NASA CONTRACTOR REPORT

NASA CR-2803

NASA CR-2

STUDY OF SYNTHESIS TECHNIQUES FOR INSENSITIVE AIRCRAFT CONTROL SYSTEMS

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Prepared by
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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1977

1. Report No. NASA CR-2803	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle STUDY OF SYNTHESIS TECHNIQUES FOR INSENSITIVE AIRCRAFT CONTROL SYSTEMS		5. Report Date April 1977
		6. Performing Organization Code
7. Author(s) C. A. Harvey and R. E. Pope		8. Performing Organization Report No. 76SRC37
9. Performing Organization Name and Address Honeywell Inc., Systems and Research Center 2600 Ridgway Parkway Minneapolis, Minnesota 55413		10. Work Unit No.
		11. Contract or Grant No. NAS1-13680
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Hampton, Virginia 23665		13. Type of Report and Period Covered Final Report December 74 - October 76
		14. Sponsoring Agency Code

15. Supplementary Notes

The authors acknowledge the contributions of Jonathan Whitlock, Donald Shaner, and Betty Kizilos in the computer program development. C. R. Stone served as a consultant in the C-5A model development task, and Ernest S. Armstrong was the Langley technical monitor. Final report.

16. Abstract

A study of synthesis techniques for designing insensitive flight control systems is described. The study consisted of 1) determining meaningful criteria for use in design and evaluation of insensitive controllers, 2) developing new techniques, 3) synthesizing controllers for a realistic aircraft flight control problem using eight different techniques, and 4) evaluating and comparing the design techniques on the basis of performance and design effort required.

Insensitive flight control system design criteria was defined in terms of maximizing performance (handling qualities, RMS gust response, transient response, stability margins) over a defined parameter range. Wing load alleviation for the C-5A was chosen as a design problem. The C-5A model was a 79-state, two-control structure with uncertainties assumed to exist in dynamic pressure, structural damping and frequency, and the stability derivative, M_W. Five new techniques (mismatch estimation, uncertainty weighting, finite dimensional inverse, maximum difficulty, dual Lyapunov) were developed. Six existing techniques (additive noise, minimax, multiplant, sensitivity vector augmentation, state dependent noise, residualization) and the mismatch estimation and uncertainty weighting techniques were synthesized and evaluated on the design example. Evaluation and comparison of these six techniques indicated that the minimax and the uncertainty weighting techniques were superior to the other six, and of these two, uncertainty weighting has lower computational requirements. Techniques based on the three remaining new concepts appear promising and are recommended for further research.

17. Key Words (Suggested by Author(s)) Synthesis Techniques, Linear Control, Optimal Control, Parameter Uncertainty, Sensitivity, Unmodeled Dynamics, Flight Control, Aircraft Modeling	18. Distribution Statement Unclassified - Unlimited. Subject Category 63

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	308	$9.25

* For sale by the National Technical Information Service, Springfield, Virginia 22161

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SECTION I

INTRODUCTION AND SUMMARY

Over the last three decades there has been considerable interest in the development of control systems that display insensitivity to variations from the design condition. During this time, a number of techniques have been proposed which promise to achieve this desired insensitivity.[1-34] No one technique, however, has received widespread acceptance by control system designers. In fact, new techniques continue to appear in the literature. One consequence of this proliferation of insensitive controller design techniques is an absence of critical evaluations as to how well these techniques perform against one another. Specifically, there has been very little comparative analysis done on insensitive controller performance for an authentic, complex control problem (for example, the design of an aircraft flight control system). Recent advances in control system technology produce greatly improved performance of certain flight systems by applying active control and by integrating control system design results into preliminary configurations of new aircraft designs.

Each control system design is based on a mathematical model of the relevant flight system. These mathematical models are approximate. They involve uncertain parameters and often neglect known dynamics. Key uncertainties often occur with respect to actuator dynamics, unsteady aerodynamics during transonic and low-speed flight, and structural dynamics. Even if it were possible to accurately determine the unsteady aerodynamics and structural dynamics, some approximation of these models would be required for mathematical tractability in the active control design. Thus, before the full benefits of active control technology can be realized, a synthesis technique is needed which produces control systems that exhibit satisfactory performance in the presence of modeling errors.

PROGRAM OBJECTIVES

In view of the extensive research devoted to insensitive control systems, the objectives of this program were to determine the capabilities of existing synthesis techniques with respect to a realistic flight control problem and to develop new techniques applicable to such a problem. The specific objectives were to:

- Determine meaningful criteria which can be used for the design and evaluation of insensitive control systems,
- Develop new techniques that can be applied to the aircraft flight control problem,
- Synthesize controllers for a realistic aircraft, using several existing and, if possible, newly developed insensitive controller design techniques, and
- Evaluate and compare the design techniques on the basis of relative performance capability and design effort required.

A distinguishing feature of these objectives is the synthesis of controllers for the same authentic example using different design techniques to provide data for a quantitative comparison of the techniques.

PROGRAM OVERVIEW

This program was performed under the cost-sharing contract NAS 1-13680, Study of Synthesis Techniques for Insensitive Aircraft Control Systems. In addition to the basic program, relevant research was conducted in Honeywell's Independent Research Program.

The approach used to achieve the program objectives utilized a unique combination of Honeywell experience and outside consultants. Three university professors were engaged primarily for the task of developing new insensitive controller design techniques. These were Professor David L. Kleinman of the University of Connecticut, Professor William A. Porter of the University of Michigan, and Professor David L. Russell* of the University of Wisconsin. Early discussions with the consultants and the contract monitor led to the following set of ground rules for the study:

1. The system to be controlled may be represented by a set of linear, constant coefficient differential equations.
2. The emphasis in this effort will be on control system sensitivity to uncertain model parameters and unmodeled dynamics.

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* Professor Russell has been a Honeywell consultant in the past. His previous research on sensitivity theory and also his research on the work described in this report were sponsored by Honeywell's Independent Research Program.

3. The uncertain model parameter variations are considered to be time invariant or represented by a time stationary random process.

4. The unmodeled dynamics can be thought of as either known or unknown dynamics. The known unmodeled dynamics will represent very low or very high frequency dynamics.

5. All controller designs will use full state feedback. It is recognized that all states are not measurable which poses a definite practical design problem. However, since all controllers perform at their "optimum" with full state feedback, the task of comparison and evaluation would be more straightforward using this approach. The limited measurement problem is an area that is highly recommended for further study.

6. Finally, the cost of complexity in the development of new concepts is not constrained, recognizing the fact that later refinement could possibly reduce complexity.

With the consultants selected and the ground rules established, the study then focused on the following five tasks:

1. Criteria Definition
2. Model Development
3. Synthesis of Existing Insensitive Controllers
4. Development and Synthesis of New Insensitive Controllers
5. Comparison and Evaluation

A summary of these tasks and the significant results of each will now be presented.

Criteria Definition

The first task required defining meaningful criteria which could be used as a measure of the level of insensitivity a control system possesses. These criteria would then be used both as a guideline for insensitive controller synthesis technique developments and finally as a "measuring stick" of the quality of the techniques developed. The derivation of these criteria is described in Section III.

The resultant criteria recognize the very real interrelationship between performance and sensitivity which is the focal point of good control system design. The criteria may be summarized as follows:

- A nominal set of performance criteria that would reflect desired performance characteristics under nominal conditions, assuming everything is known.
- A definition of the type, range, and probability of occurrence of uncertainties that the system may experience.
- A minimum acceptable performance criteria for system performance under specified worst case conditions.

These criteria were used as guidelines for the insensitive technique development. Though qualitative in nature, they can be easily reduced to quantitative measures, whether they be stability margins, statistical response criteria, or transient response criteria.

MODEL DEVELOPMENT

The C-5A in the climb flight condition was chosen to be the design and evaluation model. The C-5A was chosen for a number of reasons. First, a detailed, authenticated C-5A data base was available. Second, a set of realistic control system design specifications had been previously defined. Third, the C-5A model form facilitated the investigation of the effect of uncertain parameters and unmodeled dynamics. From the most complete C-5A representation (79 states, 3 controls, 56 responses), a set of reduced order models was constructed.

Two procedures, truncation and residualization, were used for this purpose. The truncated and residualized reduced order models were then used as control system design models. The objective was to design a controller with no regard to sensitivity using a reduced order model at the nominal conditions of the uncertain parameters. The performance of this nominal controller then would serve as a benchmark for comparison against the insensitive controllers to be designed. The criteria used for the design had been previously specified for the design of an Active Lift Distribution Control System (ALDCS).

By analyzing the performance of controllers designed using truncated and residualized models, it was determined that controllers designed with residualized models produced much more consistent results than those designed with truncated models. Hence, residualized models were used in all insensitive controller designs.

Dynamic pressure, structural frequency and damping, and the stability derivative M_w were chosen as uncertain model parameters. The range of variations on these parameters that would be investigated was determined experimentally. This was done by varying the uncertain parameters until the performance of the nominal controller violated the design specifications. Two worst case conditions, representing a combination of uncertain parameters and specification violations, were determined.

The model development is described in Section IV. Appendix A presents the numerical data for the design and evaluation models. The method used to model the parameter uncertainties is given in Appendix B. Subtleties of computing state sensitivity equations and response rates for reduced order models are described in Appendices C and D, respectively.

Synthesis of Insensitive Controllers - Existing Techniques

Five insensitive controllers were designed based on the following existing techniques:

- Additive Noise
- Minimax
- Multiplant
- Sensitivity Vector Augmentation
- State Dependent Noise

The synthesis of each technique was based on existing theory. However, in many cases, approximations had to be made in order to make the technique tractable on the C-5A design model. The details are presented in Section V.

Development and Synthesis of Insensitive Controllers - New Techniques

Nine new techniques were developed or proposed, eight of them by the consultants. The consultants' contributions were as follows:

Professor Kleinman	-	Mismatch Estimation Maximum Difficulty
Professor Porter	-	Terminal Equivalence Finite Dimensional Inverse Model Following Uncertainty Weighting
Professor Russell	-	Specific/Nonspecific Eigenvalue Sensitivity Eigenvalue/Maneuver Sensitivity

Professor Russell's developments (based on a dual Lyapunov approach presented in Appendix G) and an additional technique, re-residualization, were developed under Honeywell's Independent Research Program. The maximum difficulty, terminal equivalence, finite dimensional inverse, and model-following techniques are described in Section IX.

Three of the above techniques reached the controller design stage. These were:

- Kleinman's Mismatch Estimation
- Porter's Uncertainty Weighting
- Re-Residualization

Synthesis of controllers based on these three techniques are described in Section V. The synthesis and comparison with the other controllers was performed as part of Honeywell's Independent Research Program. Model reduction via re-residualization is compared with model reduction via the singular perturbation method in Appendix E.

Evaluation and Results

The five existing insensitive controller design concepts--additive noise, minimax, multiplant, sensitivity vector augmentation, state-dependent noise--and the three newly developed approaches--mismatch estimation, uncertainty weighting, and re-residualization--were evaluated against the nominal controller, both qualitatively and quantitatively. Qualitative judgments were made with respect to user acceptability issues on the insensitive controller's synthesis techniques themselves. Such practical considerations as computer memory and time requirements, controller implementation requirements, whether the technique provides insight into design problems, and whether the technique treats all forms of engineering design specifications were among a set of ten items that we felt would influence user acceptability. No overall conclusions were drawn based on this qualitative evaluation; however, it is recommended that a potential user consider the results presented when selecting an insensitive controller design technique.

Quantitative evaluations were made with respect to the performance of each of the insensitive controllers designed versus nominal controller performance. A varied combination of evaluation conditions and evaluation models were used to evaluate the effects of parameter uncertainty and unmodeled dynamics on controller performance. Three types of performance evaluation measures were defined. These reflected: 1) a coarse relative rating of each of the insensitive controllers with respect to the nominal, 2) a finer normalized performance and range rating with respect to the nominal, and 3) normalized performance specification rating with respect to the nominal. Although these measures produced slightly different rankings among the insensitive controllers, two consistent sets of data did appear. First, despite the measure used, the minimax controller and uncertainty weighting controller always rated better than the nominal controller. Second, the sensitivity vector augmentation controller always rated worse than the nominal controller. Section VIII presents the evaluation in detail.

OVERALL CONCLUSIONS AND RECOMMENDATIONS

The overall conclusions must be tempered by the fact that each of the insensitive controller designs were performed in a limited amount of time. It was felt in all cases that all designs were acceptable; however each technique could probably have been modified in some form to produce a different though not necessarily better controller. This is particularly true of the mismatch estimation concept, the uncertainty weighting concept, the state dependent noise concept, and the sensitivity vector augmentation concept. The results of the study do indicate significant improved performance using the minimax technique or the uncertainty weighting technique. The recommended technique is uncertainty weighting because of the reduced computational requirements. Also the sensitivity vector augmentation scheme has no promise for application to a design problem of this size. The remaining controllers are grouped more closely about the nominal controller--some indicating slightly better performance, others slightly worse.

Of the new techniques that were developed, the finite dimensional inverse concept, the maximum difficulty concept, and the dual Lyapunov approach show promise and are recommended for further research.

SECTION II

SYMBOLS*

OPERATORS

A	Integral operator
E	Mathematical expectation
G	Forward loop compensator
H, $\hat{H}$	Causal feedback compensators
$\tilde{H}$	Integral operator
I	Identity operator
P	Plant operator
$\bar{P}$	Orthogonal projection
P_L	Projection onto L
P^t	Truncation operator
Q	Open-loop compensator
R	Open-loop compensator
$Re(\cdot)$	Real part of $(\cdot)$
T	Integral operator
TR	Trace
V	Integral operator
W	Integral operator
d	Differential
δ	Linearized perturbation
Δ	Incremental perturbation

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* This list is applicable to the main body of the report. Notation in the appendices may differ.

∂	Partial differential
π	Integral operator
$\nabla_{(\cdot)}$	Gradient with respect to $(\cdot)$
$\nabla_{(\cdot)(\cdot)}$	Matrix of second partial derivatives with respect to $(\cdot)(\cdot)$
$(\dot{\ })$	Derivative with respect to time
$(\ )^T$	Transpose
$(\ )^*$	Adjoint
$(\ )^{-1}$	Inverse
$(\ )^{-R}$	Right inverse
$(\ )^\dagger$	Pseudo inverse
$(\ )'$	Closed loop
$\langle \cdot , \cdot \rangle$	Inner product

SUPERSCRIPTS

i	(1) Row vector component index
	(2) Iteration stage

SUBSCRIPTS

a	(1) Augmented
	(2) Aircraft
c	Command
f	Factor
g	Gust
HT	Horizontal tail
i	(1) Column vector component index
	(2) Iteration stage
	(3) Inboard

ij Matrix element index

j Column vector component index

o (1) Initial condition
(2) Nominal condition
(3) Outboard

ss Steady state

W Wing

(•) Partial derivative with respect to (•)

UPPER CASE

A Coefficient matrix

AT Pole of Wagner dynamics transfer function for the horizontal tail

AW Pole of Wagner dynamics transfer function for the wing

B (1) Bending moment
(2) Coefficient matrix

B^t Normalized projection of $(\eta^t)^t$

C_m Pitching moment coefficient

D Coefficient matrix of control vector in response equation

E_c Closed-loop error

E_o Open-loop error

F Plant coefficient matrix

G_1 Control input coefficient matrix

G_2 Noise input coefficient matrix

H Coefficient matrix of state vector in response equation

$\mathcal{H}$ Hamiltonian

$\underline{H}$ Hilbert space

I Identity matrix

J Performance index

J_s	Sensitivity performance index	
K	(1)	Mach number correction factor
	(2)	System gain
	(3)	Feedback gain matrix
	(4)	Solution of Lyapunov equation
L	Linear subspace	
$L_2^m(0, \infty)$	Square integrable functions in R^m	
L(t)	Continuous matrix	
L_w	Gust wavelength	
M	(1)	Pitching moment
	(2)	Observability matrix
M(t)	Measurement matrix	
NSD	Noise intensity adjustment factor	
N(t)	Grammian matrix	
P	(1)	Range of parameter vector
	(2)	Solution of Ricatti equation
Q	(1)	Weighting matrix in performance index
	(2)	Solution of Lyapunov equation
R	(1)	Weighting matrix in performance index
	(2)	Response covariance matrix
R^n	n-dimensional Euclidean space	
S	(1)	Sensitivity matrix
	(2)	Costate matrix
	(3)	System
$	Sensitivity ratio	
SF	Sensitivity factors	
$S_{(\cdot)}^{(\cdot)}$	Sensitivity index	
S(x, t)	Cost function	
T	(1)	Overall transfer function
	(2)	Torsion moment
	(3)	Gust distribution time constant

U_o	Forward velocity of aircraft
$V(x, t)$	Lyapunov function
W	(1) Sensitivity index function (2) Wagner dynamics transfer function (3) Controllability matrix
X	State covariance matrix
Z	Vertical force

LOWER CASE

$a(t)$	Function of time (vector)
b	Coefficient vector
$b(t)$	Function of time (vector)
$\bar{c}$	Wing chord
$\bar{c}_{HT}$	Horizontal tail chord
d	Coefficient vector
e	(1) Estimation error (2) Model-following error
f_1	First row of F matrix
g	(1) Forward loop gain (2) Design parameter for estimator
g_1	Coefficient vector
h	Feedback compensation
j	Square root of -1
ℓ	Dimension of subspace
n	(1) Dimension of state vector (2) Integer
n_2	Scale factor used to normalize pitch rate

p	(1) Vector of uncertain parameters	
	(2) Plant transfer function	
	(3) Kussner gust states	
q	(1) System dominant root	
	(2) Pitch rate	
$\bar{q}$	Dynamic pressure	
r	Response vector	
$\tilde{r}$, $\approx r$	Uncertainty response vectors	
$r_{i_{BIAS}}$	Bias on the standard deviation of the i^{th} component of the response vector	
s	Laplace transform variable	
t	Time	
u	(1) Control input	
	(2) Sensitivity function	
$v(t, \tau)$	Kernel of V	
w	Vertical velocity	
w_g	Vertical gust velocity	
$w(t, \beta)$	Kernel of $\hat{H}$	
$w(t, \tau)$	Kernel of W	
$\dot{w}T$	Wagner states representing tail unsteady aerodynamics	
$\dot{w}W$	Wagner states representing wing unsteady aerodynamics	
x	(1) State vector	
	(2) Plant parameter	
x_{1_R}	Residualized state vector	
$x_{1_{RR}}$	Re-residualized state vector	
x_{1_T}	Truncated state vector	
y	Output vector	
z	System zero	

GREEK SYMBOLS

Upper Case

Γ	Diagonal coefficient matrix
$\Phi(t, s)$	Open-loop transition matrix
$\psi(t, s)$	Closed-loop transition matrix

Lower Case

α	Angle of attack
γ	Reciprocal of correlation time of Γ
$\gamma(t)$	Measurement noise
δa	Aileron deflection
δe	Elevator deflection (perturbation from trim)
$\dot{\delta} aW$	Control surface Wagner state
$\dot{\delta} eT$	Control surface Wagner state
ϵ	Error criterion
ζ	Damping ratio
η	Structural displacement vector
$\eta, \tilde{\eta}$	Gaussian white noise
η^t	Normalized projection of b
$(\eta^t)^t$	Vector which satisfies $N(t)(\eta^t)^t = \eta^t$
$\eta(\cdot)$	Null space of $(\cdot)$
$\dot{\eta}T$	Wagner state representing tail unsteady aerodynamics
$\dot{\eta}W$	Wagner state representing wing unsteady aerodynamics
θ	Pitch attitude
$\ddot{\theta} T/n_2$	Wagner state representing tail unsteady aerodynamics
$\ddot{\theta} W/n_2$	Wagner state representing wing unsteady aerodynamics

λ	(1)	Perturbation parameter
	(2)	Scalar parameter
	(3)	Response sensitivity vector
	(4)	Gaussian white noise
μ		Parameter in performance index
ν	(1)	Parameter in performance index
	(2)	Lagrange multiplier
ξ	(1)	Gaussian white noise
	(2)	Filtered white noise
$\hat{\xi}$		Estimate of ξ
$\pi(t, \tau)$		Kernel of π
τ	(1)	Vector of sensitivity states
	(2)	Response standard deviation vector
σ_w		Standard deviation of vertical gust velocity, w_g
ϕ	(1)	Real valued function
	(2)	Empty set
ω		Frequency

SECTION III

CRITERIA DEFINITION

In this section some of the fundamental issues that influence the design of insensitive control systems are discussed. An alternate measure of system sensitivity, which is based on system performance, is defined.

SENSITIVITY CONSIDERATIONS

The use of mathematical models in the design of automatic control systems is an accepted practice among control engineers. The control engineer must be aware, however, that these mathematical models necessarily involve approximations. It is his task to evaluate the effects of these approximations on his prime control system design objective, i.e., system performance. In other words, he must be able to measure the "sensitivity" of the system's performance to characteristics that are not represented by the mathematical models. This control system design necessity led to the development of many concepts and theories which are generally included under the title "Sensitivity Analysis." Tomovic, one of the early contributors to the field of sensitivity analysis, has summarized the sensitivity information that is most important from the engineering point of view as follows:[1]

1. Sensitivity to small perturbations of the model parameters around a reference position,

2. Sensitivity to large displacement of the model parameters around a reference position or global sensitivity,

3. Sensitivity to the reduction of order of the mathematical model,

4. Sensitivity to transition from continuous to discrete mathematical models, and

5. Structural sensitivity or sensitivity to the influence of various functional blocks which make up a system.

The majority of the efforts extended in sensitivity analysis over the past several decades has been concentrated on the first three of Tomovic's sensitivity considerations. This study will also confine itself to the first three considerations. The first two we have chosen to lump under sensitivity to parameter uncertainties. The third we have chosen to call sensitivity to unmodeled dynamics.

The treatment of system performance sensitivity to parameter uncertainties has proceeded along several paths that are, in general, functions of the form of the performance criteria. The earliest treatments operated in the frequency domain. Feedback, at that time, was used mainly for sensitivity reduction. Performance requirements were satisfied through open-loop control. Sensitivity functions were defined to aid in feedback design. Bode,[2] for example, defined the sensitivity of the overall transfer function T to the plant parameter x as

$$S_x^T \triangleq \frac{dx/x}{dT/T} = \frac{\partial \ell nx}{\partial \ell nT} \tag{1}$$

The more accepted definition, employed by Truxal[3] and Horowitz[4] is the inverse of Bode's or

$$S_x^T \triangleq \frac{dT/T}{dx/x} = \frac{\partial \ell nT}{\partial \ell nx} \tag{2}$$

Feedback was employed if S_x^T given by Equation (2) could be reduced.

Equation (2) may be extended to apply to dominant root sensitivity as defined by Horowitz:[4]

$$S_x^i \triangleq \frac{\partial q_i}{\partial x/x} \tag{3}$$

when q_i represents the system's dominant roots and x may be system gain, a pole, or zero. Reference 5 employs a dominant root sensitivity approach; however, the sensitivity functions are defined differently as

$$S_K^i \triangleq \frac{\partial q_i}{\partial K/K} \tag{4a}$$

$$S_{p_j}^i \triangleq \frac{\partial q_i}{\partial p_j} \tag{4b}$$

$$S_{z_j}^i \triangleq \frac{\partial q_i}{\partial z_j} \tag{4c}$$

when K is the system gain, and p_j and z_j are the system poles and zeros.

If a system is represented by the signal flow diagram given in Figure 1, then the sensitivity-function defined by Equation (2) is given by

$$S_g^T = \frac{1}{1+hgp} \tag{5}$$

where g represents the forward loop gain, p is the plant, and h represents the feedback compensation. Cruz and Perkins[6] extended the single-input/single-output representation to the multivariable case where the relationship between open-loop errors, E_o, and closed-loop errors, E_c, is given by

$$E_c = S\,E_o$$

where S is the sensitivity matrix function given by

$$S = (I + P'\,GH)^{-1} \tag{6}$$

where P' differs from P because of parameter variations. This work was in turn extended by Kriendler in Reference 7.

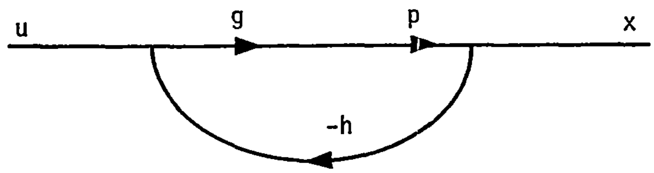


Figure 1. System Signal Flow Diagram

In the time domain, Pagurek[8] and Dorato[9] treat the sensitivity of performance indices to parameter variations. Pagurek defines performance index sensitivity functions as

$$W^i (t_o, x_o, p_o) = \left. \frac{\partial J(t_o, x_o, p, p_o)}{\partial p_i} \right|_{p = p_o} \tag{7}$$

where J is the performance index and p represents the uncertain parameters.

Another time domain approach proposed by Kreindler[10, 11, 12, 13] defines sensitivity states or

$$\sigma = \frac{\partial x}{\partial p} \tag{8}$$

These states are then included in a performance index which is to be minimized. Open-loop versus closed-loop sensitivity measures or relative sensitivity measures are typically employed for design purposes. This type of sensitivity analysis falls under the category of trajectory sensitivity analysis.

Compared to the treatment of sensitivity to parameter uncertainties, the treatment of sensitivity to unmodeled dynamics has been meager. Prime in the field has been the work of Kokotovic[14, 15] in the development of singular perturbation theory. Kokotovic examines the behavior of a sensitivity function given by

$$u_i(t, \lambda) = \frac{\partial x_i (t, \lambda)}{\partial \lambda} \tag{9}$$

where λ represents a perturbation in the form of the model equations or

$$\dot{x}_i = \phi_i (x_1, \ldots, x_n; t), \; i < n \tag{10a}$$

$$\lambda \dot{x}_n = \phi_n (x_1, \ldots, x_n; t) \tag{10b}$$

A recent application of this approach is given in Reference 16.

The above review of past development in sensitivity analysis is by no means intended to be exhaustive. Additional developments by Cruz and Perkins,[17-22] Horowitz,[23-28] and the work of Porter[29-34] are but a few of the outstanding contributions to the field.

The intent of the review was to indicate the varying sensitivity viewpoints and, in particular, the different ways in which sensitivity is measured. Four types of sensitivity were observed: 1) transfer function sensitivity, 2) dominant root or eigenvalue sensitivity, 3) performance index sensitivity, and 4) trajectory sensitivity. In designing insensitive controllers related to these types of sensitivities, relative sensitivity measures are employed. For example, if a performance index sensitivity function is used, the designer wishes to obtain a controller which reduces this sensitivity function with respect to the magnitude of a sensitivity function at some reference condition. This approach suggests the following questions:

- What magnitude of sensitivity reduction is desirable?
- How is performance affected?

Let's look at the second question first. In the early stages of sensitivity analysis, reducing system sensitivity was synonymous with increasing system performance. This was due to the fact that the open-loop system was designed to achieve the desired performance. Feedback was used only to reduce the open-loop system's sensitivity to possible variations. The design objective was to design a closed-loop system which would produce open-loop performance. Today, in many cases, open-loop performance is unsatisfactory. This is particularly true in the case of modern aircraft. In many cases it is necessary to use feedback to augment open-loop performance. Thus, if feedback is used for performance augmentation, can it, at the same time, be used for sensitivity reduction? If so, what are the tradeoffs? With this in mind, the two questions may be answered simultaneously. <u>The magnitude of sensitivity reduction is acceptable until the level of performance is unacceptable.</u> This statement is the basis for the definition of design criteria used in this study.

The first thing that becomes obvious about the above statement is that a sensitivity measure is not really needed. The prime design objective is to maintain satisfactory control system performance over both nominal and off-nominal conditions. Nominal conditions are defined to mean those real world conditions that are represented by the mathematical model. Off-nominal means that there is some variation (i.e., parameter uncertainties and/or unmodeled dynamics) between the real world and the mathematical model. We suggest that what the control engineer really wants in an insensitive controller is a controller which maximizes performance over a given type and range of model variations. He does not want a controller that minimizes some sensitivity functions unless it also maximizes performance. The sensitivity measure is important only as a tool in indicating how performance can be maximized.

With this as background, we defined the following set of design criteria and specifications that we felt should be used in the design of an insensitive controller:

1. A nominal set of performance criteria that would reflect desired performance characteristics under nominal conditions. This gives the designer a peak performance goal to shoot for.

2. A minimum acceptable set of performance criteria for system performance under off-nominal conditions. This gives the designer a lower bound on performance which he can use to determine whether or not model variations will require control system adaptation rather than insensitivity.

3. Type, range, and, if appropriate, the probability of occurrence of known variations in the model. This gives the designer a design space.

These characteristics can easily be translated into quantitative criteria since they are not dependent on any specific performance specification.

For this study, Item 1 consisted of designing an optimal controller at the nominal condition with no regard for sensitivity considerations. The performance of this controller was then defined as the desired performance.

The minimum acceptable performance criteria which the nominal controller satisfied with margin to spare were typical design specifications and are described in Section IV.

For this study Item 3 was determined experimentally. This is also described in Section IV.

Since these design specifications can be used for the design of an insensitive controller no matter what technique is used, they also provide excellent means of evaluating insensitive controller design techniques and resultant controller performance; thus they were used for that purpose in this contract.

In summary, we feel that measuring sensitivity in this strictly performance oriented fashion is realistic, practical, and is directed at the true objective of control system design. It eliminates the need for explicitly defining a sensitivity measure and the resultant complexity of evaluating that measure. We feel that we have returned the focal point of insensitive control design to where it began, i.e., performance, as opposed to pure insensitivity. Finally, this criteria allows us to evaluate a variety of techniques fairly and consistently.

SECTION IV

MODEL DEVELOPMENT

In this section, the formulation of the C-5A aircraft model is presented. The design of the nominal controller is also discussed. This controller serves as the benchmark for evaluation and comparison of the insensitive controllers.

MODEL SELECTION

The model chosen as an authentic aircraft example was the C-5A in the climb condition. The characteristics of this flight condition are given in Table 1. The C-5A was chosen for the following reasons:

1. A detailed, authenticated data base which was developed in previous programs[35] was available.

2. It is a comprehensive model including high frequency bending modes and unsteady aerodynamic effects. These features are exploited in the evaluation stage where the efficacy of designing controllers with reduced order models is determined.

3. A set of design specifications which encompassed the full range of types of flight control system design criteria was defined in a previous program.[35] These design criteria included statistical response, steady state transient response, handling quality, and stability margin specifications.

4. The structure of the model permits the modeling of variations as 1) those which affect the majority of the model elements, 2) those which affect a selected subset of the model elements, or 3) a single model element variation.

5. Although it was not realized at the time the C-5A was selected as the model, the flight control system design criteria are given for vehicle response states which have model variations not completely covered by the vehicle dynamic state model variations. Thus there can be an additional two classifications of variations: state equation variations and response equation variations. This consideration and how it affects insensitive controller design and performance will be covered in detail in this and subsequent sections.

TABLE 1. C-5A CLIMB FLIGHT CONDITION

Total weight, N (lb)	3.107×10^6 (698,400)
Mach number	0.448
Altitude, m (ft)	2.3×10^3 (7,500)
Dynamic pressure, N/m^2 (psf)	9.15×10^3 (191)
Airspeed, m/sec (fps)	1.43×10^2 (468)
Fuel, N (lb)	9.541×10^5 (14,500)
Cargo, N (lb)	7.1×10^5 (160,000)
Center of gravity (% mac)	31
Trim angle of attack (deg)	5.15×10^{-2} (2.95)
Load factor	1

MODEL DESCRIPTION

The mathematical model is a comprehensive representation of the linear, longitudinal dynamics of the C-5A. In its most complete form, the model is described by a 79th order constant coefficient differential equation:

$$\dot{x} = Fx + G_1 u + G_2 \eta$$

where x is a 79th order state vector, u is a 3rd order control vector, and η is a scalar white-noise gust model driver. A description and order of the states and controls is given in Table 2. Only two of the controls were used in this study.

As can be seen from Table 2, the aircraft states can be divided into the following sets:

1. Rigid body states x_1, x_2
2. Bending mode velocities x_3 - x_{17}
3. Bending mode displacements x_{18} - x_{32}

TABLE 2. CASE 1 STATE AND CONTROL DEFINITION
(79 STATES, 2 CONTROLS)

States		Dimensions	Definition
1	w	0.0254 m/sec (in/sec)	Vertical velocity
2	q/n_2	0.0254 m/sec (in/sec)	Normalized pitch rate n_2 = conversion factor
3-17	$\dot{\eta}_i$ i = 1, 15	0.0254 m/sec (in/sec)	Structural rate of displacement
18-32	η_i i = 1, 15	0.0254 m (in)	Structural displacement
33	δ_a	radian	Aileron deflection
34	δe_i	radian	Inboard elevator deflection
35	δe_o	radian	Outboard elevator deflection
36-41	P_1-P_6	--	Kussner gust states
42	w_g	0.0254 m/sec (in/sec)	Vertical gust
43	$\dot{w}$(Wing)	--	Wagner state representing wing unsteady aerodynamics
44	$\dot{q}$(Wing)/n_2	--	Wagner state representing wing unsteady aerodynamics
45-59	$\dot{\eta}_i$(Wing) i = 1, 15	--	Wagner states representing wing unsteady aerodynamics
60	$\dot{w}$(Tail)	--	Wagner state representing tail unsteady aerodynamics
61	$\dot{q}$(Tail)/n_2	--	Wagner state representing tail unsteady aerodynamics
62-76	$\dot{\eta}_i$(Tail) i = 1, 15	--	Wagner states representing tail unsteady aerodynamics
77	$\dot{\delta}_a$(Wing)	--	Control surface Wagner state
78	$\dot{\delta e}_i$(Tail)	--	Control surface Wagner state
79	$\dot{\delta e}_o$(Tail)	--	Control surface Wagner state
Controls		**Dimensions**	**Definition**
1	δa_c	radian	Aileron command
2	δe_{i_c}	radian	Inboard elevator command

4. Control surface displacements $x_{33} - x_{35}$

5. Gust dynamics $x_{36} - x_{42}$

6. Wagner dynamics $x_{43} - x_{79}$

Following is a brief description of the modeling associated with each set.

Rigid Body

The dynamics of the first two states represent the aircraft short period characteristics. The low frequency phugoid dynamics are not pertinent to this study and have been neglected.

