ADDITION OF HIGHER ORDER PLATE ELEMENTS TO NASTRAN

by

R. Narayanaswami
Department of Mechanical Engineering and Mechanics
Old Dominion University, Norfolk, Virginia

and

Chuh Mei
Vought Corporation, Hampton, Virginia

SUMMARY

Two plate elements, the linear strain triangular membrane element
CTRIM6 and the higher order plate bending element CTRPLT1, have
been added to NASTRAN Level 16.0. This paper describes the theo-
retical formulation, programming details, and bulk data information
pertaining to the addition of these elements. Sample problems
illustrating the use of these elements have been presented.

A triangular shallow shell element, CTRSHL, formulated from the
above mentioned plate elements utilizing shallow shell theory, is
under development. Details of formulation of the element are given.

With the addition of these elements, NASTRAN has excellent capability
for solving plate and shell problems using the statics, normal modes,
and buckling analyses rigid formats.

INTRODUCTION

NASTRAN presently (Level 16.0) has, in all, a total of ten different forms of plate elements in two different shapes (triangular and quadrilateral). The present NASTRAN triangular membrane element TRMEM, uses a linear displacement field and is found to be overly stiff; the basic bending element, TRBSC, the basic unit from which the bending properties of the other plate elements are formed, is also found to be very stiff due to the constraint of normal slope having to vary linearly on the exterior edges of the element. The higher order membrane and bending plate elements (given the mnemonic of TRIM6 and TRPLT1, respectively) available in the literature (refs. 1 and 2) are found to be more accurate and computationally more efficient than the existing NASTRAN plate elements. In this paper, the addition of these elements into the Level 16 version of NASTRAN is described.

A triangular shallow shell element formulated from the above mentioned elements utilizing shallow shell theory (mnemonic: TRSHL) is also being added into the Level 16.0 version of NASTRAN. Addition of this element is nearing completion; details of the use of this element are also given in this paper.

LIST OF SYMBOLS

a, b	lengths
a_i, c_i, d_i, h_i	coefficients
C_i	constants
D	bending rigidity
E	Young's modulus
I	moment of inertia
L	length
m, n	number of nodal lines
M	moment forces
N	membrane force
P	applied load

R	radius
T	temperature
t	thickness
X, Y, Z	global coordinates
x, y, z	element coordinates
u, v, w	displacements
α, β	rotations
δ	deflection
ϵ	strain
κ	rate of taper
λ	buckling factor
υ	Poisson's ratio
ρ	mass density
Ω	frequency parameter
ω	circular frequency

Special Symbols

,	denotes derivative

Subscripts

o	denotes value at end of beam
mn	number of nodal lines
1,2	denotes value at top and bottom of column
x,y	in x and y direction
cr	critical

TRIANGULAR MEMBRANE ELEMENT, TRIM6

The present development is based on the derivation in reference 1. The important characteristics of the element are that

1. The stresses and strains vary linearly within the element.
2. Bilinear variation in the planar co-ordinates for the thickness of the element is permitted.
3. Bilinear variation in the planar co-ordinates for the temperature in the element is provided.
4. Currently the use of the element is limited to solving problems using Rigid Formats 1 and 3.

Element Geometry and Derivation of Matrices

The geometry of the element is shown in figure 1. The element has six grid points, three at the vertices and three at the mid-points of the sides. u and v are components of displacements parallel to the x and y axes of the element co-ordinate system and are assumed to vary quadratically with position on the surface of the element.

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \tag{1}$$

$$v = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2 \tag{2}$$

The undetermined coefficients a_1 to a_{12} can be related to the 12 components of the element displacement vector.

The thickness t of the element, and also the temperature T, at any point (x,y) within the element are given by

$$t = c_1 + c_2 x + c_3 y \tag{3}$$

$$T = d_1 + d_2 x + d_3 y \tag{4}$$

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The coefficients c_1, c_2, and c_3, as also d_1, d_2, and d_3, can be evaluated from the user specified values for t and T, respectively, at the three vertices of the element.

The derivation of the element stiffness matrix, consistent mass matrix and consistent thermal load vector follows familiar lines (see, for example, ref. 3) and will not be described here. The stresses are evaluated at the three vertices and the centroid of the element. Execution time to evaluate the stiffness matrix is 0.16 sec. per element in single precision and 0.23 sec. per element in double precision on the CDC 6600 machine.

Two new bulk data cards have been added to the NASTRAN bulk data deck for using the TRIM6 element. They are the connection and property cards for the element, viz, the CTRIM6 and PTRIM6 cards, and are given in Appendix A. Some examples of the application of the element are given in Appendix B. Excellent agreement with plane stress elasticity and experimental solutions is shown even for coarse mesh subdivisions.

TRIANGUALR PLATE ELEMENT, TRPLT1

The mnemonic of TRPLT1 will be used to refer to this element. The important characteristics of the element are:

1. Bilinear variation in the planar co-ordinates for the thickness of the element is permitted.
2. Bilinear variation in the planar co-ordinates for the temperature in the element is provided for in the element dependent subroutine for the calculation of the thermal load vector. However, this capability can be operational in NASTRAN only after provision for specification of grid point temperature gradients is made.
3. Transverse shear flexibility is taken into account in the element stiffness formulation.
4. Currently, the use of the element is limited to solving problems using Rigid Formats 1 and 3.

Element Geometry and Derivation of Matrices

The geometry of the element is shown in figure 2. The element has six grid points, three at the vertices and three at the mid-points of the sides. The deflection w(x,y) within the element is assumed to vary as a quintic polynomial in the local co-ordinates, i.e.,

$$w(x,y) = a_1 + a_2 + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3$$

$$+ a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^4 + a_{12} x^3 y \qquad (5)$$

$$+ a_{13} x^2 y^2 + a_{14} xy^3 + a_{15} y^4 + a_{16} x^5 + a_{17} x^4 y$$

$$+ a_{18} x^3 y^2 + a_{19} x^2 y^3 + a_{20} xy^4 + a_{21} y^5$$

The element has 18 d.o.f., viz, the transverse displacement w, rotation about x-axis, α, and rotation about y-axis, β, at the six grid points of the element. Three constraints among the coefficients of eq. (5) are now introduced so that the normal slope varies cubically along each edge. Thus the 18 independent coefficients of eq. (5) can be related to the 18 d.o.f. of the element. The thickness, t, of the element, and the temperature, T, in the element, at any point (x,y) within the element are given by equations (3) and (4), respectively.

Examples of the application of the element for classical plate bending problems are given in reference 4 and show the excellent accuracy of the element for such problems. Additional problems showing the use of the element for variable thickness plates are given in Appendix C. Two new bulk data cards have been added to the NASTRAN bulk data deck for using the TRPLT1 element. They are the connection and property cards for the element, viz , the CTRPLT1 and PTRPLT1 cards, and are given in Appendix A. Execution time to evaluate the stiffness matrix is 0.987 sec. per element in single precision and 1.579 sec. per element in double precision on the CDC 6600 machine.

