CURVE FITTING OF AEROELASTIC TRANSIENT RESPONSE DATA WITH EXPONENTIAL FUNCTIONS

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SUMMARY

The problem of extracting frequency, damping, amplitude, and phase information from unforced transient response data is considered. These quantities are obtained from the parameters determined by fitting the digitized time-history data in a least-squares sense with complex exponential functions. The highlights of the method are described and the results of several test cases are presented. The effects of noise are considered both by using analytical examples with random noise and by estimating the standard deviation of the parameters from maximum-likelihood theory.

INTRODUCTION

One of the fundamental tasks in flutter testing is the determination of the frequency and damping of aeroelastic modes. Transient response or free decay records are often used for extracting this information and may be generated directly by a method such as the resonant dwell and cut (e.g., see ref. 1), or indirectly through the use of autocorrelation or randomdec types of data-reduction techniques (refs. 2 and 3). Graphical or manual techniques have often been used to determine frequency and damping, but, with the widespread use of automated data-reduction procedures, numerical curve-fitting techniques of complex exponential functions or damped sine waves are frequently used. There may be strong interactions between the curve-fitting method and the data-collection process, especially in the areas of record length requirements and specifications of noise level and distortion. Several procedures are currently available for the curve-fitting process (refs. 4 to 6). The purpose of this paper is to describe a method that takes a somewhat different approach from the previous works. The emphasis here is on developing a nearly real-time digital technique that is not only computationally fast but is also stable in the presence of real-world noise or contamination effects. A simple direct search technique for obtaining a least-squares fit using exponential functions has been developed and is presented. The application to several test cases is presented and discussed. Some effects of measurement noise are evaluated by comparing test-case results for different signal-to-noise ratios, and by developing estimates of the standard deviations of the parameters from maximum-likelihood theory (ref. 7, e.g.).

It should be kept in mind that although in a practical engineering sense the use of exponential functions for the analysis of data may be satisfactory, the aeroelastic equations are not strictly constant-coefficient ordinary differential equations (ref. 8) and may involve other functions. Furthermore, the extrapolation of damping measured at subcritical conditions to flutter may also have shortcomings. For example, a case presented in reference 9 indicated a slope and curvature away from a flutter crossing in a plot of damping against velocity, even up to within 2 percent of the flutter speed.

SYMBOLS

a_0	coefficient in curve fit, the offset or static value (eq. (1))
a_k	coefficient of kth cosine term in curve fit (eq. (1))
b_k	coefficient of kth sine term in curve fit (eq. (1))
E	mean-squared error (eq. (2))
$\hat{E}$	expected value (eq. (3))
f	frequency, Hz
f_i	ith data point of digitized time history
i	data point index, 1 to N
j	parameter index
K	number of modes in curve fit
k	modal index, 1 to K
N	number of data points in digitized time history
R_1	output error covariance matrix
$\hat{S}$	parameter sensitivity matrix
t	time, seconds
V	velocity
V_f	flutter velocity
Y	curve-fitting expression (eq. (1))
ζ	fraction of critical damping
η	damping coefficient (eq. (1))
ω	frequency, rad/sec

ANALYSIS

Least-Squares Fitting Procedure

Given a free decay record containing the response of one or more vibration modes in the form of a digitized time history, the problem is to determine the modal damping, frequency, amplitude, and phase of each mode. A least-squares curve fit is made with complex exponential functions (or damped sine waves) in the form

$$Y(t) = a_0 + \sum_{k=1}^{K} e^{-\eta_k t} (a_k \cos \omega_k t + b_k \sin \omega_k t) \tag{1}$$

by minimizing the squared-error difference between the output fit $Y(t_i)$ and the input time history f_i. The error is given by

$$E = \sum_{i=1}^{N} [Y(t_i) - f_i]^2 \tag{2}$$

Inspection of equation (1) shows that if η_k and ω_k are preassigned, it is possible to compute a_0, a_k, and b_k by solving a linear least-squares problem. The nonlinear parameters η_k and ω_k must be determined by some type of search algorithm. Although this is a standard nonlinear, unconstrained optimization problem for which several methods are available for trial, for simplicity a direct search technique is used to search the coordinate space (η_k, ω_k) until the values that minimize equation (2) are obtained. At each step, values for η_k and ω_k are determined, the small linear system solved, and the error recomputed.

The technique has been programed for the Xerox Sigma 5 digital computer. In the program, the coordinate stepping process proceeds as follows:

(1) A starting set of coordinates η_k, ω_k $(k = 1, \ldots, K)$ and a starting step size are furnished to the program.

(2) The error E is computed at (η_k, ω_k) and at $4K$ additional points obtained by adding and subtracting the step size to or from each value of η_k and ω_k. If the central error E is less than any of the $4K$ peripheral values of E, the step size is reduced by 75 percent, and the calculations are repeated.

(3) Otherwise, the point that gave the lowest value of E is taken to be the new central point, and the step size is increased by 10 percent.

(4) The procedure is terminated when either the step size has been reduced below a preassigned threshold or a preassigned number of steps have been executed.

The method requires starting values for η_k and ω_k. For a single mode the starters can be arbitrary. However, for the multiple-mode case, the computer time can be significantly reduced by choosing good starters. The following procedure has been found to be a reasonable way of getting starters for multiple-mode cases:

(a) Generate a one-mode solution using arbitrary starters.

(b) Compute the difference between the one-mode solution and the input data, that is, the output error. Then generate a one-mode fit to the error.

(c) Use the η_k and ω_k values from steps (a) and (b) as the starters for the two-mode solution.