Bending Dynamics

A modal representation of the structural dynamics was provided by Lockheed. The equations of motion were formulated in terms of generalized coordinates. A detailed description is given in Reference 35.

Control Surfaces and Actuations

The dynamics of the control surface actuators are represented by first order lags given by

$$\delta_a = \frac{6.0}{s + 6.0} \delta_{a_{command}}$$

for the ailerons, and

$$\delta_{e_i} = \frac{7.5}{s + 7.5} \delta_{e_{i_{command}}}$$

for the inboard elevator.

Gust Dynamics

Gusts are applied in the vertical axis and are represented in Dryden form[42] by

$$w_g = \sigma_w \sqrt{\frac{L_w}{U_o}} \quad \frac{1 + \sqrt{3}\,\frac{L_w}{U_o}\,s}{\left(1 + \frac{L_w}{U_o}\,s\right)^2} \quad \eta$$

where

w_g = gust velocity, m/sec (fps)

L_w = wavelength, 576 m (1890 ft)

U_o = forward velocity, 143 m/sec (468 fps)

σ_w = gust magnitude RMS, 0.3048 m/sec (1.0 fps)

For analysis and synthesis the gust magnitude is normalized to unity and the corresponding specifications on gust response characteristics are normalized accordingly. This may be represented in state space by

$$\begin{bmatrix} \dot{x}_{41} \\ \dot{x}_{42} \end{bmatrix} = \begin{bmatrix} -2\,\frac{U_o}{L_w} & -\left(\frac{U_o}{L_w}\right)^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{41} \\ x_{42} \end{bmatrix} + \begin{bmatrix} \sigma_w\,(1 - 2\sqrt{3})\left(\frac{U_o}{L_w}\right)^{3/2} \\ \sigma_w \sqrt{\frac{3U_o}{L_w}} \end{bmatrix} \eta$$

First order Kussner dynamics[43] representing gust distribution on the wing and tail are given by

$$\dot{x}_{36} = -\frac{U_o}{K\bar{c}_{HT}}\,x_{36} + \frac{U_o}{K\bar{c}_{HT}}\,w_g$$

$$\dot{x}_{40} = -\frac{U_o}{K\bar{c}}\,x_{40} + \frac{U_o}{K\bar{c}}\,w_g$$

where

$\bar{c}_{HT}$ = horizontal tail chord = 4.66 m (15.28 ft)

$\bar{c}$ = wing chord = 9.429 m (30.934 ft)

K = Mach number correction factor = 1.38

Zero-th over first and first over second Pade approximation are used for wing and tail transport delays, respectively:

$$\dot{x}_{37} = \frac{-1}{T_w} x_{37} + \frac{1}{T_w} x_{40}$$

$$\dot{x}_{38} = x_{39} - \frac{2}{T_{HT}} x_{36}$$

$$\dot{x}_{39} = -\frac{6}{T^2_{HT}} x_{38} - \frac{4}{T_{HT}} x_{39} + \frac{14}{T^2_{HT}} x_{36}$$

where

$$T_w = \frac{54.7}{U_o} \quad \text{sec}$$

$$T_{HT} = \frac{183.7}{U_o} \quad \text{sec}$$

Wagner Dynamics

Wagner dynamics[43] represent the lift buildup on the wing, tail, and control surfaces and are represented by first order lags given by

$$W_w(s) = 1 - \frac{0.5}{s + AW}$$

for the wing and

$$W_{HT}(s) = 1 - \frac{0.5}{s + AT}$$

for the tail where

$$AW = \frac{U_o}{K\bar{c}}$$

$$AT = \frac{U_o}{K\bar{c}_{HT}}$$

This describes the form of the Wagner dynamics; however, the process required for transformation to a state space representation is quite complex[35] and will not be discussed here.

A set of vehicle responses may also be defined. These may be states or controls or linear combinations of states and controls. The vector of such responses is given by

$$r = Hx + Du \tag{12}$$

The responses of general interest in our example constitute a 56th order response vector which is defined in Table 3.

The most complete dynamic and response model, which is given by Equations (1) and (2), will hereafter be referred to as Case 1. The nominal values for matrices F, G_1, G_2, H, and D are given in Appendix A together with Case 1 eigenvalues and statistical gust response data for the free aircraft.

MODEL VARIATIONS

As discussed in Sections I and III, one of the primary objectives of this contract was the synthesis and evaluation of insensitive controllers which maximize performance over given types and ranges of model variations. The types of model variations that are observed in flight control system designs generally fall into one of the following categories:

1. Uncertainties associated with model elements of a known model structure.
2. Dynamics which are known but neglected in constructing a design model.
3. Dynamics which are unknown and which, therefore, cannot be modeled.

TABLE 3. CASE 1 RESPONSES (56 RESPONSES)

Response	Dimensions	Definition
1-10 B_i, T_i i = 1, 5	0.113 Nm (in-lbs)	Bending and torsion moments at 5 wing stations
11-20 $\dot{B}_i$, $\dot{T}_i$ i = 1, 5	0.113 Nm/sec (in-lb/sec)	Rate of change of bending and torsion moments at 5 wing stations
21-35 $\dot{\eta}_i$ i = 1, 15	0.0254 m/sec (in/sec)	Structural rate of displacement for each flexure mode
36-50 η_i i = 1, 15	0.0254 m (in)	Structural displacement for each flexure mode
51 $\dot{\delta}_a$	rad/sec	Aileron rate
52 $\dot{\delta}e_i$	rad/sec	Inboard elevator rate
53 δ_a	rad	Aileron displacement
54 δe_i	rad	Inboard elevator displacement
55 w	0.0254 m/sec (in/sec)	Vertical velocity
56 q/n_2	0.0254 m/sec (in/sec)	Normalized pitch rate $n_2 = 0.606 \times 10^{-3}$ rad/sec

In general, the known model structure may be linear or nonlinear as may be the neglected dynamics or unknown dynamics.

Only linear structures and dynamics were treated in this study. In addition, model variations from the real world produced by such effects as digital flight control system considerations (e.g., quantization, transport lag, sampling rates, etc.) were not treated. We treated what we considered to be realistic, comprehensive variations that are routinely experienced in flight control system design.

Beginning with Category 1, model variations in the form of parameter uncertainties, three types of parameter uncertainties were considered. These were:

1. Uncertainties in dynamic pressure, $\bar{q}$. Dynamic pressure which is a function of air density and relative velocity can vary as much as 25 percent from nominal at a particular flight condition. The magnitude of $\bar{q}$ is directly proportional to the frequency content of the vehicle dynamics. Since flight control systems are generally designed on the basis of designs at a set of fixed flight conditions, dynamic pressure is an extremely important design parameter.

2. Structural frequency, ω, and structural damping, ζ. Determining accurate structural mode shapes and frequency distribution is a very complex problem. It is very difficult to duplicate flight conditions on the ground for simulating structural loads and forcing functions. Structural damping ratios of flexure modes are generally assumed to be small and the same for all modes ($\zeta = 0.02$ sec^{-1} for all 15 C-5A bending modes). The relationship between bending mode frequencies and rigid body frequencies is an important issue in flight control system design, particularly if notch filters are used to suppress bending modes with frequencies in the rigid body range. If active control is to be applied for maneuver load control or gust load alleviation, variations in structural frequency and damping must be considered.

3. Stability derivative, M_w. The stability derivative M_w represents the change in pitching moment due to a change in vertical velocity or, equivalently, angle of attack. Together with Z_q, the change in vertical force due to pitch rate, it determines the aircraft's short period frequency. It is typically one of the most difficult derivatives to predict from wind tunnel tests. During the early design stages of the C-5A, M_w (or more correctly C_{m_α}) experienced the largest variations in magnitude as the aircraft went through design modifications. Because short period frequency is directly related to handling qualities and because of the difficulty to predict C_{m_α}, variations in M_w are considered to be truly representative of significant uncertainties encountered in flight control design.

In addition to being representative of realistic model variations, the choice of dynamic pressure, structural frequency and damping, and M_w as uncertain parameters covers a wide range of types of variations. Variations in dynamic pressure affect the majority of the elements in both the dynamic equations and the response equations. Structural frequency and damping variations affect particular subsets of the elements of both equations. Variations in M_w are limited to single model element effects.

These uncertain parameter variations were incorporated into the aircraft model given by Equations (11) and (12) in the following form. The matrix F may be represented in the form:

$$F = F_o + \bar{q}_o \bar{q}_f F_{\bar{q}} + \bar{q}_o \bar{q}_f (M_{w_f} - 1) F_{M_w}$$
$$+ \zeta_o \zeta_f \omega_f \sum_{i=1}^{15} \omega_{i_o} F_{\zeta\omega_i} + (\omega_f)^2 \sum_{i=1}^{15} \omega_{i_o}^{\ 2} F_{\omega_i 2} \tag{13}$$

where

F_o is independent of $\bar{q}$, ζ, ω_i, and M_w

$\bar{q}_o$ = nominal dynamic pressure

$\bar{q}_f$ = dynamic pressure uncertainty factor

$F_{\bar{q}}$ = dynamic pressure dependent elements with $\bar{q}$ factored out

M_{w_f} = M_w uncertainty factor

F_{M_w} = M_w dependent elements with M_w factored out

ζ_o = nominal structural damping ratio

ζ_f = structural damping ratio uncertainty factor

ω_{i_o} = nominal structural frequency for ith mode

ω_f = structural frequency uncertainty factor

$F_{\zeta\omega_i}$ = structural damping and frequency dependent elements with $\zeta\omega_i$ factored out

$F_{\omega_i^{\ 2}}$ = structural frequency dependent elements with $(\omega_i)^2$ factored out

The matrix F_o includes actuator, gust, and Wagner dynamics and pure integrations. The matrix H will have a similar representation:

$$H = H_o + \bar{q}_o \bar{q}_f H_{\bar{q}} + \zeta_o \zeta_f \omega_f \sum_{i=1}^{15} \omega_{i_o} H_{\zeta\omega_i}$$
$$+ (\omega_f)^2 \sum_{i=1}^{15} \omega_{i_o}^{\ 2} H_{\omega_i^{\ 2}} \tag{14}$$

The parameter uncertainties related to M_w should also appear in Equation (14). However the form of the available data did not permit the M_w related elements to be separated easily. Consequently, M_w uncertainty effects were not evaluated in the response equations.

The G_1 matrix contains only surface actuator gains which in general can have uncertainties associated with them. Uncertainties in G_1 were not treated in this study.

The G_2 matrix contains the magnitude of the gust intensity which is currently scaled for a 0.3048 m/sec (1.0 ft/sec) RMS condition. The gust model depends on airspeed and hence on dynamic pressure. However, uncertainties in the gust model were not applied in this study.

Appendix B contains more specific details on the computation of the uncertain parameter matrices.

The second category of model variations is neglected dynamics. Standard flight control system design procedure is to eliminate as many structural modes as possible in the design process. The C-5A model provides the capability of analyzing the effects of neglected dynamics through judicious manipulation of the structural modes. This will be discussed in more detail under Reduced Order Models.

Finally, the third category of model variations is unknown dynamics. Uncertainties in G_1 were not treated in this study since this would have involved defining an additional set of uncertain parameters which would have increased computational requirements. It is felt that the methods for treating F uncertainties would be applicable to the treatment of uncertainties in G_1. Hence, to keep computational requirements at a workable level, G_1 was assumed to be known. However, the unsteady aerodynamics, included in the C-5A model, could be treated as unknown dynamics for evaluation purposes. No attempt was made to include the effect of unsteady aerodynamics in the development of reduced order models. Thus, for systems designed with reduced order models that do not include the effect of unsteady aerodynamics, these dynamics can be interpreted as unknown.

REDUCED ORDER MODELS

When the size of a model does not facilitate control system design, an attempt is usually made to approximate the model with a reduced order model that retains the characteristics which significantly influence the control system design. The Case 1 form of the C-5A model with its 79th order dynamics exemplifies a model that does not facilitate control system design. Two techniques have been used to reduce the size of unwieldy models. These are:

- Truncation
- Residualization

To demonstrate these techniques, consider the partitioned form of Equations (11) and (12) given by

$$\dot{x} = \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} G_{1_1} \\ G_{1_2} \end{bmatrix} u + \begin{bmatrix} G_{2_1} \\ G_{2_2} \end{bmatrix} \eta \tag{15}$$

where

$\tilde{x}_1$ = states to be retained in reduced order model

$\tilde{x}_2$ = states to be eliminated

A truncated reduced order model is constructed on the premise that

$$\tilde{x}_2 = 0 \tag{16}$$

thus producing a truncated model given by

$$\dot{\tilde{x}}_1 = F_{11}\tilde{x}_1 + G_{1_1} u + G_{2_1} \eta \tag{17}$$

A residualized model attempts to retain a steady state effect of the eliminated states by setting

$$\dot{\tilde{x}}_2 = 0 \tag{18}$$

in Equation (15) and solving for $\tilde{x}_2$, which gives

$$\tilde{x}_2 = -F_{22}^{-1} [F_{21}\tilde{x}_1 + G_{1_2} u + G_{2_2} \eta] \tag{19}$$

Substituting the right-hand side of Equation (19) for $\tilde{x}_2$ in the $\dot{\tilde{x}}_1$ equation produces the residualized reduced order model

$$\dot{\tilde{x}}_1 = [F_{11} - F_{12} F_{22}^{-1} F_{21}] \tilde{x}_1 + [G_{1_1} - F_{12} F_{22}^{-1} G_{1_2}] u$$
$$+ [G_{2_1} - F_{12} F_{22}^{-1} G_{2_2}] \eta \tag{20}$$

Both the truncation and residualization techniques were investigated in this study. The following reduced order models were constructed:

Case 2 - Truncated 42nd order model

Case 3T - Truncated 24th order model

Case 3R - Residualized 24th order model

Case 4T - Truncated 16th order model

Case 4R - Residualized 16th order model

For the first reduced order model constructed, Case 2, the vector x of Case 1 was partitioned into $\tilde{x}_1$ and $\tilde{x}_2$ in the notation of Equation (15) as follows:

$$\tilde{x}_1 = [x_1 \ldots x_{42}] \triangleq x^2$$

$$\tilde{x}_2 = [x_{43} \ldots x_{79}]$$

The truncation technique was used to construct Case 2. It eliminated all the unsteady aerodynamics but retained all 15 bending modes, all seven gust states, all three actuator states, and the two rigid body states. Truncation rather than residualization was used to generate Case 2 to provide the capability of realistically evaluating the effects of "unknown" dynamics. The residualization process attempts to retain some of the characteristics of the states that are not included in the reduced order model. It requires knowing the dynamics of these states. In most cases this is a desirable feature. However, for purposes of this study, we assumed that the unsteady aerodynamics were "unknown" so that Case 1 could be used to evaluate the effects of unknown dynamics in insensitive controller performance. Had we residualized to Case 2, we would have included the characteristics of "unknown" dynamics which would have compromised our evaluation procedure. Case 2, then, becomes our most complete "known" model. The remaining truncation and residualization procedures were applied to Case 2. (With respect to truncation, this point is immaterial.) For both Case 3T and Case 3R, $\tilde{x}_1^{\,2}$ and $\tilde{x}_2^{\,2}$ are defined as follows:

$$\tilde{x}_1^{\,2} = [x_1^{\,2} \ldots x_8^{\,2}, x_{18}^{\,2} \ldots x_{23}^{\,2}, x_{33}^{\,2} \ldots x_{42}^{\,2}] \triangleq x^3$$

$$\tilde{x}_2^{\,2} = [x_9^{\,2} \ldots x_{17}^{\,2}, x_{24}^{\,2} \ldots x_{32}^{\,2}]$$

Case 3T and 3R do not contain the nine highest frequency structural modes but retain all remaining states. For Case 4T and 4R, $\tilde{x}_1^{\,2}$ and $\tilde{x}_1^{\,2}$ are

$$\tilde{x}_1^{\,2} = [x_1^{\,2} \ldots x_3^{\,2}, x_5^{\,2}, x_{18}^{\,2}, x_{20}^{\,2}, x_{33}^{\,2} \ldots x_{42}^{\,2}] \triangleq x^4$$

$$\tilde{x}_2^{\,2} = [x_4^{\,2}, x_6^{\,2}, \ldots x_{17}^{\,2}, x_{19}^{\,2}, x_{21}^{\,2}, \ldots x_{32}^{\,2}]$$

In Case 4T and 4R, only the first and third structural modes are retained. All other structure modes were eliminated. The third mode was retained over the second because of stability problems associated with that mode observed in previous studies.

Appendix A contains the F, G_1, G_2, H, and D matrices for each of the reduced order models and "free aircraft" eigenvalues and statistical response data for each case.

DESIGN SPECIFICATIONS

The design criteria, which were used for the design of the nominal controller and for the evaluation of the insensitive controllers, were initially specified for the design of the Active Lift Distribution Control System (ALDCS). ALDCS was the product of a Honeywell-Lockheed study whose objective was to reduce wing loading on the C-5A through an active application of control surfaces. The design specifications, as supplied by Lockheed for the ALDCS study, are given in Table 4. As can be seen from Table 4, the criteria cover the spectrum of flight control design objectives. The maneuver load control specification is a transient response criterion. The gust load alleviation specifications are statistical response criteria. Handling qualities are specified by closed-loop root locations. And, finally, there are standard stability margin specifications.

These criteria, as specified, provide a realistic foundation for nominal control system design coupled with the scope and flexibility needed for insensitive controller evaluation.

TABLE 4. ALDCS DESIGN CRITERIA

Maneuver load control	30% steady state bending moment reduction at wing root
Gust load alleviation	30% RMS bending moment reduction at the wing root No more than 5% increase in RMS torsional moment at the wing root
Handling qualities	Acceptable as measured by closed-loop short period roots and transient response to pilot step input
Stability margins	Gain margin ≥ 6 dB Phase margin ≥ 0.7854 rad (45 degrees)

DESIGN OF NOMINAL CONTROLLER

The design of the nominal controller utilized Honeywell's developments in "quadratic" methodology and computer software. "Quadratic" methodology refers to optimal control system design procedures that minimize a quadratic performance index. For the case of the time-invariant stochastic control problem that the C-5A design represents, the performance index is given by

$$J = E\,[r_d^T\ Q\ r_d] \tag{21}$$

where E [ ] is the expected value operator, r_d is a vector of design responses whose form is given by Equation (12), and Q is a symmetric weighting matrix. The r_d vector that was used to design an optimal controller which satisfied the ALDCS design specifications is given in Table 5.

Only the ailerons and the inboard elevator were used in the controller design. The outboard elevator is reserved for pilot command inputs. The design approach consists of the following steps:

TABLE 5. RESPONSE VECTOR AND QUADRATIC WEIGHTS

Response vector	Physical quantity	Weight
r_{d_1}	B_1 = bending moment at wing root	1×10^{-10}
r_{d_2}	T_1 = torsion moment at wing root	1×10^{-9}
r_{d_3}	$\dot{B}_1$ = rate of change of bending moment at wing root	5.5×10^{-13}
r_{d_4}	$\dot{T}_1$ = rate of change of torsion moment at wing root	1×10^{-11}
r_{d_5}	δ_a = aileron displacement	0.32×10^{8}
r_{d_6}	δ_{e_i} = inboard elevator displacement	0
r_{d_7}	$\dot{\delta}_{a_m}$ = function of aileron displacement and aileron command	1×10^{8}
r_{d_8}	$\dot{\delta}_{e_i}$ = inboard elevator rate	1×10^{6}
r_{d_9}	r_{CF} = control follower response	2×10^{7}

<u>Step 1:</u> Design a full state optimal controller for Case 3T, the 6 bending mode truncated model, which satisfies the bending and torsion moment statistical criteria, places short period roots at a frequency greater than 1.6 radians/second and damping ratio between 0.7 and 0.8, satisfies bending moment response to step elevator criteria, satisfies gain and phase margin criteria, and maintains actuator roots, surface displacements, and surface rates at realizable values.

<u>Step 2:</u> Using quadratic weights determined in Step 1, obtain optimal gains for Case 3R and validate that design criteria are satisfied.

Step 3: Evaluate Case 3T optimal controller on higher order models, Case 1 and Case 2.

Step 4: Evaluate Case 3R optimal controller on higher order models, Case 1 and Case 2.

The same four steps are also applicable to the design of the nominal controller for Case 4 (T and R), the 2 bending mode truncated and residualized model. Controller design utilized full state feedback or

$$u = Kx \tag{22}$$

One of the early optimal controller designs, while meeting all other design specifications, did not meet the 6 dB gain margin criteria for the aileron loop. In fact, frequency response analysis showed that the controller was very close to a low frequency instability in the aileron loop, the gain margin being only 1.2 dB. An analysis of the aileron loop root loci indicated that a root equivalent to the aileron actuator was proceeding into the right half plane. This was attributed to the presence of positive feedback (Gain $K_{1,\,15} = 0.7$) of aileron displacement to aileron command in the full state feedback controller.

By arbitrarily setting $K_{1,\,15}$ to zero and maintaining the remaining gains at their optimally designed values, a gain margin of 15 dB was realized. Since one of the agreed upon ground rules of this contract was to design the nominal controller using full state feedback optimal control, an attempt was made to force $K_{1,\,15}$ to zero through manipulation of the quadratic weights. The optimal gains K are obtained via the algebraic relationship

$$K = -(D^TQD)^{-1}\ (D^TQH + G_1{}^TP) \tag{23}$$

Where matrixes D and H have been defined in Equation (13), Q is the weighting matrix of Equation (21) and P is the solution to the Ricatti equation. The Ricatti equation may be written as

$$\begin{aligned} PG_1(D^TQD)^{-1}\,G_1{}^TP - PF - F^TP &= H^TQH \\ &-G_1(D^TQD)^{-1}\,D^T\,QHP - PG_1(D^TQD)^{-1}\,D^TQH \\ &-H^T\,QD\,(D^TQD)^{-1}\,D^TQH \end{aligned} \tag{24}$$

It was determined that the 0.7 value of gain $K_{1,\ 15}$ was produced in the following manner:

$$K_{1,\ 15} = 0.7 = -\frac{(D^TQH)_{1,15}}{(D^TQD)_{(1,15)}} - \frac{(G_1{}^TP)_{1,15}}{(D^TQD)_{(1,15)}} \tag{25}$$

where

$$\frac{(D^TQH)_{1,15}}{(D^TQD)_{1,15}} = -1.0 \qquad \frac{(G_1{}^TP)_{1,15}}{(D^TQD)_{1,15}} = 0.3$$

where the subscript 1, 15 is used to indicate the terms of the respective matrix operations which contribute to the $K_{1,\ 15}$ gain computation.

The first term of Equation (25) can be set to 0.3 by defining a new response variable which is a modification of the aileron rate response or

$$\dot{\delta a}_m = -2.0\,\delta a + 6.0\ \delta a_c$$

This produces a modification of the H matrix and the term D^TQH. The second term will be held at 0.3 if, as a first approximation, it is assumed that the major component of Equation (24) in the determination of P is the term H^TQH. By modifying the weighting matrix Q such that $(H^TQH)_{1,15}$ remains the same, this approximation can be achieved. Specifically this requires placing a weight on the aileron response to compensate for the modified H. This modification led to a satisfactory design.

Another unique feature of the design involved the manner in which the handling quality criteria were satisfied.

The C-5A aircraft is augmented with a simple SAS system to enhance the handling qualities of the aircraft. It consists of pitch rate feedback to the inboard elevator through a gain of 0.5 or

$$\delta_{e_{i_{c_m}}} = 0.5\ q \tag{26}$$

where the c subscript indicated commanded position.

This has the effect of moving the short period dynamics from $\omega_{sp} \cong 1.57$ radians/second, $\xi_{sp} \cong 0.57$ to $\omega \cong 1.78$ radians/second, $\xi_{sp} \cong 0.75$.

The incorporation of this SAS system was included in the optimal controller design by using a control model-following approach. Specifically, one of the system responses was chosen to be

$$r_{CF} = \delta_{e_{i_{c_m}}} - \delta_{e_{i_c}} \tag{27}$$

where $\delta_{e_{i_{c_m}}}$ is the SAS system of the C-5A aircraft. Substituting (24) into (25), the equation reads

$$r_{CF} = 0.5q - \delta_{e_{i_c}}$$

Thus by varying the weights on r_{CF}, the short period roots could be controlled. The final set of weights is given in Table 5.

CONTROLLER PERFORMANCE

The performance of the controllers designed on the basis of the 6 bending mode and 2 bending mode models is given in Table 6. For comparison purposes, the performance of the ALDCS controller on the Case 1 model is also given. The progression of results, as shown in Table 6 for both the truncated and residualized model controllers, are as follows:

Truncated (Residualized) Results (Reduced Model)	Presents performance results of controllers designed using the 6-mode and 2-mode truncated (residualized) models and evaluated on those same models.
Truncated (Residualized) Results (Case 2)	Presents performance results of controllers designed using the 6-mode and 2-mode truncated (residualized) models, but evaluated on the Case 2 model.
Truncated (Residualized) Results (Case 1)	Presents performance results of controllers designed using the 6-mode and 2-mode truncated (residualized) models, but evaluated on the Case 1 model.

TABLE 6. COMPARISON OF CANDIDATE DESIGN MODELS

Specification description		Criteria	ALDCS	Truncated results (reduced model)		Truncated results (case 2)		Truncated results (case 1)		Residualized results (reduced model)		Residualized results (case 2)		Residualized results (case 1)	
				6 Mode	2 Mode	6 Mode	2 Mode	6 Mode	2 Mode	6 Mode	2 Mode	6 Mode	2 Mode	6 Mode	2 Mode
Maneuver load control % change	B	< -30%	-30%	-34%	-31%	-34%	-30%	-36%	-33%	-40%	-41%	-40%	-40%	-43%	
Gust load alleviation % change	B	< -30%	-36%	-33%	-34%	-31%	-27%	-36%	-33%	-35%	-35%	-35%	-35%	-39%	-40%
	T	< + 5%	- 1%	-27%	-25%	-27%	-22%	-29%	-24%	-31%	-31%	-31%	-31%	-33%	-32%
Handling qualities	w_{sp} (rad/sec)	> 1.6	2.30	2.05	2.01	2.00	1.90	2.25	2.08	2.12	2.12	2.13	2.13	2.42	2.40
	3 (sec)$^{-1}$	0.7-0.8	0.76	0.72	0.70	0.72	0.69	0.74	0.70	0.72	0.72	0.72	0.72	0.73	0.73
Stability margins	Gain	> 6 dB	δ_a 17.2 dB	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
			δ_e 15.4 dB	28 dB	43 dB	29 dB	∞	28 dB	15 dB	29 dB	29 dB	29 dB	29 dB	28 dB	36 dB
	Phase	> (45°) 0.7854 rad	δ_a 1.75 rad (100°)	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
			δ_e 2.50 rad (143°)	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

A number of conclusions may be drawn based on the data presented in Table 6. First, the residualization technique leads to much more consistent results than truncation. This can be demonstrated graphically by considering the deviation in performance that controllers which are designed and evaluated on lower order models experience when applied to higher order models. The results are given in Figure 2 in bar graph form.

Shown plotted on the y axes are design criteria deviations in the units of the specification. The x axis is separated into individual comparison cases. For example, Comparison 1, Case 3, 3 versus Case 4, 4 represents a comparison in performance between a controller designed using Case 3 and evaluated using Case 3 and a controller that is designed using Case 4 and evaluated using Case 4. The first number represents the model case that was used to design the controller. The second number indicates the model case that it was evaluated on. Both truncated and residualized results are shown. The results to the left of the dashed line clearly indicate the consistency in performance of residualization over truncation. As an example, on Comparison 1 it can be seen that the "residualized" controllers show a deviation in performance only for the maneuver load performance. The performance for all other criteria is identical. Truncated results, on the other hand, show deviations between cases. Similar results are shown for Comparisons 2, 3, and 4.

The deviations in performance between Comparison 3 and Comparison 4 indicate a reduction in deviation for "truncated" controllers applied to higher order systems if more dynamics are retained in the "truncated" controller design model.

The cases shown to the right of the dashed line do not demonstrate comparison evaluations of truncation versus residualization per se. The two comparisons shown represent Case 3 and 4 controllers applied to Case 1. As was discussed earlier, Case 3R and Case 4R were residualized from Case 2, not Case 1. Thus evaluation of the two techniques with respect to consistency in performance should be limited to the Comparisons 1 through 4. Comparisons 5 and 6, however, do provide additional information. First, "residualized" controllers, no matter what reduced order model they were designed with, exhibit the same trend in performance when applied to higher order models. For example, bending RMS deviations are positive (i.e., better performance on higher order system) for Comparisons 5 and 6. On the other hand, Controller 3T demonstrates better performance when applied to Case 1 while Controller 4T does worse.

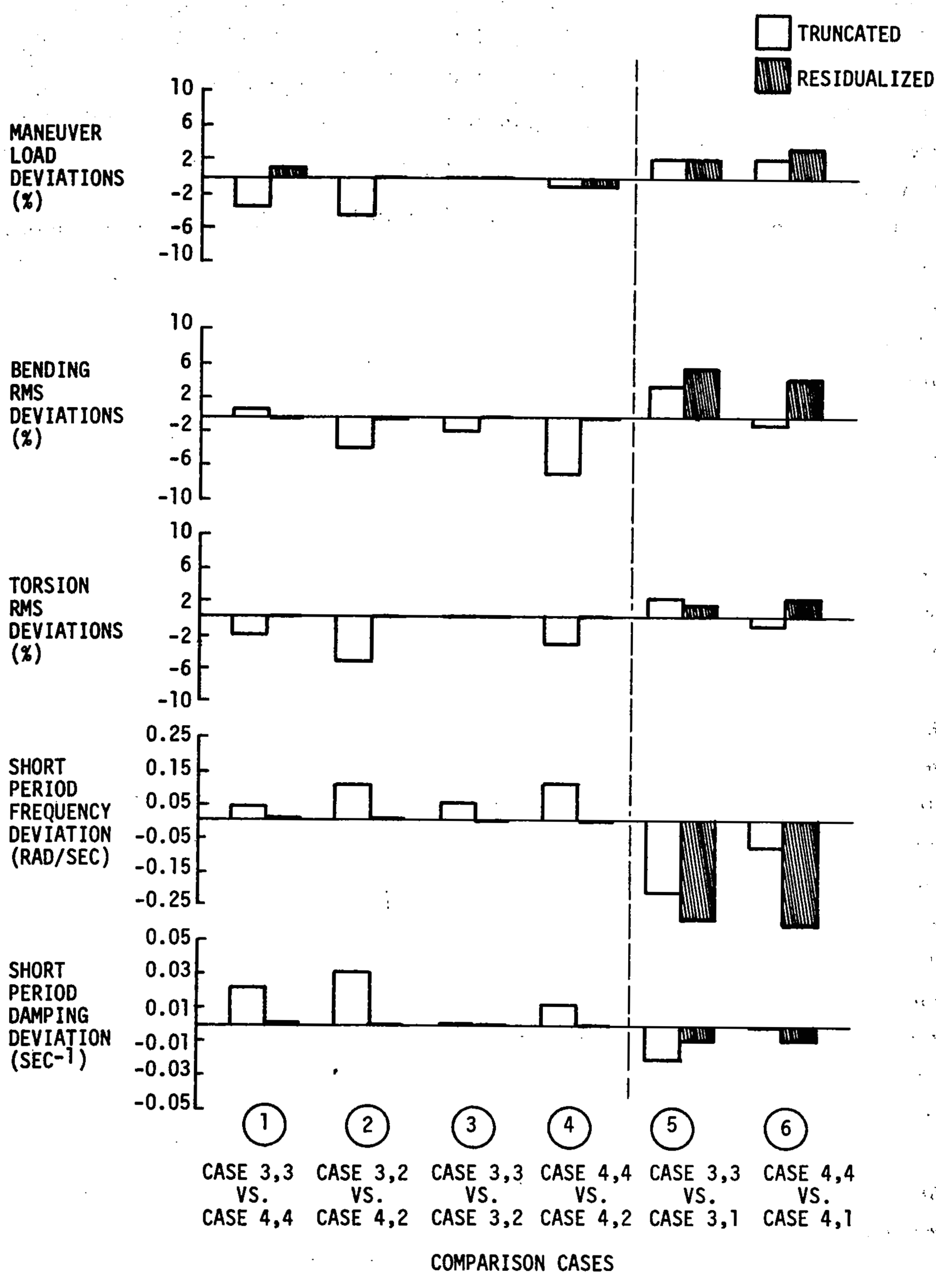


Figure 2. Truncation versus Residualization

Second, there is very little difference in magnitude between "residualized" controller performance in Comparisons 5 and 6. This is not true for the "truncated" controllers.

Based on the second observation and also on the "residualized" controller performances shown in Comparisons 3 and 4, it was concluded that nothing was gained by including the higher dynamics of Case 3R in a design model. The Case 4R controller did just as well. Consequently, we decided to use Case 4R, i.e., the residualized 2-mode model, as the nominal design model and also the model for all insensitive controller designs. The optimal controller designed with Case 4R is defined as the nominal controller. Its performance was used as a benchmark in all comparisons and evaluations of the insensitive controllers designed. The gains for the nominal controller are given in Table 7.

RANGE OF MODEL VARIATIONS

As discussed in Section III, the ideal problem statement for use in the design of an insensitive control system should include a specification of the types and ranges of variations from real world conditions that the model may exhibit. With respect to the three categories of types of variations that were discussed previously, we defined 1) a set of uncertain parameters ($\bar{q}$, ω, ζ, M_w), 2) a set of known unmodeled dynamics (high frequency structural modes), and 3) a set of assumed "unknown" unmodeled dynamics (unsteady aerodynamics). The last two categories require no more definition. The first category, however, still requires a specification on the range of uncertainty that the uncertain parameters may experience. In general, if the range of parameter uncertainties is not defined a priori, the control system designer will rely on his experience and whatever information he can obtain on model efficacy to define a realistic uncertainty range. For the purposes of this study, we added a third factor. We wished to define an uncertainty range in which, for some point or points in that range, the performance of the nominal controller produced a design specification violation. This is not a necessary feature for insensitive controller design. It merely provides an additional quantitative reference that can be used in subsequent evaluation of insensitive controller performance.