TRIANGULAR SHALLOW SHELL ELEMENT, TRSHL

The mnemonic of TRSHL will be used to refer to this element. The important characteristics of the element are:

1. Bilinear variation in the planar co-ordinates for the thickness of the element is permitted.
2. Bilinear variation in the planar co-ordinates for the temperature in the element is provided for in the element dependent subroutine for calculation of the thermal load vector. However, only the calculations for membrane effect utilize this feature at present; the calculations for the bending effect will be operational only after provision for specification of grid point temperature gradients is made.

Element Geometry and Derivation of Matrices

The geometry of the element is shown in fig. 3. The shell shape is defined by the height, $z(x,y)$, above the base plane in which x,y are taken as local co-ordinates and X,Y,Z are the global co-ordinates. The shell surface over each element is approximated as a quadratic polynomial

$$z(x,y) = h_1 + h_2 x + h_3 y + h_4 x^2 + h_5 xy + h_6 y^2 \qquad (6)$$

The membrane displacements u and v are assumed to vary as quadratic polynomials as in equations (1) and (2). The normal displacement w is assumed to vary as a quintic polynomial as in equation (5). From the shallow shell theory of Novozhilov (ref. 5), the membrane strains are given by

$$\left.\begin{array}{l} \varepsilon_{xx} = u_{,x} - z_{,xx} w \\[12pt] \varepsilon_{yy} = v_{,y} - z_{,yy} w \\[12pt] \varepsilon_{xy} = u_{,y} + v_{,x} - 2z_{,xy} w \end{array}\right\} \qquad (7)$$

The element has 6 grid points and 5 d.o.f. per grid point in the element co-ordinate system. As in the case of TRPLT1, the three constraint equations are introduced so that the normal slope varies cubically along each edge. Additional details of formulation of the element and application of the element to solve shell problems are given in reference 6. Two new bulk data cards for the element, viz , CTRSHL and PTRSHL cards, that have been added to the NASTRAN bulk data deck are given in Appendix A. Two example problems illustrating the use of the shell element is given in Appendix D.

CONCLUDING REMARKS

Two plate type elements, a linear strain membrane element (TRIM6) and a higher order bending element (TRPLT1), have been added to a proprietary version of NASTRAN Level 16.0. The addition of a triangular shallow shell element is in advanced stages of completion. The theoretical formulation, bulk data information and example problems pertaining to these elements are discussed in this paper. The example problems illustrate the excellent accuracy of these elements for solution of plate and shell problems for statics, vibration and buckling rigid formats.
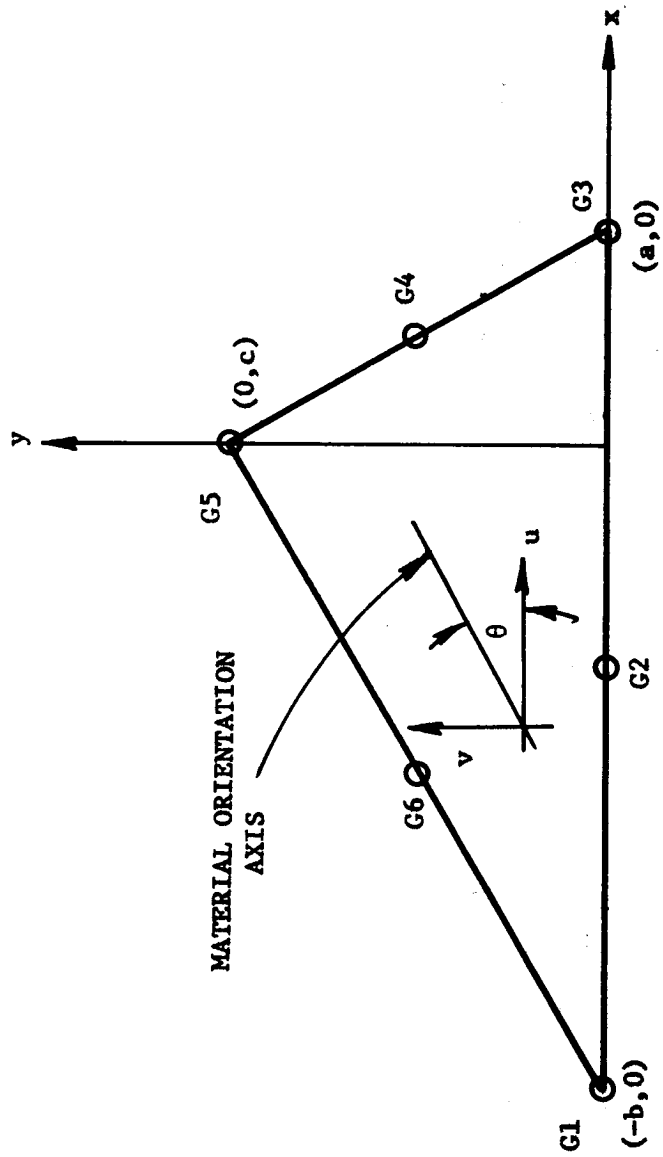
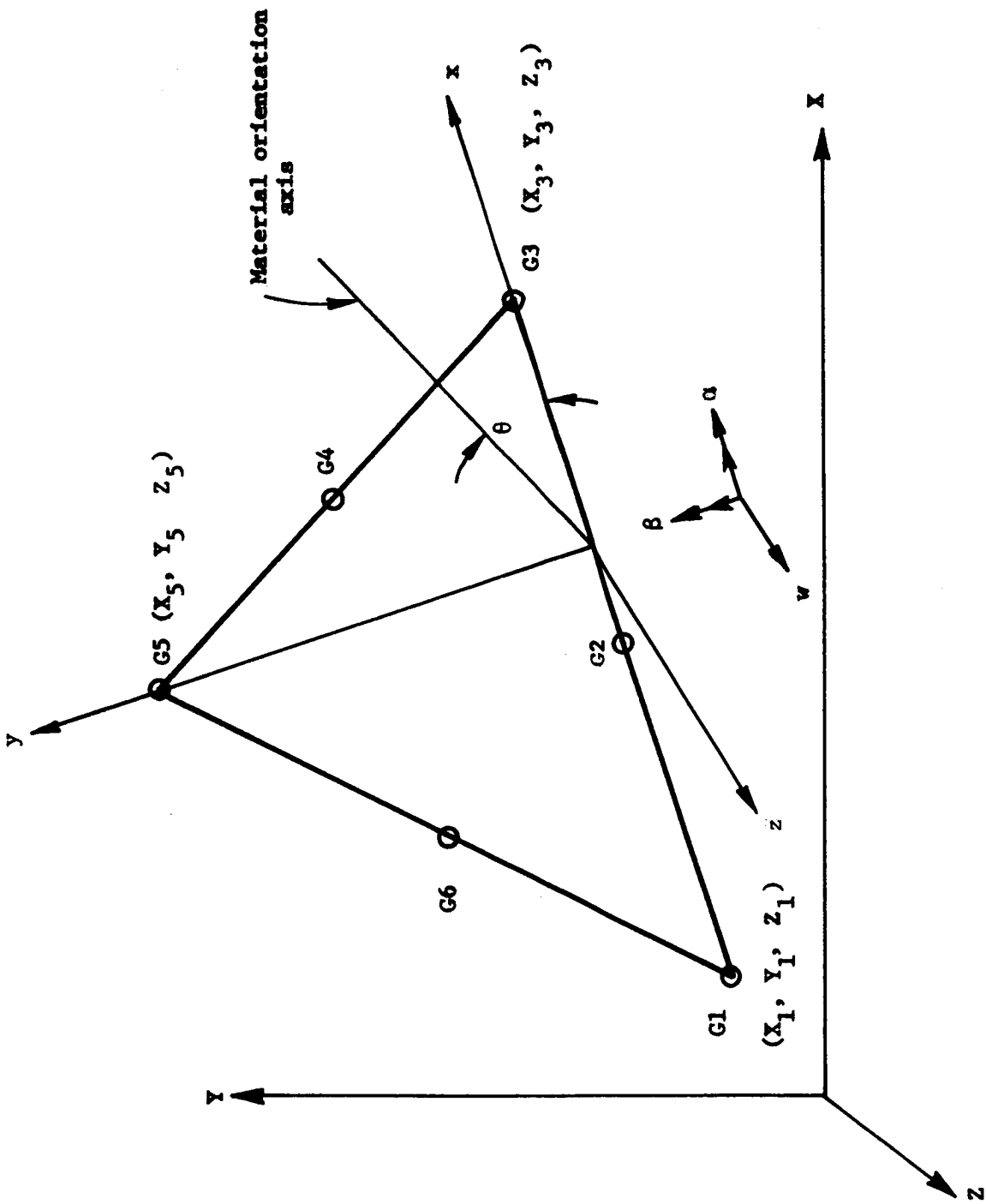**Figure 1.** TRIM6 membrane element in element coordinate system.