(d) For higher modes, steps (b) and (c) are repeated using the difference between the current multiple-mode solution and the original data to estimate the next higher mode.

Although this procedure is cumbersome, it appears to be stable and, at least for the single-mode case, surprisingly fast. It would also be very helpful to set the method up on an interactive basis similar to the technique described in reference 10.

One of the schemes in the literature is referred to as Prony's method (ref. 4). It computes η_k and ω_k by solving a 2K-order polynomial equation whose coefficients are determined from a least-squares process. The solution for the coefficients a_0, a_k, and b_k is then determined by a linear least-squares procedure, as is done here. Since this method is elegant and computationally efficient, it was examined during the present study. However, it has been the authors' experience that although Prony's method works well for perfect data, it is so sensitive to real-world noise that it is essentially useless even for generating starters for the search algorithm.

Uncertainty Levels of Estimated Parameters

The standard deviations of the estimated parameters, or uncertainty levels, can be determined from maximum-likelihood theory (ref. 7, e.g.). This type of estimate has provided some useful results in the field of stability and control (e.g., ref. 11). Assuming only measurement noise that is Gaussian and white, the expected variance of the parameter vector is

$$\hat{E}[\Delta\vec{p}\ \Delta\vec{p}^T] = \left\{ \sum_{i=1}^{N} \left[\hat{S}^T(t_i) R_1^{-1}\ \hat{S}(t_i) \right] \right\}^{-1} \tag{3}$$

where $\hat{S}$ is the parameter sensitivity matrix, R_1 is the output error covariance matrix, here a constant, and T denotes matrix transpose. The parameter vector $\vec{p}$ is made up of a_0, a_k, b_k, η_k, and ω_k, and the sensitivity matrix is given by $S_j(t_i) = \frac{\partial Y(t_i)}{\partial p_j}$. These elements of the sensitivity matrix can be calculated by directly differentiating equation (1), and the variance is a normal output parameter. Thus, for a single channel of data, as considered here, these parameter uncertainty levels can be readily calculated after the curve-fitting process is completed.

RESULTS AND DISCUSSION

The curve-fitting method has been applied to three sets of data as test cases. The first case is a calculated damped sine wave with noise added with a random-number generator. The true answer is thus known. The second case is wind-tunnel data from the dynamic calibration of an aircraft gust vane. The third case is a set of data consisting of the subcritical randomdec signatures of the response to input noise of a two-dimensional flutter model that was implemented on an analog computer.

Analytical Test Case

The calculated data for the analytical test case with no added noise are shown in figure 1(a) and are compared with the fitted curve, which is exact in this case. For this case, the analytical input function was a single mode with offset and is given by

$$Y(t) = 1 - e^{-5t} \cos 30t$$

The curve fits for various levels of random noise are shown in figures 1(b) to 1(d). The noise level is defined as the rms level of the Gaussian noise and is given as the fraction of the maximum amplitude of the mode that is 1. The results of the curve fit are summarized in figure 2. Only modest degradation of the results is shown for reasonable values of noise level of up to 0.10 or 0.20. Also shown, as brackets on the points, are the standard deviations of the parameters, or uncertainty levels, calculated from equation (3) using the results output from the curve-fit procedure. In this case the exact modal parameters are known and it is possible to calculate a predicted uncertainty level from the exact parameters by assuming that the output error covariance is the value for the noise only. These predicted levels are shown as dashed lines. Both results give a good indication of the actual scatter, and thus the confidence level, with noise level. It might be noted that the effect of noise is larger on the coefficients a_1 and b_1 than on the damping, frequency, or offset. Thus, one must be more cautious in using the magnitude and phase information from such procedures.

These results amply demonstrate that the algorithm works well in the presence of random measurement noise. It has been the authors' experience, however, that a test case of this type does not indicate that a method will be satisfactory in practice. The noise here is random with zero mean, whereas in the real world, the effects of frequency drift, meandering means, and harmonic distortion are more severe.

Gust Vane Data

The two cases considered are from a wind-tunnel dynamic calibration of a light balsa vane used to sense atmospheric turbulence on an aircraft. The vane was mechanically displaced and released repetitively. Since the response to tunnel turbulence was a sizable fraction of the total response and the release conditions somewhat ill defined, the transients were ensemble averaged. The background noise was thus diminished, but a pure step response was not obtained. The two cases considered are called the low-damping case and the high-damping case (although the low-damping case is relatively highly damped by structural standards). The data for the low-damping case and the one-mode fit are shown in figure 3(a). The fit is reasonable, but there is some systematic deviation, particularly near the rightmost portion of the data. A two-mode fit was computed and, as shown in figure 3(b), gives a significantly improved result. The results of the one- and two-mode fits are summarized as follows:

One-mode fit

$$Y(t) = 0.0444 - e^{-39.9t}(0.239 \cos 140t - 0.452 \sin 140t)$$

Two-mode fit

$$Y(t) = 0.0450 - e^{-35.1t}(0.137 \cos 133t - 0.372 \sin 133t) - e^{-34.6t}(0.086 \cos 190t - 0.026 \sin 190t)$$

As compared with the one-mode results, the two-mode data indicate that the offset is nearly the same, the frequency of the first mode is reduced by about 5 percent, and the damping is reduced by about 10 percent, along with sizable changes in the coefficients of the first mode. The physical significance of the second mode is not clear in this case; it may be low-frequency noise that has not completely averaged out in the ensembling process. However, it is thought that the results for the first or principal mode obtained in the two-mode fit are more representative of the system response.