The range of parameter uncertainties was determined experimentally by varying the uncertain parameters and calculating the performance of the nominal controller under these off-nominal conditions. Table 8 presents the results of the evaluation of the effects of the parameter uncertainties on nominal controller performance. Run 66 represents performance under nominal conditions. The factors $\bar{q}_f$, ζ_f, ω_f, and M_{w_f} were defined earlier to represent the parameter variations. As a reminder, $\bar{q}_f = 1.1$ represents a 10

TABLE 7. NOMINAL CONTROLLER GAINS

```
      GAINS MATRIX

ROW   1
-.32077E-04  .18016E-04  .11200E-03  .32268E-04 -.34501E-03 -.70350E-03 -.21382E+00  .11248E-02  .11472E-04 -.27191E-03
 .47293E-04  .61141E-05 -.20954E-03  .13560E-03 -.13574E-04
ROW   2
 .13446E-03  .40718E-03 -.18809E-03  .26898E-03  .46278E-03 -.62506E-02  .34005E-02 -.15227E+00 -.17730E-05  .67391E-03
-.12350E-02 -.69637E-04  .43997E-03  .85089E-03  .13845E-02
```

TABLE 8. RESULTS OF PARAMETER UNCERTAINTY EVALUATION

Run #	$\bar{q}_f$	ζ_f	ω_f	M_{W_F}	ω_{sp}	ζ_{sp}	B dec.	T dec.	B_{ss} dec.	δ_A GM	δ_A PM	δ_e GM	δ_e PM
66	1.00	1.0	1.00	1.0	2.12	0.72000	35.00%	31.00%	41.00%	∞	∞	29.00	∞
73	1.10	1.0	1.00	1.0	2.35	0.72000	31.00%	28.00%	34.00%	∞	∞	30.00	∞
74	1.20	1.0	1.00	1.0	2.60	0.73000	27.00%	24.00%	28.00%	∞	∞	30.00	∞
75	0.90	1.0	1.00	1.0	1.89	0.71000	40.00%	36.00%	46.00%	∞	∞	30.00	∞
76	0.80	1.0	1.00	1.0	1.67	0.71000	44.00%	40.00%	53.00%	∞	∞	∞	∞
84	1.00	0.9	1.00	1.0	2.12	0.72000	35.00%	37.00%	40.00%	∞	∞	30.00	∞
90	1.00	1.0	1.10	1.0	2.14	0.71000	35.60%	31.50%	40.50%	∞	∞	42.00	∞
91	1.00	1.0	0.90	1.0	2.10	0.72900	34.70%	31.17%	39.50%	∞	∞	27.00	∞
92	1.00	1.0	1.00	1.1	2.17	0.69976	37.00%	35.00%	40.50%	∞	∞	27.80	∞
93	1.00	1.0	1.00	0.9	2.07	0.73690	32.76%	27.53%	37.80%	∞	∞	37.35	∞
94	1.20	0.9	0.90	0.9	2.51	0.76000	23.40%	18.20%	23.60%	∞	∞	25.30	∞
95	1.00	0.5	1.00	1.0	2.12	0.71700	34.50%	31.30%	40.10%	∞	∞	35.80	∞
96	0.50	1.0	1.00	1.0	1.03	0.70100	59.80%	54.80%	70.70%	∞	∞	∞	∞
98	0.50	1.0	1.00	1.2	1.08	0.67100	61.30%	51.50%	71.50%	∞	∞	∞	∞
100 Case 1	1.20	0.9	0.90	0.9	3.13	0.77600	28.10%	19.10%	35.70%	∞	∞	28.56	∞
101 Case 1	0.50	1.0	1.00	1.2	1.16	0.68000	68.20%	57.30%	74.80%	∞	∞	∞	∞
102 Case 2	1.00	0.5	1.00	1.0	2.13	0.71500	34.15%	30.07%	40.12%	∞	∞	28.51	∞
103	1.20	0.5	0.75	0.8	2.38	0.83000	19.00%	9.70%	17.13%	∞	∞	19.00	∞
104	1.25	0.5	0.75	0.8	2.50	0.84300	16.87%	6.97%	13.24%	∞	∞	19.56	∞
105 Case 1	1.25	0.5	0.75	0.8	3.75	0.81000	21.80%	5.68%	28.10%	∞	∞	21.32	∞

percent increase in $\bar{q}$ from the nominal value. All evaluations were done on Case 4R unless otherwise designated.

An analysis of Table 8 led to the following conclusions:

1. No combination of uncertainties produced a violation of all five criteria. Run 104 represents a combination which violated three of the five criteria. The torsion moment and the stability margins are still within specification.

2. The nominal controller was insensitive to variations in structural damping as demonstrated by runs 95 and 102.

3. The effect of unmodeled dynamics as seen when the same perturbations were applied to Case 4R and Case 1 is to improve performance in all cases.

4. Two worst case conditions, shown in runs 101 and 105, were defined. Run 101 produced a violation in only the handling quality specification but it shows the effect of large variation. Run 105 is worst case in that the largest number of violations are produced.

Based on the results given in Table 8 the following range on the uncertain parameters was defined.

Dynamic Pressure Uncertainty	-	$0.5 \le \bar{q}_f \le 1.25$
Structural Damping Uncertainty	-	$0.5 \le \zeta_f \le 1.50$
Structural Frequency Uncertainty	-	$0.75 \le \omega_f \le 1.25$
Stability Derivative, M_w, Uncertainty	-	$0.8 \le M_{w_f} \le 1.20$

Except for the lower limit on $\bar{q}_f$, it is felt that the magnitude of the range of uncertainties is realistic. The lower $\bar{q}_f$ limit was selected to obtain a compromised specification violation, i.e., a condition which violates one design specification but gives improved performance on the others.

SECTION V

INSENSITIVE CONTROLLER DESIGN--
EXISTING TECHNIQUES

In this section the design of insensitive controllers, based on synthesis techniques that have appeared in the recent control literature, will be described.

Five synthesis techniques for the design of insensitive control systems that are based on existing theory were investigated in this study. These have been identified by the following descriptive titles:

1. Additive Noise
2. Minimax
3. Multiplant
4. Sensitivity Vector Augmentation
5. State and Control Dependent Noise

All of these techniques can be formulated as time-invariant stochastic control problems with the standard quadratic performance index. This formulation introduces the difficulties associated with selecting quadratic weights. To alleviate these difficulties, the approach taken was to structure the synthesis technique in such a way that quadratic weight manipulation was minimal or, at worst, straightforward.

The design of the insensitive controllers utilized full state feedback, as did the design of the nominal controller. As discussed earlier, the use of full state feedback was a ground rule for this study.

A description of each synthesis technique and the resulting controller design will now be presented.

ADDITIVE NOISE

The additive noise concept has been used extensively to design controllers for systems subject to uncertain inputs such as gusts and pilot commands. In this study, however, the effects of uncertainties in aircraft dynamics will be approximated by additional noise disturbances to the nominal system. Thus, the ensemble of disturbances which the design system is expected to encounter is "shaped" by the uncertainties in system parameters.

The formulation of the concept for this purpose is as follows. The dynamical system is represented by the differential equation

$$\dot{x} = F(x, u, p, \eta) \tag{28}$$

where x is the state vector, u is the control vector, p is a vector of parameters whose values may be uncertain, and η is a (disturbance) noise vector. The responses of interest comprise the response vector

$$r = H(x, u, p) \tag{29}$$

The nominal value of p is p_o. The nominal value of η is zero. It is assumed that a nominal solution (x_o, u_o) is known which satisfies

$$\dot{x}_o = F(x_o, u_o, p_o, 0) \tag{30}$$

Then perturbation equations are written as

$$\delta\dot{x} = F_x \delta x + F_u \delta u + F_p \delta p + F_\eta \eta \tag{31}$$

$$\delta r = H_x \delta x + H_u \delta u + H_p \delta p \tag{32}$$

where F_x denotes the matrix of partial derivatives of F evaluated at $(x_o, u_o, p_o, 0)$ and the remaining coefficient matrices are defined similarly. The vector δp is then assumed to satisfy the differential equation

$$\delta\dot{p} = A\,\delta p + B\tilde{\eta}, \quad \delta p(0) = \delta p_o \tag{33}$$

where δp_o is a random variable, and $\tilde{\eta}$ is a white Gaussian noise input vector. A, B, and the statistics of δp_o and $\tilde{\eta}$ are chosen such that $E\{\delta p(\delta p)'\} = 3\,\Sigma$ with Σ denoting the covariance of expected parameter variations and the frequency content of δp being consistent with physical expectations.

In our specific example, the matrices F_p and H_p are not known and the cost of determining them would be excessive. This is probably typical of many control problems. Therefore, we developed the following formulation.

Consider the nominal plant represented by

$$\dot{x}_o = F(p_o)x_o + G_1 u + G_2 \eta \tag{34}$$

where p_o represents a vector of uncertain parameters at their nominal values. Equation (34) may be partitioned to demonstrate the influence of the gust states, x_g, on the aircraft states, x_{o_a}.

$$\begin{bmatrix} \dot{x}_{o_a} \\ \dot{x}_g \end{bmatrix} = \begin{bmatrix} F_{11}(p_o) & F_{12}(p_o) \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} x_{o_a} \\ x_g \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ G_2 \end{bmatrix} \eta \tag{35}$$

where $x_o^T = [x_{o_a}^T, x_g^T]$.

The response equation may be written as

$$r_o = H(p_o)\, x_o + D(p_o)\, u \tag{36}$$

For a perturbed system, the plant may be represented in the same partitioned form by

$$\begin{bmatrix} \dot{x}_{p_a} \\ \dot{x}_g \end{bmatrix} = \begin{bmatrix} F_{11}(p) & F_{12}(p) \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} x_{p_a} \\ x_g \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ G_2 \end{bmatrix} \eta \tag{37}$$

and the response by

$$r_p = H(p)x_p + D(p)u \tag{38}$$

where

$$p = p_o + \Delta p$$

Equation (37) may be rewritten as

$$\begin{bmatrix} \dot{x}_{p_a} \\ \dot{x}_g \end{bmatrix} = \begin{bmatrix} F_{11}(p_o) & F_{12}(p_o) \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} x_{p_a} \\ x_g \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ G_2 \end{bmatrix} \eta +$$

$$\begin{bmatrix} F_{11}(p) - F_{11}(p_o) & F_{12}(p) - F_{12}(p_o) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{p_a} \\ x_g \end{bmatrix}$$

The additive noise concept may be viewed as approximating this equation with one of the form

$$\begin{bmatrix} \dot{x}_{p_a} \\ \dot{x}_g \end{bmatrix} = \begin{bmatrix} F_{11}(p_o) & F_{12}(p_o) \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} x_{p_a} \\ x_g \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \tilde{G}_2 \\ G_2 \end{bmatrix} \eta \qquad (39)$$

where the parameter variations are lumped into the $\tilde{G}_2 \eta$ term. Thus $\tilde{G}_2$ should be chosen in such a manner that the noise produces the same effects (in some sense) as parameter variations produce on system responses.

We made three major assumptions in defining $\tilde{G}_2$. They were:

1. *The magnitude of $\tilde{G}_2$ should reflect 3σ magnitudes of parameter variations,*

2. $\tilde{G}_2$ should also reflect the "worst case" condition found in the evaluation of the nominal controller, and

3. The effect of the uncertain parameter variations should be referenced to the closed-loop system response. Thus $\tilde{G}_2$ should be chosen in such a manner that the noise produces the same effects as parameter variations produce on closed-loop system responses.

Given those assumptions, the solution for $\tilde{G}_2$ proceeds by first selecting a worst case set of uncertain parameter variations. The case selected was taken from the worst case conditions discussed in the previous section. It consists of a 25 percent increase in dynamic pressure, a 50 percent decrease in structural damping, a 25 percent decrease in the structural frequency at all modes, and a 20 percent decrease in the stability derivative, M_w. The covariance, X_1, of the plant under these worst case conditions is the solution of the Lyapunov equation which is given by

$$F'(p_1)X_1 + X_1F'(p_1)^T + G_2G^T{}_2 = 0$$

$$F'(p_1) = F(p_1) + G_1K = \text{closed-loop state matrix} \tag{40}$$

where p_1 is the "worst case" uncertain parameter vector, and K represents nominal system optimal gains.

Equation (40) may also be written as

$$(F'(p_o) + \Delta F'_1)X_1 + X_1(F'(p_o) + \Delta F'_1)^T + G_2G^T{}_2 = 0$$

$$\Delta F'_1 = F'(p_1) - F'(p_o) \tag{41}$$

Partitioning (41) into the gust and non-gust covariances yields

$$\begin{bmatrix} F'_{o_{11}} + \Delta F'_{1_{11}} & F'_{o_{12}} + \Delta F'_{1_{12}} \\ F'_{o_{21}} + \Delta F'_{1_{21}} & F'_{o_{22}} + \Delta F'_{1_{22}} \end{bmatrix} \begin{bmatrix} X_{1_{11}} & X_{1_{12}} \\ X^T_{1_{12}} & X_{1_{22}} \end{bmatrix} +$$

$$\begin{bmatrix} X_{1_{11}} & X_{1_{12}} \\ X^T_{1_{12}} & X_{1_{22}} \end{bmatrix} \begin{bmatrix} F'_{o_{11}} + \Delta F'_{1_{11}} & F'_{o_{21}} + \Delta F'_{1_{21}} \\ F'_{o_{12}} + \Delta F'_{1_{12}} & F'_{o_{22}} + \Delta F'_{1_{22}} \end{bmatrix} + \tag{42}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & G_2G_2^T \end{bmatrix} = 0$$

where $F_o = F(p_o)$.

The covariance of the states associated with Equation (30), X_2, is similarly the solution of the Lyapunov equation given by

$$\begin{bmatrix} F'_{o_{11}} & F'_{o_{12}} \\ F'_{o_{21}} & F'_{o_{22}} \end{bmatrix} \begin{bmatrix} X_{2_{11}} & X_{2_{12}} \\ X^T_{2_{12}} & X_{2_{22}} \end{bmatrix} + \begin{bmatrix} X_{2_{11}} & X_{2_{12}} \\ X^T_{2_{12}} & X_{2_{22}} \end{bmatrix} \begin{bmatrix} F'_{o_{11}} & F'_{o_{21}} \\ F'_{o_{12}} & F'_{o_{22}} \end{bmatrix} + \begin{bmatrix} \tilde{G}_2\tilde{G}_2^T & \tilde{G}_2 G_2^T \\ (\tilde{G}_2 G_2^T)^T & G_2 G_2^T \end{bmatrix} = 0 \tag{43}$$

We can solve for $\tilde{G}_2\tilde{G}_2^T$ by requiring that $X_1 = X_2$ and equating Equations (42) and (43) and simplifying for

$$\tilde{G}_2\tilde{G}_2^T = \Delta F'_{1_{11}} X_{1_{11}} + \Delta F'_{1_{12}} X^T_{1_{12}} + X_{1_{11}} \Delta F'_{1_{11}} + X_{1_{12}} \Delta F'_{1_{12}} \tag{44}$$

Similarly,

$$\tilde{G}_2 G_2^T = \Delta F'_{1_{11}} X_{1_{12}} + \Delta F'_{1_{12}} X_{1_{22}} + X_{1_{11}} \Delta F'_{1_{21}} + X_{1_{12}} \Delta F'_{1_{22}} \tag{45}$$

Equation (44) defines the magnitude but not the direction of $\tilde{G}_2$. However, for design purposes, only the products $\tilde{G}_2\tilde{G}_2^T$ and $\tilde{G}_2 G_2^T$ are needed since it is the products that are used in calculating the covariance.

The uncertainties in the response equations are treated in a like manner. The response covariance, R_1, associated with the worst case condition is given by

$$R_1 = [H(p_1) + D(p_1)K]\, X_1\, [H(p_1) + D(p_1)K]^T \tag{46}$$

The response covariance associated with the nominal is

$$R_o = [H(p_o) + D(p_o)K]\, X_o\, [H(p_o) + D(p_o)K]^T \tag{47}$$

where

$$F'(p_o)\,X_o + X_o\,F'(p_o)^T + G_2 G_2^T = 0$$

and

$$F'(p_o) = F(p_o) + G_1 K$$

An RMS bias was formed as

$$r_{i_{BIAS}} = \sqrt{(r_{ii})_1} - \sqrt{(r_{ii})_o} \tag{48}$$

for each of the components of r. This is used to include the effects of the parameter uncertainties in the design response equations on the RMS responses as

$$\sigma_i = \sqrt{(r_{ii})_p} + r_{i_{BIAS}} \tag{49}$$

where σ_i is the design RMS value for the ith component.

Design Details and Results

The design strategy for this approach was to consider the system given by

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_g \end{bmatrix} = \begin{bmatrix} F_{11}(p_o) & F_{12}(p_o) \\ 0 & F_{22}(p_o) \end{bmatrix} \begin{bmatrix} x_a \\ x_g \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \tilde{G}_2 \\ G_2 \end{bmatrix} \eta$$

$$r = H(p_o)x + D(p_o)\,u$$

as the design system. The $\tilde{G}_2$ represents the effects of parameter uncertainties in the state equations. An RMS bias was used to represent the effect of parameter uncertainties in the response equations. The weighting factors were then varied so as to drive the RMS values of bending and torsion moments back to their nominal values.

Table 9 presents a comparison of RMS bending and torsion moments. Run 66 is the nominal controller at the nominal condition, i.e., $\tilde{G}_2 = 0$, $r_{BIAS} = 0$. Run 142 and 143 include nonzero $\tilde{G}_2$ and r_{BIAS}. Run 142 is the nominal controller, and run 143 is the additive noise controller. The weights and gains associated with run 143 are given in Tables 10 and 11, respectively.

TABLE 9. ADDITIVE NOISE DESIGN RESULTS

Run #	Weights	Bending RMS 0.113 Nm (in-lb)	Torsion RMS 0.133 Nm (in-lb)
66	Nom	0.7210×10^6	1.090×10^5
142	Nom	0.8627×10^6	1.008×10^5
143	Table 10	0.7710×10^6	1.026×10^5

MINIMAX DESIGN

The minimax technique is based on a worst case approach. The design goal is to minimize the worst performance that a system may experience for a given set of parameters and parameter ranges by appropriate choice of the controller. The technique then requires that the performance of each candidate controller be evaluated at each permissible set of parameter values in the expected range. Hence, even for a small parameter set, a direct application of this technique could prove to be computationally infeasible. However, many systems possess fortunate extreme properties so that the evaluation may be restricted to the boundary of the parameter range. We relied on this characteristic in our design.

For this approach, assume that the system is represented by

$$\dot{x} = F(p)x + G_1 u + G_2 \eta \tag{50}$$

with responses

$$r = H(p)x + D(p)u \tag{51}$$

where p is a vector containing the uncertain parameters, and x, u, and η are the state, control, and noise vectors, respectively. The control is to be of the form

TABLE 10. ADDITIVE NOISE WEIGHTS

```
Q  MATRIX

ROW    1
 .10000E-08 0.         0.         0.         0.         0.         0.         0.         0.         0.
ROW    2
0.          .10000E-07 0.         0.         0.         0.         0.         0.         0.         0.
ROW    3
0.         0.          .55000E-11 0.         0.         0.         0.         0.         0.         0.
ROW    4
0.         0.         0.          .10000E-09 0.         0.         0.         0.         0.         0.
ROW    5
0.         0.         0.         0.          .32000E+08 0.         0.         0.         0.         0.
ROW    6
0.         0.         0.         0.         0.         0.         0.         0.         0.         0.
ROW    7
0.         0.         0.         0.         0.         0.          .10000E+07 0.         0.         0.
ROW    8
0.         0.         0.         0.         0.         0.         0.          .10000E+05 0.         0.
ROW    9
0.         0.         0.         0.         0.         0.         0.         0.         0.         0.
ROW   10
0.         0.         0.         0.         0.         0.         0.         0.         0.          .20000E+07
```

TABLE 11. ADDITIVE NOISE CONTROLLER GAINS

```
GAINS MATRIX

ROW    1
-.19743E-03  .10892E-03  .74498E-03  .43550E-03 -.16935E-02 -.53315E-02 -.53693E+00 -.25895E-01  .59888E-04 -.15004E-02
 .86001E-04  .29408E-04 -.11748E-02  .93911E-03 -.22855E-03
ROW    2
 .12716E-03  .43213E-03 -.12977E-03  .22622E-03  .23581E-03 -.56483E-02 -.10307E-01 -.13650E+00 -.26717E-04  .52193E-03
-.99827E-03 -.59490E-04  .33436E-03  .89826E-03  .13289E-02
TIME=    7.9220
```

$$u = Kx \tag{52}$$

with K chosen to minimize the performance index

$$J(K) = \max_{p \in P} E\{r^T Q r\} \tag{53}$$

where P is the set of allowable values of the parameter vector, p. In practice, one might also impose constraints on the set of permissible K's to be allowed in the minimization process.

Generally, the computational requirements for the solution of this approach are severe. Salmon[36] has given a general algorithm for the numerical solution of such a problem. Let J(K,p) denote the value of $E\{r'Qr\}$ for the controller $u = Kx$, and let the parameter vector equal p. Then Salmon's algorithm may be expressed for our problem as given in Figure 3.

Two theorems in Appendix F give some hope that only one iteration of this algorithm is required if K° is chosen as the optimal control law for the nominal parameter vector p_o. In fact, if the range of parameter variations is sufficiently small and $\nabla_p J^*(p_o) \neq 0$ and $\nabla_{pp} J^*(p_o) > 0$, the theorems guarantee that only one iteration is required. Here $J^*(p) = \min_K J(K, p)$. A good a priori estimate of the range of validity is lacking.

So, for our problem, we chose to limit the set P of admissible parameter vectors to those corresponding to the nominal plant and the eight combinations of extreme variations corresponding to vertices of the "cube" of parameter variations. They are shown in Figure 4. The set of permissible controls was also restricted to those which are optimal for some one of the nine admissible parameter vectors. These two restrictions were imposed to limit the possible computational expense.

With the sets of admissible parameters and controllers so restricted, the algorithm could require nine controller calculations (Ricatti equation solutions) and 81 controller evaluations (Lyapunov equation solutions). Fortunately, only one iteration was required.

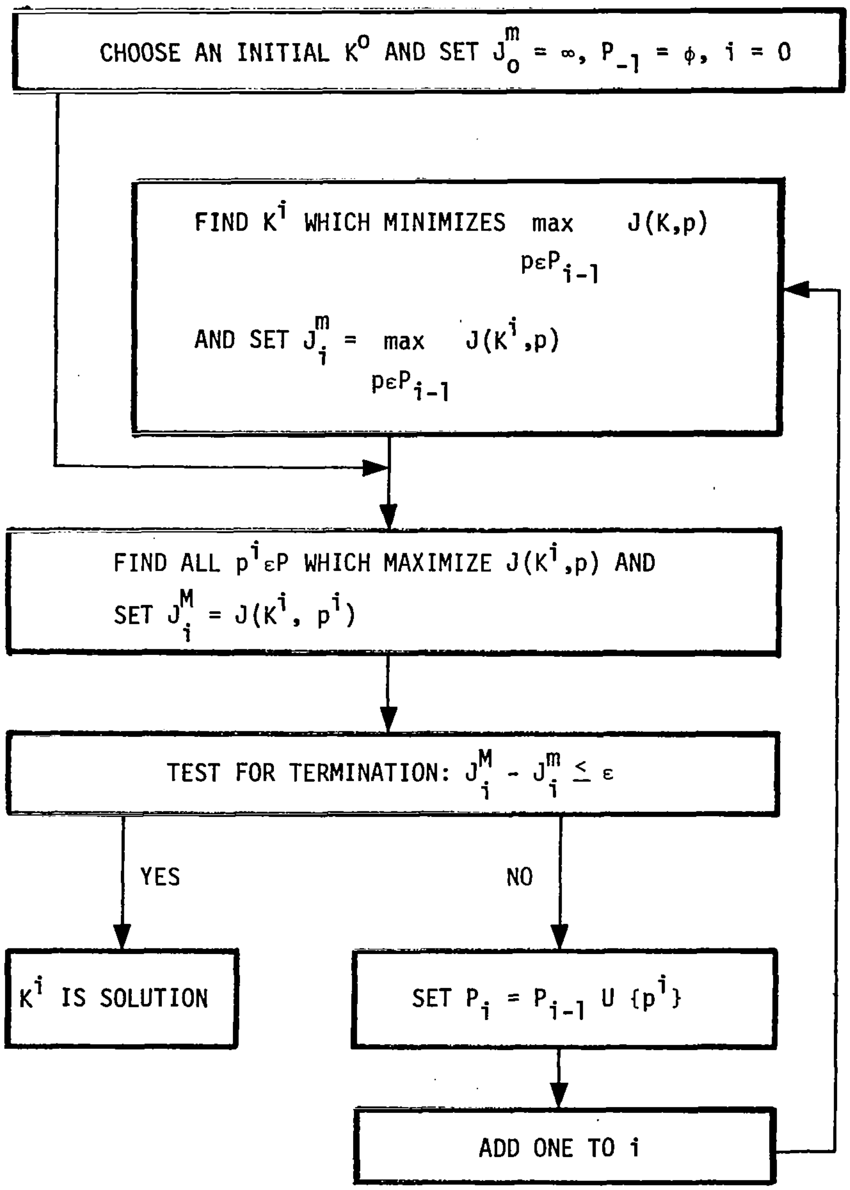


Figure 3. Minimax Algorithm

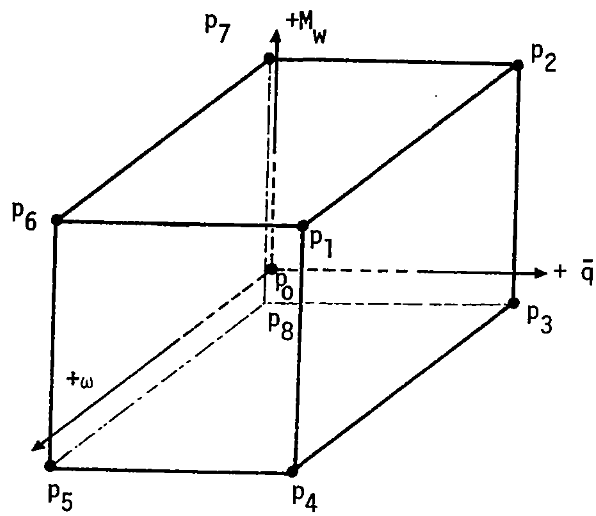


Figure 4. "Cube" of Parameter Variations

Design Details and Results

Table 12 presents the results of the controller design process which was completed with one iteration. The weights used for the design of the u_3 controller and also for evaluating the performance indices were the nominal controller design weights.

TABLE 12. MINIMAX INSENSITIVE CONTROLLER DESIGN

Run No.	Cube location	$\bar{q}_f$	ω_f	M_{w_f}	Controller	J
66	p_0	1.00	1.00	1.0	u_o	88.470
111	p_3	1.25	0.75	0.8	u_3	131.948
113	p_0	1.00	1.00	1.0	u_3	94.693
118	p_1	1.25	1.25	1.2	u_3	101.783
117	p_2	1.25	0.75	1.2	u_3	102.039
115	p_4	1.25	1.25	0.8	u_3	127.266
116	p_5	0.50	1.25	0.8	u_3	64.470
119	p_6	0.50	1.25	1.2	u_3	52.023
120	p_7	0.50	0.75	1.2	u_3	53.378
114	p_8	0.50	0.75	0.8	u_3	66.597

As can be seen, u_3, the optimal controller designed with p_3 uncertainties and evaluated with p_3 uncertainties, had the maximum cost when evaluated over all other sets of uncertainties. Thus u_3 is the minimax controller. The u_3 controller gains are given in Table 13.

MULTIPLANT DESIGN

The system is assumed to be represented by

$$\dot{x}(p) = F(p)\, x(p) + G_1(p)u + G_2\eta \tag{54}$$

and the responses are given by

$$r(p) = H(p)\, x(p) + D(p)u \tag{55}$$

where, again, p is a vector of uncertain parameters. The performance index of interest is

$$\tilde{J} = E\{r(p)^T\, Qr(p)\} \tag{56}$$

To achieve insensitivity to parameter uncertainty, one may choose M values of the parameter vector, p, say p^1, p^2, p^3, ... p^M. Let $F^i = F(p^i)$, $G_1{}^i = G_1(p^i)$, $H^i = H(p^i)$, $D^i = D(p^i)$, $x^i = x(p^i)$, and $r^i = (p^i)$. We will be interested in feedback controls of the form $u^i = Kx^i$ for each of the resulting plants. Thus the M systems may be represented as

$$\dot{x}^i = F^i x^i + G_1{}^i u^i + G_2\eta \tag{57}$$

The performance index will be defined as the average of $\tilde{J}$ over the M plants; i.e.,

$$J = \sum_{i=1}^{M} E\{r^{iT}\, Qr^i\} \tag{58}$$

This optimization problem can be quickly reduced to an algebraic problem. Assuming that a gain matrix can be found that will stabilize all M plants, the covariance equation for each plant is then

$$0 = (F^i + G_1{}^i K)X^i + X^i(F^i + G_1{}^i K)^T + C$$

TABLE 13. u_3 CONTROLLER GAINS - MINIMAX CONTROLLER

```
GAINS MATRIX

ROW   1
-.10543E-04  .15677E-04  .35670E-04  .26535E-04 -.82708E-04 -.24019E-03 -.18081E+00 -.95071E-02  .26388E-05 -.89573E-04
-.59500E-04 -.29503E-05 -.77972E-04  .10406E-03  .47539E-04
ROW   2
 .17972E-03  .56839E-03 -.57965E-04  .13998E-03  .26293E-03 -.24839E-02 -.36217E-01 -.15133E+00 -.49036E-04  .28258E-03
-.13289E-02 -.94086E-04  .30875E-03  .11669E-02  .22408E-02
TIME=  22.8630
```

where η is white noise and $E[\eta(t)\,\eta(\tau)^T] = C\ \delta\ (t-\tau)$ and $X^i = E(x^i x^{iT})$.

The cost is

$$J = \sum_{i=1}^{M} TR[(H^i + D^i K)^T\, Q^i (H^i + D^i K) X^i]$$

where TR is the trace operator. Appending the covariance equations to J via Lagrange multipliers, S^i yields the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{M} TR[(H^i + D^i K)^T\, Q^i (H^i + D^i K) X^i]$$

$$+ \sum_{i=1}^{M} TR\{S^i[(F^i + G_1{}^i K) X^i + X^i (F^i + G_1{}^i K)^T + C]\}$$

Equating derivatives to zero yields

$$\frac{\partial \mathcal{H}}{\partial S^i} = 0 = (F^i + G_1{}^i K) X^i + X^i (F^i + G_1{}^i K)^T + C \tag{59}$$

$$\frac{\partial \mathcal{H}}{\partial X^i} = 0 = (F^i + G_1{}^i K)^T\, S^i + S^i (F^i + G_1{}^i K) + (H^i + D^i K)^T Q^i (H^i + D^i K) \tag{60}$$

$$\frac{\partial \mathcal{H}}{\partial K} = 0 = \sum_{i=1}^{M} (D^{iT} Q^i\, (H^i + D^i K) + G_1{}^{iT} S^i) X^i \tag{61}$$

The algebraic problem is to solve these 2M+1 matrix equations for K. A satisfactory method of solving Equations (59) through (61) is not available.

An alternate method which could be applied is to treat the problem as a fixed-form control problem. For this formulation, consider the system

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \vdots \\ \dot{x}^M \end{bmatrix} = \begin{bmatrix} F^1 & 0 & \cdots & 0 \\ 0 & F^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & F^M \end{bmatrix} x + \begin{bmatrix} G_1^{\ 1} & 0 & \cdots & 0 \\ 0 & G_1^{\ 2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & G_1^{\ M} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^M \end{bmatrix} + \begin{bmatrix} G_2 \\ G_2 \\ \vdots \\ G_2 \end{bmatrix} \eta \tag{62}$$

The fixed-form desired for the controller is

$$u = \begin{bmatrix} K & 0 & \cdots & 0 \\ 0 & K & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & K \end{bmatrix} x \tag{63}$$

We can introduce a single parameter λ and write the control as

$$u(\lambda, x) = \begin{bmatrix} K(\lambda) & 0 & \cdots & 0 \\ 0 & K(\lambda) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & K(\lambda) \end{bmatrix} x + \lambda \begin{bmatrix} K^1 - K(1) & 0 & \cdots & 0 \\ 0 & K^2 - K(1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & K^M - K(1) \end{bmatrix} x \tag{64}$$

where K^i denotes the optimal gain matrix for the ith plant. We may choose K(1) arbitrarily. One possibility is to take K(1) to be the optimal gain matrix for the nominal plant.

The control, $u(\lambda, x)$, given by Equation (64) may be expressed as

$$u(\lambda, x) = \bar{K}(\lambda)x \tag{65}$$

with

$$\overline{K}(\lambda) = \lambda K^{o} + \text{diag}\,[K(\lambda)] \tag{66}$$

We have chosen $\overline{K}(1)$ so that the control $u(1, x)$ minimizes the performance index J of Equation (58); that is, $\partial J/\partial K(\lambda) = 0$ at $\lambda = 1$. Now, we may proceed to calculate $K(\lambda)$ such that $\partial J/\partial K(\lambda) \equiv 0$ for $1 \geq \lambda \geq 0$. The resulting $\overline{K}(0)$ is of the desired fixed-form and satisfies the necessary condition for optimality. The matrix $K(\lambda)$ may be obtained by integrating the differential equation

$$\frac{dK(\lambda)}{d\lambda} = - \left[\frac{\partial^2 J}{\partial K(\lambda)^2} \right]^{-1} \left[\frac{\partial^2 J}{\partial K(\lambda)\partial\lambda} \right] \tag{67}$$

This equation is derived by use of the implicit function theorem and the constraint equation

$$\frac{d}{d\lambda} \left[\frac{\partial J}{\partial K(\lambda)} \right] = 0 \, , \quad 1 \geq \lambda \geq 0 \tag{68}$$

In these equations, $K(\lambda)$ may be thought of as a vector. The major drawback to this approach is the computational requirements associated with computing $\partial J/\partial K$ and especially $\partial^2 J/\partial K^2$. This latter requirement can be avoided by using the incremental gradient method. In this method, one starts with $\overline{K}(1)$ as defined above. An increment, $\Delta\lambda > 0$, is chosen. Then one considers the controller gain matrix of the form $\overline{K} = K + (1 - \Delta\lambda)K^{o}$ and, starting with $K = K(1)$, uses gradient corrections to achieve $\partial J(\overline{K})/\partial K = 0$. Then another incremental step in λ is taken toward zero, and the procedure is continued until $\lambda = 0$.