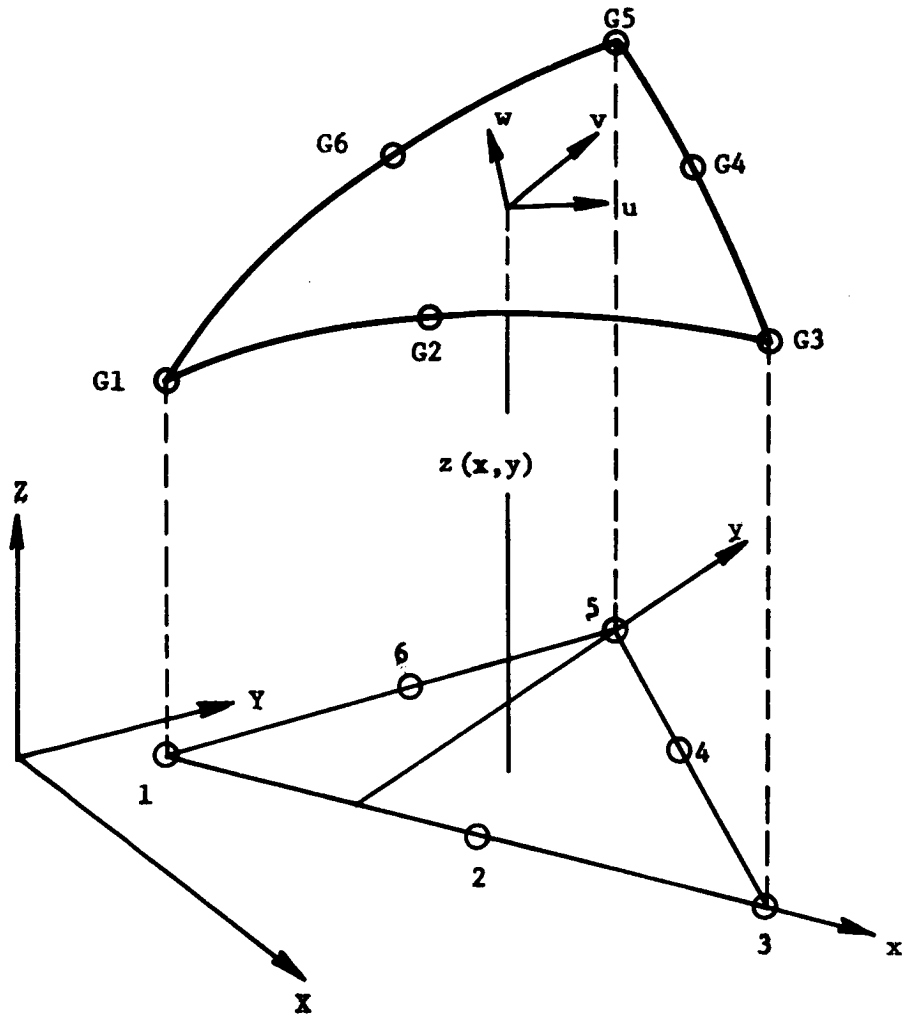Figure 2.   TRPLT1 triangular bending element geometry.

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Figure 3. TRSHL   shell element geometry and coordinate systems.

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BULK DATA DECK

Input Data Card **CTRIM6**   Triangular Element Connection

Description: Defines a linear strain triangular membrane element
(TRIM6) of the structrual model.

Format and Examples:

1	2	3	4	5	6	7	8	9	10
CTRIM6	EID	PID	G1	G2	G3	G4	G5	G6	+abc
CTRIM6	220	666	100	110	120	210	220	320	+C3

+abc	TH								
+C3	1.2								

Field	Contents
EID	Element identification number (Integer > 0)
PID	Identification number of PTRIM6 property card (Default is EID) (Integer > 0)
G1, G2, G3, G4, G5, G6	Grid point identification numbers of connection points (Integers > 0; G1 ≠ G2 ≠ G3 ≠ G4 ≠ G5 ≠ G6)
TH	Material property orientation angle in degrees (Real) The sketch below gives the sign convention for TH.
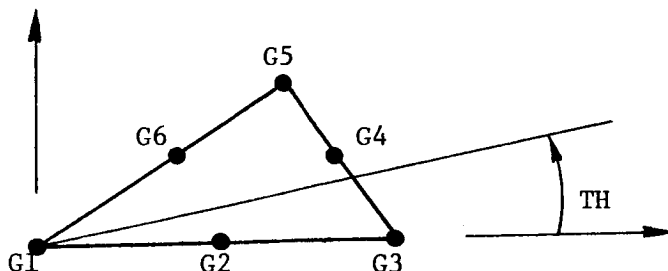
Remarks:

1. Element identification numbers must be unique with respect to <u>all</u> other element identification numbers.
2. Interior angles must be less than 180°.
3. The gridpoints must be listed consecutively going around the perimeter in an anticlockwise direction and starting at a vertex.

4. G2, G4, and G6 are assumed to lie at the midpoints of the sides. The locations of these gridpoints (on GRID Bulk Data cards) are used only for global coordinate system definition, GPWG (weight generator module), centrifugal forces, and deformed structure plotting.
5. Continuation card must be present.

Input Data Card <u>PTRIM6</u>   .Triangular Membrane Property

<u>Description</u>: Used to define the properties of a linear strain trian-
gular membrane element. Referenced by the CTRIM6 card. No bending
properties are included.