The results for the highly damped case are presented in figures 4(a) and 4(b). The results and trends are similar to those of the low-damping case. This case is a particularly difficult one to analyze, as it has high damping, a large offset, and a low-frequency distortion. The algorithm of this paper appears to give a reasonable result for this case.

Randomdec-Analog Flutter Data

Some subcritical randomdec signatures of the response of a two-dimensional, two-degree-of-freedom flutter model to input noise on an analog computer are also treated. The mathematical model and test setup were the same as those of the investigation of reference 9. The signatures and a one-mode curve fit are shown in figures 5(a) to 5(d) as the velocity approaches flutter. As flutter is approached the signatures show little scatter or distortion, in contrast to the lower velocities. The signatures contain two modes, but the lower frequency mode is apparently unconverged in the randomdec procedure and could not be adequately resolved by the curve-fit procedure. The results for the flutter mode are compared with the exact solution in figure 6. The agreement is quite good in both frequency and damping, with the flutter speed underpredicted by less than 1 percent, which is within the expected accuracy of the analog setup. Thus, the curve-fit procedure appears to be a practical means of analyzing randomdec signatures.

CONCLUDING REMARKS

A least-squares curve-fitting procedure to extract frequency, damping, amplitude, and phase information from free decay records has been presented. The method appears to be stable and to give reasonable results in the presence of noise. Some of the effects of noise on the parameter estimates can be assessed by calculating the uncertainty levels from maximum-likelihood theory. The method is relatively fast for a one-mode fit, generally requiring 5 to 15 seconds on a Xerox Sigma 5 computer (which would be about 1 second on a CDC 6600 computer) and thus is a candidate for a real-time method. The two-mode solution, however, requires 2 to 5 minutes, and a three-mode solution is very long to calculate. Further work is needed to accelerate the multiple-mode calculations. It would also be very helpful to set the method up on an interactive basis. Currently, the only multiple-channel capability is to fit each channel of data separately, determine a weighted mean for frequency and damping, and then recalculate the coefficients for each channel. This procedure may be satisfactory for engineering purposes, but the development to a true multiple-channel method may be desirable.

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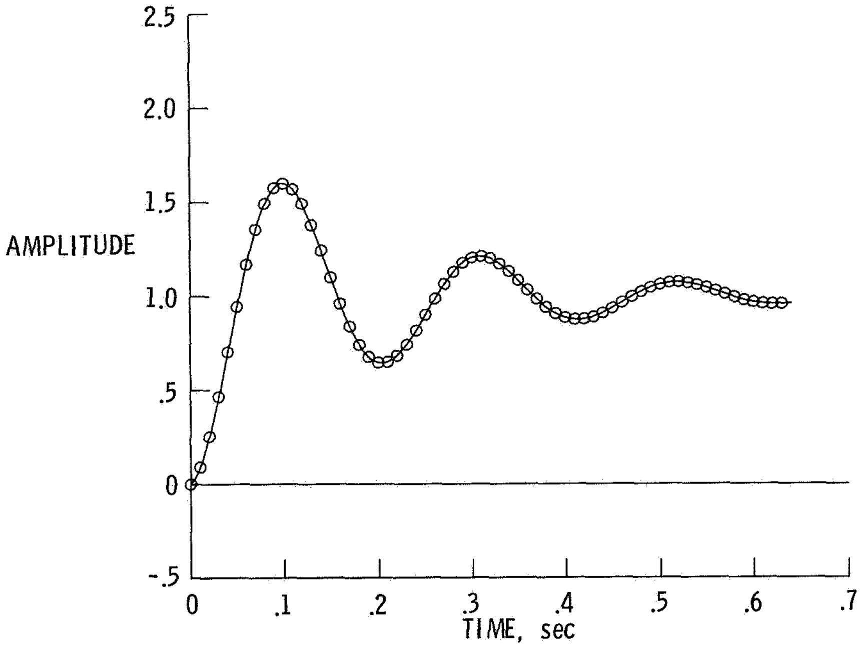
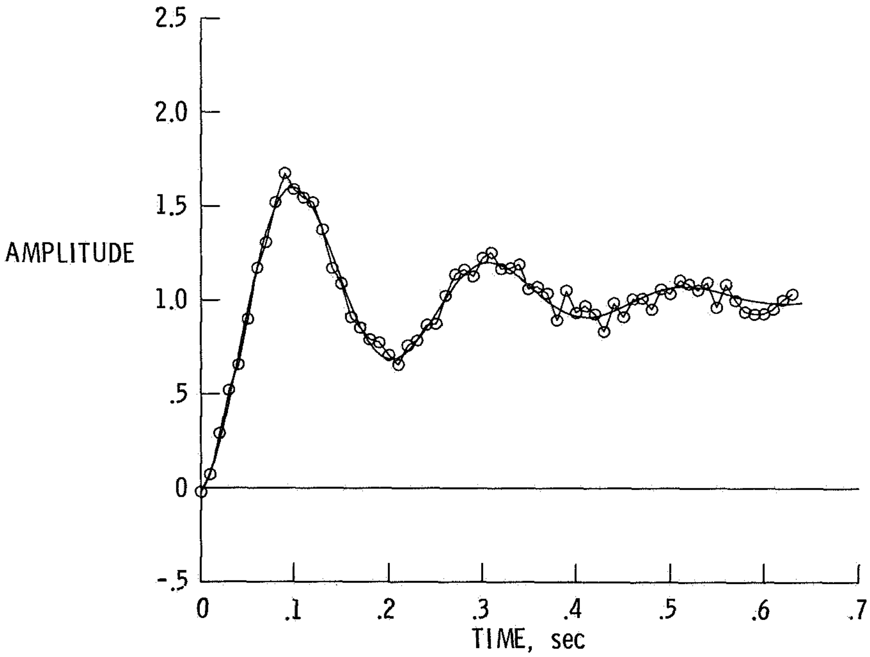
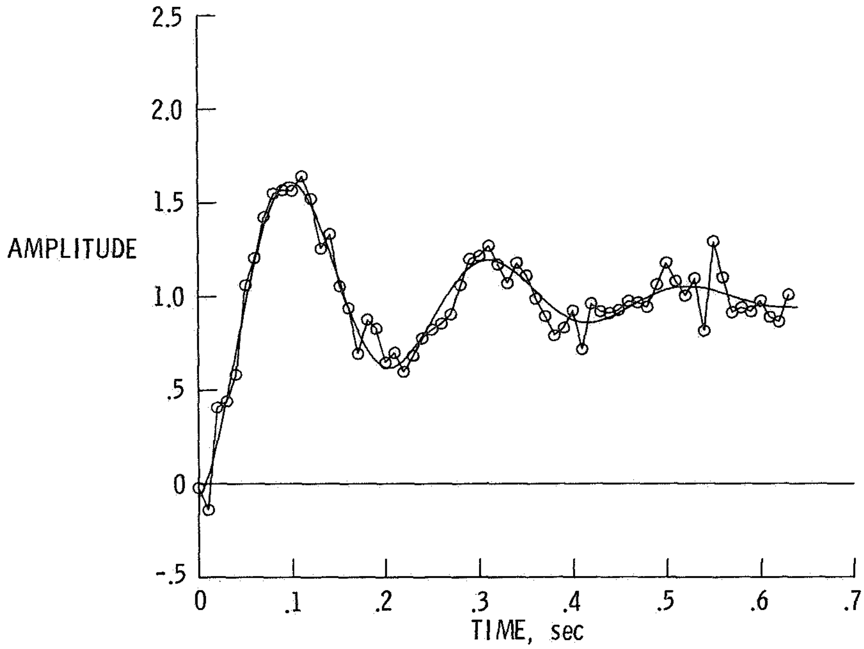