Design Details and Results

For this design we considered two plants, the nominal plant and a perturbed plant. The plants are described by

$$\dot{x}^{i} = F^{i} + G_1 u^{i} + G_2 \eta$$

where $i = 1$ for the nominal plant and $i = 2$ for the perturbed plant. The associated responses are given by

$$r^i = H^i x^i + D^i u^i$$

The performance index is chosen to be

$$J = E\{r^{1T} Q r^1\} + E\{r^{2T} Q r^2\}$$

where Q is the weighting matrix used to define the optimal nominal controller. The controller of the form

$$u^i = Kx^i$$

is sought which minimizes J.

This problem may be reformulated by combining the two plants into one system as

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} F^1 & 0 \\ 0 & F^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + \begin{bmatrix} G_2 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$\begin{bmatrix} r^1 \\ r^2 \end{bmatrix} = \begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} D^1 & 0 \\ 0 & D^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

$$J(K) = E\left\{ [r^{1T} \; r^{2T}] \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} r^1 \\ r^2 \end{bmatrix} \right\}$$

where we wish to minimize J with respect to K.

A gradient procedure was used to perform the minimization. The initial choice for the matrix K was taken to be the optimal gain matrix for the nominal plant. For the combined system, the cost, J, for this initial gain matrix was 234.59. The individual costs for these two plants were previously computed for the nominal controller. They were 88.47 for the nominal plant and 144.28 for the perturbed plant, the sum being 232.75. Also the cost for the perturbed plant with its optimal control had been computed as 131.95. Thus the sum of the two optimal costs, 220.42, is a lower bound for the optimal multiplant controller. The multiplant optimization was accomplished in four gradient steps giving a cost of 228.93. The initial and final gain matrices are shown in Table 14.

SENSIVITITY VECTOR AUGMENTATION

Design of insensitive controllers based on the sensitivity vector augmentation concept has been extensively investigated by Kreindler.[10, 11, 12] The procedure involves the introduction of the sensitivity vector, the partial derivative of the state vector with respect to the parameters. The original system is then augmented by appending the dynamics associated with the sensitivity vector. A controller is then designed for this augmented system with the magnitude of the sensitivity vector included in the performance index.

The information given in References 10 through 12 was used as a basis for the following formulation of the synthesis and design approach using the sensitivity vector augmentation concept. Suppose a plant is represented by

$$\dot{x} = F(p)\,x + G_1 u + G_2 \eta \tag{69}$$

with responses

$$r = H(p)x + D(p)u \tag{70}$$

where p is a vector containing the uncertain parameters of the system.

Define a sensitivity vector, σ, by

$$\sigma = \frac{\partial x}{\partial p} \tag{71}$$

TABLE 14. MULTIPLANT CONTROLLER GAINS

INITIAL GAINS MATRIX

ROW 1									
-3.2077E-05	1.8016E-05	1.1200E-04	3.2268E-05	-3.4501E-04	-7.0350E-04	-2.1382E-01	1.1248E-03	1.1472E-05	-2.7191E-04
4.7293E-05	6.1141E-06	-2.0954E-04	1.3560E-05	-1.3574E-05	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
ROW 2									
1.3446E-04	4.0718E-04	-1.8809E-04	2.6898E-04	4.6278E-04	-6.2506E-03	3.4005E-03	-1.5227E-01	-1.7730E-06	6.7391E-04
-1.2350E-03	-6.9637E-05	4.3997E-04	8.5089E-04	1.3845E-03	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
ROW 3									
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	-3.2077E-05	1.8016E-05	1.1200E-04	3.2268E-05	-3.4501E-04
-7.0350E-04	-2.1382E-01	1.1248E-03	1.1472E-05	-2.7191E-04	4.7293E-05	6.1141E-06	-2.0954E-04	1.3560E-05	-1.3574E-05
ROW 4									
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.3446E-04	4.0718E-04	-1.8809E-04	2.6898E-04	4.6278E-04
-6.2506E-03	3.4005E-03	-1.5227E-01	-1.7730E-06	6.7391E-04	-1.2350E-03	-6.9637E-05	4.3997E-04	8.5089E-04	1.3845E-03

FINAL GAINS MATRIX

ROW 1									
-2.9527E-05	2.3996E-05	9.8050E-05	3.1828E-05	-3.4103E-04	-7.0349E-04	-2.1382E-01	1.1248E-03	1.0859E-05	-2.7287E-04
4.5784E-05	4.5708E-06	-2.1024E-04	1.3744E-05	-1.4141E-05	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
ROW 2									
1.6317E-04	4.0768E-04	-1.8179E-04	2.6929E-04	4.6129E-04	-6.2506E-03	3.4005E-03	-1.5227E-01	-2.9534E-06	6.7238E-04
-1.2367E-03	-7.6517E-05	4.3866E-04	8.5130E-04	1.3835E-03	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
ROW 3									
0	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	-2.9527E-05	2.3996E-05	9.8050E-05	3.1828E-05	-3.4103E-04
-7.0349E-04	-2.1382E-01	1.1248E-03	1.0859E-05	-2.7287E-04	4.5784E-05	4.5708E-06	-2.1024E-04	1.3744E-05	-1.4141E-05
ROW 4									
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	1.6317E-04	4.0768E-04	-1.8179E-04	2.6929E-04	4.6129E-04
-6.2506E-03	3.4005E-03	-1.5227E-01	-2.9534E-06	6.7238E-04	-1.2367E-03	-7.6517E-05	4.3866E-04	8.5130E-04	1.3835E-03

The dynamics of that sensitivity vector may then be represented by

$$\dot{\sigma} = \frac{\partial \dot{x}}{\partial p} = \frac{\partial F}{\partial p}\bigg|_{p_o} x + F(p_o)\sigma \tag{72}$$

where

$$\frac{\partial G_1}{\partial p} = \frac{\partial G_2}{\partial p} = \frac{\partial u}{\partial p} = \frac{\partial \eta}{\partial p} = 0$$

The approach then proposed by Kreindler is to augment Equation (69) at the nominal condition with Equation (72) or

$$\dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} F(p_o) & 0 \\ \frac{\partial F}{\partial p}\Big|_{p_o} & F(p_o) \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta \tag{73}$$

where p_o represents the nominal value of the uncertain parameter. An optimal controller

$$u = K^* \tilde{x} \tag{74}$$

could then be designed which minimized a performance index, J, given by

$$J = E(\tilde{x}^T Q\tilde{x} + u^T Ru) = TR[(Q + K^T RK)\tilde{X}] \tag{75}$$

where $\tilde{X} = E\{\tilde{x}\,\tilde{x}^T\}$ and Q, R = weighting matrices.

At this point, three problems are apparent. First, an optimal controller in the form of Equation (74) places feedback gains on the sensitivity vector states, σ, which are not physical quantities. Second, the formulation given by Equations (73), (74), and (75) does not address the uncertainties that are present in the response equations which, generally, contain the design parameters. Third, sensitivity as represented by Equations (71) and (72) reflects open-loop sensitivity as opposed to the preferred measure of closed-loop sensitivity. These problems were treated as follows.

Sensitivity State Feedback

The theoretical size of the sensitivity vector is n x q where n is the order of plant equations (Equation (64)) and q is the number of parameter uncertainties. An augmented system is then n(q+1) in size.

To minimize the dimensionality problem, we took the following approach:

1. Design would be limited to use of Case 4R model (n = 15). Of those 15 states, seven are gust states which were assumed to be independent of the parameter uncertainties. This approximation reduced the order of the augmented system to $n + (n - n_{gust})q$.

2. As shown in Section IV, the nominal controller was insensitive to variations in structural damping. Thus structural damping was eliminated as an uncertainty, leaving dynamic pressure, structural frequency, and M_w comprising the uncertainty vector p. The order of the augmented design system is then $15 + (15 - 7)\,3 = 39$.

   The 39-state augmented system contains 24 sensitivity states, all of which will have gains in the optimal controller design. It was decided that for the purposes of this contract the sensitivity states could be computed by numerically integrating Equation (72) with the system states as forcing functions. Kreindler proposes the same approach in Reference 11.

Response Sensitivity

The effect of parameter uncertainties in the response equations was handled by augmenting the response variables with response sensitivity states; i.e., define

$$\lambda = \frac{\partial r}{\partial p} \tag{76}$$

or

$$\lambda = \frac{\partial H}{\partial p}\Big|_{p_o} x + H \frac{\partial x}{\partial p} + \frac{\partial D}{\partial p} \quad u$$

Substituting (71) into (76) gives

$$\lambda = \frac{\partial H}{\partial p}\Big|_{p_o} x + H\sigma + \frac{\partial D}{\partial p} u \tag{77}$$

With this approach, not only are response equation uncertainties treated, but the designer now has an excellent way of weighting performance versus sensitivity in selection of the weighting matrices. The order of the augmented response variables is r(1 + q) where r is the number of design responses.

Open-Loop versus Closed-Loop Sensitivity

With respect to sensitivity considerations, the designer's ultimate objective is that the closed-loop system be insensitive. However, the sensitivity vector augmentation concept requires selecting gains on sensitivity states which are themselves functions of the plant dynamics. This may result in a closed-loop system that is unrelated to the system which defined the sensitivity vector. In Reference 11, Kreindler proposes an iterative solution to the problem by estimating the closed-loop dynamics for sensitivity vector definition on each iteration. This requires expanding the size of the augmented system to n(3q+1) which is clearly unrealistic for an aircraft control system design problem.

The approach taken in this formulation is to consider the nominal closed-loop system as a reference for sensitivity vector definition. Since the nominal controller produces optimum performance for the nominal system, it is hypothesized that sensitivity considerations should be referenced to that nominal controller optimum performance.

Using this approach, Equation (73) becomes

$$\dot{\tilde{x}} = \begin{bmatrix} F(p_o) & 0 \\ \frac{\partial F}{\partial p}\Big|_{p_o} & F(p_o) + G_1 K_o^* \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta \tag{78}$$

where $u = K_o^* x$ represents the optimal controller for the nominal system.

The controller design was based on the following state and response equations:

$$\dot{\tilde{x}} = \begin{bmatrix} F(p_0) & 0 \\ \frac{\partial F}{\partial p}\Big|_{p_0} & F(p_0) + G_1 K_0^* \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta \tag{79}$$

where x = 15th order state vector - Case 4R

σ = 24th order sensitivity vector

$$= \frac{\partial x_1}{\partial \bar{q}} \cdots \frac{\partial x_8}{\partial \bar{q}}, \frac{\partial x_1}{\partial \omega_f} \cdots \frac{\partial x_8}{\partial \omega_f}, \frac{\partial x_1}{\partial M_w} \cdots \frac{\partial x_8}{\partial M_w}$$

u = 2nd order control vector

η = scalar noise driver

$p_0 = (\bar{q}_0, \omega_{f_0}, M_{w_0})^T$

$$\tilde{r} = \begin{bmatrix} r \\ \lambda \end{bmatrix} = \begin{bmatrix} H(p_0) & 0 \\ \frac{\partial H}{\partial p}\Big|_{p_o} & H(p_o) \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} + \begin{bmatrix} D \\ \frac{\partial D}{\partial p}\Big|_{p_o} \end{bmatrix} u \tag{80}$$

where r = 9th order design response (Table 5)

λ = 27th oraer response sensitivity vector

$$= \left[\frac{\partial r_1}{\partial \bar{q}} \cdots \frac{\partial r_9}{\partial \bar{q}}, \frac{\partial r_1}{\partial \omega_f} \cdots \frac{\partial r_9}{\partial \omega_f}, \frac{\partial r_1}{\partial M_w} \cdots \frac{\partial r_9}{\partial M_w} \right]^T$$

The performance index is

$$J = TR\, \tilde{Q}\, \tilde{R} \tag{81}$$

where $\tilde{R} = E(\tilde{r}\, \tilde{r}^T)$ and $\tilde{Q}$ will be defined in the following manner:

$$\tilde{Q} = \begin{bmatrix} Q_{nom} & & & 0 \\ & SF_1\, Q_{nom} & & \\ & & SF_2\, Q_{nom} & \\ 0 & & & SF_3\, Q_{nom} \end{bmatrix}$$

Q_{nom} is the set of weights determined for the design of the nominal control system and is given in Table 5. SF_1, SF_2, and SF_3 are scalar sensitivity factors which are used to weight sensitivity versus performance. $SF_1 = SF_2 = SF_3 = 1.0$ could be thought of as weighting sensitivity equally with performance. Table 15 presents closed-loop bending and torsion RMS values for different sensitivity factors.

Table 16 shows the closed-loop roots for run 126 ($SF_1 = SF_2 = SF_3 = 1.0$). As can be seen, it is impossible to distinguish the aircraft roots (e.g., short period) from the sensitivity roots. This case was chosen as the sensitivity vector augmentation controller.

Gains for run 126 are given in Table 17.

TABLE 15. SENSITIVITY VECTOR INSENSITIVE CONTROLLER PERFORMANCE

Run No.	Sensitivity factor SF_1	SF_2	SF_3	Bending RMS 0.113 Nm (in-lb)	Torsion RMS 0.113 Nm (in-lb)
66	0.0	0.0	0.0	0.7210×10^6	0.1092×10^6
126	1.0	1.0	1.0	0.5878×10^6	0.1120×10^6
127	0.1	0.1	0.1	0.6765×10^6	0.1097×10^6
128	10.0	10.0	10.0	0.4381×10^6	0.1172×10^6
129	10.0	1.0	1.0	0.5879×10^6	0.1119×10^6
130	1.0	10.0	1.0	0.4321×10^6	0.1174×10^6

TABLE 16. RUN 126 ROOTS

EIGENVALUES

REAL	IMAG	FREQ	DAMP
-.24680489	0.00000000	.24680489	-1.00000000
-.24843511	0.00000000	.24843511	-1.00000000
-22.18500000	0.00000000	22.18500000	-1.00000000
-8.54920000	0.00000000	8.54920000	-1.00000000
-10.98300000	0.00000000	10.98300000	-1.00000000
-5.09600000	3.60330182	6.24123385	-.81650522
-6.90305895	0.00000000	6.90305895	-1.00000000
-6.90305895	0.00000000	6.90305895	-1.00000000
-6.22636018	0.00000000	6.22636018	-1.00000000
-6.81540970	0.00000000	6.81540970	-1.00000000
-6.22643472	0.00000000	6.22643472	-1.00000000
-6.70646623	0.00000000	6.70646623	-1.00000000
-7.03939441	0.00000000	7.03939441	-1.00000000
-5.88527701	0.00000000	5.88527701	-1.00000000
-1.63168154	1.58998461	2.27825299	-.71619870
-1.63393795	1.56905073	2.26531971	-.72128360
-1.62395229	1.63000052	2.30089607	-.70579124
-1.58588053	1.48346996	2.17156629	-.73029340
-.77244034	5.40698905	5.46188563	-.14142375
-.77275387	5.40733478	5.46227224	-.14147114
-.97193560	4.46214088	4.56676692	-.21282794
-1.46883259	6.42677522	6.59248883	-.22280396
-1.06015530	13.86500602	13.90547810	-.07624012
-1.06015511	13.86500586	13.90547792	-.07624011
-2.02826871	11.82782515	12.00047173	-.16901575
-2.56963122	15.90052027	16.10681687	-.15953687

TABLE 17. RUN 126 GAINS - SENSITIVITY VECTOR AUGMENTATION

```
GAINS MATRIX

ROW   1
-.18541E-03  .37071E-04  .77147E-03  .35761E-03 -.14648E-02 -.85623E-02 -.52548E+00 -.38342E-01  .48437E-04 -.14822E-02
-.83843E-04  .14232E-04 -.10538E-02  .45416E-03  .18388E-03  .19880E-05  .13924E-05  .82572E-05 -.20889E-05  .87879E-05
-.13638E-04 -.25243E-02  .88528E-03  .16441E-04  .21755E-04  .16072E-03  .10830E-03  .64851E-03  .57352E-03 -.27624E-01
 .82545E-02 -.45724E-06  .98493E-06 -.65169E-06  .37337E-07 -.94164E-05  .34339E-05 -.30331E-03 -.28432E-03
ROW   2
 .31918E-03  .17662E-03 -.67123E-03  .34341E-02  .23643E-02 -.42657E-01 -.14607E+00 -.82179E+00  .18244E-03  .49749E-02
-.49769E-02 -.13902E-03  .20740E-02  .54163E-03  .61830E-03 -.83960E-06  .22662E-04  .74410E-04 -.82628E-05 -.35959E-04
 .21604E-04 -.27926E-01  .79702E-02  .28119E-05  .38910E-04 -.15362E-03  .60711E-03 -.55579E-03  .41974E-02  .30425E-01
-.33612E-01 -.50925E-05  .92005E-05 -.41247E-05  .14214E-05 -.53149E-04  .12753E-04 -.19674E-02 -.36919E-02
TIME=  57.4700
```

STATE AND CONTROL DEPENDENT NOISE[47]

In this concept the parameter uncertainty is modeled as noise. If the parameter appears in the F matrix, the noise is multiplied by the state; therefore the noise is state dependent. Similarly if the parameter appears in the G_1 matrix, the noise is control dependent. The modeling of the parameter uncertainty as noise requires a modified form of the state equation. The quadratic optimization formulation for controller synthesis then leads to solution of a modified Riccati equation. The design of an insensitive controller based on the state and control dependent noise approach was performed by Professor Kleinman. Since the control matrix, G_1, is independent of parameter variations, this technique reduces to a state dependent noise problem. Consider, again, the representation

$$\dot{x} = F(p)x + G_1 u + G_2 \eta \tag{82}$$

where p is the vector of uncertain parameters. Equation (82) may be written

$$\dot{x} = F(p_o)x + [F(p) - F(p_o)]x + G_1 u + G_2 \eta \tag{83}$$

where p_o = vector of uncertain parameters at their nominal value. Consider the definition of a partial derivative

$$\left.\frac{\partial F}{\partial p}\right|_{p_o} = \lim_{\Delta p \to o} \frac{F(p) - F(p_o)}{\Delta p} \tag{84}$$

where $\Delta p = p - p_o$.

Equation (83) may now be written

$$\dot{x} = F(p_o)x + \left.\frac{\partial F}{\partial p}\right|_{p_o} \Delta p x + G_1 u + G_2 \eta \tag{85}$$

If the perturbation in the uncertain parameter, Δp, is treated as white noise, Equation (85) represents a state dependent noise formulation. The state dependent noise approach does not consider the effect of uncertainties in the response equations. No satisfactory representation was found for the response uncertainties and consequently they were neglected in the design process.

Details of State Dependent Noise Design

In the state dependent noise model approach, the constant Δp is replaced by a zero-mean, white Gaussian noise ξ_i with standard deviation

$$\sigma = M/NSD \tag{86}$$

NSD is a fudge-factor for adjusting the noise intensity in proportion to $|\Delta p|_{max}$. Typically NSD $\simeq$ 3. It is hypothesized that a system which functions well for arbitrary time variations in its parameters will also perform well for any constant variation within the prespecified range.

In order to solve the state dependent noise problem, a suitable Ito equation must be written for the state equation (83):

$$dx = F\, x\, dt + \sum_{i=1}^{3} F_i\, x\, d\xi_i + G_1 u\, dt \tag{87}$$

where i is the index on the vector of uncertain parameters. $(p_1, p_2, p_3) = (\bar{q}_f, \omega_f, M_{w_f})$, and $F_i = \partial F/\partial p_i$.

At this point, two different F matrices are possible, depending on how a certain stochastic integral is interpreted. The two possibilities are:[44]

$$F = \begin{cases} F_o & \text{(Fisk-Stratonovitch formula)} \quad (88a) \\ F_o + \sum_{i=1}^{3} F_i^2 \sigma_i^2/2 & \text{(Wong-Zakai formula)} \quad (88b) \end{cases}$$

In the present effort Equation (88b) will be used since it has been found to be more consistent with observed system motions in cases when $\xi(t)$ is wide-band (non-white). For the C-5A dynamics, this F matrix is slightly more stable than the original F_o matrix.

The optimal linear control for the state dependent noise problem of minimizing the performance index J given by

$$J = E\,\{x^T Q x + u^T R u\} \tag{89}$$

is

$$u = -R^{-1} G_1^T P x = Kx$$

where P satisfies

$$0 = PF + F^T P + Q - PG_1 R^{-1} G_1^T P + \sum_{i=1}^{3} F_i^T P F_i \sigma_i^2 \tag{90}$$

An algorithm for solving this equation is given in Reference 39. This algorithm has been programmed and has been found effective for the present effort.

Numerical Results

The parameter partials $\partial F_o/\partial p_3$ were computed numerically for i = 1, 2. For i = 3, $\partial F_o/\partial p_3 = 0$ except for element 2, 1. The maximum parameter variations M_i were

M_1 = 0.25 (i.e., $\pm$ 25 percent change in dynamic pressure)

M_2 = 0.25 (i.e., $\pm$ 25 percent change in natural frequencies)

M_3 = 0.20 (i.e., $\pm$ 20 percent change in f_{21} or M_w)

The NSD_i were all set equal to a common parameter, NSD. Optimal, noise-dependent gains were computed for NSD = 2, 3, 4, 6, ∞ (corresponding to maximum parameter deviations of 2σ, 3σ, 4σ, and 6σ, respectively, and nominal gains alone). Each set of gains was evaluated in a crude manner--to determine which NSD results in the least sensitive system--by computing the steady state RMS responses of the perturbed (residualized) system at all 2^3 = 8 maximum deviation possibilities or cube vertices.

The perturbed system is

$$\dot{x} = \left[F_o + \sum_{i=1}^{3} \frac{\partial F_o}{\partial p_i} \Delta p_i \right] x + G_1 u + G_2 \eta \tag{91}$$

$$= F_{mod}\, x + G_1 u + G_2 \eta$$

and

$$r = \left[H_o + \sum_{i=1}^{3} \frac{\partial H_o}{\partial p_i} \Delta p_i \right] x + \left[D_o + \sum_{i=1}^{3} \frac{\partial D_o}{\partial p_i} \Delta p_i \right] u \tag{92}$$

$$= H_{mod} x + D_{mod} u$$

where the response partials $\partial H_o / \partial p_3 = \partial H_o / \partial p_2 = \partial H_o / \partial p_3 = 0$. Note that no consideration of the response vector sensitivities are included in the state dependent noise approach that finds the gains K. This is a very definite drawback with the scheme as presently formulated.

In order to determine the "best" NSD, it was found that, with NSD = 1, no linear feedback control would effectively stabilize the system. With NSD = 2, the optimal gains were quite large (compared to the nominal gains), indicating that this choice of NSD is too small, i.e., too pessimistic. On the other hand, NSD = 4, 6 were too large since closed-loop performance was not very far removed from the case NSD = ∞. A value NSD = 3 thus appears to be "best." This is intuitively appealing as the maximum deviations are thus $\pm 3\sigma$.

Plots of RMS r_1 and r_2 (i.e., bending and torsion) for the eight cube vertices as well as the cube center (nominal case) are given in Figures 5 and 6. As can be seen, NSD = 3 does result in less sensitivity in these measures than does the nominal (NSD = ∞) gains. Over the range of parameter variations the quantity

$$s = \frac{(r_1)_{max} - (r_1)_{min}}{(r_1)_{nominal}}$$

is smaller for NSD = 3 than for NSD = ∞. Note that this is accomplished by an across-the-board improvement in performance.

The optimal gains for the case NSD = 3 are given in Table 18. These gains represent the state dependent noise controller that was used in subsequent evaluations.

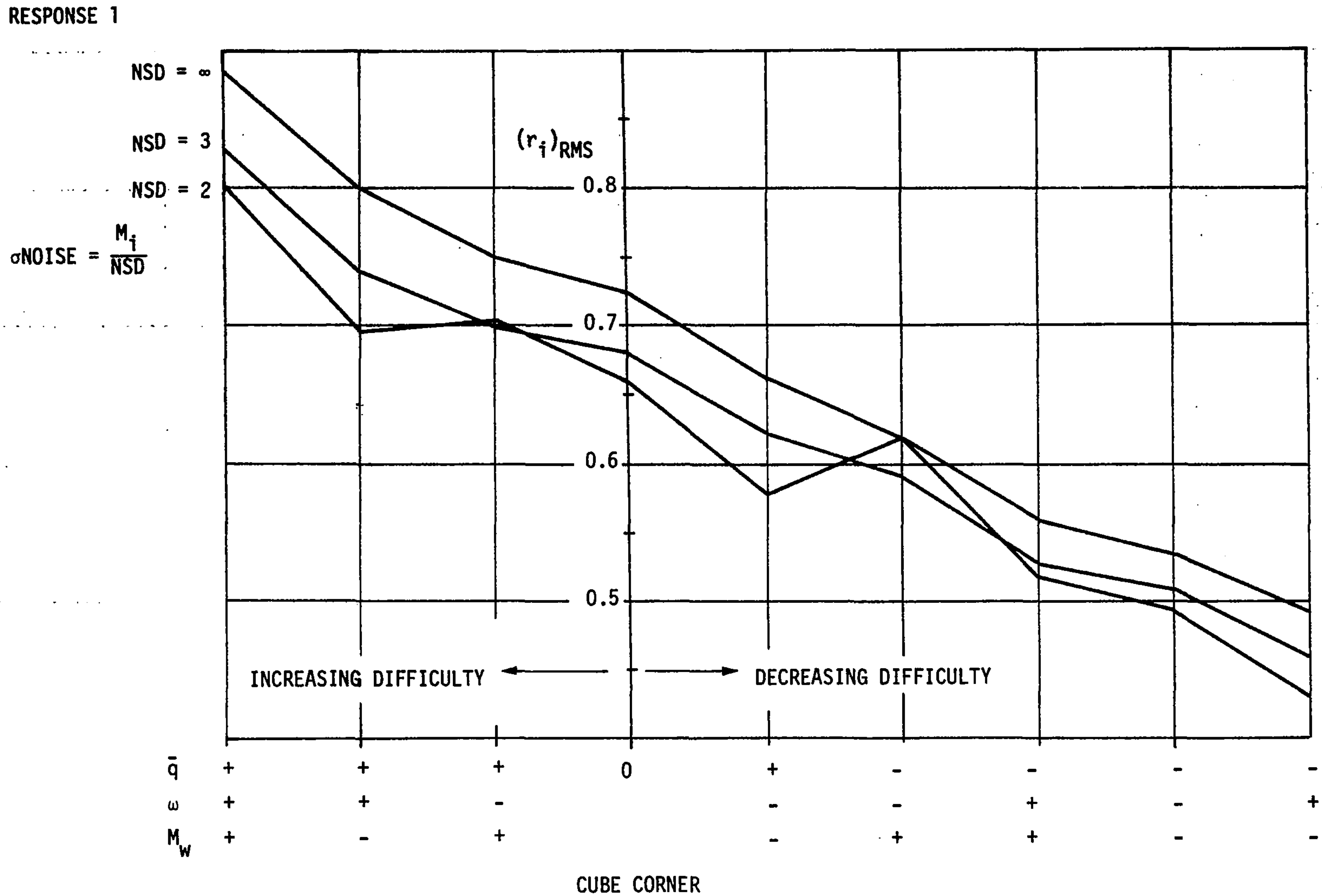


Figure 5. Bending RMS Response

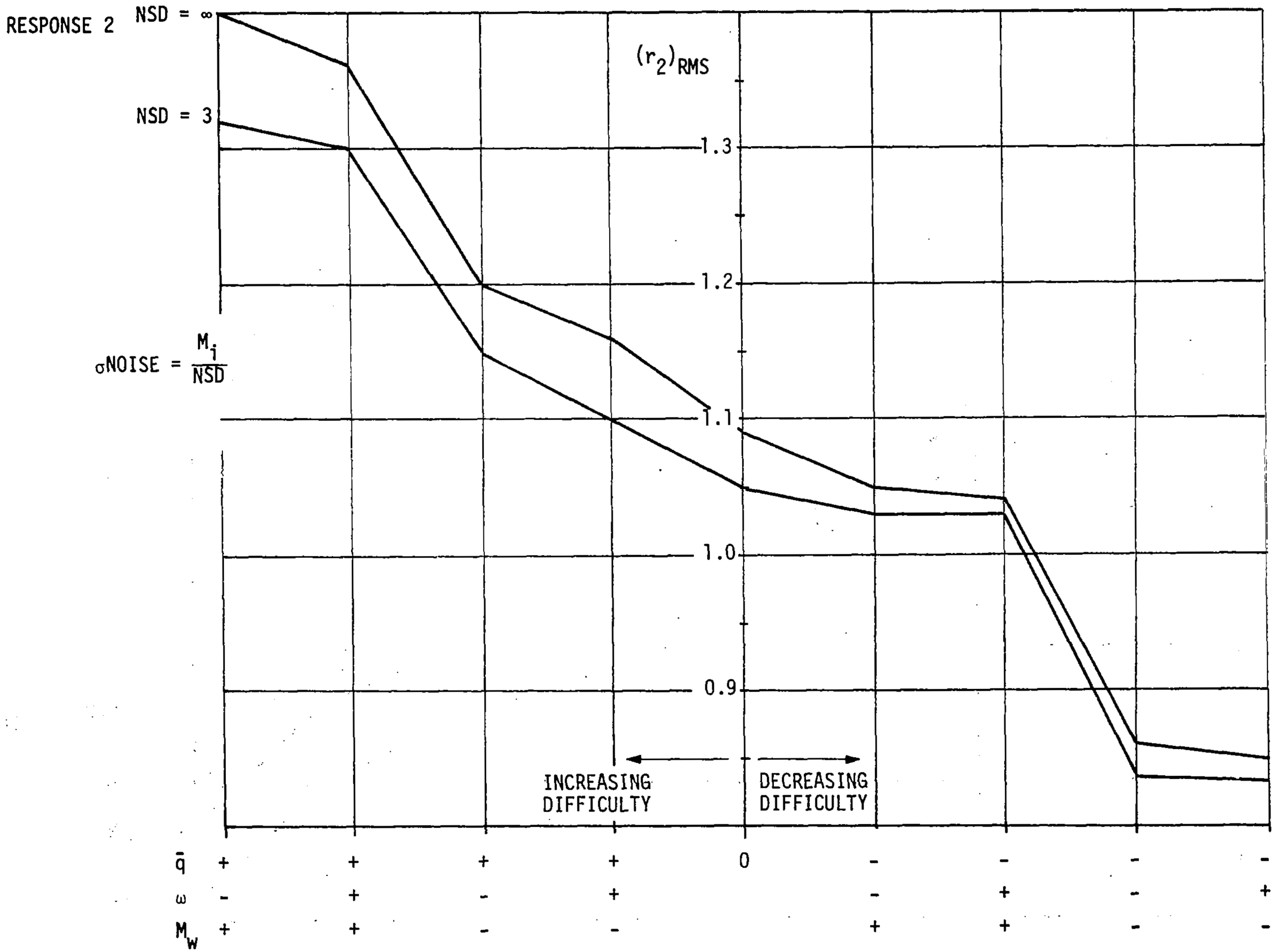


Figure 6. Torsion RMS Response

TABLE 18. STATE DEPENDENT NOISE CONTROLLER GAINS

1 1	-.37134E-04	1 2	-.37242E-04	1 3	.14729E-03	1 4	.36490E-03	1 5	-.36043E-03
1 6	-.44409E-02	1 7	-.28498E+00	1 8	-.79049E-01	1 9	.66251E-05	110	.56979E-04
111	-.45715E-03	112	-.13671E-04	113	-.92992E-04	114	.11236E-03	115	-.56034E-04
2 1	.18674E-03	2 2	.23950E-03	2 3	-.18006E-03	2 4	.25367E-02	2 5	.95321E-03
2 6	-.26202E-01	2 7	-.30204E+00	2 8	-.66785E+00	2 9	.76260E-04	210	.31014E-02
211	-.42872E-02	212	-.15782E-03	213	.12537E-02	214	.89272E-03	215	.13438E-02

SECTION VI

INSENSITIVE CONTROLLER DESIGN--
NEW TECHNIQUES

This section contains a description of the new techniques for the design of insensitive controllers that were synthesized during this study.

Three of the eight new techniques that were developed for the design of insensitive controllers were also synthesized in this study. These were:

- Mismatch Estimation - developed by Professor Kleinman
- Uncertainty Weighting - suggested by Professor Porter
- Re-Residualization

Each of these techniques, and the resulting insensitive controller designs, will now be described.

MISMATCH ESTIMATION

In some cases it may be possible to estimate the effects of parameter variations in key dynamic equations and cancel out their effect. Consider a single-input system

$$\dot{x} = Fx + g_1 u + d\xi(t) + G_2 \eta \tag{93}$$

where $\xi(t)$ are the combined effects of parameter variations in, say, the F_{1j} elements of F so that d looks like $[1\ 0\ 0\ \dots\ 0]^T$. The question is one of estimating $\xi(t)$ on-line and improving the feedback control.

Let us approximate $\xi(t)$ for control modeling purposes as

$$\dot{\xi}(t) = -\gamma\xi(t) + \lambda(t) \tag{94}$$

where $\lambda(t)$ is white noise and $1/\gamma$ is an approximation of the correlation time of $\xi(t)$, which will relate to closed-loop time constants in the case of parameter variations. To solve the optimal control problem of minimizing

$$J = E\{x^T Q x + u^T R u\} \tag{95}$$

we augment the state vector to work with $x_a = [x, \xi]^T$ and

$$\dot{x}_a = \left[\begin{array}{c|c} F & d \\ \hline 0 & -\gamma \end{array}\right] x_a + \begin{bmatrix} g_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \lambda \tag{96}$$

and

$$x^T Q x \rightarrow x_a^T \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 0 \end{array}\right] x_a \tag{97}$$

The optimal control is

$$u = -Kx + k_2 \xi \tag{98}$$

where K is the gain matrix for the unaugmented system; i.e.,

$$K = R^{-1} g_1^T P_{11} \tag{99}$$

where

$$0 = P_{11} F + F^T P_{11} + Q - P_{11} g_1 R^{-1} g_1^T P_{11} \tag{100}$$

and k_2 are obtained from

$$k_2 = R^{-1} g_1^T (F^{-T} - \gamma I)^{-1} P_{11} d \tag{101}$$

where

$$\overline{F} = F - g_1 R^{-1} g_1^T P_{11} \tag{102}$$

These results can be obtained by writing out the matrix partitions of the associated (n+1) x (n+1) Riccati equation.