<u>Format and Example</u>:

1	2	3	4	5	6	7	8	9	10
PTRIM6	PID	MID	T1	T3	T5	NSM			
PTRIM6	666	999	1.17	2.52	3.84	8.3			

<u>Field</u>	<u>Contents</u>
PID	Property identification number (integer > 0)
MID	Material identification number (integer .> 0)
T1, T3, T5	Membrane thickness at the vertices of the element (Real)
NSM	Nonstructural mass per unit area (Real)

<u>Remarks</u>:

1. All PTRIM6 cards must have unique property identification
    numbers.

2. For structural problems, the material may be MAT1 or MAT2.

3. The thickness varies linearly over the triangle. If T3 or
    T5 is specified 0.0 or blank, it will be set equal to T1.

Input Data Card   <u>CTRPLT1</u>   Triangular Element Connection

<u>Description</u>:   Defines a triangular bending element (TRPLT1) of the structural model.
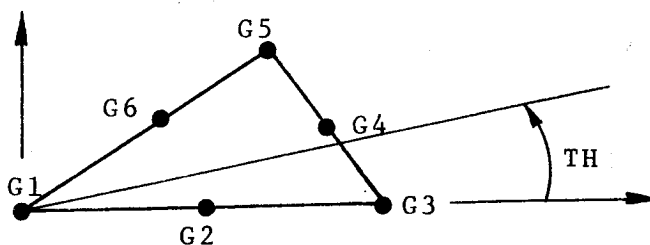
<u>Format and Example</u>:

1	2	3	4	5	6	7	8	9	10
CTRPLT1	EID	PID	G1	G2	G3	G4	G5	G6	+abc
CTRPLT1	160	20	120	10	30	40	70	110	+ABC

+abc	TH								
+ABC	16.2								

Field	Contents

EID   Element identification number (Integer > 0)

PID   Identification number of PTRPLT1 property card (Default is EID) (Integer > 0)

G1, G2, G3,   Grid point identification numbers of connection
G4, G5, G6   points (integer > 0: G1 ≠ G2 ≠ G3 ≠ G4 ≠ G5 ≠ G6)

TH   Material property orientation angle in degrees (Real)-
The sketch below gives the sign convention for TH.

<u>Remarks</u>:

1. Element identification numbers must be unique with respect to all other element identification numbers.

2. Interior angles must be less than 180°.

3. The grid points must be listed consecutively going around the perimeter in an anticlockwise direction and starting at a vertex.

4. Continuation card must be present.

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BULK DATA DECK

Input Data Card   <u>PTRPLT1</u>   Triangular Plate Property

<u>Description</u>: Used to define the bending properties of a triangular plate element. Referenced by the CTRPLT1 card. No membrane properties are included.

<u>Format and Example</u>:

1	2	3	4	5	6	7	8	9	10
PTRPLT1	PID	MID1	I1	I3	I5	MID2	TS1	TS3	+abc
PTRPLT1	15	25	20.0	30.0	40.0	35	3.0	1.15	+PQR

+abc	TS5	NSM	Z11	Z21	Z13	Z23	Z15	Z25	
+PQR	1.0	9.0	1.5	-1.5	2.0	-2.0	2.5	-2.5	

Field	Contents
PID	Property identification number (Integer > 0)
MID1	Material identification number for bending (Integer > 0)
I1, I3, I5	Bending area moment of inertia per unit width at the grid points G1, G3, G5, respectively (Real > 0.0) ; $I1 = T_1^3/12$, $I3 = T_3^3/12$, $I5 = T_5^3/12$ where T_1, T_3, T_5 are the thickness of the element at the vertices, respectively.
MID2	Material identification number for transverse shear (Integer > 0)
TS1, TS3, TS5	Transverse Shear thickness at the grid points G1, G3, G5, respectively.
NSM	Nonstructural mass per unit area (Real)
Z11, Z21, Z13 Z23, Z15, Z25	Fiber distances for stress computation at grid points G1, G3, G5, respectively, positive according to the right-hand sequence defined on the CTRPLT1 card (Real)

<u>Remarks</u>:

1. All PTRPLT1 cards must have unique property identification numbers.

2. If TS1 is zero, the element is assumed to be rigid in transverse shear.

3. If TS3 or TS5 is 0.0 or blank, it will be set equal to TS1.

4. IF I3 or I5 is 0.0 or blank, it will be set equal to I1.

5. The stresses at the centroid are computed at the top and bottom fibers.

Input Data Card **CTRSHL** Triangular Shell Element Connection

Description: Defines a triangular thin shallow shell element (TRSHL) of the structural model.
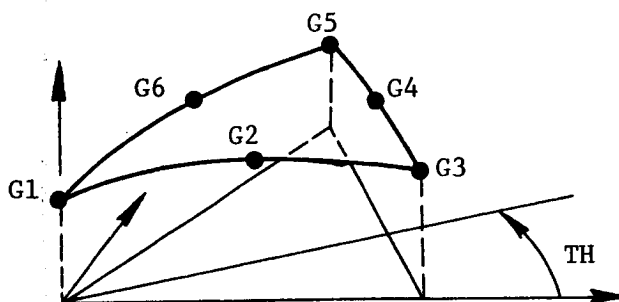
Format and Example:

1	2	3	4	5	6	7	8	9	10
CTRSHL	EID	PID	G1	G2	G3	G4	G5	G6	+abc
CTRPLT	160	20	120	10	30	40	70	110	+ABC

+abc	TH								
+ABC	16.2								

Field	Contents
EID	Element identification number (Integer > 0)
PID	Identification number of PTRSHL property card (Default is EID) (Integer > 0)
G1, G2, G3, G4, G5, G6	Grid point identification numbers of connection points (Integer > 0: G1 ≠ G2 ≠ G3 ≠ G4 ≠ G5 ≠ G6)
TH	Material property orientation angle in degrees (Real) – The sketch below gives the sign convention for TH.

Remarks:

1. Element identification numbers must be unique with respect to all other element identification numbers.

2. Interior angles must be less than 180°.

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3. The grid points must be listed consecutively going around the perimeter in an anticlockwise direction and starting at a vertex.

4. Continuation card must be present.

Input Data Card <u>PTRSHL</u>   Triangular Shell Property

<u>Description</u>: Used to define the bending properties of a triangular shell element. Referenced by the CTRSHL card.

<u>Format and Example</u>:

1	2	3	4	5	6	7	8	9	10
PTRSHL	PID	MID1	T1	T3	T5	MID2	I1	I3	+abc
PTRSHL	10	20	3.0	6.0	4.0	30	2.25	18.0	+PQR

+abc	I5	MID3	TS1	TS3	TS5	NSM	Z11	Z21	+def
+PQR	5.33	40	2.5	5.0	3.5	50	1.5	-1.5	+STU

+def	Z13	Z23	Z15	Z25					
+STU	3.0	-3.0	2.0	-2.0					

Field	Content
PID	Property Identification number (Integer > 0)
MID	Material identification number for membrane effect (Integer > 0)
T1, T3, T5	Thickness for membrane action at vertices 1,3,5 of the elements (Real > 0.0)
MID2	Material identification number for bending effects (Integer > 0)
I1, I3, I5	Area moments of inertia of the element at the vertices 1,3,5 of the element. (Real > 0.0)
MID3	Material identification number for transverse shear (Integer > 0)
TS1, TS3, TS5	Transverse shear thickness (Real > 0.0) at the vertices 1, 3, 5, of the element.
NSM	Non-structural mass per unit area (Real)
Z11, Z12, Z13, Z23, Z15, Z25	Fiber distances for stress computation at grid points G1, G3, G5, respectively, positive according to the right-hand sequence defined on the CTRSHL card (Real > 0.0)