(a) No added noise.

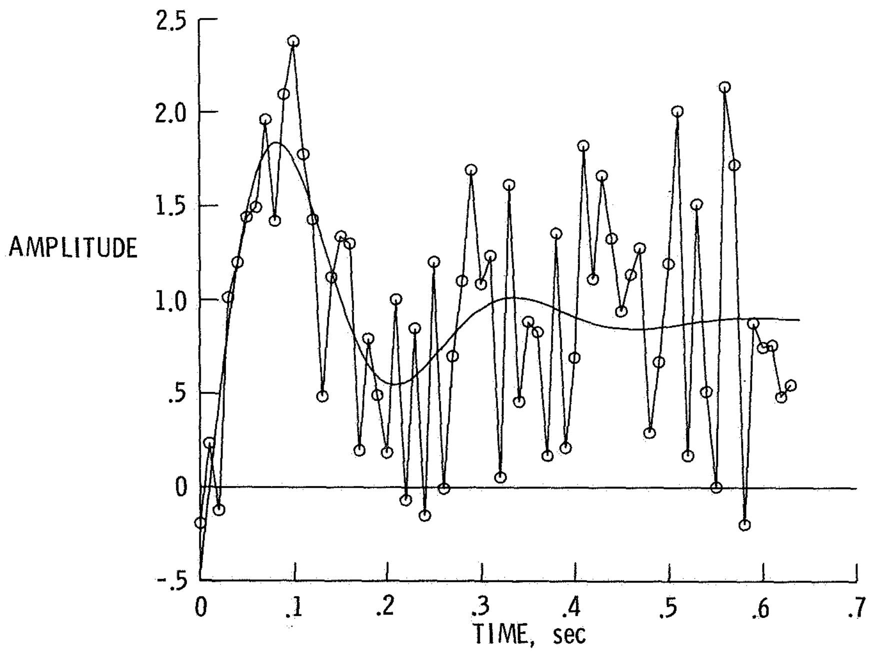


(b) Noise, 0.05.

Figure 1.- Curve fit for analytical case with random noise.

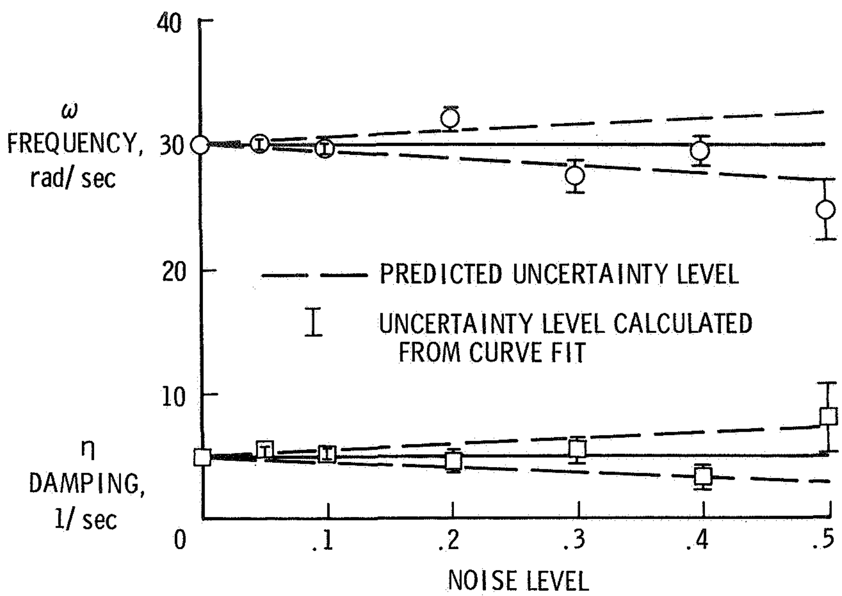


(c) Noise, 0.10.