The problem is to estimate $\xi(t)$ on-line by generating a signal $\hat{\xi}(t)$ that "tracks" $\xi(t)$. If it is assumed that all states are measurable (except for $\xi(t)$), then observer theory can be used. Taking the first component of vector equation (93) and neglecting the white-noise driving term $G_2\eta(t)$, then

$$\dot{x}_1 = f_1 x + \tilde{g}_1 u + \xi(t) \tag{103}$$

where f_1 and $\tilde{g}_1$ are the first rows of F and g_1, respectively. Letting $\hat{\xi}$ be the estimate of ξ, then

$$e = \text{error} = \xi(t) - \hat{\xi}(t) = -\hat{\xi} + \dot{x}_1 - f_1 x - \tilde{g}_1 u \tag{104}$$

The estimator equation is obtained by using e(t) as a correction term to Equation (104); i.e.,

$$\begin{aligned} \dot{\hat{\xi}} &= -\gamma\hat{\xi} + g\,[-\hat{\xi}(t) + \dot{x}_1 - f_1 x - \tilde{g}_1 u] \\ &= -\gamma\xi + g\,[-\hat{\xi}(t) + \xi(t)] = -\gamma\hat{\xi} + ge(t) \end{aligned} \tag{105}$$

The error thus satisfies

$$\dot{e}(t) = -(\gamma + g)\, e(t) + \lambda(t) \tag{106}$$

Since $\dot{x}_1$ is not available, we implement Equation (105) by defining

$$s(t) = \hat{\xi}(t) - gx_1 \tag{107}$$

Then

$$\begin{aligned} \dot{s} &= -\gamma\hat{\xi} - g\,[\hat{\xi} + f_1 x + \tilde{g}_1 u] \\ &= -(\gamma + g)\hat{\xi} - g[f_1 x + \tilde{g}_1 u] \\ &= -(\gamma + g)\, s(t) - g[\gamma + g)x_1 + f_1 x + \tilde{g}_1 u] \end{aligned} \tag{108}$$

For convenience, define $s^* = -s/g$; so $s = -gs^*$ and

$$\hat{\xi} = g\,[x_1 - s^*] \tag{109}$$

and

$$\dot{s}^*(t) = -(\gamma + g)\, s^*(t) + [(\gamma + g)x_1 + f_1 x + \tilde{g}_1 u) \tag{110}$$

Application to Insensitivity

In actual situations, the first row of F and G_1 are assumed to deviate from the nominal values. In this case, $\xi(t)$ represents the deviations

$$\xi(t) \sim \Delta f_1 x + \Delta g_1 u \tag{111}$$

The actual system is

$$\dot{x} = F_a x + G_{1_a} u + G_2 \eta \tag{112}$$

$$= Fx + G_1 u + d\xi(t) + G_2 \eta$$

where $F_a = F(p)$, $G_{1_a} = G_1(p)$, $F = F(p_o)$, and $G_1 = G_1(p_o)$ with control given by Equation (98) and computed on the assumption of nominal F, G_1. The estimator is as given in Equation (110).

Defining an augmented state vector $\underline{x}$ = col [x, s*], we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{s}^* \end{bmatrix} = \left[\begin{array}{c|c} F_a & 0 \\ \hline \hat{f}_1 & -(\gamma+g) \end{array}\right] \begin{bmatrix} x \\ s^* \end{bmatrix} + \begin{bmatrix} G_{1_a} \\ g_{1_1} \end{bmatrix} u + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta \tag{113}$$

where

$$\hat{f}_1 = f_1 + [\gamma + g \quad 0 \quad \dots \quad 0] \tag{114}$$

with

$$u(t) = Kx + k_2 \hat{\xi} = Kx + k_2 g\, [x_1 - s^*] = \hat{K}\underline{x} \tag{115}$$

where

$$\hat{K} = [K \mid -k_2 g] + [\underset{\substack{\uparrow \\ \text{first column}}}{k_2 g} \quad 0 \quad 0 \quad \dots \quad 0] \tag{116}$$

Substituting into Equation (113) gives the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{s}^* \end{bmatrix} = \left\{ \begin{bmatrix} F_a & 0 \\ \hat{f}_1 & -(g+\gamma) \end{bmatrix} + \begin{bmatrix} G_{1_a} \\ g_{1_1} \end{bmatrix} \hat{K} \right\} \begin{bmatrix} x \\ s^* \end{bmatrix} + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta(t) \qquad (117)$$

Thus, the closed-loop covariance can be computed with and without the use of mismatch estimation. Note that we are only considering parameter variations in the x_1 equation. The above method is summarized as follows:

a. Compute gains K, k_2 by setting up augmented matrices

$$F_o = \left[\begin{array}{c|c} F & d \\ \hline 0 & -\gamma \end{array}\right]; \qquad G_{1_o} = \left[\begin{array}{c} G_1 \\ \hline 0 \end{array}\right]$$

and solving associated (n+1) Riccati equation to get K, k_2.

b. Apply parameter variations to F, G_1 to get F_a, G_{1_a}.

c. Set up F, G_1 matrices as in Equation (113) with f_1, g_{1_1} the nominal system values.

d. Substitute

$$u = Kx + k_2\hat{\xi} = [K \mid -k_2 g]\,\underline{x} + k_2 g x_1 = \hat{K}\underline{x}$$

and obtain "closed-loop" system (117).

e. Compute steady state covariance $\underline{X}$ and output covariance. (Note: If $g \gg \gamma$, the choice of γ becomes unimportant.)

Multi-State Insensitivity

The above concept is extended easily to the case where there are $\ell > 1$ states where parameters are subject to variation. However, a separate first order filter must be associated with each state variable. For example, suppose we are interested in "desensitizing" the first r states. Then write

$$
F_o = \begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{\ell} \end{array}
\left[\begin{array}{cc} F & I_\ell \\ & 0 \\ \hline 0 & -\Gamma \end{array}\right]
\begin{array}{l} \}n \\ \\ \}\ell \end{array}
\qquad
G_{1_o} = \left[\begin{array}{c} G_1 \\ \\ \hline 0 \end{array}\right]
\begin{array}{l} \}n \\ \\ \}\ell \end{array}
$$

where $\Gamma = \text{diag}\ (\gamma_i)$. Solve the associated $(n \times \ell) \times (n + \ell)$ Riccati equation to get the feedback gains

$$u = Kx + K_2\xi$$

where ξ is an ℓ-dimensional vector. (This can also be done more efficiently by solving the original $n \times n$ Riccati equation and then solving an associated $n \times \ell$ linear matrix equation to get K_2.)*

The estimator is then

$$\dot{s}^*(t) = -(\Gamma+G)\ s^*(t) + \{[\Gamma+G]x_\ell + F_\ell x + G_{1_\ell} u\} \tag{118}$$

$$\hat{\xi}(t) = G[x'_\ell - p^*] \tag{119}$$

where $G = \text{diag}\ (g_i)$ and F_ℓ, G_{1_ℓ} are the first ℓ rows of nominal F and G_1, respectively. $x_\ell = \text{col}\ (x_1 \ldots x_\ell)$. The closed-loop system for purposes of covariance propagation, etc., is then

$$
\begin{bmatrix} \dot{x} \\ \dot{s}^* \end{bmatrix} = \left\{ \begin{bmatrix} F_a & 0 \\ \hat{F}_{a_\ell} & -(\Gamma+G) \end{bmatrix} + \begin{bmatrix} G_1 \\ G_{1_\ell} \end{bmatrix} \hat{K} \right\} \begin{bmatrix} x \\ s^* \end{bmatrix} + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta(t) \tag{120}
$$

where F_a, $G1_a$ are the actual system matrices (i.e., with parameter variations) and

$$\hat{F}_{a_\ell} = F_\ell + [\underbrace{\Gamma+G}_{\ell} \mid \underbrace{0 \ \ldots \ 0}_{n}] \tag{121}$$

---

* This is done by expanding the sub-matrix blocks of the $(n + \ell)$ Riccati equation).

$$\hat{K} = [\underbrace{K}_{n} \mid \underbrace{-k_2 G}_{\ell}] + [\underbrace{k_2 G}_{\ell} \mid \underbrace{0 \ \ldots \ 0}_{n}] \tag{122}$$

<u>Details of Mismatch Estimation Design</u>

The actual design is a two-step procedure. The first step consists of computing optimal gains for a design model given by

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \left[\begin{array}{c|c} F(p_o) & D \\ & 0 \\ \hline 0 & -\Gamma \end{array}\right] \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} G_1 \\ \hline 0 \end{bmatrix} u + \begin{bmatrix} G_2 \\ \hline 0 \end{bmatrix} \eta \tag{123}$$

where

p_o = vector of uncertain parameters at their nominal values

D = a diagonal $\ell \times \ell$ matrix used to modify the magnitude of the perturbations generated by certain dynamics

ℓ = the order of state equations affected

Initially $D = I$

For this design, the state equations that are affected by parameter uncertainties are the two rigid body equations and the two flexure mode equations. (As earlier stated, all insensitive controller designs were done on the Case 4R model.) ℓ, then, is equal to 4. Γ also is an $\ell \times \ell$ diagonal matrix. The values chosen for the diagonal elements reflect the dominant frequency characteristics of the affected state equations. The values chosen were

$\gamma_{1,1} = 2.5$

$\gamma_{2,2} = 2.5$

$\gamma_{3,3} = 15.0$

$\gamma_{4,4} = 15.0$

where the value of 2.5 is directed at the closed-loop short period frequency and the value 15.0 is directed at the highest bending mode frequency in the Case 4R model.

The second step of the process involved evaluating the performance of the gains computed in the first step on an "actual" system. The evaluation model is given by

$$\begin{bmatrix} \dot{x} \\ \dot{s}* \end{bmatrix} = \left\{ \begin{bmatrix} F(p_1) & 0 \\ \hat{F}(p_o) & -\Gamma+G \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \hat{K} \right\} \begin{bmatrix} x \\ s* \end{bmatrix} + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta(t) \qquad (124)$$

where

p_1 = vector of uncertain parameters at a worst case condition ($\bar{q}_f = 1.25$, $\omega_f = 0.75$, $M_{w_f} = 0.8$)

$$\hat{F}(p_o) = F(p_o) + [\Gamma+G \;\vdots\; 0 \;\ldots\; 0]$$

$$\hat{K} = [K \;\vdots\; -K_2G] + [K_2G \;\vdots\; 0 \;\ldots\; 0]$$

The estimator gains, G, are left to the designer's choice. The procedure followed in selecting G was based on trial and error. The objective was to select G such that system performance returned to specifications. It was found, however, that this could not be achieved without creating unrealistically high bandwidth estimators for the case of D = I. It was also found that the choice of D = 10I and G = 30I would produce specification satisfying performance under worst case conditions. The resulting K gains are given in Table 19.

TABLE 19. MISMATCH ESTIMATOR GAINS

ROW 1									
.77981E-03	.21450E-02	.26534E-02	.83483E-03	-.34501E-03	-.70350E-03	-.21382E+00	.11248E-02	.11472E-04	-.27191E-03
.47293E-04	.61141E-05	-.20954E-03	.13560E-03	-.13574E-04	-.81188E-03	-.21270E-02	-.25414E-02	-.80256E-03	
ROW 2									
.11506E-01	.15162E-01	-.38538E-02	.71624E-02	.46278E-03	-.62506E-02	.34005E-02	-.15227E+00	-.17730E-05	.67391E-03
-.12350E-02	-.69637E-04	.43997E-03	.85089E-03	.13845E-02	-.11371E-01	-.14755E-01	.36657E-02	-.68935E-02	

It must be noted that an extensive investigation of the effects of varying Γ, D, and G was not performed in this study. Although we did obtain a controller which satisfied our design rules, we recognize the fact that this controller may not be the "optimum" mismatch estimation controller. It is our hope that future refinement, based on the results of this study, could produce the methodology necessary for an "optimal" controller design.

UNCERTAINTY WEIGHTING

Let us assume that the system of interest is described by

$$\dot{x} = F(p)x + G_1 u + G_2 \eta \tag{125}$$

$$r = H(p)x + Du \tag{126}$$

where p represents an uncertain parameter. The case of multiple parameters may be treated by a straightforward extension. Suppose that p_o is the nominal value of p and that a weighting matrix Q has been found which defines a good nominal controller, i.e., one that minimizes

$$J = E\{r^T_o Q r_o\} \tag{127}$$

Here the subscript indicates the nominal value is used:

$$\dot{x}_o = F(p_o)x_o + G_1 u_o + G_2 \eta, \quad r_o = H(p_o)x_o + Du \tag{128}$$

The variational equations for perturbations in states and responses caused by variation in the parameter are

$$\delta\dot{x} = F(p)\,\delta x + \delta F x_o \tag{129}$$

$$\delta r = H(p)\,\delta x + \delta H x_o \tag{130}$$

where $\delta F = F(p) - F(p_o)$ and $\delta H = H(p) - H(p_o)$.

To keep δ_r small, we may ask that δx be small and that $\delta H x_o$ be small. To keep δx small, we may ask that $\delta F x_o$ be small. To accomplish this in the original framework of the optimization problem, let us introduce

$$\tilde{r} = \delta F x \text{ and } \tilde{\tilde{r}} = \delta H x. \tag{131}$$

Then instead of minimizing $J = E\{r_o^T Q r_o\}$ let us minimize

$$\hat{J} = E\{r_o Q r_o + \lambda \tilde{r}_o^T \tilde{r}_o + \mu \tilde{\tilde{r}}_o^T \tilde{\tilde{r}}_o\} \tag{132}$$

The parameters λ and μ are design parameters which are to be suitably selected. We may rewrite $\hat{J}$ in our standard form of

$$\hat{J} = E\{\hat{r}_o^T \hat{Q} \hat{r}_o\} \tag{133}$$

by defining

$$\hat{r} = \begin{bmatrix} r \\ \tilde{r} \\ \tilde{\tilde{r}} \end{bmatrix} \text{ and } \hat{Q} = \begin{bmatrix} Q & 0 & 0 \\ 0 & \lambda\tilde{I} & 0 \\ 0 & 0 & \mu\tilde{\tilde{I}} \end{bmatrix} \tag{134}$$

where $\tilde{I}$ and $\tilde{\tilde{I}}$ are appropriately dimensioned identity matrices. There are many variations that could be made on this theme. For instance, one might choose $\tilde{r} = [F(p)]^{-1}\,\delta F x$ to reflect the steady state variation in the states, or one might wish to weight the variations in responses with Q to reflect the weighting of the nominal responses.

Details of Uncertainty Weighting Design

The augmented responses, $\tilde{r}$ and $\tilde{\tilde{r}}$, were computed by defining

$$\delta F = \tilde{F}(p_1) - \tilde{F}(p_o) \tag{135}$$

$$\delta H = \tilde{H}(p_1) - \tilde{H}(p_o) \tag{136}$$

where

p_1 = vector of parameter uncertainties at a worst case condition ($\bar{q}_f = 1.25$, $\omega_f = 0.75$, $M_{w_f} = 0.8$)

p_o = vector of parameter uncertainties at the nominal condition

$$\tilde{F} = F_{ij}(p_1) - F_{ij}(p_o) \qquad \begin{array}{l} i = 1, 4 \\ j = 1, 15 \end{array}$$

$$\tilde{H} = H_{ij}(p_1) - H_{ij}(p_o) \qquad \begin{array}{l} i = 1, 4 \\ j = 1, 15 \end{array}$$

Then the augmented response vector consists of the nine nominal design responses and the eight augmented uncertainty responses.

The design objective was to reduce the RMS bending response, r_1, to be less than

$$[r_{1_{design}}]_{RMS} = [r_1(p_o)]_{RMS} - [\Delta r_1]_{RMS}$$

where

$$[\Delta r_1]_{RMS} = [r_1(p)]_{RMS} - [r_1(p_o)]_{RMS}$$

where $[r_1(p)]_{RMS}$ denotes the RMS response with the nominal controller while at the same time maintaining RMS torsion response and short period roots. The value μ, given in Equation (134), was actually chosen to reflect the same weighting magnitudes as the nominal design responses. λ was maintained at 1. The final set of weights used on the augmented responses which satisfied the design objective were the following:

$Q_{10,10} = 1.0$

$Q_{11,11} = 1.0$

$Q_{12,12} = 1.0$

$Q_{13,13} = 1.0$

$Q_{14,14} = 0.1 \times 10^{-9}$

$Q_{15,15} = 0.1 \times 10^{-8}$

$Q_{16,16} = 0.1 \times 10^{-12}$

$Q_{17,17} = 0.1 \times 10^{-11}$

The nominal design weights, $Q_{1,1}$ through $Q_{9,9}$ remained the same, as shown in Table 5. The optimal controller gains are given in Table 20.

TABLE 20. UNCERTAINTY WEIGHTING GAINS

```
GAINS MATRIX

ROW  1
-.46570E-04  .22290E-05  .15210E-03  .11826E-03 -.39155E-03 -.16790E-02 -.24800E+00 -.20040E-01  .73026E-05 -.26357E-03
-.87190E-04 -.22504E-06 -.23670E-03  .19021E-03 -.74677E-04
ROW  2
 .27093E-03  .58764E-03 -.25292E-03  .86606E-03  .67232E-03 -.11254E-01 -.77229E-01 -.37732E+00  .15859E-05  .13323E-02
-.27544E-02 -.14704E-03  .78156E-03  .17743E-02  .32957E-02
```

RE-RESIDUALIZATION

Of the new techniques developed, re-residualization is the only one which specifically treats the unmodeled dynamics problem. This process was developed under Honeywell's Independent Research Program. In this study, we have defined unmodeled dynamics as either neglected known dynamics or unknown dynamics. The re-residualization process is an extension of residualization in that it too attempts to include the characteristics of known unmodeled dynamics in the construction of reduced order models. Consider a linear system described by the differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u \qquad (137)$$

where x is the state vector $[x^T{}_1, x^T{}_2]^T$ and u is the control vector. The state vector is partitioned into two sub-vectors x_1 and x_2. It is assumed that x_1 contains all the states which we wish to retain in the design or reduced order model and that x_2 contains the states to be eliminated. The coefficient matrixes are partitioned into dimensionally consistent sub-matrices. We have taken the control coefficient matrix to have a special structure, the bottom sub-matrix being zero. This is based on the assumption that at least first order actuator dynamics will be included in the model and that these dynamics will be retained in the reduced order model. The assumption is motivated by the fact that, without it, the re-residualization method gives rise to a control rate term in the reduced order model.

For simplicity of discussion, Equation (137) can be written as two equations:

$$\dot{x}_1 = F_1 x_1 + F_2 x_2 + G_1 u \tag{138}$$

$$\dot{x}_2 = F_3 x_1 + F_4 x_2 \tag{139}$$

The truncation method consists of setting $x_2 = 0$ in Equation (138) and ignoring Equation (139). This yields

$$\dot{x}_{1_T} = F_1 x_{1_T} + G_1 u$$

The residualization is more sophisticated. It consists of assuming $\dot{x}_2 = 0$. Then if F_4 is nonsingular from Equation (139), $x_2 = -F_4^{-1} F_3 x_1$. This expression is then used to eliminate x_2 in Equation (138) giving

$$\dot{x}_{1_R} = (F_1 - F_2 F_4^{-1} F_3) x_{1_R} + G_1 u \tag{140}$$

The re-residualization method goes one step further. Assuming initially that $\dot{x}_2 = 0$ leads to $x_2 = -F_4^{-1} F_3 x_1$ and computing $\dot{x}_2$ from this expression yields

$$\dot{x}_2 = -F_4^{-1} F_3 \dot{x}_1 = -F_4^{-1} F_3 (F_1 x_1 + F_2 x_2 + G_1 u) \tag{141}$$

Now equating the right-hand sides of Equation (141) and Equation (139) gives

$$F_3 x_1 + F_4 x_2 = -F_4^{-1} F_3 (F_1 x_1 + F_2 x_2 + G_1 u) \tag{142}$$

Solving Equation (142) for x_2 yields

$$x_2 = -(F_4 + F_4^{-1} F_3 F_2)^{-1} [(F_3 + F_4^{-1} F_3 F_1) x_1 + F_4^{-1} F_3 G_1 u] \tag{143}$$

Substituting this expression for x_2 into Equation (138) yields

$$\begin{aligned} \dot{x}_{1_{RR}} &= [F_1 - F_2 (F_4 + F_4^{-1} F_3 F_2)^{-1} (F_3 + F_4^{-1} F_3 F_1)] x_{1_{RR}} \\ &\quad + [G_1 - F_2 (F_4 + F_4^{-1} F_3 F_2)^{-1} F_4^{-1} F_3 G_1] u \end{aligned} \tag{144}$$

The same substitution may be made in the response equation

$$r = [H_1, H_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du \tag{145}$$

which gives

$$\begin{aligned} r = {} & [H_1 - H_2 (F_4 + F_4^{-1} F_3F_2)^{-1} (F_3 + F_4^{-1} F_3F_1)]x_{1_{RR}} \\ & + [D - F_2 (F_4 + F_4^{-1} F_3F_2)^{-1} F_4^{-1} F_3G_1]\, u \end{aligned} \tag{146}$$

The following alternate derivation of the re-residualization procedure was suggested by the contract monitor, Dr. Ernest S. Armstrong. Computing $\ddot{x}_2$ from Equations (138) and (139) yields

$$\ddot{x}_2 = (F_3F_1 + F_4F_3)\, x_1 + (F_3F_2 + F_4^2)\, x_2 + F_3G_1\, u \tag{147}$$

Setting $\ddot{x}_2 = 0$ gives

$$x_2 = -(F_3F_2 + F_4^2)^{-1} [(F_3F_1 + F_4F_3)\, x_1 + F_3G_1\, u] \tag{148}$$

which is equivalent to (143) if F_4 is nonsingular. This alternate form is meaningful if F_4 is singular and $F_3F_2 + F_4^2$ is nonsingular. Furthermore, this derivation yields a clearer interpretation of the relationship between truncation, residualization, and re-residualization as corresponding to the approximations $x_2 = 0$, $\dot{x}_2 = 0$, and $\ddot{x}_2 = 0$, respectively.

For more details on how re-residualization relates to singular perturbation as far as the degree to which the characteristics of unmodeled dynamics are reproduced in a reduced order model, see Appendix E.

Details of Re-Residualization Design

The design of a controller based on a re-residualized model was straightforward. First Case 4RR, a re-residualized version of Case 4R, was constructed. Secondly, an optimal controller was designed using the same response vector and the same set of weights that was used in the nominal controller design. The re-residualized model and the optimal gains computed are given in Table 21.

TABLE 21. RE-RESIDUALIZED MODEL

```
PERTURBED F4R MATRIX

ROW    1
-.69865E+00  .32643E+01 -.35860E-01 -.29374E-01 -.64406E+00 -.18660E+01 -.21813E+03 -.22866E+03 -.50669E+00 -.71108E+01
-.10853E+01  .17090E-02 -.17602E-01 0.           .19559E-02
ROW    2
-.51436E+00 -.11334E+01  .33044E-01 -.30620E+00 -.10387E+00 -.10172E+02 -.59909E+03 -.24794E+04  .25883E+01 -.98057E+00
-.11444E+02  .26813E-01  .46879E-01 0.           .25836E-01
ROW    3
-.16708E+01 -.64174E-01 -.97884E+[illegible] -.21393E+00 -.29892E+02 -.17998E+02 -.32470E+04  .10469E+04  .60428E+00 -.23714E+02
 .48379E+01 -.21299E-01 -.18718E+00 0.          -.28026E-01
ROW    4
 .11349E+00 -.78860E+00  .47860E-01 -.12482E+01  .14116E+01 -.19278E+03 -.14591E+04 -.24684E+04 -.23842E+01  .11303E+02
-.11204E+02  .25979E-01  .96230E-01 0.           .14800E-01
ROW    5
0.          0.           .10000E+01 0.          0.          0.          0.          0.          0.          0.
0.          0.          0.          0.          0.
ROW    6
0.          0.          0.           .10000E+01 0.          0.          0.          0.          0.          0.
0.          0.          0.          0.          0.
ROW    7
0.          0.          0.          0.          0.          0.          -.60000E+01 0.          0.          0.
0.          0.          0.          0.          0.
ROW    8
0.          0.          0.          0.          0.          0.          0.          -.75000E+01 0.          0.
0.          0.          0.          0.          0.
ROW    9
0.          0.          0.          0.          0.          0.          0.          0.          -.22185E+02 0.
0.          0.          0.          0.           .22185E+02
ROW   10
0.          0.          0.          0.          0.          0.          0.          0.          0.          -.85492E+01
0.          0.           .85492E+01 0.          0.
ROW   11
0.          0.          0.          0.          0.          0.          0.          0.          -.50960E+01 0.
0.           .10000E+01 0.          0.          0.
ROW   12
0.          0.          0.          0.          0.          0.          0.          0.           .90891E+02 0.
-.38953E+02 -.10192E+02 0.          0.          0.
ROW   13
0.          0.          0.          0.          0.          0.          0.          0.          0.          0.
0.          0.          -.10983E+02 0.           .10983E+02
ROW   14
0.          0.          0.          0.          0.          0.          0.          0.          0.          0.
0.          0.          0.          -.49524E+00 -.61315E-01
ROW   15
0.          0.          0.          0.          0.          0.          0.          0.          0.          0.
0.          0.          0.           .10000E+01 0.
```

TABLE 21. Continued

G14R MATRIX

Row		
ROW 1	.62439E+00	.22582E+01
ROW 2	-.51428E+01	.37786E+02
ROW 3	.14549E+02	-.32527E+02
ROW 4	-.43429E+01	.35003E+02
ROW 5	0.	0.
ROW 6	0.	0.
ROW 7	.60000E+01	0.
ROW 8	0.	.75000E+01
ROW 9	0.	0.
ROW 10	0.	0.
ROW 11	0.	0.
ROW 12	0.	0.
ROW 13	0.	0.
ROW 14	0.	0.
ROW 15	0.	0.

G24R MATRIX

Row	
ROW 1	0.
ROW 2	0.
ROW 3	0.
ROW 4	0.
ROW 5	0.
ROW 6	0.
ROW 7	0.
ROW 8	0.
ROW 9	0.
ROW 10	0.
ROW 11	0.
ROW 12	0.
ROW 13	0.
ROW 14	-.30360E+00
ROW 15	.86190E+00

TABLE 21. Continued

```
PERTURBED H4R MATRIX

ROW   1
-.26695E+05 -.15675E+05  .13997E+05 -.12851E+05  .12666E+07 -.34441E+07  .65725E+07 -.40756E+08 -.10203E+05 -.18921E+06
-.18021E+06  .56208E+03  .62620E+03 0.           .69612E+03
ROW   2
-.22020E+05 -.60152E+04 -.14910E+04 -.32144E+04 -.82316E+05 -.62363E+06  .38779E+07 -.10375E+08  .45578E+04 -.23603E+06
-.47198E+05  .65911E+02 -.23662E+03 0.          -.25845E+02
ROW   3
 .19420E+06  .38663E+06  .12452E+07 -.32958E+07 -.38249E+06  .63413E+07  .17273E+09  .13322E+10  .81923E+05  .16706E+07
 .47719E+07 -.18949E+06 -.16282E+07 0.          -.15893E+06
ROW   4
 .21842E+05 -.33905E+05 -.73622E+05 -.61588E+06  .52277E+05  .83632E+06 -.76727E+07  .12602E+09  .10637E+06  .21565E+07
 .22679E+06 -.47407E+05 -.20242E+07 0.           .10671E+06
ROW   5
0.          0.          0.          0.          0.          0.           .10000E+01 0.          0.          0.
0.          0.          0.          0.          0.
ROW   6
0.          0.          0.          0.          0.          0.          0.           .10000E+01 0.          0.
0.          0.          0.          0.          0.
ROW   7
0.          0.          0.          0.          0.          0.          -.20000E+01 0.          0.          0.
0.          0.          0.          0.          0.
ROW   8
0.          0.          0.          0.          0.          0.          0.          -.75000E+01 0.          0.
0.          0.          0.          0.          0.
ROW   9
 .10000E+01 0.          0.          0.          0.          0.          0.          0.          0.          0.
0.          0.          0.          0.          0.
ROW  10
0.          -.22748E-02 0.          0.          0.          0.          0.          0.          0.          0.
0.          0.          0.          0.          0.
```

TABLE 21. Concluded

```
      D4R MATRIX


ROW   1
-.12560E+06  .79503E+06
ROW   2
 .23301E+05  .78615E+05
ROW   3
 .40983E+08 -.31237E+09
ROW   4
 .23938E+08 -.77433E+08
ROW   5
0.          0.
ROW   6
0.          0.
ROW   7
 .60000E+01 0.
ROW   8
0.           .75000E+01
ROW   9
0.          0.
ROW  10
0.           .75000E+01


      GAINS MATRIX


ROW   1
-.31992E-04  .18264E-04  .11144E-03  .42629E-04 -.34628E-03 -.76062E-03 -.21435E+00 -.11345E-02  .10573E-04 -.28480E-03
 .43255E-04  .69739E-05 -.18776E-03  .13560E-03 -.22736E-04
ROW   2
 .13704E-03  .40411E-03 -.18182E-03  .20930E-03  .53537E-03 -.73355E-02  .38475E-02 -.16539E+00  .15800E-04  .72479E-03
-.13286E-02 -.73634E-04  .39039E-03  .84898E-03  .14882E-02
```

SECTION VII

EVALUATION

This section describes the results of a qualitative and quantitative evaluation of the insensitive controller design techniques that were investigated.

If an insensitive controller design technique is to be acceptable, it must satisfy two requirements. First, and most obvious, the design technique must produce a controller that is insensitive. Second, the level of effort that is required in the design, i.e., the ease with which the designer can apply the technique, should not compromise the usefulness of the technique. This second requirement is directed at techniques which require, for example, excessive trial and error computations or large computer memory and time requirements. Techniques which possess characteristics such as these would have limited acceptance among control system designers. Hence, the approach that was used to evaluate the insensitive controller synthesis techniques consisted of the following:

- A qualitative evaluation of the techniques with respect to critical criteria defined specifically for the controller synthesis stage, and

- A quantitative evaluation of the performance of the insensitive controllers designed.

SYNTHESIS EVALUATION

Ten criteria, which reflect important capabilities that a synthesis technique should possess, were defined for evaluation purposes. These ten criteria versus the nine techniques that were investigated are shown in Table 22. Each of these criteria and significant benefits or deficiencies in the synthesis techniques will now be discussed.

Treats Unmodeled Dynamics--Item 1

This item refers to whether or not a technique explicitly treats unmodeled dynamics in the synthesis process. Table 22 indicates that only residualization and re-residualization treat unmodeled dynamics. Both techniques, of course, treat only known unmodeled

TABLE 22. SYNTHESIS EVALUATION

Technique	(1) Treats Unmodeled Dynamics	(2) Treats Response Uncertainty	(3) A Priori Range Required	(4) Additional Modeling Required	(5) Computational Load	(6) Treats Engineering Design Criteria: Transient	RMS	Stability	(7) Provides Insight Into Design Problems	(8) Weigh Performance versus Sensitivity	(9) Implementation Requirements	(10) Limits To Growth Potential	Remarks
Additive Noise	*	Coarse approximately developed	Yes	None if nonlinear model given otherwise minimal (+)	Low		Yes	No	No	Essentially No	**	Response Uncertainty only changes weights	Very simplistic approach
Minimax	*	Yes	Yes	Substantial	High to very high		Yes	Only in a gross sense	Some	Essentially No	**	Computational load	Can get "lucky" to reduce comp. load
Multiplant	*	Yes	Yes	Substantial	High to very high		Yes	Only in a gross sense	Some	Yes in theory	**	Computational load	May require many plants
Sensitivity Vector Augmentation	*	Response sensitivity states defined	No	Substantial	Very high	Yes with model response included	Yes	No	No	Yes	Unrealistic full state plus sensitivity state feedback	Computational and Implementation load	No additional comments needed
Residualization	Explicit first order	N/A	N/A	Minimal	Minimal		Indirect	Undetermined	N/A	No	Decrease complexity	Known linear unmodeled dynamics	Proven technique
State Dependent Noise	*	No	Yes	Low	Low		Yes	No	Some	Yes	**	Response uncertainty	Required new algorithm
Kleinman's Mismatch Estimation	Implicit	No	No	Substantial	High		Yes	Implicit	Some	Yes	Increased complexity	Computational and Implementation load	
Re-Residualization	Explicit Second Order	N/A	N/A	Minimal (+)	Minimal		Indirect	Undetermined	N/A	No	Decreased complexity	Same as residual	
Porter's Uncertainty Weighting	*	Yes	Not necessary but desirable	Minimal (+)	Low		Yes	No	Some	Yes	**	Only changes weights	

* No inherent treatment
** Full state but possibly could be reduced to limited state

dynamics. The mismatch estimation technique implicitly treats unmodeled dynamics, both known and unknown, by, in effect, forcing the real world plant to perform as the reduced order model. The remaining techniques are designed specifically to handle parameter uncertainties.

Treats Response Uncertainty--Item 2

As discussed in Section VI, some of the existing techniques have not been formulated to treat uncertainties in the response equation. A modification to handle this problem was developed for the additive noise concept and the sensitivity vector augmentation technique. Neither the state dependent noise nor mismatch estimation concepts treat response uncertainty as currently formulated.

A Priori Range Required--Item 3

This item refers to whether or not the technique requires that the range of the uncertain parameter variation be specified before design can begin. The sensitivity vector augmentation concept, since it uses partial derivatives, is the only concept that requires no knowledge of the range of variations. The mismatch estimation concept and uncertainty weighting technique do not require the complete range of variations. However, they do require an off-nominal condition for design purposes.

Additional Modeling Required--Item 4

This item represents the first of several qualitative judgments. It refers to the manner in which a technique constructs its design model. We have defined the level of additional modeling required as minimal if the order of the design model remains the same as the "free" aircraft model and a minimum of support computations are required. The residualization technique is the only concept that falls in this level since all other techniques used residualized models. The additive noise, uncertainty weighting, and re-residualization concepts require only a slight increase in support computations with the order of the design model remaining the same as the "free" aircraft model. The order of the design model for the state dependent noise concept remains the same as the "free" aircraft model; however, since it requires the computation of partial derivatives, we felt that the additional modeling required could not be judged to be minimal. The remaining techniques all require a design model of higher order than the "free" aircraft.