Remarks:

1. All PTRSHL cards must have unique property identification numbers.

2. If T3 or T5 equal to 0.0, or blank, they will be set equal to T1.

3. If I3 or I5 equal to 0.0, or blank, they will be set equal to I1.

4. If TS3 or TS5 equal to 0.0, or blank, they will be set equal to TS1.

5. If TS1 is 0.0, or blank, the element is assumed to be rigid in trasnverse shear.

6. The stresses at the centroid will be computed at the top and bottom fibers.

RESULTS OF TRIANGULAR MEMBRANE ELEMENT

The triangular membrane element is used to solve two static beam problems. The first one is a cantilever beam subjected to linear thermal gradient over the depth as shown in figure B1(a). The temperature is $188.71^\circ K$ $(-120^\circ F)$ at the top surface, and $322.04^\circ K$ $(120^\circ F)$ at the bottom surface of the beam. This problem was analyzed in reference 7 for demonstration of the isoparametric solid elements. The same material properties which were given in reference 7 are used in the analysis. The beam was modeled with eight TRIM6 elements. Results for the tip displacement using TRIM6, IHEXi and theoretical solution are shown in table B1. It is demonstrated that excellent accuracy has been obtained using TRIM6 with as few as eight elements.

The second problem is a cantilever beam of variable cross section under a concentrated force of 133.85 lb. (595.39 N) at the free end as shown in figure B1(b). The exact deflection is given by

$$
\delta_y = 3.2727 \frac{P}{E} \left\{ \frac{1}{b^2} \left[ax + \frac{bx^2}{2} - \frac{a}{b} \left\{ (a + bx) \log (a + bx) \right. \right. \right.
$$

$$
\left. \left. \left. - (a + bx) \right\} \right] + C_1 x + C_2 \right\}
$$

where

$\quad$ a = 100.0

$\quad$ b = 1.0

$\quad$ $C_1 = -(150.0 - 100 \log 150)$

$\quad$ $C_2 = -(6250.0 - 100.0 (150 \log 150 - 150)) - 50 C_1$

(B1)

Four TRIM6 elements were used to model the beam. NASTRAN and theoretical results for lateral deflections along the length of the beam are shown in table B2.

TABLE B1. TIP DEFLECTIONS FOR A CANTILEVER BEAM SUBJECTED TO LINEAR
THERMAL GRADIENT

	Tip deflection δ_y, cm
IHEX1 (216 Elements)	3.668
IHEX2 (36 Elements)	3.932
IHEX3 (8 Elements)	3.894
TRIM6 (8 Elements)	3.858
Theoretical Solution	3.762

TABLE B2. DEFLECTIONS FOR A CANTILEVER BEAM OF VARIABLE CROSS
SECTION UNDER A CONCENTRATED FORCE AT THE FREE END

$\dfrac{x}{L}$	Deflection, inch (cm)	
	NASTRAN TRIM6	Exact
0.25	.006722 (0.0171)	.006878 (0.0175)
0.50	.025601 (0.0650)	.025658 (0.0652)
0.75	.053129 (0.1349)	.053141 (0.1350)
1.00	.085662 (0.2176)	.085410 (0.2169)
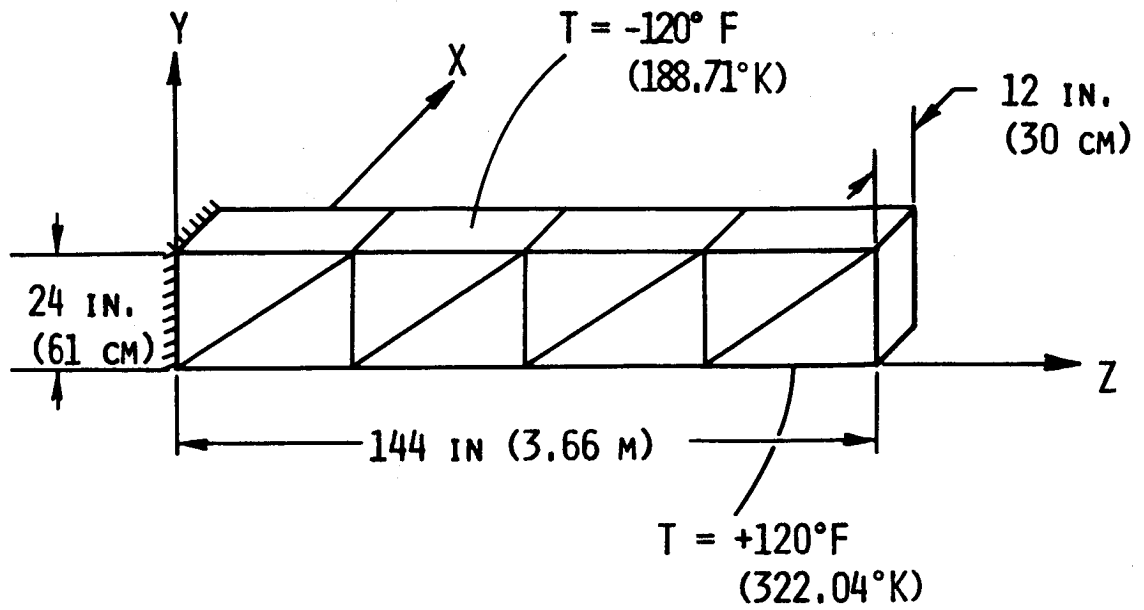
(A) UNIFORM CANTILEVER BEAM UNDER THERMAL LOADING.
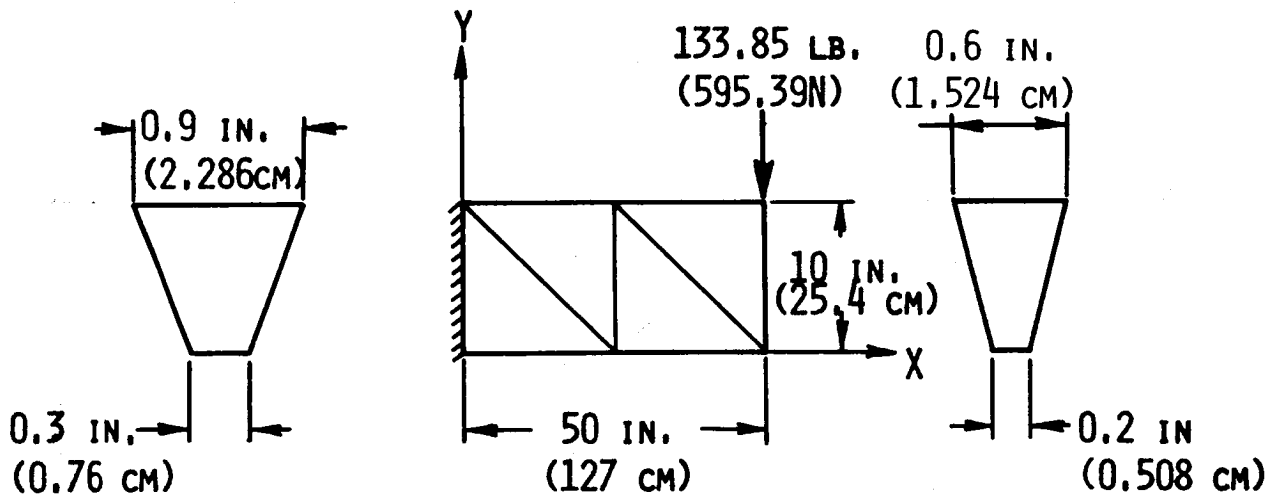
(B) TAPERED CANTILEVER BEAM UNDER CONCENTRATED LOAD.