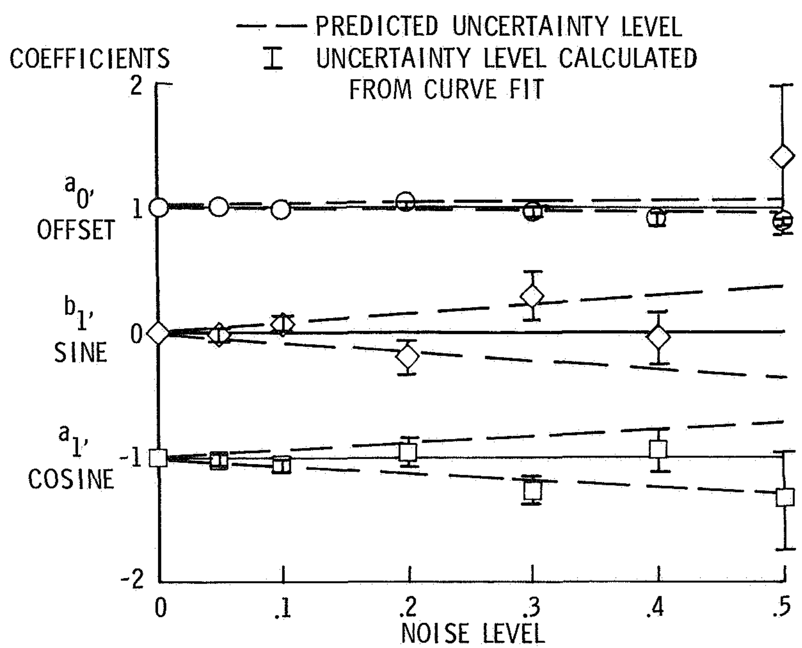


(d) Noise, 0.50.

Figure 1.- Concluded.

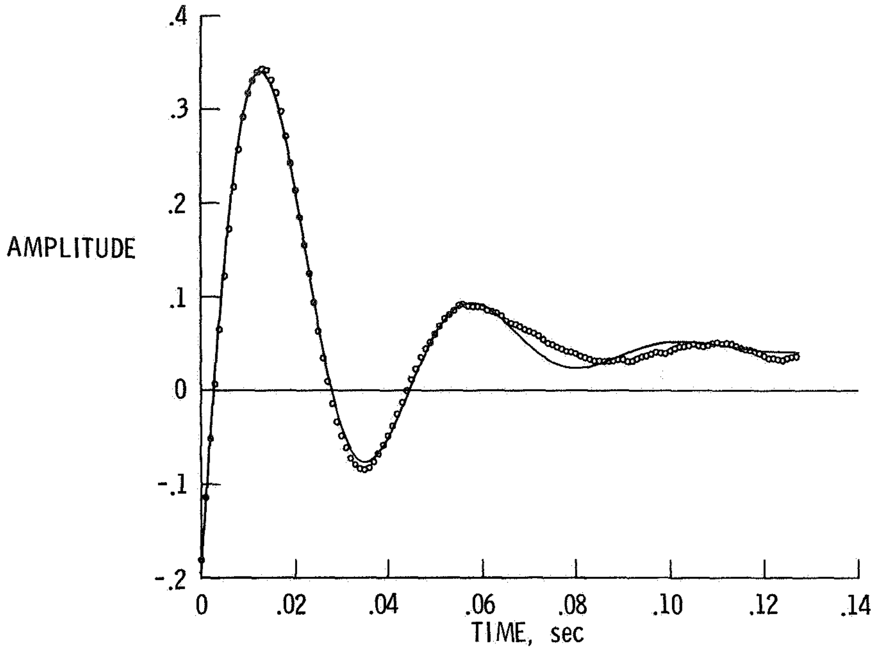


(a) Frequency and damping.

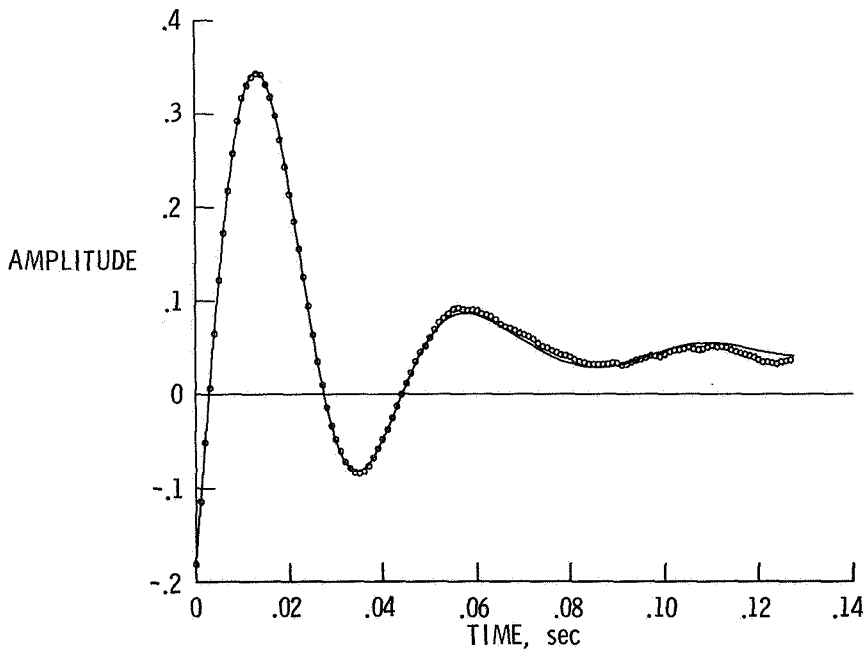


(b) Amplitude coefficients.