(Note: This is not precisely true for the minimax concept. However it does require modeling additional off-nominal conditions.) For our example, the sensitivity vector augmentation concept had, by far, the largest additional modeling demands. In addition to having to compute partial derivatives, the sensitivity vector augmentation concept required augmenting the aircraft model with 24 additional states. The 24 states were, in themselves, an approximation to the 45 states that the technique, in theory, requires.

Computational Load--Item 5

Since all insensitive controllers were designed via computer using the quadratic methodology software, this item refers to the computer requirements of both core size and time. It is closely related to Item 4 in that higher order design models will generally require large core requirements and computation times. In this case, too, the sensitivity vector augmentation concept had the largest computational requirements. One important distinction must be made however. The computational requirements for the sensitivity vector augmentation concept were very high, but they were bounded. This is not the case for the multiplant or minimax concepts. For this study, we made approximations which resulted in bounding the computer requirements for the two concepts. However, this may not be possible in other design problems. In short, although the computational requirements for the multiplant and minimax concepts were not excessive for this study, they have the potential for creating a very high computational load.

Treats Engineering Design Criteria--Item 6

The preceding items dealt with the preliminaries to the synthesis process, i.e., data that must be known before design can begin. This item is directed at the heart of the control design process--the design criteria. As discussed earlier, there are three types of design criteria specifications: transient response, statistical response, and stability criteria. The design criteria used in this study included all three types of criteria, though this does not necessarily have to be the case. Ideally, however, a synthesis technique should have the capability of treating all engineering design criteria since the type of criteria may vary from problem to problem. Item 6 shows the performance of the synthesis techniques versus the "ideal" capability.

Since all techniques utilized quadratic performance minimization designs, then the inclusion of a response term directed at transient response criteria is a sufficient approach for this type of criteria. This capability, however, is more consistent with the quadratic approach than with the synthesis techniques themselves.

The statistical response, criteria, or RMS response is similarly satisfied by specifying appropriate response terms. The residualization and re-residualization processes represent an indirect approach in that the characteristics of higher order model statistical response are approximated with lower order models.

None of the techniques specifically consider typical stability margin criteria in the design process. The mismatch estimation concept may be considered an implicit treatment since it forces performance to follow a model which may have stability margins designed into it. The minimax and multiplant approaches have received a qualifier in the sense that both techniques produce stable controllers at a number of off-nominal points. We have judged this to be treating stability in a gross sense. The effect of residualization and re-residualization on stability is unknown at this time. It is an area recommended for further study.

Provides Insight into Design Problems--Item 7

This is another extremely qualitative item. We have included this item to possibly identify techniques that provide data in a form that could lead the designer to critical design problems or provide him with a greater awareness of system operating capability. This item was not evaluated extensively; however, we felt that it is a necessary characteristic of a synthesis technique. All techniques, except for additive noise and sensitivity vector augmentation, provide some insight into critical design problems. This results mainly from the fact that these techniques require analyzing performance at off-nominal conditions or at different magnitudes of parameter uncertainties. Since the additive noise concept, as we formulated it, is limited to one design point, we felt it provided no additional insight. We also felt that the sensitivity vector augmentation concept had the potential for providing valuable design information. However, as formulated, it was not in the form that a designer could easily translate.

Weigh Performance versus Sensitivity--Item 8

It has been stated that an insensitive control system design requires trading off performance versus sensitivity. This item evaluates the makeup of the synthesis technique as to whether it contains an explicit means of achieving this tradeoff. The sensitivity vector augmentation concept and the uncertainty weighting scheme define sensitivity terms for inclusion in the performance index. These terms can be weighed directly against performance terms. Thus these two concepts are highly desirable from this point of view. The other concepts are not as direct. However, except for the additive noise concept, they all provide a means of measuring the performance variation produced through sensitivity reduction.

Implementation Requirements--Item 9

This item refers to the practical concern of implementing insensitive controllers designs in system hardware. The most recognizable constraint lies in the requirement of full state feedback. All techniques require full state feedback; however, if combined with residualization or re-residualization, the difficulties of this constraint can be reduced. In addition, no attempt has been made in this study to investigate the use of limited state feedback. This would be a worthwhile endeavor and is recommended for future study. The sensitivity vector augmentation concept and the mismatch estimation concept require adding states. This does not appear to be a significant problem except in the case of the sensitivity vector augmentation concept when a 24-state augmentation may be excessive.

Limits to Growth Potential--Item 10

This item attempts to identify those characteristics of a synthesis technique that limit its growth and probability of acceptance. Computational load problems have been observed in four techniques: minimax, multiplant, sensitivity vector augmentation, and mismatch estimation. Although they are not as severe in mismatch estimation as the others, this problem, if not resolved, does limit the concept. One other consideration not directly treated is the manner in which the insensitivity is achieved. The additive noise approach and the uncertainty weighting approach reduce to a weight change in the performance index. This approach may prove unacceptable to designers in that it really does not consider the operating range of the system.

In summary, no attempt was made to rank the above ten considerations in terms of relative importance even though some are obviously more important than others. Table 22 has been constructed to provide a means of quantitizing what are in most respects very qualitative criteria. We do not try to rate one technique over another based on this criteria. However, it is felt that the results presented should be weighed against the quantitative performance results that follow.

EVALUATION PROCEDURE

The insensitive controllers designed with the techniques described in Sections V and VI were evaluated with respect to the following considerations:

1. The effect of parameter uncertainties on controller performance,
2. The effect of unmodeled dynamics on controller performance, and
3. The effect of parameter uncertainties and unmodeled dynamics on controller performance.

In order to evaluate the effect of parameter uncertainties, critical system response parameters were computed at six evaluation conditions for each of the insensitive controllers with the Case 4R model. The critical system response parameters are those associated with the design criteria, i.e., steady state bending moment due to maneuvers (maneuver load), RMS bending moment, RMS torsion moment, short period frequency, short period damping, and phase and gain margins for both elevator and aileron loop. In addition, the control activity requirements were monitored through computation of aileron and elevator RMS deflections. The evaluation conditions chosen were defined and designated as follows:

- Nominal (N) - ($\bar{q}_f = 1.0$, $\omega_f = 1.0$, $M_{w_f} = 1.0$)
- Worst Case 1 (WC1) - ($\bar{q}_f = 1.25$, $\omega_f = 0.75$, $M_{w_f} = 0.8$)
- Worst Case 2 (WC2) - ($\bar{q}_f = 0.5$, $\omega_f = 1.0$, $M_{w_f} = 1.2$)
- Independent Variation 1 (p_1) - ($\bar{q}_f = 1.0$, $\omega_f = 1.0$, $M_{w_f} = 0.8$)
- Independent Variation 2 (p_2) - ($\bar{q}_f = 1.0$, $\omega_f = 0.75$, $M_{w_f} = 1.0$)
- Independent Variation 3 (p_3) - ($\bar{q}_f = 1.25$, $\omega_f = 1.0$, $M_{w_f} = 1.0$)

The nominal condition was chosen to provide a reference for insensitive controller performance versus nominal controller performance at the nominal condition and insensitive controller performance at the other evaluation conditions. The two worst case conditions were defined through trial and error procedures as described in Section IV. Worst Case 1 represents the most severe condition in that nominal controller performance violates the largest number of design specifications at this condition. Worst Case 2 is a compromised condition which produces improved nominal controller performance with respect to some design specifications but specification violations for others. The three independent variations were chosen to provide the capability of analyzing the effect of independent variations and also to provide additional evaluation conditions. Tables 23 through 28 present the results of the critical response parameter computations. Note that, since the purpose of this data is the evaluation of uncertain parameter effects, the Case 4R model was used for the computations. Comparison of the nominal controller performance with respect to maneuver load response as given in Tables 23 through 28 versus that given in Section IV shows a discrepancy in maneuver load response values. The values shown in Section IV represent the steady state maneuver load reduction computed with respect to a step inboard elevator command. The percentage reduction was computed by forming the ratio of the open-loop maneuver load response to a step elevator command to the closed-loop maneuver load response to a step elevator command. Since a step elevator command is essentially a pitch rate command, the critical parameter than must be included in the evaluation is steady state pitch rate. Unfortunately, each insensitive control system has a different pitch rate to elevator gain factor. This could have been avoided only through the use of a complex constrained gain design or the use of integral control to force the steady state pitch rate to some specified constant value common for all controllers. An alternate, less complex approach, which was employed in this study, is to evaluate all maneuver load responses at a prescribed pitch rate. The pitch rate chosen was the steady state pitch rate achieved by the nominal controller at the nominal condition 0.2164 rad/sec (12.4°/sec). Using this approach instead of the one used to generate Section IV data produces the discrepancies in the two tables. In fact, a close observation of the two tables shows that, for the Worst Case 1 condition, the nominal controller now satisfies the design specification. (Note: The results given in Section IV also assumed the $\dot{q}$ to δ_e ratio was the same for both the open-loop and closed-loop systems. Application of the same pitch rate criteria would produce different values for the nominal maneuver load response performance. However, since all insensitive controller performance is measured relative to the nominal, it was felt that it was unnecessary to modify these values.)

TABLE 23. INSENSITIVE CONTROLLER PERFORMANCE - CASE 4R NOMINAL

Evaluation Condition - Nominal $(\bar{q}_f = 1.0,\ \omega_f = 1.0,\ M_{w_f} = 1.0)$

Evaluation Model - Case 4R

Specification description		Criteria	Nominal	Re-Residual-ization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensitivity vector	Mismatch Estimation
Maneuver load % change	B	< -30%	-40.1%	-40.1%	-42.5%	-39.8%	-40.0%	-40.6%	-40.2%	-40.7%	-40.2%
Gust load alleviation % change	B	< -30%	-35.0%	-35.4%	-44.5%	-37.8%	-37.7%	-47.6%	-39.1%	-47.0%	-35.3%
	T	< + 5%	-31.5%	-31.6%	-21.9%	-37.9%	-34.9%	-40.2%	-33.9%	-29.6%	-31.5%
Handling qualities	ω_{sp}	< 1.6 rad/sec	2.12	2.12	2.13	2.30	2.17	2.39	1.99	2.29	2.12
	ζ_{sp}	0.7-0.8 sec^{-1}	0.718	0.72	0.729	0.720	0.689	0.701	0.636	0.665	0.717
Stability margins Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			*	*	*	∞	∞	∞	∞	∞	*
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	1.85 rad (106°)	∞	∞	∞	∞	1.80 rad (103.1°)	∞
elevator			∞	∞	∞	3.08 rad (176.2°)	∞	2.79 rad (160°)	∞	∞	∞
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00018	0.00019	0.0010	0.00011	0.00017	0.00027	0.00023	0.00075	0.00019
	$\dot{\delta}_a$		0.00080	0.00082	0.0035	0.00032	0.00075	0.00110	0.00100	0.00350	0.00083
	δ_e		0.00150	0.00150	0.0014	0.00180	0.00170	0.00210	0.00170	0.00190	0.00150
	$\dot{\delta}_e$		0.00370	0.00380	0.0031	0.00430	0.00370	0.00640	0.00470	0.00670	0.00370
Run number		NA	249	241	234	239	240	243	242	244	247

* Not evaluated

TABLE 24. INSENSITIVE CONTROLLER PERFORMANCE - CASE 4R WC1

Evaluation Condition - Worst Case 1 **Evaluation Model - Case 4R**

($\bar{q}_f = 1.25$, $\omega_f = 0.75$, $M_{w_f} = 0.8$)

Specification description		Criteria	Nominal	Re-Residual-ization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensivitiy vector	Mismatch estimation
Maneuver load % change	B	< -30%	-31.3%	-31.3%	-30.9%	-31.6%	-31.3%	-31.7%	-31.7%	-28.4%	-33.7%
Gust load alleviation % change	B	< -30%	-17.2%	-17.3%	-22.0%	-25.2%	-23.2%	-37.3%	-21.6%	-11.7%	-29.8%
	T	< + 5%	-7.1%	-7.3%	1.2%	-19.4%	-14.8%	-24.7%	-11.1%	+1.1%	-14.5%
Handling qualities	ω_{sp}	> 1.6 rad/sec	2.51	2.51	2.54	2.63	2.51	3.17	3.16	3.8	2.39
	ζ_{sp}	0.7-0.8 sec^{-1}	0.84	0.861	0.741	0.749	0.794	0.824	0.823	0.49	0.687
Stability margins											
Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	20 dB	∞
elevator			21 dB	19.4 dB	24.4 dB	∞	*	*	∞	63 dB	∞
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	∞	∞	∞	∞	∞	1.27 rad (73.0°)	∞
elevator			∞	∞	∞	2.97 rad (170.1°)	∞	2.87 rad (164.2°)	∞	*	2.35 rad (134.4°)
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00031	0.00032	0.0014	0.00012	0.00029	0.00038	0.00039	0.0016	0.00056
	$\dot{\delta}_a$		0.00140	0.00140	0.0059	0.00048	0.00130	0.00170	0.00180	0.0062	0.00170
	δ_e		0.00180	0.00180	0.0017	0.00210	0.00200	0.00250	0.00190	0.0030	0.00220
	$\dot{\delta}_e$		0.00450	0.00460	0.0034	0.00460	0.00450	0.00720	0.00550	0.0100	0.00480
Run number		NA	249	241	234	239	240	243	242	245	246

* Not evaluated

TABLE 25. INSENSITIVE CONTROLLER PERFORMANCE - CASE 4R WC2

Evaluation Condition - Worst Case 2 **Evaluation Model - Case 4R**

($\bar{q}_f = 0.5$, $\omega_f = 1.0$, $M_{wf} = 1.2$)

Specification description		Criteria	Nominal	Re-Residualization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensitivity vector	Mismatch estimation
Maneuver load % change	B	< -30%	-68.4%	-68.4%	-70.5%	-68.1%	-68.3%	-68.6%	-68.5%	-71.2%	-70.9%
Gust load alleviation % change	B	< -30%	-61.3%	-61.6%	-69.7%	-61.2%	-62.2%	-66.9%	-63.3%	-71.2%	-56.3%
	T	< +5%	-57.5%	-57.5%	-51.7%	-60.0%	-58.6%	-60.1%	-57.8%	-53.9%	-44.4%
Handling qualities	ω_{sp}	> 1.6 rad/sec	1.08	1.08	1.06	1.16	1.11	1.18	1.02	1.07	1.51
	ζ_{sp}	0.7-0.8 sec^{-1}	0.67	0.670	0.701	0.700	0.653	0.663	0.574	0.56	0.237
Stability margins Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			∞	∞	∞	∞	*	∞	∞	∞	24 dB
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			∞	∞	∞	∞	∞	2.91 rad (166.5°)	3.02 rad (173.3°)	2.98 rad (171°)	∞
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00020	0.00022	0.0012	0.00013	0.00021	0.00032	0.00025	0.00088	0.00066
	$\dot{\delta}_a$		0.00057	0.00059	0.0027	0.00025	0.00055	0.00081	0.00073	0.00025	0.00130
	δ_e		0.00140	0.00140	0.0013	0.00170	0.00160	0.00200	0.00150	0.00180	0.00500
	$\dot{\delta}_e$		0.00300	0.00310	0.0025	0.00380	0.00300	0.00550	0.00410	0.00530	0.00830
Run number		NA	249	241	234	239	240	243	242	245	247

* Not evaluated

TABLE 26. INSENSITIVE CONTROLLER PERFORMANCE - CASE 4R P1

Evaluation Condition - P1 Evaluation Model - Case 4R

($\bar{q}_f = 1.0$, $\omega_f = 1.0$, $M_{w_f} = 0.8$)

Specification description		Criteria	Nominal	Re-Residualization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensitivity vector	Mismatch estimation
Maneuver load % change	B	< -30%	-42.5%	-42.5%	-44.8%	-42.2%	-42.4%	-43.1%	-42.7%	-43.2%	-42.0%
Gust load alleviation % change	B	< -30%	-29.9%	-30.2%	-39.1%	-35.0%	-33.9%	-45.2%	-33.6%	-41.1%	-36.6%
	T	< +5%	-22.7%	-23.0%	-13.9%	-32.5%	-28.2%	-35.6%	-24.8%	-20.9%	-31.5%
Handling qualities	ω_{sp}	> 1.6 rad/sec	2.02	2.02	2.02	2.20	2.07	2.30	1.89	2.08	2.12
	ζ_{sp}	0.7-0.8 $\sec^{-1}$	0.758	0.760	0.770	0.751	0.725	0.733	0.674	0.805	0.695
Stability margins Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			*	*	*	∞	*	∞	∞	∞	12.7 dB
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	1.95 rad (112°)	∞	∞	∞	∞	1.34 rad (77°)	∞
elevator			∞	∞	∞	2.90 rad (166°)	∞	2.72 rad (156°)	3.05 rad (174.7°)	∞	3.04 rad (174.3°)
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00018	0.00019	0.0010	0.00011	0.00017	0.00027	0.00023	0.00076	0.00021
	$\dot{\delta}_a$		0.00081	0.00082	0.0035	0.00032	0.00075	0.00110	0.00100	0.00350	0.00082
	δ_e		0.00170	0.00170	0.0016	0.00200	0.00190	0.00240	0.00180	0.00210	0.00200
	$\dot{\delta}_e$		0.00360	0.00380	0.0031	0.00440	0.00380	0.00640	0.00470	0.00670	0.00390
Run number		NA	249	241	234	239	240	243	242	245	247

* Not evaluated

TABLE 27. INSENSITIVE CONTROLLER PERFORMANCE - CASE 4R P2

Evaluation Condition - P2 Evaluation Model - Case 4R

($\bar{q}_f = 1.0$, $\omega_f = 0.75$, $M_{w_f} = 1.0$)

Specification description		Criteria	Nominal	Re-Residual-ization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensitivity vector	Mismatch estimation
Maneuver load % change	B	< -30%	-41.9%	-41.9%	-42.2%	-42.0%	-41.9%	-42.3%	-42.2%	-39.9%	-44.0%
Gust load alleviation % change	B	< -30%	-34.6%	-34.7%	-40.8%	-37.4%	-37.2%	-46.6%	-38.1%	-40.0%	-38.1%
	T	< +5%	-30.2%	-30.3%	-21.0%	-35.9%	-33.5%	-38.7%	-32.6%	-26.7%	-28.1%
Handling qualities	ω_{sp}	> 1.6 rad/sec	2.05	2.05	2.10	2.19	2.08	2.38	2.03	3.85	2.09
	ζ_{sp}	0.7-0.8 sec^{-1}	0.755	0.764	0.721	0.717	0.722	0.750	0.728	0.643	0.610
Stability margins Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			24 dB	21.6 dB	25 dB	∞	20.0 dB	∞	∞	16 dB	∞
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	*	∞	∞	∞	∞	1.92 rad (110°)	∞
elevator			∞	∞	∞	∞	∞	3.05 rad (174.5°)	∞	1.71 rad (98°)	2.86 rad (163.6°)
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00027	0.00027	0.0012	0.00012	0.00024	0.00035	0.00034	0.0012	0.00044
	$\dot{\delta}_a$		0.00110	0.00110	0.0047	0.00041	0.00110	0.00140	0.00150	0.0050	0.00130
	δ_e		0.00160	0.00160	0.0014	0.00180	0.00180	0.00230	0.00170	0.0025	0.00180
	$\dot{\delta}_e$		0.00400	0.00410	0.0031	0.00440	0.00410	0.00670	0.00510	0.0083	0.00430
Run number		NA	249	241	234	239	240	243	242	245	247

* Not evaluated

TABLE 28. INSENSITIVE CONTROLLER PERFORMANCE - CASE 4R P3

Evaluation Condition - P3 Evaluation Model - Case 4R

($\bar{q}_f = 1.25$, $\omega_f = 1.0$, $M_{w_f} = 1.0$)

Specification description		Criteria	Nominal	Re-Residual-ization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensitivity vector	Mismatch estimation
Maneuver load % change	B	< -30%	-25.6%	-25.9%	-27.7%	-25.7%	-25.7%	-26.5%	-26.0%	-25.2%	-26.9%
Gust load alleviation % change	B	< -30%	-24.9%	-25.2%	-34.0%	-28.8%	-28.2%	-40.7%	-30.0%	-37.6%	-25.6%
	T	< +5%	-21.8%	-22.1%	-9.99%	-29.7%	-26.0%	-33.2%	-25.4%	-19.7%	-20.2%
Handling qualities	ω_{sp}	> 1.6 rad/sec	2.73	2.72	2.74	2.93	2.76	3.13	2.56	3.69	2.39
	ζ_{sp}	0.7-0.8 sec^{-1}	0.728	0.732	0.715	0.704	0.695	0.705	0.658	0.795	0.834
Stability margins Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			*	*	*	∞	*	∞	∞	∞	∞
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	1.24 rad (71°)	∞	∞	∞	∞	1.99 rad (114°)	∞
elevator			∞	∞	∞	∞	∞	2.93 rad (168°)	∞	1.33 rad (76°)	∞
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00019	0.00020	0.0010	0.00011	0.00018	0.00028	0.00025	0.00085	0.00028
	$\dot{\delta}_a$		0.00095	0.00096	0.0040	0.00035	0.00088	0.00120	0.00120	0.00420	0.00110
	δ_e		0.00150	0.00150	0.0014	0.00180	0.00170	0.00210	0.00170	0.00200	0.00150
	$\dot{\delta}_e$		0.00400	0.00410	0.0032	0.00450	0.00410	0.00670	0.00500	0.00750	0.00440
Run number		NA	249	241	234	239	240	243	242	245	247

* Not evaluated

The data contained in Tables 23 through 28 have been plotted to better illustrate the performance of the insensitive controllers versus the design criteria and the evaluation conditions. These plots are given in Figures 7 through 11.

The effect of unmodeled dynamics on controller performance was evaluated with respect to two considerations. First, only two of the techniques are specifically directed at unmodeled dynamics effects. These are 1) residualization, which was used in the design of the nominal controller, and 2) re-residualization, a newly developed technique. Both techniques attempt to approximate higher order dynamic systems with lower order systems. Of course, the higher order dynamic system must be known. The higher order model that was used for both techniques was the Case 2 model which included 15 bending modes. To generate comparison data, the nominal controller, which was designed on a Case 4R residualized model, and the re-residualized controller, which was designed on a Case 4RR re-residualized model, were evaluated on the Case 2 model. Table 29 presents the results of these evaluations at three evaluation conditions. As can be seen, the performance of the two controllers is almost identical. This indicates that a residualized model was sufficient for design purposes for this aircraft example. Since a comparison of this nature is the true test of the re-residualization process, the re-residualization techniques will not be included in the comparative evaluations that follow.

The second consideration is how well the other insensitive controllers handle unmodeled dynamics since the synthesis technique with which each controller was designed does not explicitly treat the problem. For this evaluation, both known and unknown unmodeled dynamics can be included. Hence, it is necessary to compare the performance of each of the insensitive controllers on the Case 4R models versus their performance on the Case 1 model for the same evaluation condition. Critical response data generated on the Case 1 model at three evaluation conditions are given in Tables 30 through 32. The performance of the insensitive controllers with respect to unmodeled dynamics may be evaluated by comparing the performance of each at the Case 4R nominal condition (Table 23) versus the performance of each at the Case 1 nominal condition (Table 30).

The deviation in performance between Case 1 and Case 4R for each of the insensitive controllers is shown in Figure 12. Plotted is the difference between insensitive controller performance evaluated on Case 1 and insensitive controller performance evaluated on Case 4R for each of the design criteria. As can be seen, all the insensitive controllers behave, in general, much the same as the nominal controller except for the mismatch

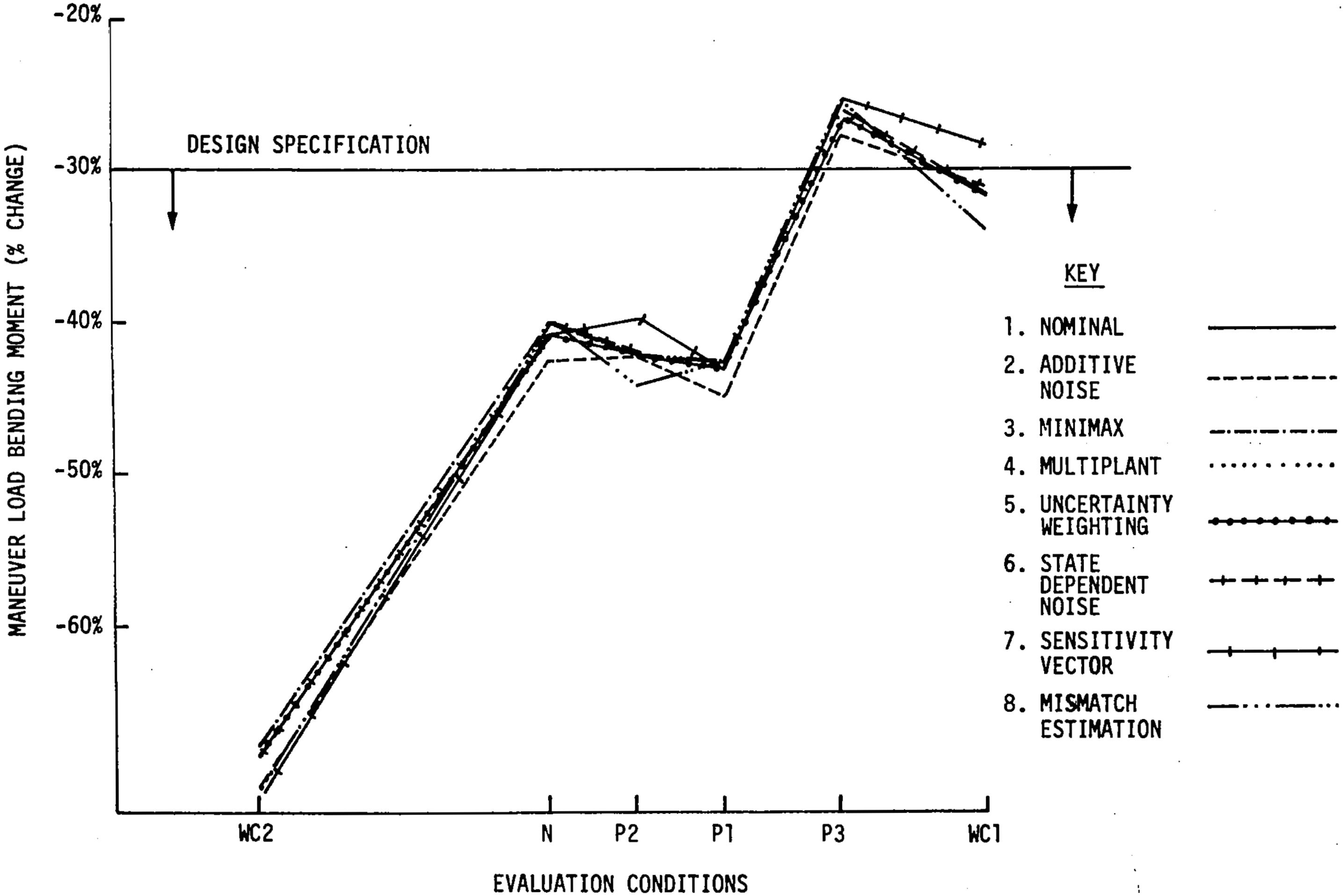


Figure 7. Case 4R Maneuver Load Performance

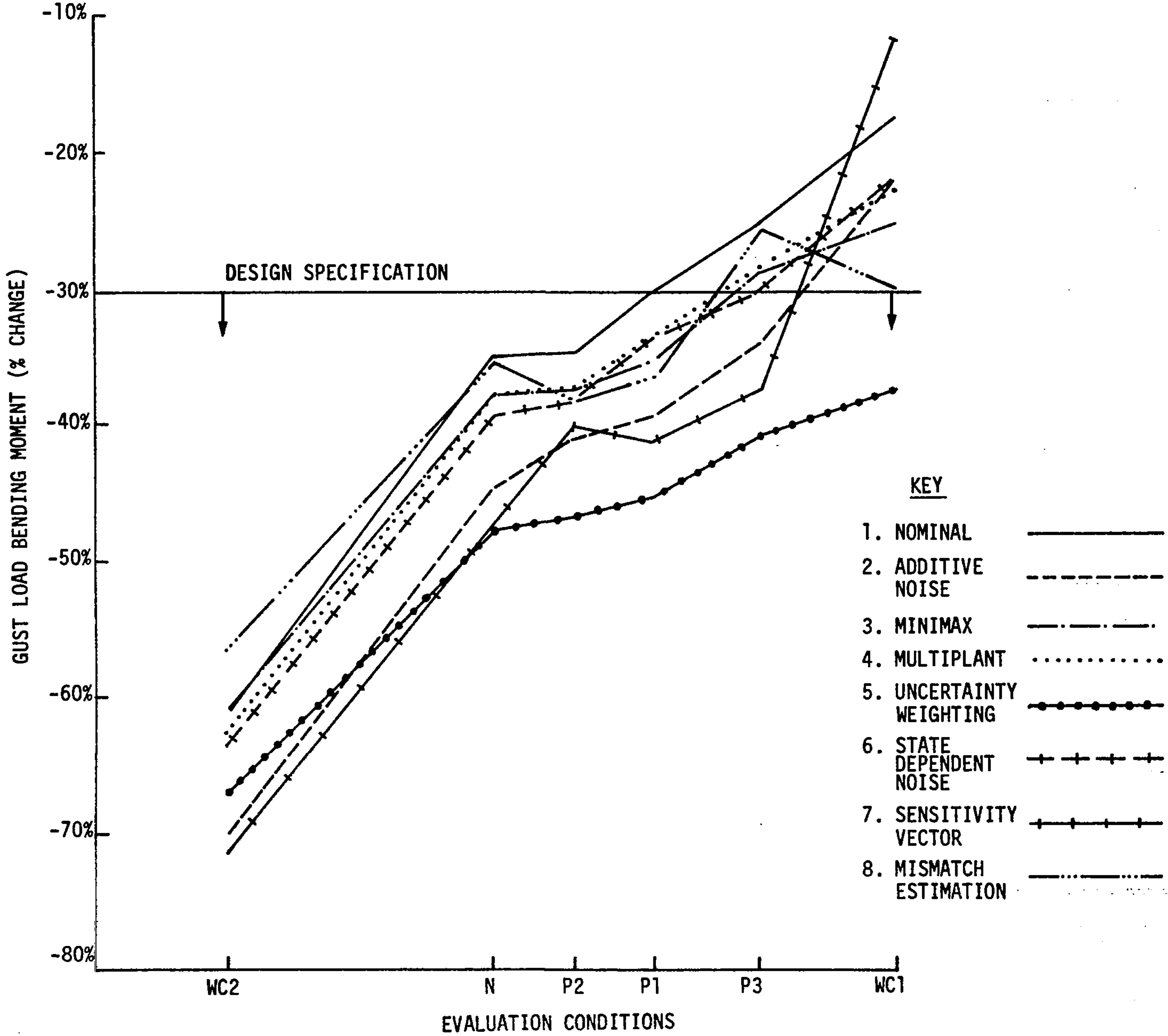


Figure 8. Case 4R Gust Load Performance (Bending Moment)

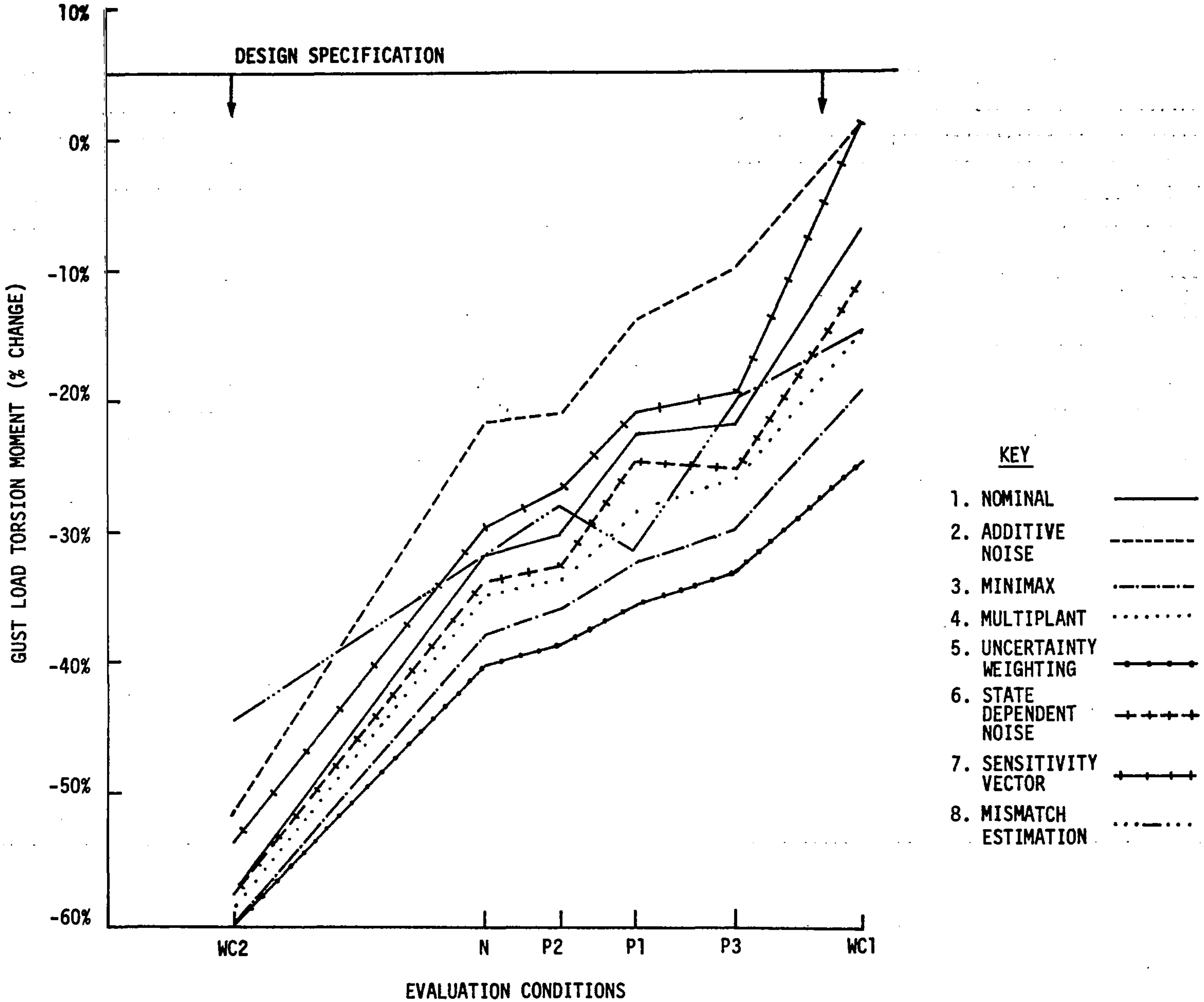


Figure 9. Case 4R Gust Load Performance (Torsion Moment)