FIGURE B1. BEAM GEOMETRY AND FINITE ELEMENT IDEALIZATION FOR THE MEMBRANE ELEMENT (TRIM6) TEST PROBLEMS.

APPENDIX C

RESULTS OF TRIANGULAR PLATE ELEMENT

The triangular bending element is used to solve problems in vibration and buckling of thin isotropic plates.

Vibration Problems

Two vibration problems were analyzed. The first one is a linearly tapered simply supported rectangular plate shown in figure C1 (a). The plate thickness is given by

$$t = t_o \left(1 + \kappa \frac{x}{a}\right) \qquad (C1)$$

where κ is a constant determining the rate of taper. Two different mesh sizes of the finite element model, 1 x 2 and 2 x 4, are used. Due to symmetry, only half of the plate is used in the analysis. Nondimensional fundamental frequencies for rectangular plates for three different aspect ratios $\frac{a}{b}$ and $\kappa = 0.5$ and 0.8 are presented in table C1. The frequency parameter is defined as:

$$\Omega = \omega a^2 \sqrt{\frac{\rho t_o}{D_o}} \qquad (C2)$$

where ω is the circular frequency, a is the length, ρ is the mass density, t_o is thickness and D_o is the bending rigidity. Analytical results from reference 8 are also shown for comparison.

The second problem is vibration of a linearly tapered rectangular

cantilever plate. The plate is idealized with a mesh size of 2 x 4,

or 16 elements, as shown in figure Cl (b). Results of frequency

parameters Ω_{mn} as defined in equation (C2), where m and n represent

the number of nodal lines perpendicular and parallel to the support,

respectively, using TRIA2 and TRPLT1 are shown in table C2. Constant

thicknesses of 0.0405 in (0.1029 cm) and 0.1215 in (0.3086 cm) was used

when modeling with TRIA2 element. Experimental data obtained by

Plunkett in reference 9 are also given. Tables Cl and C2 showed that

very good results have been obtained using the higher order plate

element. For lower modes, NASTRAN TRIA2 also gave good results.

Buckling Problems

The out-of-plane buckling of plate elements is evaluated from the

differential stiffness matrix of bending plate element TRPLT1 due

to membrane prestress effects obtained from a membrane analysis

using TRIM6 elements. To solve out-of-plane buckling of plates, a

membrane-bending combination element is necessary. TRSHL is such a

combination element with the added feature of membrane bending coupl-

ing for shell problems. For pure plate problems, where the curvature

is zero, there is no coupling between membrane and bending effects

and for such cases TRSHL reduces to a combination element. The

results of plate buckling in this section are obtained using TRSHL

elements. Since the addition of TRSHL into the NASTRAN program is

not yet complete, the results for problems in this section have been

obtained from a stand-alone version.

Three buckling problems were investigated using the triangular plate and membrane elements. The first one is buckling of a tapered column or truncated pyramid shown in figure C2 (a). The area moment of inertia at any cross section can be expressed in the form

$$I_x = I_1 \left(\frac{x}{a}\right)^4 \qquad (C3)$$

where I_1 is the moment of inertia at the top of the column ($x = a$). Results for the buckling factor for a tapered column of $\frac{I_1}{I_2} = 0.2$ from NASTRAN using TRIA2 and TRPLT1, and an analytical solution from reference 10 are given in table C3 for comparison. Monotonic convergence characteristics of the buckling factor are observed with TRSHL element although quite slow. Results given by NASTRAN TRIA2 elements appear to diverge from the exact solution. The slow convergence of TRSHL results and the apparent divergence of TRIA2 results may be due to the modeling of the beam using these plate elements instead of using beam type finite elements. The buckling factor is defined by the expression:

$$P_{cr} = \lambda \ \frac{EI_2}{L^2} \qquad (C4)$$

where I_2 is the moment of inertia at the lower end of the column.

The second problem is a simply supported square plate subjected to uniform compression in one direction. Owing to symmetry, only one

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quarter of the plate (modeled with 2 x 2 mesh size) is used as shown in figure C2 (b). Results of the buckling factor from NASTRAN TRIA2 and TRSHL elements and the exact solution are shown in table C4. The nondimensional buckling factor λ is represented by the formula

$$N_{cr} = \lambda \; \frac{\pi^2 D}{b^2} \tag{C5}$$

The third problem considered is buckling of·a simply supported rectangular plate of aspect ratio $\frac{a}{b}$ = 0.8 under inplane bending loading shown in figure C2 (c). Due to symmetry, only half of the plate is used in the analysis. NASTRAN results using TRIA2 and TRSHL with different mesh sizes are shown in table C5, along with analytical results from reference 10. Table C5 clearly shows that the TRSHL elements gave a much better prediction of the critical buckling load than the TRIA2 elements.

TABLE C1. FUNDAMENTAL FREQUENCY FOR LINEARLY TAPERED RECTANGULAR
PLATES SIMPLY SUPPORTED ON ALL EDGES: $\nu = 0.3$

Aspect ratio $\dfrac{a}{b}$	NASTRAN TRPLT1 Finite element layout	Frequency parameter $\Omega = \omega a^2 \left(\dfrac{\rho t_0}{D_0}\right)^{1/2}$	
		Taper rate $\kappa = 0.5$	Taper rate $\kappa = 0.8$
0.5	1 x 2	14.662	16.242
	Theory	15.304	16.994
1.0	1 x 2	24.171	26.901
	2 x 4	24.454	—
	Theory	24.556	27.354
2.0	1 x 2	58.560	64.770
	2 x 4	60.346	—
	Theory	60.982	67.500

TABLE C2. FREQUENCY PARAMETERS FOR A LINEARLY TAPERED RECTANGULAR CANTILEVER PLATE; $\nu = 0.3$

Mode		Frequency parameter $\Omega_{mn} = \omega_{mn} \, a^2 \left(\dfrac{\rho t_0}{D_0}\right)^{1/2}$		
		NASTRAN		Experiment
m	n	TRIA2	TRPLT1	
0	0	2.28	2.25	2.47
1	0	9.8	10.0	10.6
0	1	14.5	13.6	14.5
1	1	23.8	27.0	28.7
0	2	35.9	32.8	34.4
0	3	51.5	47.3	47.4
2	0	31.0	53.3	52.5
1	2	64.0	57.7	54.0