Figure 2.- Results of analytical case with random noise.

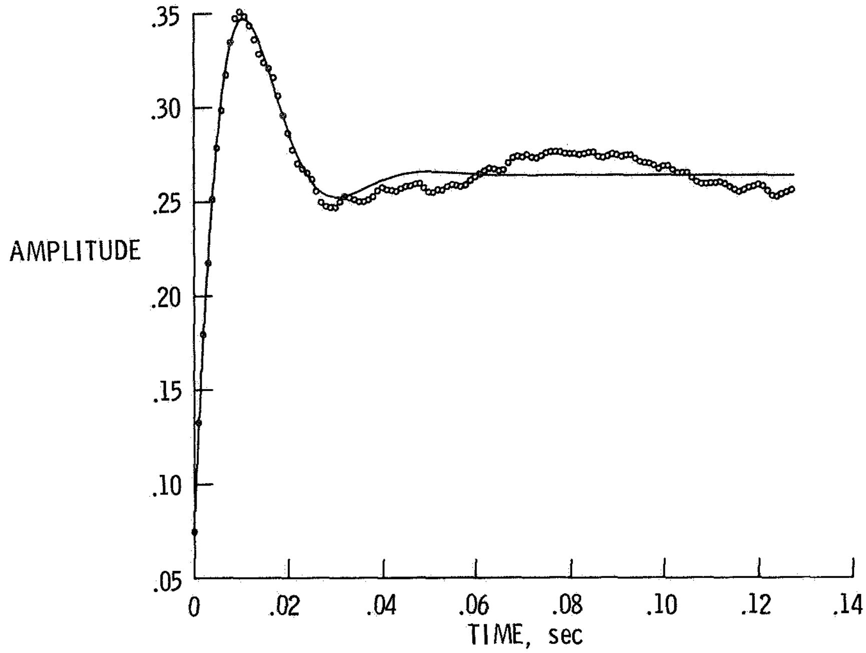


(a) One-mode fit.

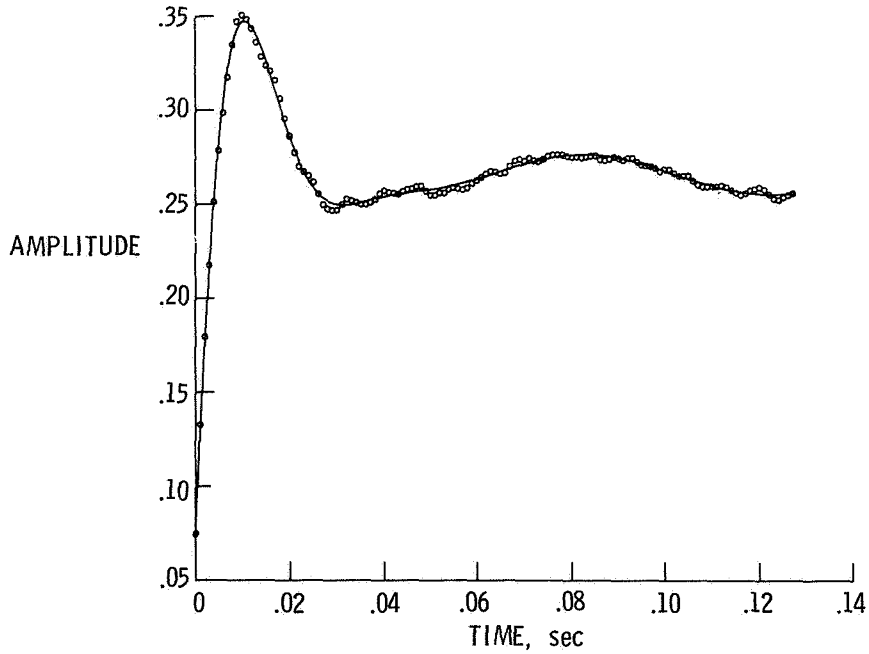


(b) Two-mode fit.

Figure 3.- Gust-vane curve fit for low-damping case.

