > 0, in which case the eigenvectors for the first ND positive roots are found. (Real F1 < F2). If METHØD = "FEER", F1 is the center of range of interest (Default is F1 = 0.0) (Real ≥ 0.0), and F2 is the acceptable relative error tolerance on frequency-squared, in percent (Default is .001/n where n is the order of the stiffness matrix) (Real > 0.0)
NE	Estimate of number of roots in range (Required for METHØD = "DET", "INV", "UDET", or "UINV", ignored for METHØD = "FEER") (Integer > 0)

Figure 1. Modifications to the EIGR bulk data card for the Tridiagonal Reduction method.

ND — Desired number of roots for METHØD = "DET", "INV", "UDET", or "UINV", (Default is 3 NE) (Integer > 0). Desired number of eigenvectors for METHØD = "GIV" (Default is zero) (Integer ≥ 0). Desired number of roots and eigenvectors for METHØD = "FEER" (Default is automatically calculated to extract at least one accurate mode) (Integer > 0)

NZ — Number of free body modes (Optional - used only if METHØD = "DET" or "UDET") (Integer ≥ 0)

E — Mass orthogonality test parameter (Default is 0.0 which means no test will be made) (Real ≥ 0.0).

NØRM — Method for normalizing eigenvectors, one of the BCD values "MASS", "MAX" or "PØINT"

MASS - Normalize to unit value of the generalized mass

MAX - Normalize to unit value of the largest component in the analysis set

PØINT - Normalize to unit value of the component defined in fields 3 and 4 - defaults to "MAX" if defined component is zero

G — Grid or scalar point identification number (Required if and only if NØRM = "PØINT") (Integer ≥ 0)

C — Component number (One of the integers 1-6) (Required if and only if NØRM = "PØINT" and G is a geometric grid point)

Remarks:

1. Real eigenvalue extraction data sets must be selected in the Case Control Deck (METHØD = SID) to be used by NASTRAN.

2. The units of F1 and F2 are cycles per unit time. If METHØD = "FEER", F2 represents the maximum upper bound, in percent, on $\left|\omega^2_{FEER}/\omega^2_{EXACT} - 1\right|$ for acceptance of a computed eigensolution.

3. The continuation card is required.

4. If METHØD = "GIV", all eigenvalues are found.

5. If METHØD = "GIV", the mass matrix for the analysis set must be positive definite. This means that all degrees of freedom, including rotations, must have mass properties. ØMIT cards may be used to remove massless degrees of freedom.

Figure 1. Continued.

6. A nonzero value of E in field 9 also modifies the convergence criteria. See Sections 10.3.6 and 10.4.4.2 of the Theoretical Manual for a discussion of convergence criteria.

7. If NØRM = "MAX," components that are not in the analysis set may have values larger than unit.

8. If NØRM = "PØINT," the selected component must be in the analysis set.

9. If METHØD = "GIV" and rigid body modes are present, F1 should be set to a small negative number rather than zero if the rigid body eigenvectors are desired.

10. The desired number of roots (ND) includes all roots previously found, such as rigid body modes determined with the use of the SUPØRT card, or the number of roots found on the previous run when restarting and APPENDing the eigenvector file.

Figure 1. Concluded.

BULK DATA DECK

Input Data Card EIGB Buckling Analysis Data

Description: Defines data needed to perform buckling analysis.

Format and Example:

1	2	3	4	5	6	7	8	9	10
EIGB	SID	METHØD	L1	L2	NEP	NDP	NDN	E	+abc
EIGB	13	DET	0.1	2.5	2	1	1	0.0	ABC

+abc	NØRM	G	C						
+BC	MAX								

Field	Contents
SID	Set identification number (Unique integer > 0)
METHØD	Method of eigenvalue extraction, one of the BCD values "INV", "DET", "FEER", "UINV", or "UDET" INV - Inverse power method, symmetric matrix operations DET - Determinant method, symmetric matrix operations FEER - Tridiagonal reduction method, symmetric matrix operations UINV - Inverse power method, unsymmetric matrix operations UDET - Determinant method, unsymmetric matrix operations
L1, L2	Eigenvalue range of interest (Real; L1 < L2 > 0.0) For METHØD = "FEER", L1 is ignored and L2 is the acceptable relative error tolerance on eigenvalues, in percent (Default is .001/n where n is the order of the stiffness matrix) (Real > 0.0)
NEP	Estimate of number of roots in positive range. Desired number of eigenvalues of smallest magnitude for METHØD = "FEER" (Default is automatically calculated to extract at lease one accurate mode) (Integer > 0)
NDP, NDN	Desired number of positive and negative roots (Default = 3 NEP) (Integer >0). Ignored for METHØD = "FEER"

Figure 2. Modifications to the EIGB bulk data card for the Tridiagonal Reduction method.

E — Convergence criteria (optional) (Real > 0.0)

NØRM — Method for normalizing eigenvectors, one of the BCD values "MAX" or "PØINT"

MAX - Normalize to unit value of the largest component in the analysis set

PØINT - Normalize to unit value of the component defined in fields 3 and 4 defaults to "MAX" if defined component is zero

G — Grid or scalar point identification number (Integer > 0) (Required if and only if NØRM = "PØINT")

C — Component number (One of the integers 1-6) (Required if and only if NØRM = "PØINT" and G is a geometric grid point)

Remarks

1. Buckling analysis root extraction data sets must be selected in the Case Control Deck (METHØD = SID) to be used by NASTRAN.

2. The quantities L1 and L2 are dimensionless and specify a range in which the eigenvalues are to be found. An eigenvalue is a factor by which the prebuckling state of stress (first subcase) is multiplied to produce buckling. If METHØD = "FEER", L1 is ignored and L2 represents the maximum upper bound, in percent, on $|\lambda_{FEER}/\lambda_{EXACT} - 1|$ for acceptance of a computed eigensolution.

3. The continuation card is required.

4. See Sections 10.3.6 and 10.4.2.2 of the Theoretical Manual for a discussion of convergence criteria.

5. If METHØD = "DET", L1 must be greater than or equal to 0.0.

6. If NØRM = "MAX", components that are not in the analysis set may have values larger than unity.

7. If NØRM = "PØINT", the selected component must be in the analysis set.

Figure 2. Concluded.