TABLE C3. BUCKLING FACTOR FOR A TAPERED COLUMN
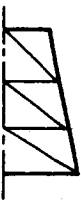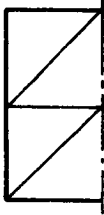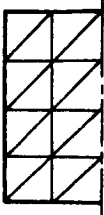
	Buckling factor $\lambda = \dfrac{P_{cr}L^2}{EI_2}$		
	Finite Element layout		
TRIA2	1.4242	1.3618	1.3420
TRSHL	1.6437	1.6050	1.5853
Theory	1.505		

TABLE C4. BUCKLING FACTOR FOR SIMPLY SUPPORTED SQUARE PLATE UNIFORMLY COMPRESSED IN ONE DIRECTION; $\nu = 0.3$

	Buckling factor $\lambda = \dfrac{N_{cr}b^2}{\pi^2 D}$
TRIA2	4.0356
TRSHL	3.9779
Exact	4.0000

TABLE C5. BUCKLING FACTOR FOR A SIMPLY SUPPORTED RECTANGULAR PLATE
OF ASPECT RATIO 0.8 UNDER INPLANE BENDING; $\nu = 0.3$

	Buckling factor $\lambda = \dfrac{(N_o)cr \, b^2}{\pi^2 D}$		
	Finite element layout		
TRIA2	29.7815	35.3289	23.8702
TRSHL	24.5507	24.1103	24.1708
Theory	24.4		

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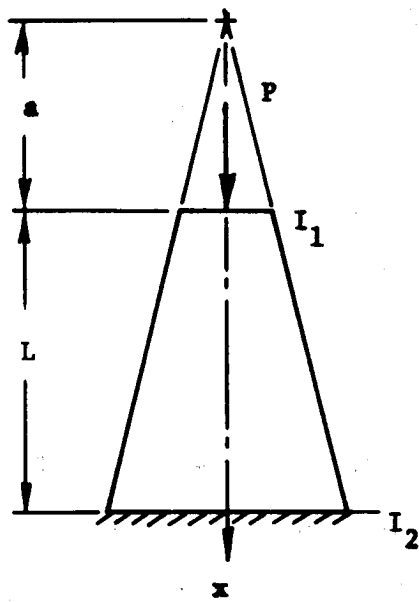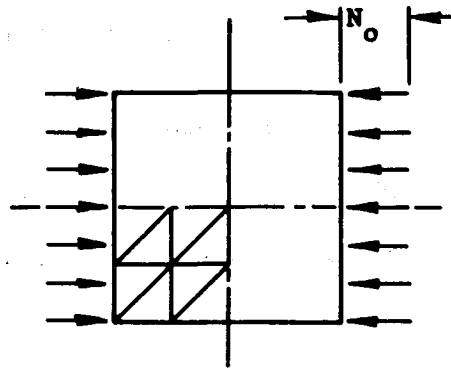**(a) Simply supported plate.**
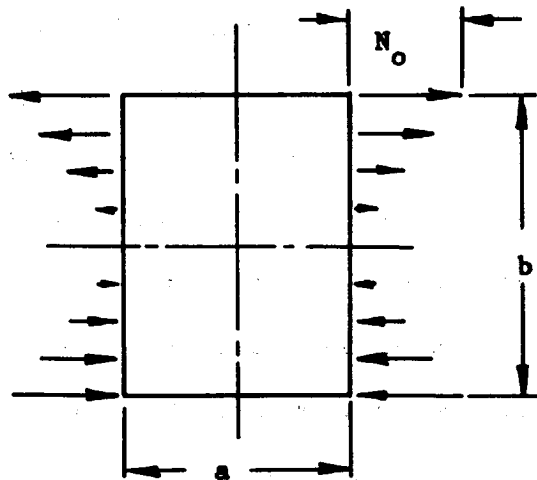
(b) Cantilever plate.
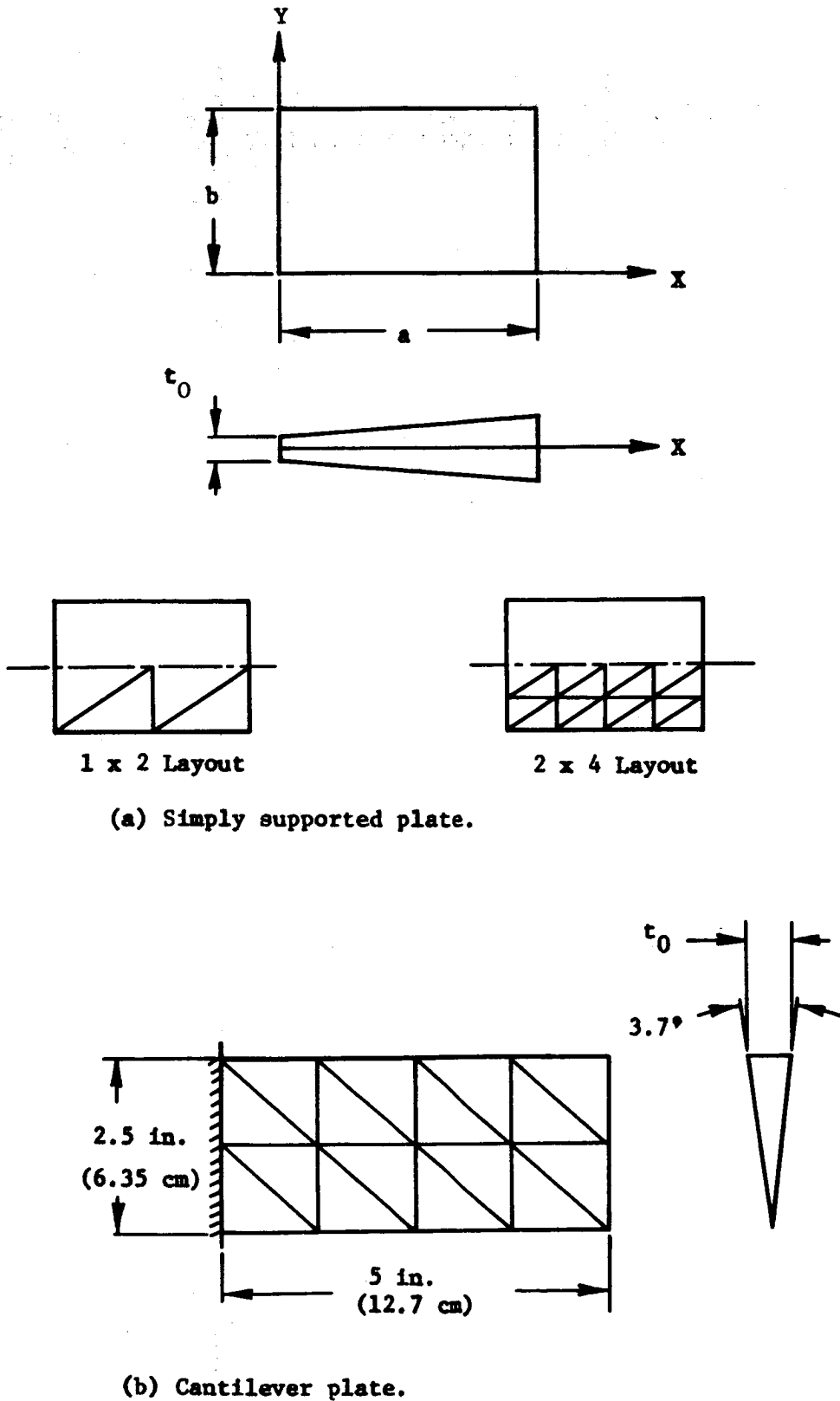
Figure C1. Plate geometry and finite element idealization for the TRPLT1 element test problems.