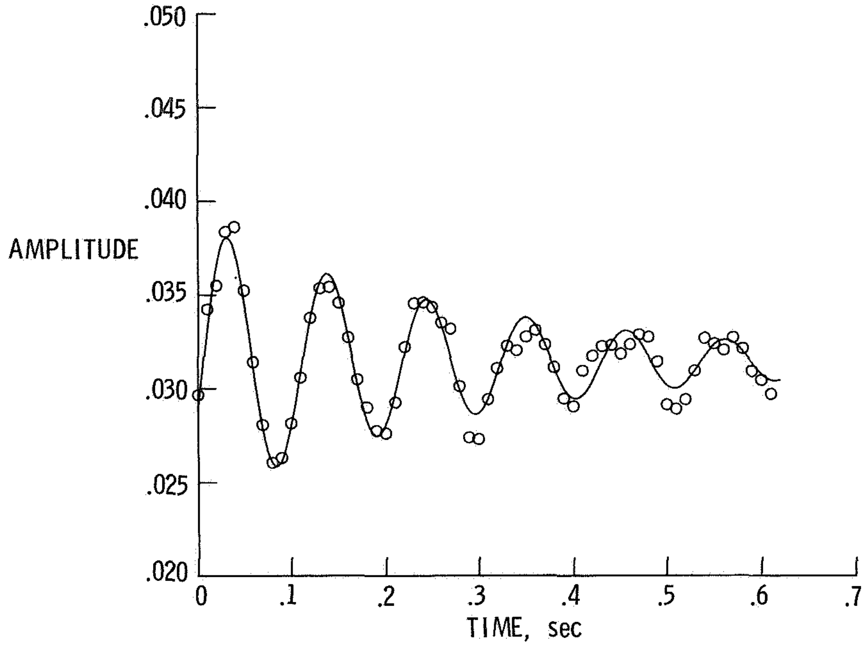


(a) One-mode fit.

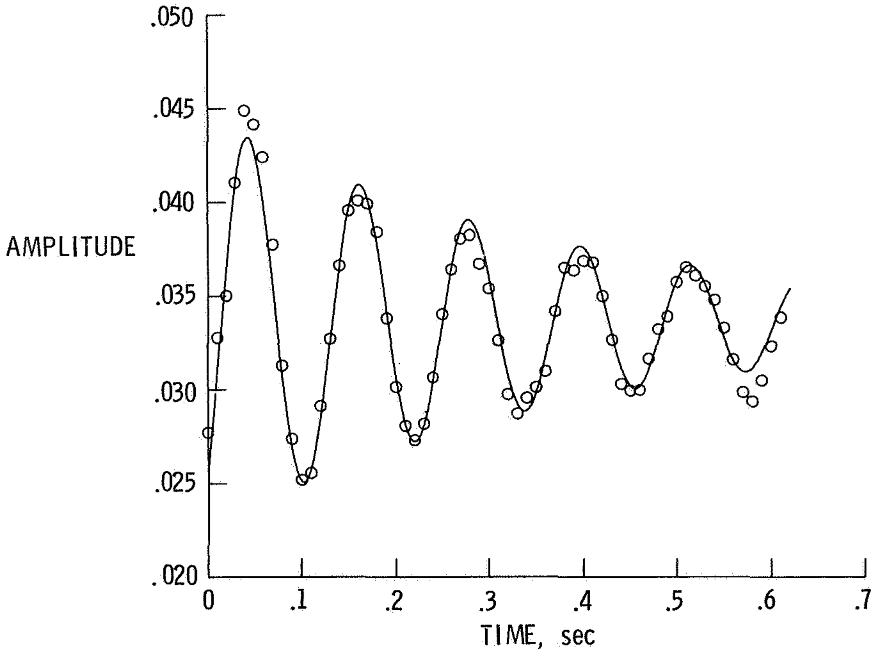


(b) Two-mode fit.

Figure 4.- Gust-vane curve fit for high-damping case.