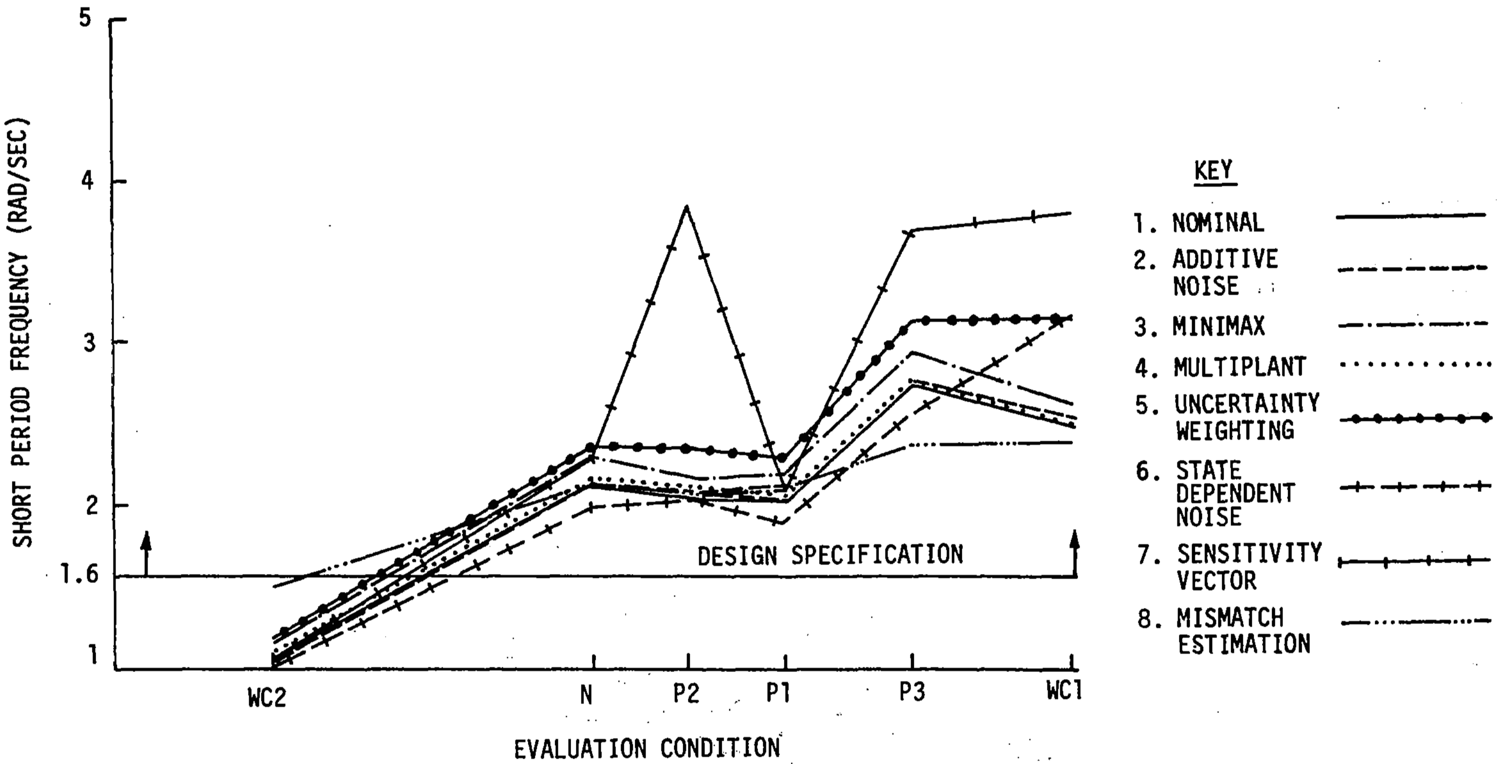


Figure 10. Case 4R - Short Period Frequency

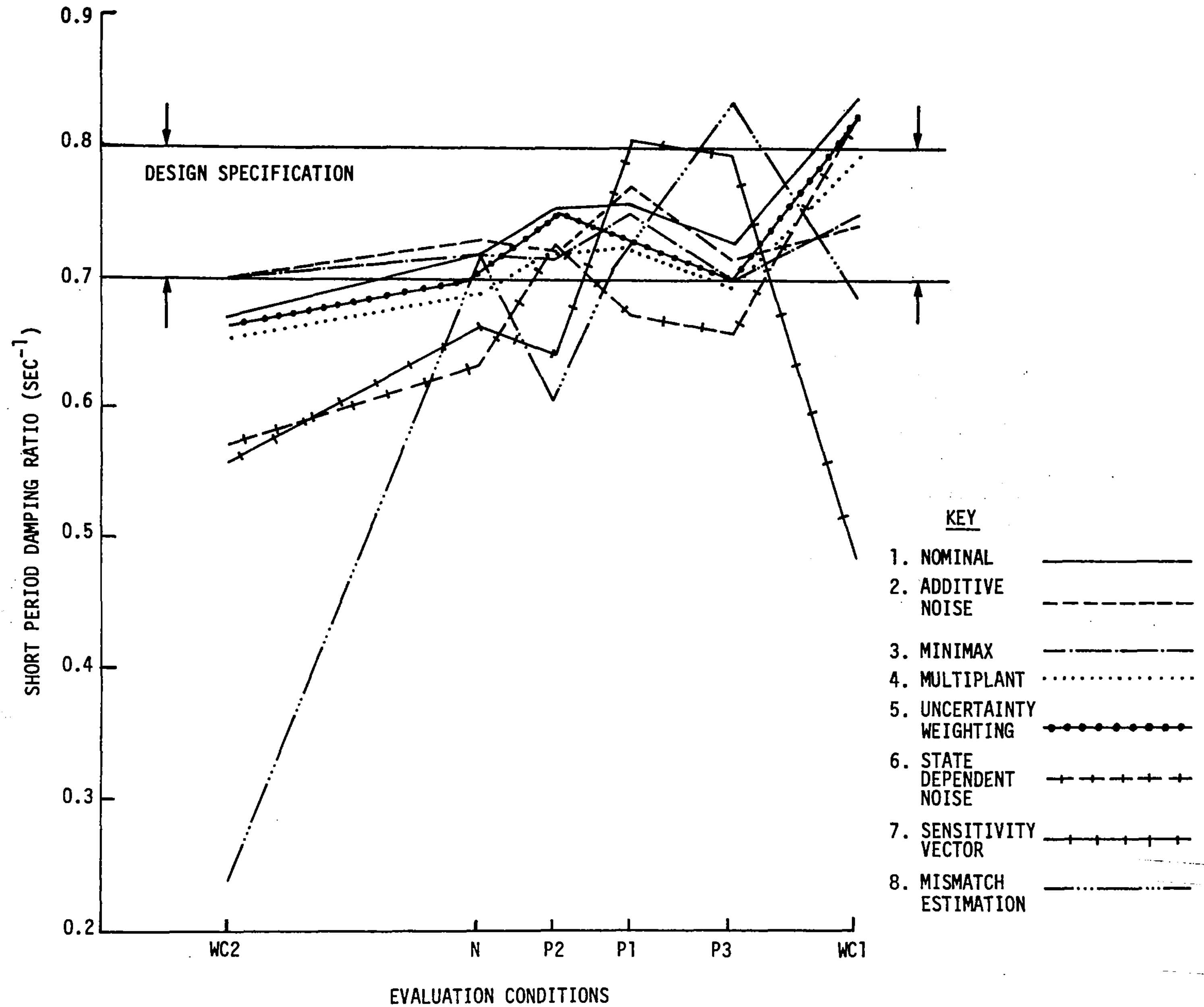


Figure 11. Case 4R Short Period Damping

TABLE 29. CASE 2 EVALUATION

Specification description		Criteria	Nominal ($\bar{q}_f$=1.0, ω_f=1.0, M_{w_f}=1.0) Nominal	Re-Residual-ization	Worst Case 1 ($\bar{q}_f$=1.25, ω_f=0.75, M_{w_f}=0.8) Nominal	Re-Residual-ization	Worst Case 2 ($\bar{q}_f$=0.5, ω_f=1.0, M_{w_f}=1.2) Nominal	Re-Residual-ization
Maneuver load % change	B	< -30%	-40.1%	-40.1%	-31.4%	-31.3%	-68.4%	-68.4%
Gust load alleviation % change	B	< -30%	-34.7%	-35.1%	-15.7%	-15.8%	-61.3%	-61.6%
	T	< +5%	-30.5%	-30.7%	+0.5%	-0.19%	-57.4%	-57.6%
Handling qualities	ω_{sp}	> 1.6 rad/sec	2.13	2.13	2.53	2.55	1.08	1.08
	ζ_{sp}	0.7-0.8 sec^{-1}	0.715	0.717	0.835	0.852	0.670	0.669
Stability margins								
Gain - aileron		≥ 6 dB	∞	*	∞	∞	∞	∞
elevator			*	*	*	20 dB	∞	∞
Phase - aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	∞	∞	∞	∞
elevator			∞	∞	∞	∞	∞	∞
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00018	0.00019	0.00033	0.00034	0.0002	0.00022
	$\dot{\delta}_a$		0.09800	0.09400	0.30000	0.29000	0.0480	0.04600
	δ_e		0.00150	0.00150	0.00180	0.00180	0.0014	0.00140
	$\dot{\delta}_e$		0.00370	0.00380	0.00460	0.00470	0.0030	0.00310
Run number		NA	253	255	254	255	254	255

* Not evaluated

TABLE 30. INSENSITIVE CONTROLLER PERFORMANCE CASE 1 NOMINAL

Evaluation Condition - Nominal Evaluation Model - Case 1

($\bar{q}_f = 1.0$, $\omega_f = 1.0$, $M_{w_f} = 1.0$)

Specification description		Criteria	Nominal	Re-Residual-ization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensitivity vector	Mismatch estimation
Maneuver load % change	B	< - 30%	-40.2%	-40.2%	-43.4%	-39.8%	-40.0%	-40.9%	-40.5%	-41.1%	-39.9%
Gust load alleviation % change	B	< -30%	-39.8%	-40.0%	-49.8%	-41.3%	-42.2%	-51.4%	-45.2%	-53.6%	-35.0%
	T	< +5%	-32.3%	-32.6%	-19.9%	-38.5%	-35.4%	-39.9%	-35.3%	-29.6%	-28.4%
Handling qualities	ω_{sp}	>1.6 rad/sec	2.41	2.41	2.37	2.59	2.44	2.83	2.25	3.37	2.04
	ζ_{sp}	0.7-0.8 sec^{-1}	0.730	0.736	0.737	0.720	0.695	0.707	0.636	0.657	0.714
Stability margins											
Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			35.8 dB	30.3 dB	39.1 dB	∞	26 dB	∞	∞	16.8 dB	20.0 dB
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	1.80 rad (103°)	∞	∞	∞	∞	1.77 rad (101.7°)	∞
elevator			∞	∞	∞	1.87 rad (107.2)	∞	1.82 rad (104°)	1.62 rad (92.7°)	1.30 rad (74.4°)	2.11 rad (121°)
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00017	0.00018	0.0010	0.00011	0.00017	0.00027	0.00022	0.00070	0.00019
	$\dot{\delta}_a$		0.00074	0.00075	0.0032	0.00030	0.00069	0.00099	0.00095	0.00310	0.00073
	δ_e		0.00150	0.00150	0.0014	0.00170	0.00160	0.00210	0.00170	0.00200	0.00120
	$\dot{\delta}_e$		0.00370	0.00380	0.0031	0.00440	0.00380	0.00640	0.00480	0.00680	0.00330
Run number		NA	256	3L	2L	5L	6L	7L	4L	8L	9L

TABLE 31. INSENSITIVE CONTROLLER PERFORMANCE - CASE 1 WC1

Evaluation Condition - Worst Case 1 Evaluation Model - Case 1

($\bar{q}_f = 1.25$, $\omega_f = 0.75$, $M_{w_f} = 0.8$)

Specification description		Criteria	Nominal	Re-Residual-ization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensitivity vector	Mismatch estimation
Maneuver load % change	B	< -30%	-33.2%	-33.2%	-32.7%	-33.6%	-33.1%	-33.7%	-33.7%	-29.1%	-35.8%
Gust load alleviation % change	B	< -30%	-22.2%	-22.2%	-27.3%	-28.1%	-27.3%	-39.7%	-29.1%	-21.7%	-28.6%
	T	< +5%	-6.5%	-6.9%	+7.7%	-17.1%	-13.1%	-19.2%	-12.5%	+3.3%	-4.5%
Handling qualities	ω_{sp}	>1.6 rad/sec	3.77	3.91	2.78	3.11	3.41	4.35	3.56	4.22	2.35
	ζ_{sp}	0.7-0.8 sec^{-1}	0.807	0.801	0.713	0.744	0.810	0.645	0.562	0.482	0.667
Stability margins Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	11.4	∞
elevator			19.2 dB	17.3 dB	20 dB	31.2 dB	16.7 dB	22.4 dB	18.6 dB	4.5 dB	∞
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	2.09 rad (120.0°)	∞	∞	∞	∞	1.49 rad (85.2°)	∞
elevator			∞	∞	∞	1.88 rad (107.7°)	∞	1.66 rad (95.1°)	1.52 rad (87.2°)	1.40 rad (80.0°)	1.91 rad (109.2°)
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00030	0.00031	0.0014	0.00012	0.00028	0.00037	0.00037	0.0014	0.00057
	$\dot{\delta}_a$		0.00130	0.00130	0.0054	0.00046	0.00120	0.00160	0.00170	0.0060	0.00170
	δ_e		0.00180	0.00180	0.0016	0.00200	0.00190	0.00250	0.00200	0.0036	0.00180
	$\dot{\delta}_e$		0.00460	0.00470	0.0034	0.00470	0.00460	0.00750	0.00590	0.0120	0.00440
Run number		NA	1L	3L	2L	5L	6L	7L	4L	263	262

TABLE 32. INSENSITIVE CONTROLLER PERFORMANCE CASE 1 WC2

Evaluation Condition - Worst Case 2 Evaluation Model - Case 1

($\bar{q}_f = 0.5$, $\omega_f = 1.0$, $M_{w_f} = 1.2$)

Specification description		Criteria	Nominal	Re-Residual-ization	Add noise	Minimax	Multi-plant	Uncertainty weighting	State dependent noise	Sensitivity vector	Mismatch estimation
Maneuver load % change	B	< -30%	-66.3%	-66.3%	-69.2%	-65.9%	-66.2%	-66.7%	-66.4%	-68.6%	-65.5%
Gust load alleviation % change	B	< -30%	-68.2%	-68.4%	-76.2%	-66.7%	-68.8%	-72.6%	-71.4%	-81.0%	-64.7%
	T	< +5%	-57.3%	-57.3%	-50.0%	-59.9%	-58.4%	-59.8%	-58.0%	-53.4%	-51.9%
Handling qualities	ω_{sp}	>1.6 rad/sec	1.16	1.16	1.13	1.25	1.19	1.28	1.09	1.16	1.51
	ζ_{sp}	0.7-0.8 sec^{-1}	0.681	0.680	0.718	0.714	0.660	0.674	0.570	0.568	0.336
Stability margins Gain: aileron		≥ 6 dB	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			∞	∞	∞	∞	41.9 dB	∞	∞	∞	4.3 dB
Phase: aileron		≥ 0.7854 rad ≥ (45°)	∞	∞	∞	∞	∞	∞	∞	∞	∞
elevator			∞	∞	∞	2.71 rad (155.2°)	∞	1.85 rad (105.8°)	1.77 rad (101.7°)	1.30 rad (78.2°)	∞
Surface activity RMS (rad, rad/sec)	δ_a	NA	0.00019	0.00021	0.0012	0.00013	0.00020	0.00031	0.00024	0.00085	0.00045
	$\dot{\delta}_a$		0.00053	0.00056	0.0026	0.00024	0.00052	0.00078	0.00068	0.00330	0.01000
	δ_e		0.00140	0.00140	0.0013	0.00170	0.00150	0.00200	0.00150	0.00180	0.00330
	$\dot{\delta}_e$		0.00300	0.00310	0.0025	0.00380	0.00300	0.00560	0.00410	0.00540	1.44000
Run number		NA	1L	3L	2L	5L	6L	7L	4L	263	262

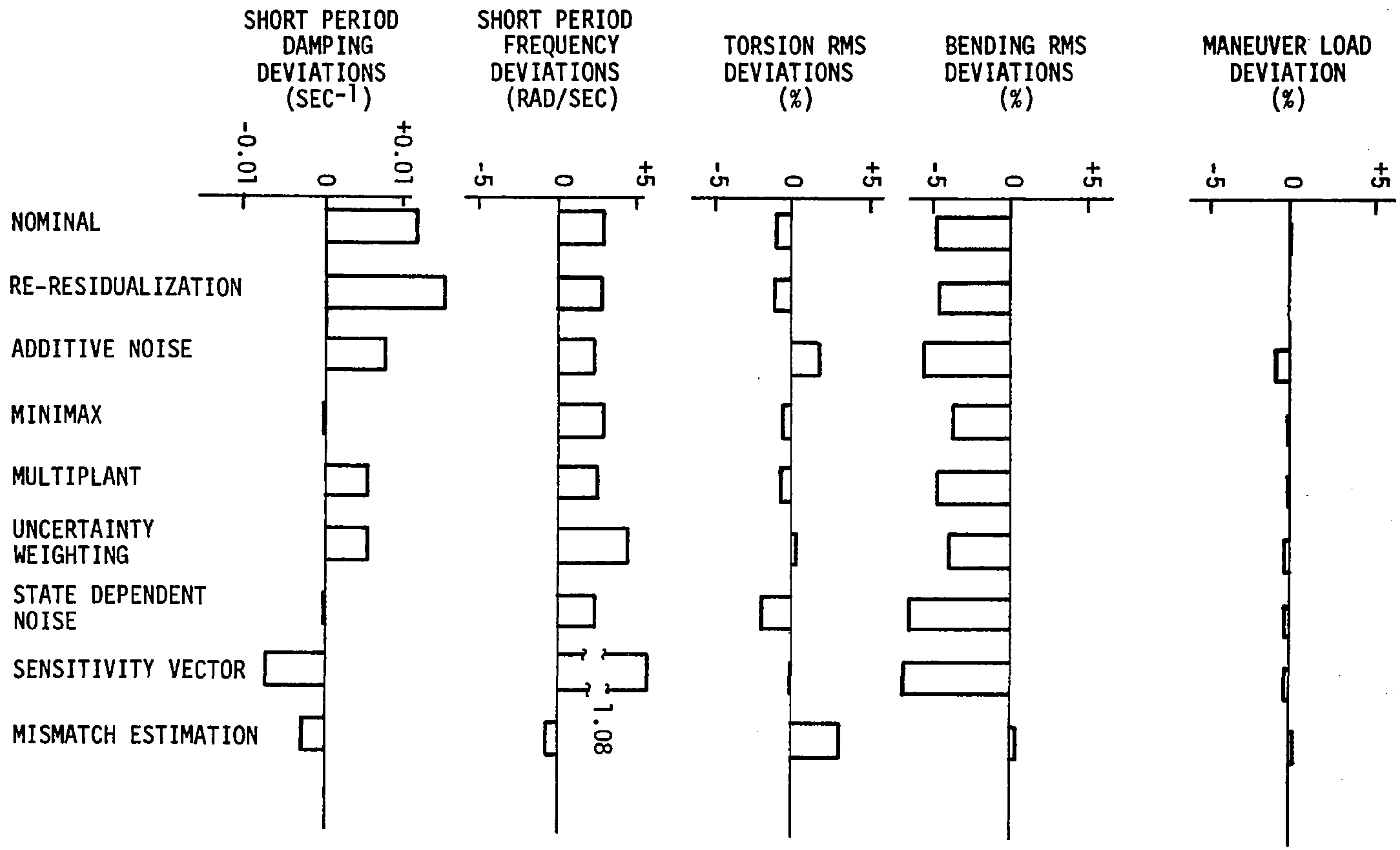


Figure 12. Effect of Unmodeled Dynamics

estimation controller. In all cases, the mismatch estimation deviation shows a sign reversal in deviation when compared to most of the other insensitive controllers. This is especially noticeable in the RMS bending deviation plot. The conclusion that can be drawn from this data is that the mismatch estimation controller is attempting to compensate for the unmodeled dynamics while the others are merely reacting.

Tables 30 through 32 also provide the necessary data for evaluating the third consideration mentioned above, i.e., the effect of parameter uncertainties and unmodeled dynamics on controller performance. Only three of the evaluation conditions were retained for Case 1 evaluation primarily because of the large computer costs accompanying the data generation runs. In addition, it was felt that the same characteristic trends that were observed in Case 1 at the nominal, Worst Case 1, and Worst Case 2 conditions versus their Case 4R counterparts would have been repeated with respect to the other conditions. Plots of Tables 30 through 32 data are given in Figures 13 through 17.

Tables 23 through 32 provide the necessary data for a comparative evaluation of insensitive controller performance. Before proceeding to the next subsection we should point out that there was no attempt to include stability margin criteria in the evaluation. An analysis of Tables 23 through 32 indicates that for all controllers at all conditions except 1 (mismatch estimation controller, Case 1, WC2) gain and phase margin criteria are more than satisfied. It was concluded, therefore, that for this example satisfying gain and phase margin criteria was not a critical design consideration.

COMPARISON PROCEDURE AND RESULTS

There are several methods that may be used to quantitatively compare the insensitive controllers. Two methods that historically have been used are the performance index and trajectory sensitivities. We introduce a normalized performance measure which encompasses the performance index method. We also introduce a normalized range measure which is akin to the trajectory sensitivity method. A combination of these two measures is also introduced. Two other methods for comparison will be introduced which relate more closely to the objective of insensitive control stated in Section III as "maximizing performance over a given type and range of model variation." One of these methods uses a coarse overall relative scoring system. The other method is based on normalized specification violation.

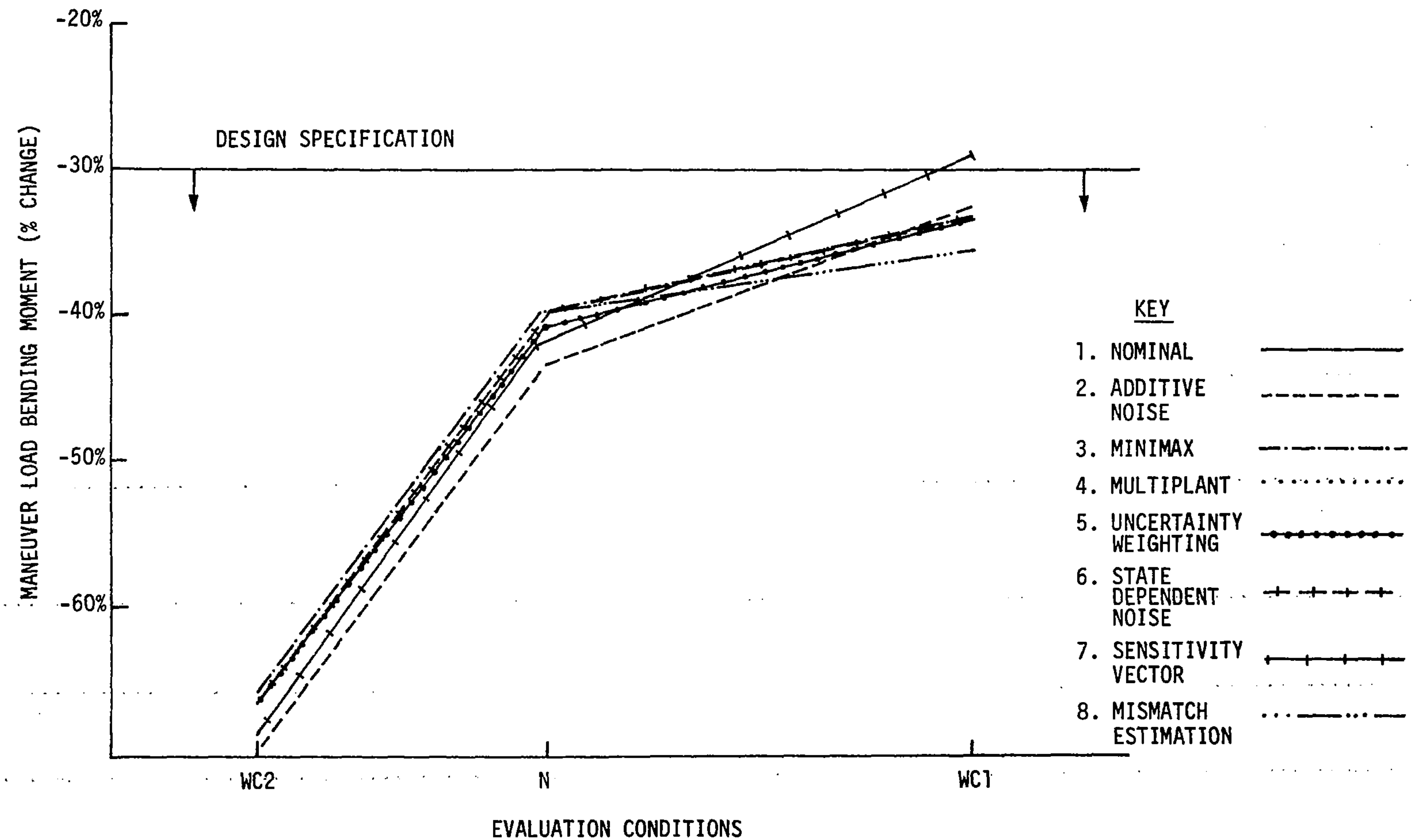


Figure 13. Case 1 Maneuver Load Performance

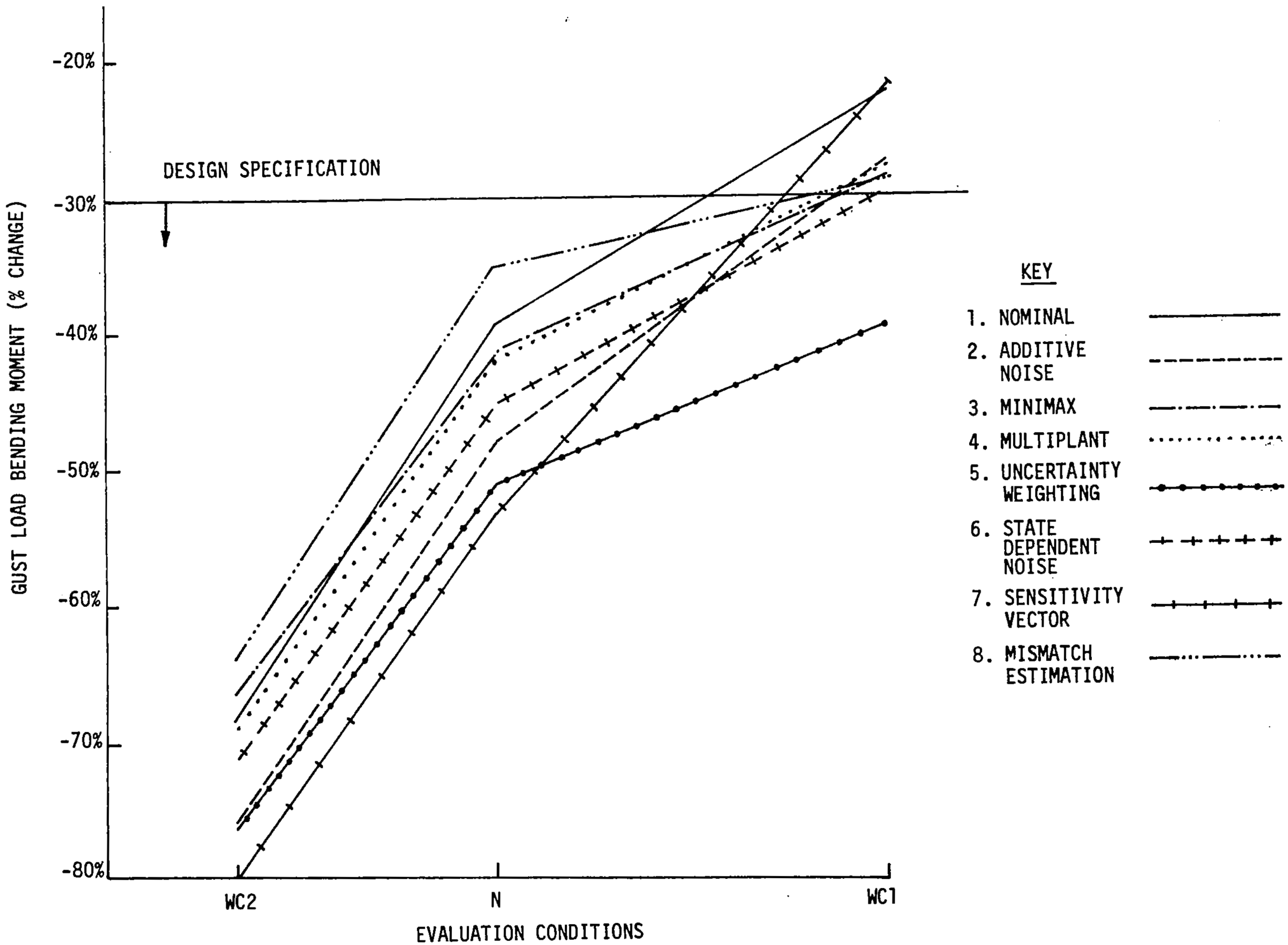


Figure 14. Case 1 Gust Load Performance (Bending)

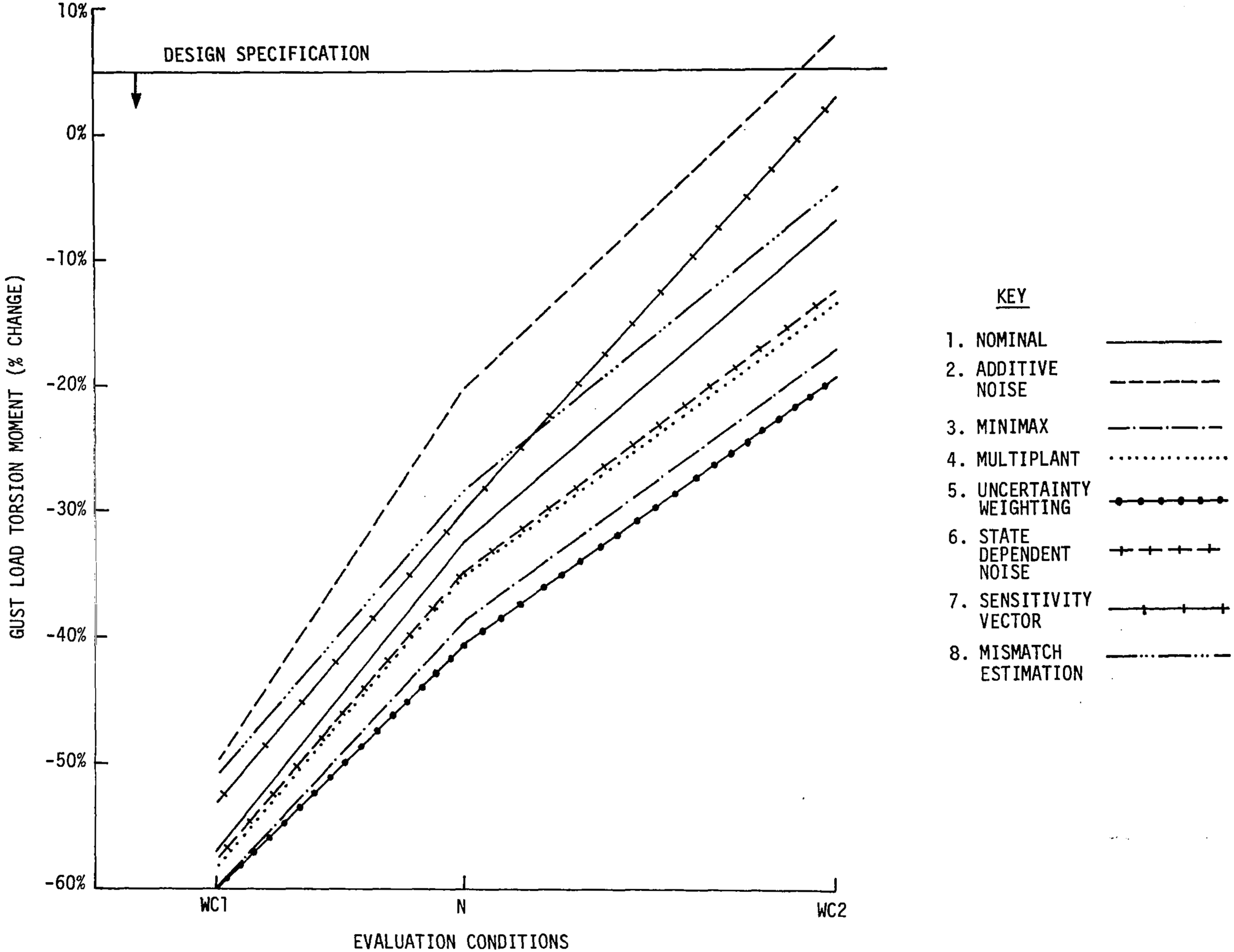


Figure 15. Case 1 Gust Load Performance (Torsion)