(a) Tapered column.

(b) Simply supported square plate under uniform compression.

(c) Simply supported plate under inplane bending.

Figure C2. Column and plate geometry for TRSHL element buckling test problems.

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APPENDIX D

RESULTS OF TRIANGULAR SHALLOW SHELL ELEMENT

Two problems, (1) that of a spherical cap, and (2) that of a cylindrical shell roof, are considered. These are the same two example problems analyzed in reference 11.

Good agreement in deflections at the center of the cap is obtained even with relatively coarse mesh sizes as shown in table D1. Even though the results appear to be oscillating about the exact value, the percentage error in the converged solution is very negligible.

The geometry and the finite element idealization of the cylindrical shell roof used in the analysis is shown in figure D1. Results for the shell roof problem and the exact solution reported by Cowper et. al. (ref. 11) are given in table D2. Reasonable agreement is seen between the finite element and the exact solutions in u_A, w_B and N_{xxB}. The convergence of the deflection at the central point C appears to be very slow; this, in turn affects the convergence of M_{xxC} and M_{yyC}. A non-uniform finer mesh near the center of the shell, C, is likely to yield converging value to the deflection w_C; the moments M_{xxC} and M_{yyC} may then approach the exact values.

TABLE D1. CENTER DEFLECTIONS FOR SPHERICAL CAP

Finite Element Grids	Spherical Cap Problem	
	$Rt/L^2 = 0.02$	$Rt/L^2 = 0.005$
	Values of $\dfrac{Etw_c}{P_o R^2}$	Values of $\dfrac{Etw_c}{P_o R^2}$
1 x 1	1.151069	1.139514
2 x 2	1.007740	0.991776
3 x 3	1.004518	1.001766
4 x 4	1.004365	1.000842
Exact	1.009785	1.000429

TABLE D2. RESULTS FOR A CYLINDRICAL SHELL ROOF

| Finite Element Grids	$10u_A$ (in.)	w_B (in.)	$10v_B$ (in.)	$10w_C$ (in.)	$10^{-3}N_{xxB}$ (lb./in.)	$10^{-3}M_{yyC}$ (lb. in./in.)	$10^{-2}M_{xxC}$ (lb. in./in.)
1 x 1	-0.45168	-0.29100	-2.48424	-4.0700	2.4659	0.7685	2.8520
2 x 2	-0.7812	-12516	-4.77312	-2.1344	4.2801	-0.9395	-0.8896
3 x 3	-1.09590	-2.49876	-7.12872	-1.3606	5.4948	-2.0283	-1.1136
4 x 4	-1.2939	-3.4332	-8.57580	2.2224	6.0277	-2.3828	-1.7912
5 x 5	-1.4160	-3.88152	-9.29000	2.8182	6.3279	-2.3538	-1.9770
6 x 6	-1.4733	-4.09176	-9.76992	3.0900	6.4444	-2.3242	-2.0638
Exact	-1.51325	-4.09916	-8.76147	5.2494	6.4124	-2.0562	-0.9272
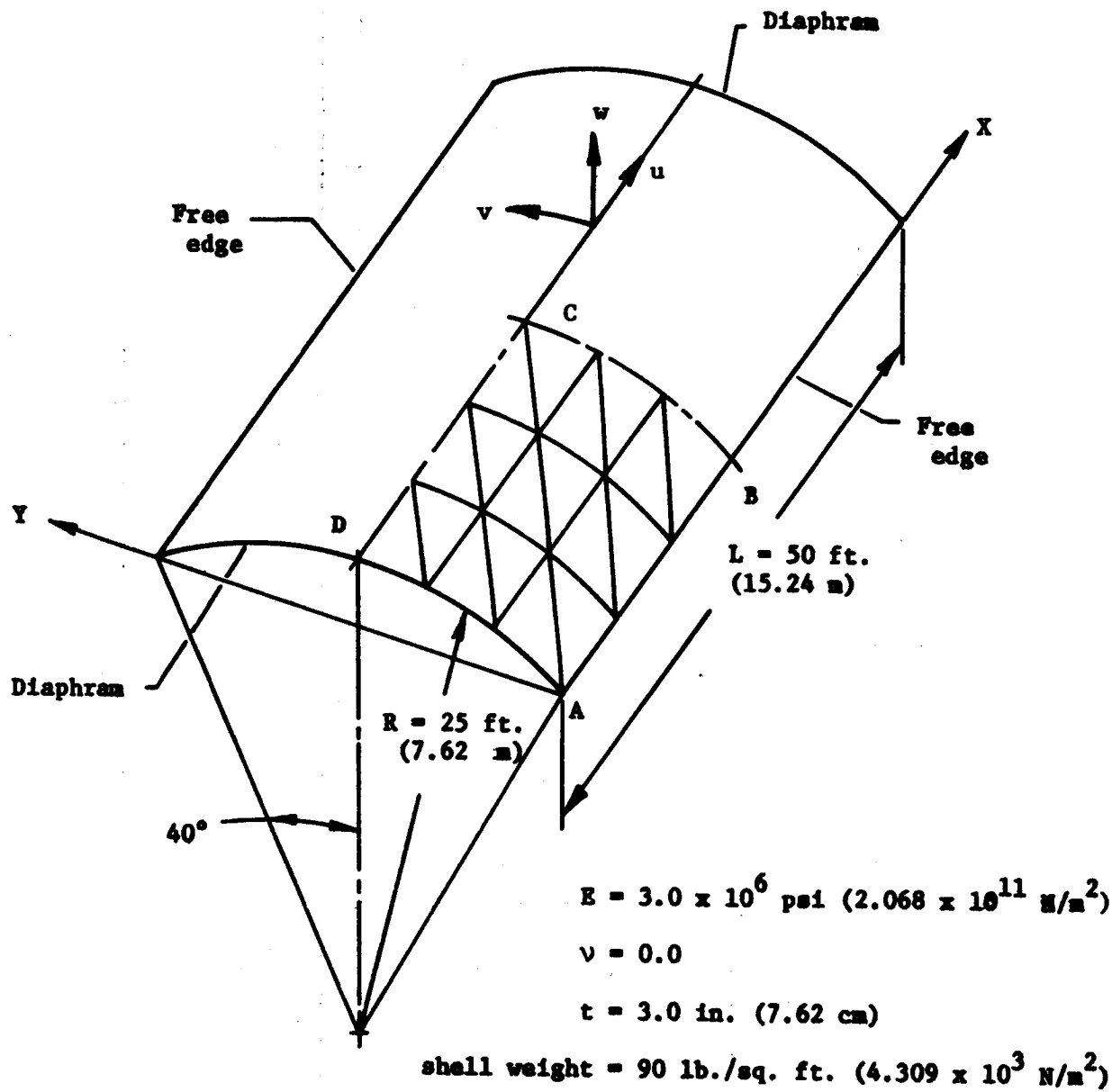
Figure D1. Geometry of cylindrical shell roof and finite element idealization.

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