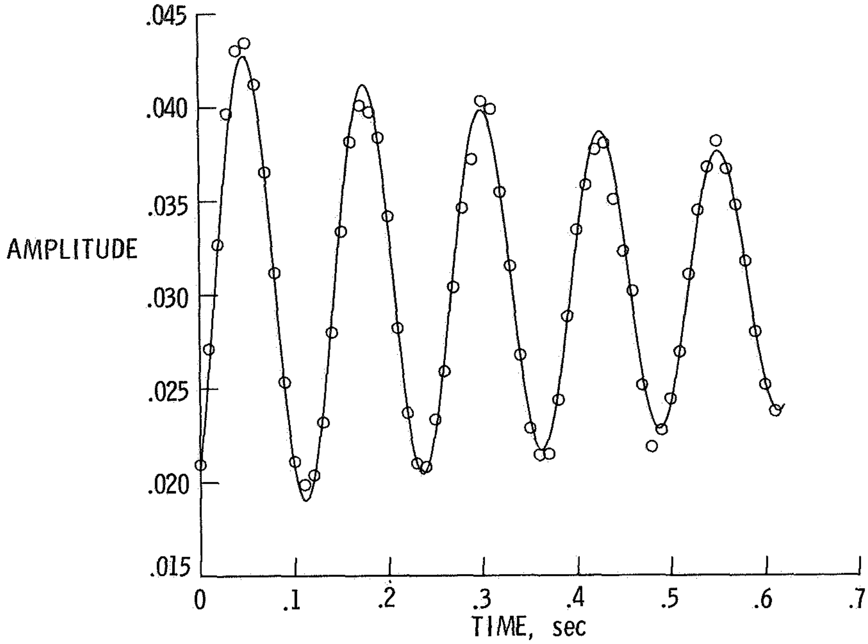


(a) $V/V_f = 0.684.$

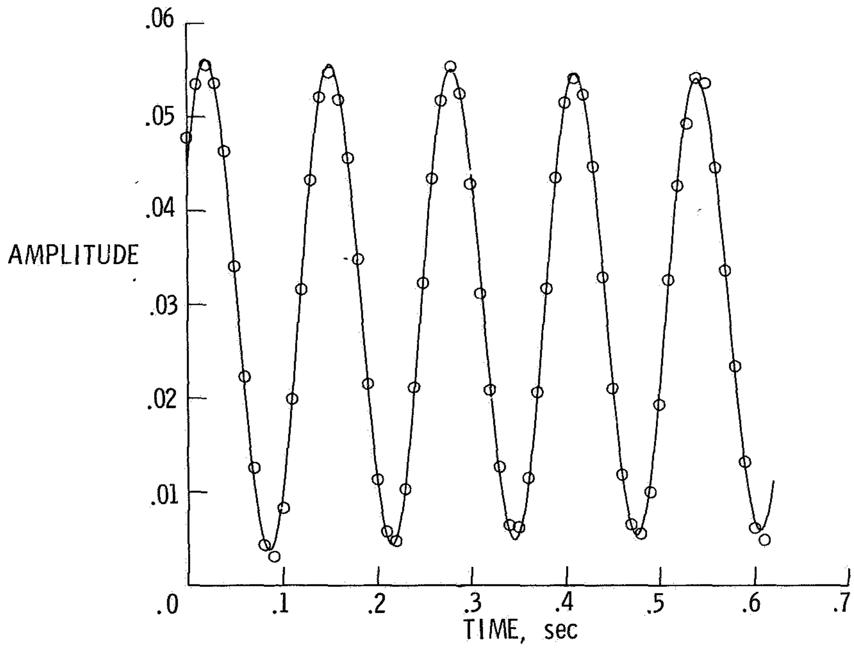


(b) $V/V_f = 0.855.$

Figure 5.- Curve fit of randomdec-analog data.



(c) V/V_f = 0.941.



(d) V/V_f = 0.992.

Figure 5.- Concluded.

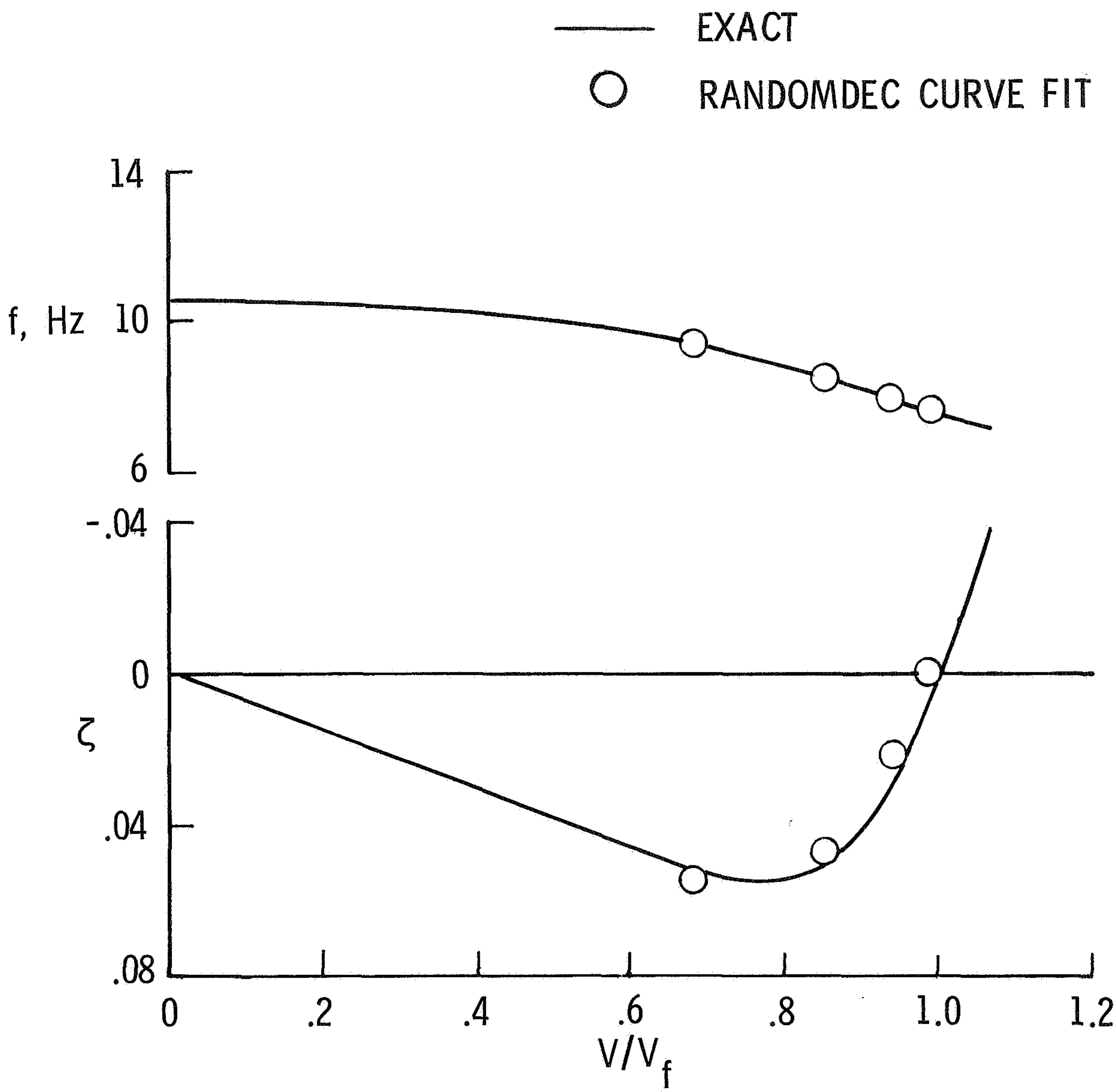


Figure 6.- Comparison of curve fit of randomdec-analog data with exact calculations.