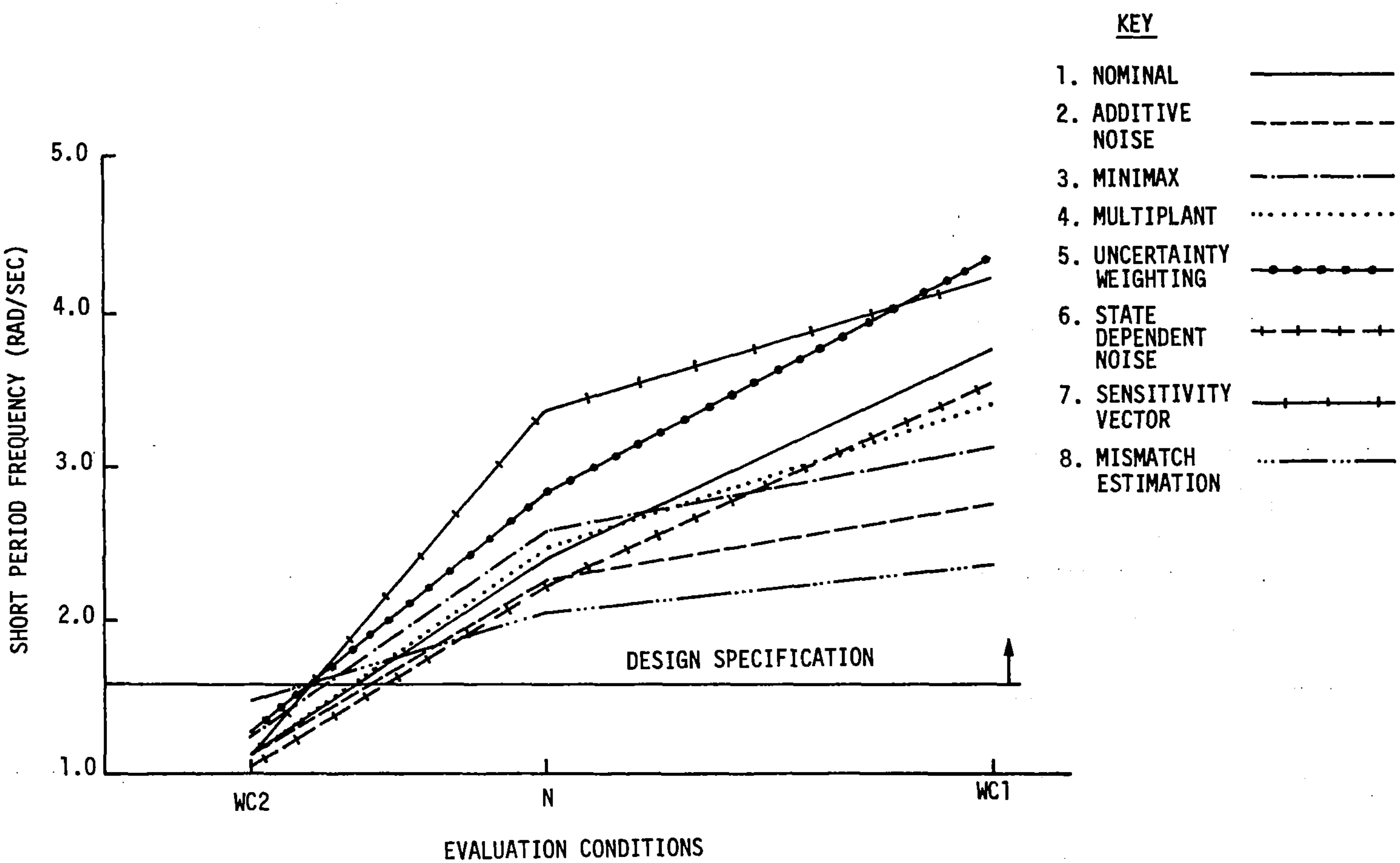


Figure 16. Case 1 Short Period Frequency

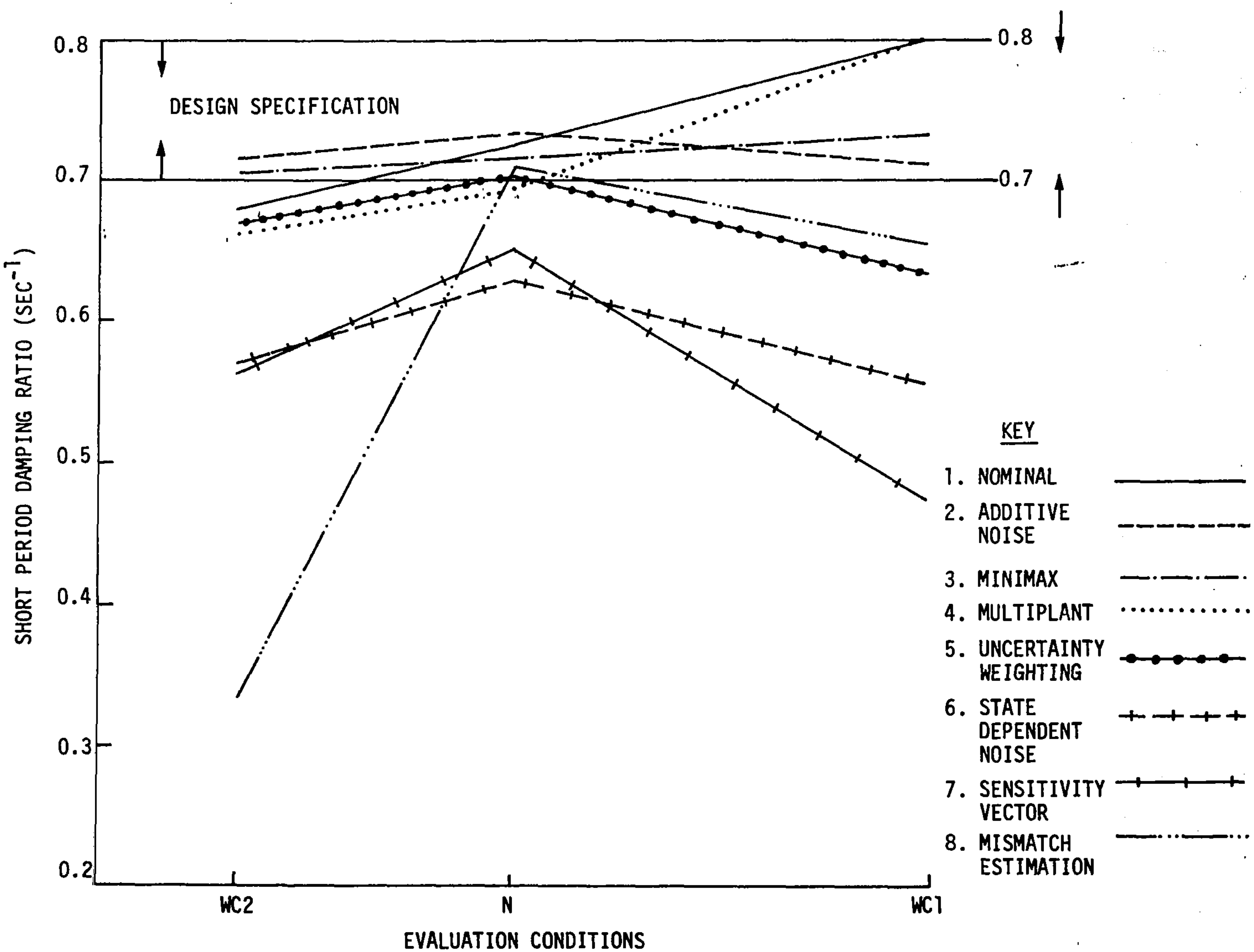


Figure 17. Case 1 Short Period Damping

Overall Relative Scoring

The score given an insensitive controller is based on its performance relative to the performance of the nominal controller with respect to each specification. The score is assigned as follows. For each of the criteria (maneuver load bending, RMS bending, RMS torsion, short period damping, and short period frequency), the insensitive controller is given a base score of

+2 if the nominal controller is out of spec and the insensitive controller is in spec.

+1 if the nominal controller is out of spec and the insensitive controller is out of spec, but not as far out as the nominal.

0 if the nominal controller and insensitive controller are both in spec, or both out of spec by the same amount.

-1 if the nominal controller is out of spec and the insensitive controller is further out of spec.

-2 if the nominal controller is in spec and the insensitive controller is out of spec.

The raw score for each insensitive controller is the sum of the base scores for all conditions considered. An "ideal" controller would receive a base score of (+2) whenever the nominal controller is out of spec. The overall relative score for an insensitive controller is then defined to be

$$S(IC) = \frac{\text{raw score of "ideal" - raw score of insensitive controller}}{\text{raw score of "ideal" - raw score of nominal controller}}$$

Thus a score less (more) than 1 indicates less (more) sensitivity than the nominal. The resulting overall relative scoring of the insensitive controllers is shown in Figure 18 based on the evaluations with Case 1 and Case 4R models. The results for the Case 4R model using the same conditions as used for Case 1 are shown with shaded bars. The striped bars indicate Case 4R results using all six evaluation conditions. Also shown in Figure 18 is the effect of adding a fictitious spec on RMS control surface activity. The hypothesized spec is three times the nominal controller surface activity at the nominal condition for Case 4R. Adding this criteria causes the overall relative scores of the additive noise, mismatch estimation, and sensitivity vector controllers to increase leaving the remainder unchanged. This would cause the additive noise controller to shift significantly in the ranking.

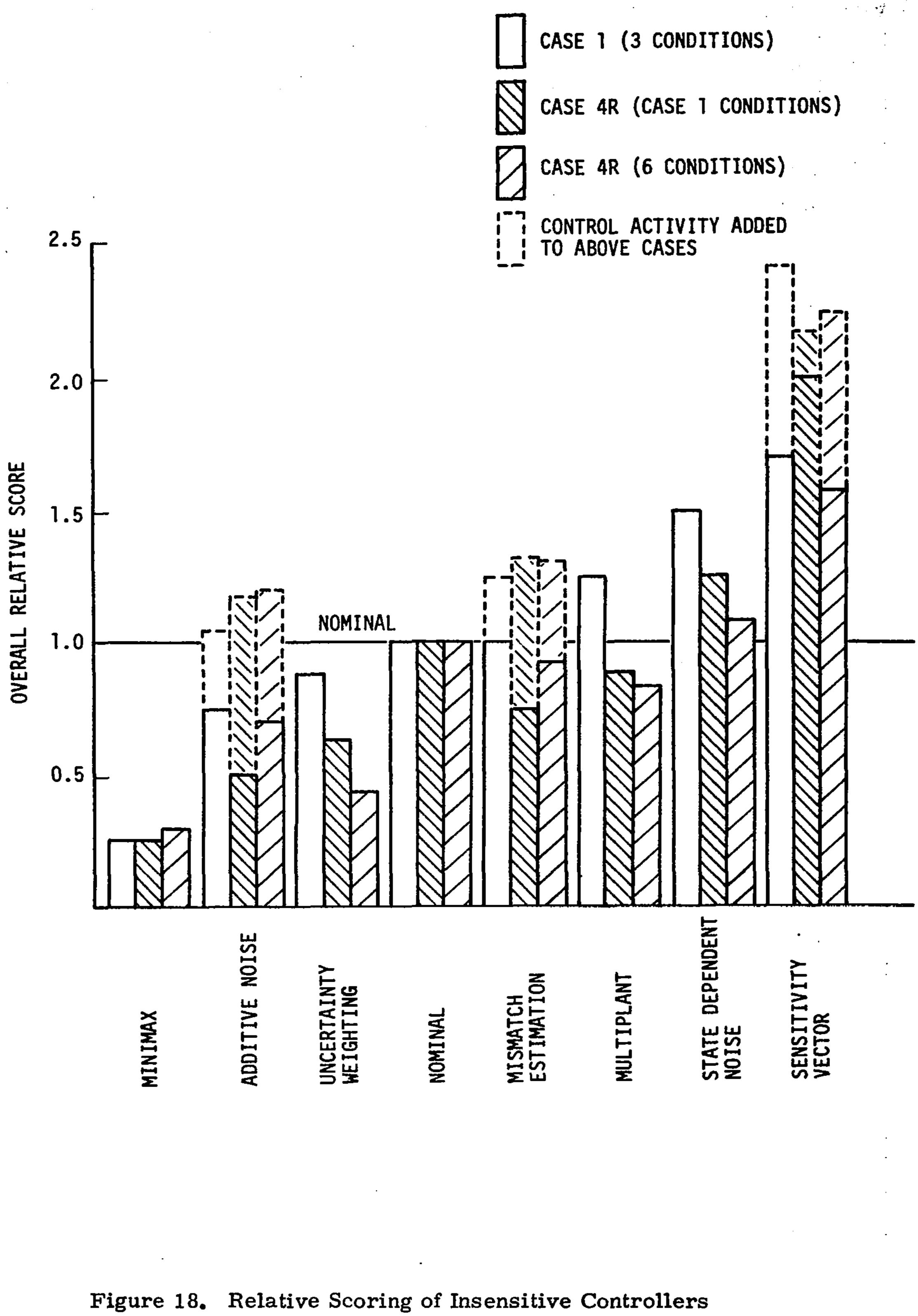


Figure 18. Relative Scoring of Insensitive Controllers

Major conclusions to be drawn from this relative overall scoring as indicated by Figure 18 are:

1. The minimax and uncertainty weighting controllers are less sensitive than the nominal in all instances.

2. The state dependent noise and sensitivity vector controllers are more sensitive than the nominal in all instances.

3. The effect of unmodeled dynamics (Case 1 versus Case 4R with the Case 1 conditions) is least for the minimax controller.

4. The ranking of controllers for Case 1 and Case 4R is generally consistent, with mismatch estimation and multiplant being the exceptions.

5. The fictitious surface activity criteria most seriously degrade the ranking of the additive noise controller.

Normalized Performance and Normalized Range

The insensitive controllers may be compared on the basis of normalized performance which is similar to the method of performance index sensitivity. Comparison may also be made on the basis of normalized range which is similar to the method of trajectory sensitivity.

The normalized performance of an insensitive controller with respect to each individual criterion is defined as follows:

The normalized performance of an insensitive controller with respect to maneuver load bending is the average of the percent reduction in maneuver load bending for the nominal controller divided by the average percent reduction in maneuver load for the insensitive controller. The average is taken over the conditions considered. With this definition, a value less than 1 corresponds to improvement in performance relative to the nominal controller and a value greater than one indicates degraded performance.

The normalized performance with respect to RMS bending is defined in the same manner as used for maneuver load bending with averages of percent reduction in RMS bending instead of percent reduction of maneuver load bending.

The normalized performance with respect to RMS torsion is defined similar to that for RMS bending except that the torsion averages are biased by 40 percent to achieve similar scales. Thus, the normalized torsion performance is the ratio

$$\frac{-40 + (\text{average percent reduction in RMS torsion for nominal controller})}{-40 + (\text{average percent reduction in RMS torsion for insensitive controller})}$$

The normalized performance with respect to short period frequency, ω, required a somewhat different definition. Whereas increasing the percent reduction in bending and torsion indicates improvement, increasing ω indefinitely is not an improvement. Thus, the normalized frequency performance is defined in terms of spec violation as the ratio

$$\frac{\text{spec violation of the insensitive controller}}{\text{spec violation of the nominal controller}}$$

$$= \frac{\sum_{\omega_i < 1.6} (1.6 - \omega_i) \text{ for the insensitive controller}}{\sum_{\omega_i < 1.6} (1.6 - \omega_i) \text{ for the nominal controller}}$$

where ω_i indicates the frequency for the i^{th} condition.

The normalized performance with respect to short period damping, ζ, was also defined in terms of spec violation. In this instance it was decided not to normalize by dividing the spec violation of the insensitive controller by the spec violation of the nominal controller because doing so would have made the range of this measure of performance much larger than the range of the previous measures. It was decided instead to choose the normalization such that, for Case 1, the value of the normalized damping performance attained by the insensitive controller with the greatest spec violation was approximately equal to 2. This was accomplished by defining the normalized performance with respect to short period damping as

$$5 \left[\sum_{\zeta_i > 0.8} (\zeta_i - 0.8) + \sum_{\zeta_j < 0.7} (0.7 - \zeta_i) \right]$$

for each controller. The normalized damping performance of the nominal controller for Case 1 based on this definition has a value of 0.13.

The normalized performance of the controllers is defined as the RSS of the individual normalized performance measures; that is, the normalized performance is the square root of one-fifth of the sum of the squares of the five performance measures just defined. A different normalization procedure would have the effect of changing the relative weighting in summing the squares. Unfortunately, there is not an obviously "correct" weighting of the performances with respect to the various criteria.

Again, as in the overall relative scoring it is of interest to include control surface activity in evaluating comparative performance. The normalized performance with respect to surface deflections is defined to be the ratio of the average RMS deflections for the insensitive controller to the average RMS deflections for the nominal controller.

The normalized range of an insensitive controller with respect to each of the variables corresponding to a specific criterion is the range of the insensitive controller divided by the range for the nominal controller. The range is defined as the maximum of the variable minus the minimum of the variable. The normalized range of the insensitive controller is the RSS of the individual normalized ranges. With these definitions the normalized range (individual and total) of the nominal controller is unity. A value less (more) than 1 for an insensitive controller indicates less (more) sensitivity than the nominal.

The normalized performance and range with respect to the individual criteria are shown for Case 1 in Figure 19. Perfect performance with respect to bending and torsion is represented. This corresponds to a 100 percent reduction for all conditions. The spec requirement is also shown for bending and torsion which corresponds to a controller that just meets spec for all conditions. Spec performance and perfect performance with respect to short period frequency and damping coincide and correspond to a value of zero.

Several conclusions may be drawn from the data displayed in Figure 19. First, there is little difference between the controllers with respect to maneuver load bending. Second, that is about the only instance of consistency. The mismatch estimation controller is the least sensitive with respect to short period frequency but is the most sensitive with respect to short period damping. The normalized range is sometimes larger and sometimes smaller than the normalized performance. Finally, the surface activity performance clearly indicates that the additive noise controller utilizes the aileron significantly more than do the other controllers.

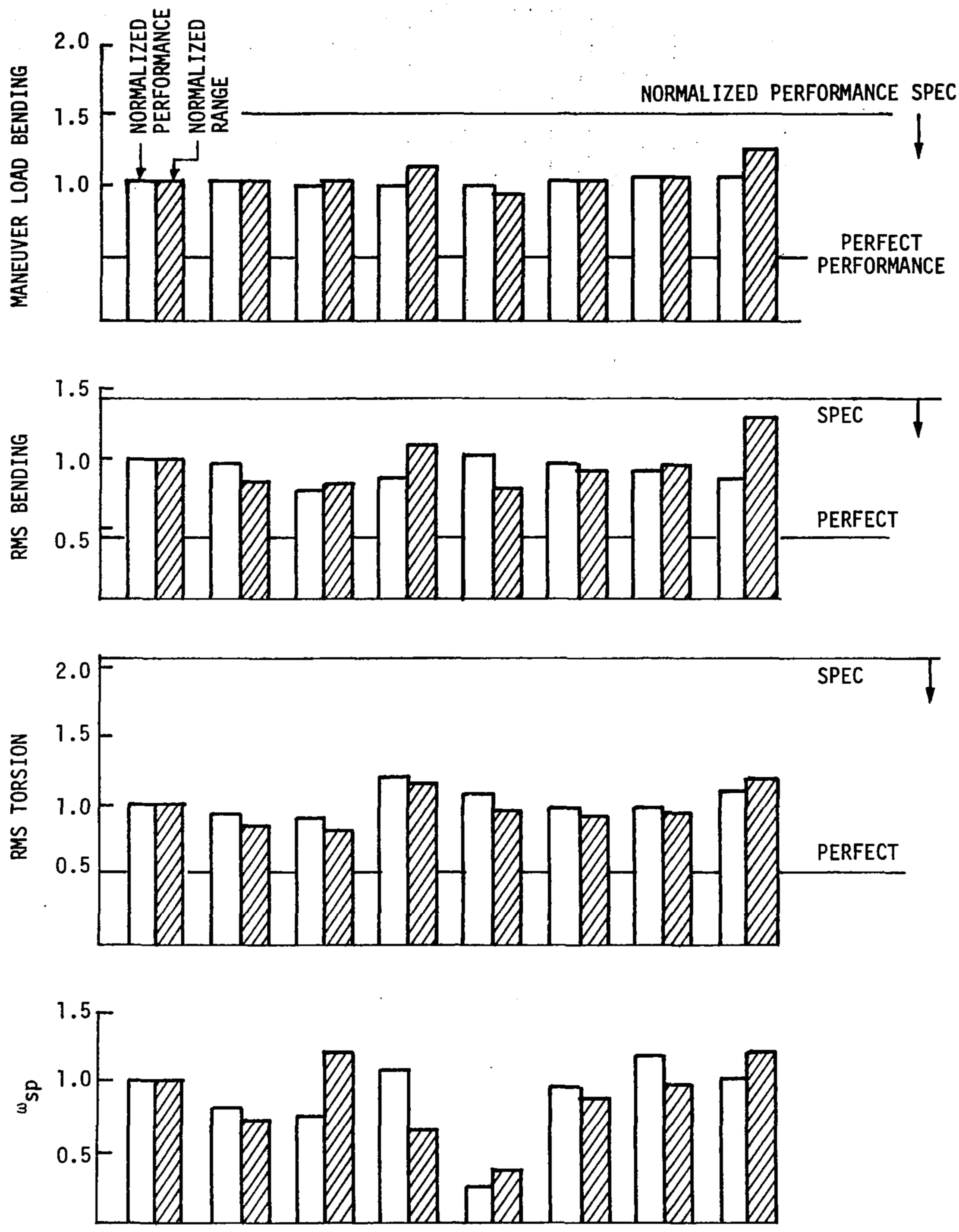


Figure 19. Normalized Performance and Range with Respect to Individual Criterion for Case 1

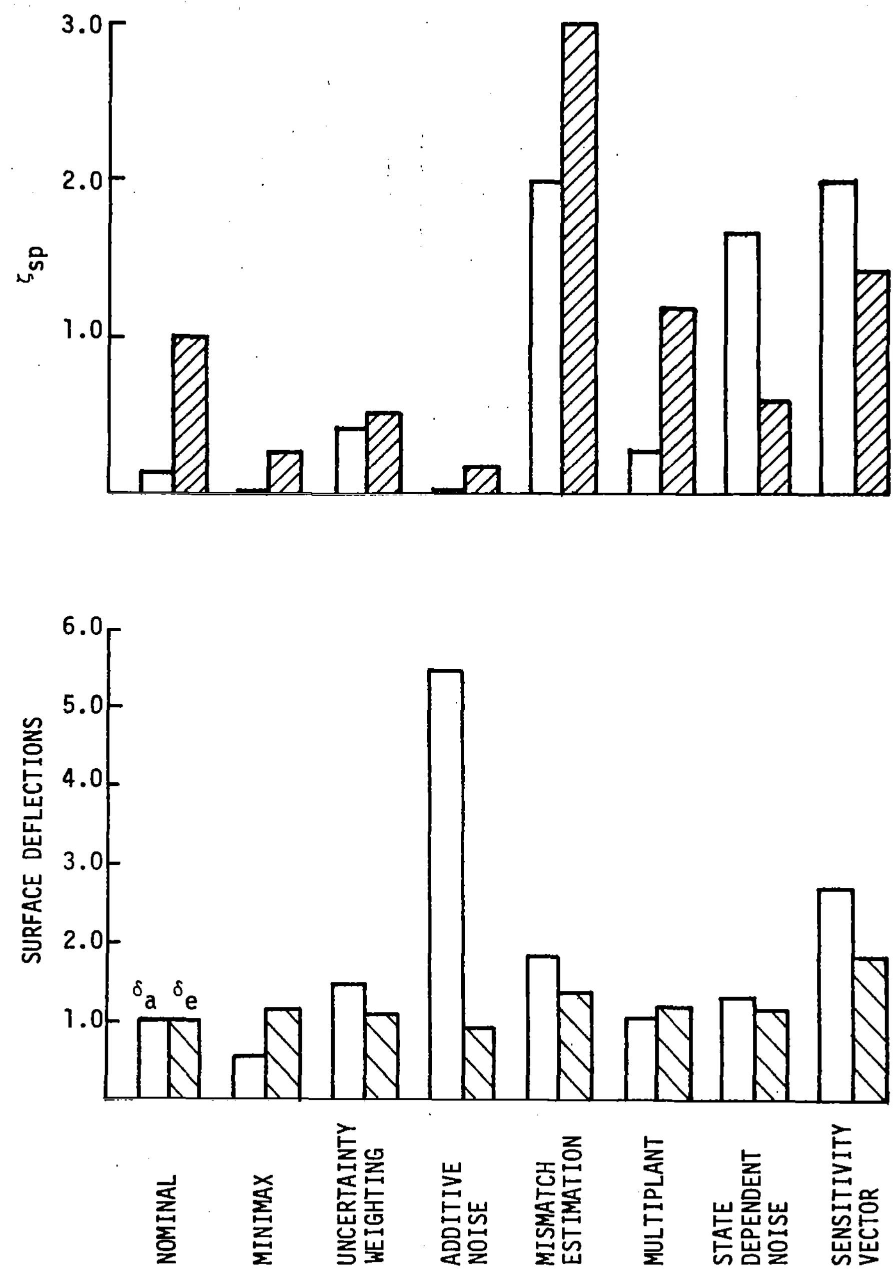


Figure 19. Concluded

The data shown in Figure 19, combined via the RSS process, is shown in Figure 20. Also the performance and range are combined in the same RSS manner to arrive at a single measure. With respect to these measures, the minimax controller is generally less sensitive than the nominal, and the mismatch estimation and sensitivity vector controllers are generally more sensitive. The state dependent noise controller displays much less range sensitivity than performance sensitivity. The surface activity increment is most notable for the additive noise controller, which is less sensitive than the nominal without control surface contribution but more sensitive than the nominal with control surface activity considered. The multiplant and uncertainty weighting controllers are slightly less sensitive than the nominal, and without control surface activity, the uncertainty weighting controller is less sensitive than the multiplant controller.

Figure 21 shows the same kind of comparison for the Case 4R evaluations on the Case 1 conditions. The results are very similar to the Case 1 results except that the control surface contribution is more pronounced for Case 4R, especially for the uncertainty weighting, additive noise, and sensitivity vector controllers. The same measures are shown in Figure 22 for Case 4R with all six evaluation conditions included. Comparing Figures 21 and 22 shows that the added evaluation conditions have a negligible effect in these measures of insensitivity.

Normalized Specification Violation

The final method of comparison is based on a measure of normalized spec violation. This measure is a refinement of the overall relative scoring. The total amount of spec violation with respect to a given criterion is computed for each controller for all conditions considered. This total is normalized by dividing by the maximum attained to give the normalized spec violation with respect to the individual criterion. An overall normalized spec violation is then the RSS of the normalized individual spec violations. In Case 4R none of the controllers violated the torsion spec, so this individual component was omitted in the RSS calculations. In Case 1 the additive noise controller slightly violated the torsion spec for one condition by 0.9 percent. Thus, RSS values for Case 1 were computed with and without the torsion contribution. The results are shown in Figure 23. The uncertainty weighting and minimax controller were the least sensitive, and the sensitivity vector controller was the most sensitive with respect to this measure.

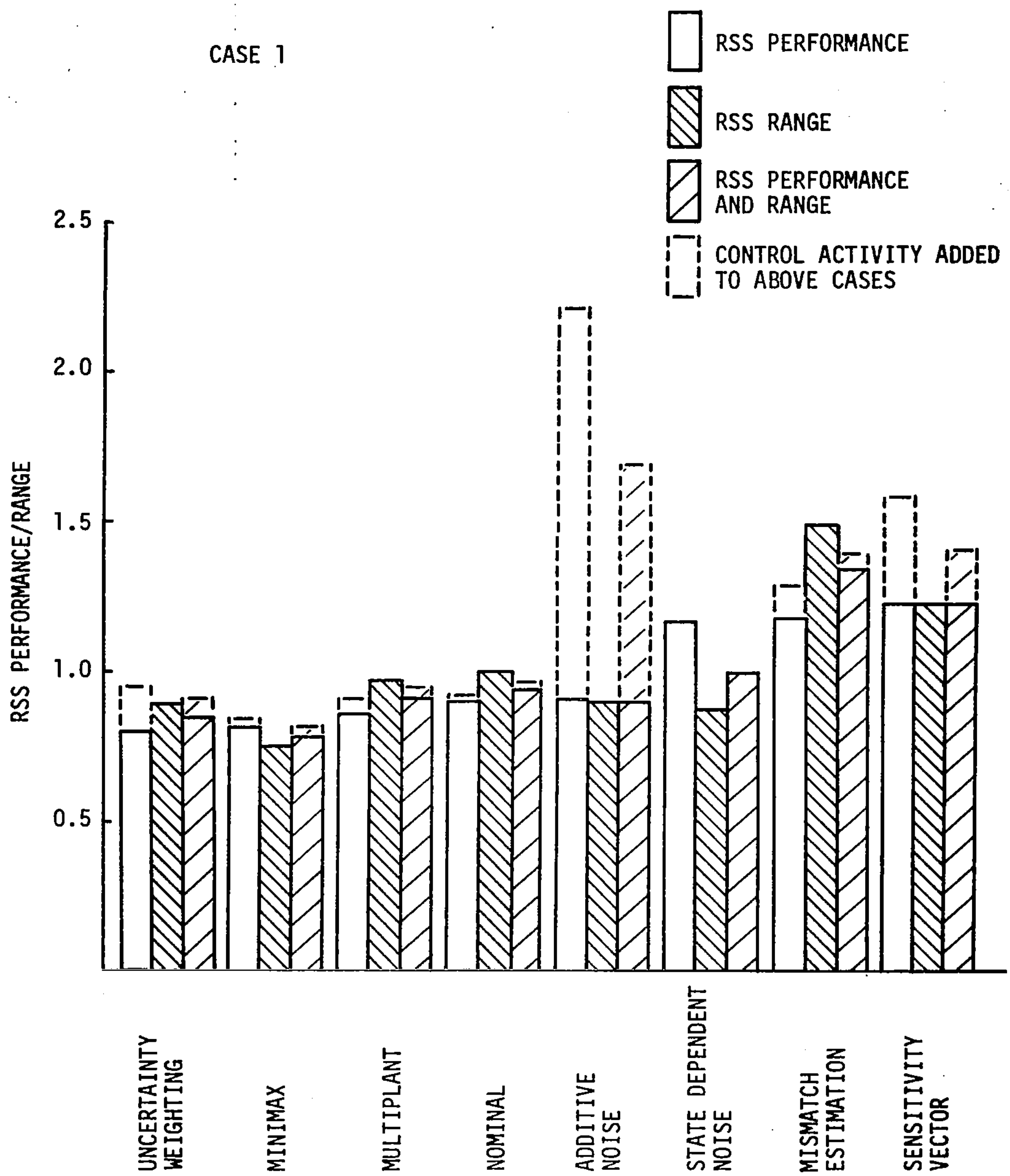


Figure 20. Normalized Performance and Range for Case 1

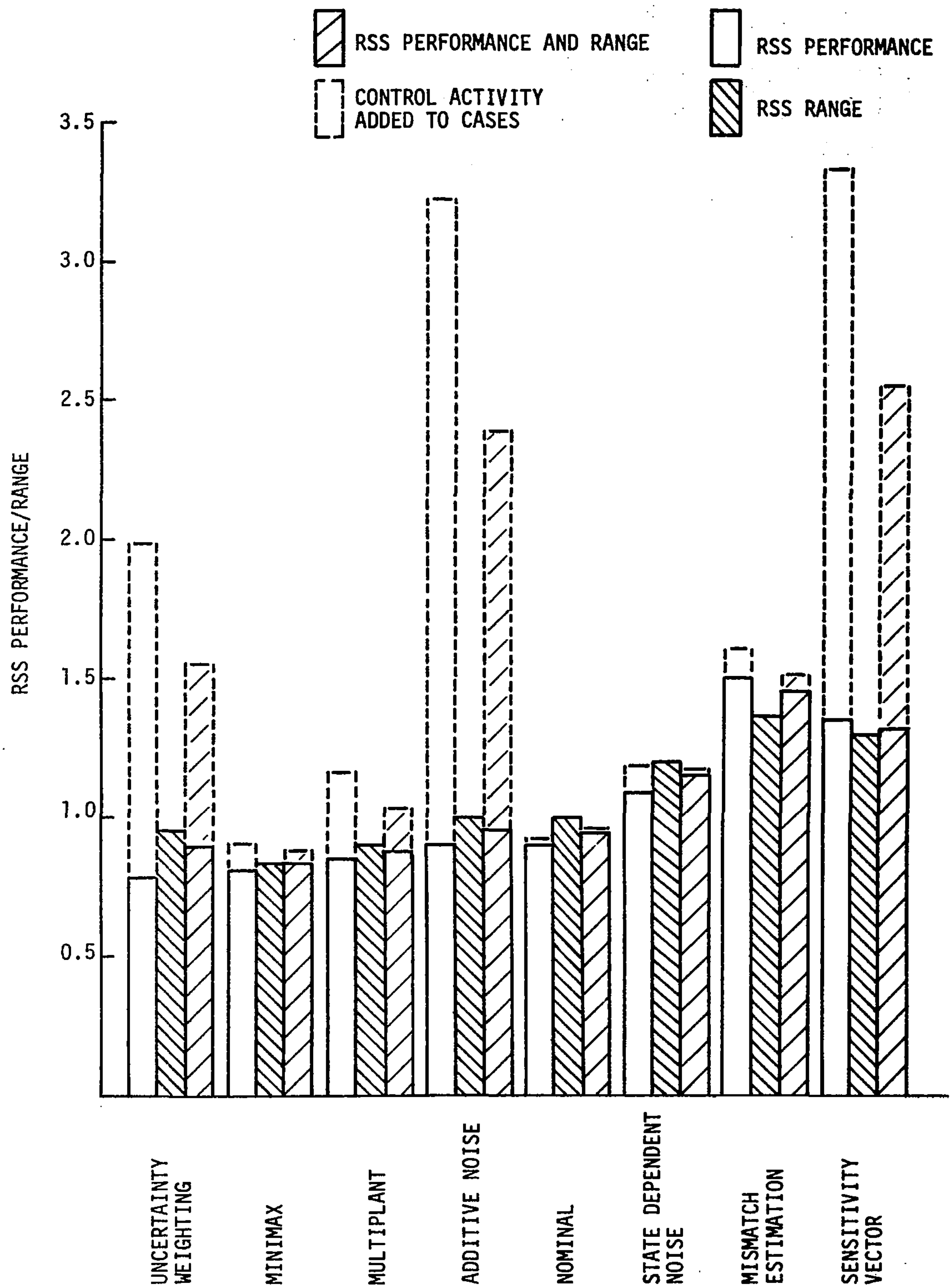


Figure 21. Normalized Performance and Range for Case 4R with Case 1 Conditions

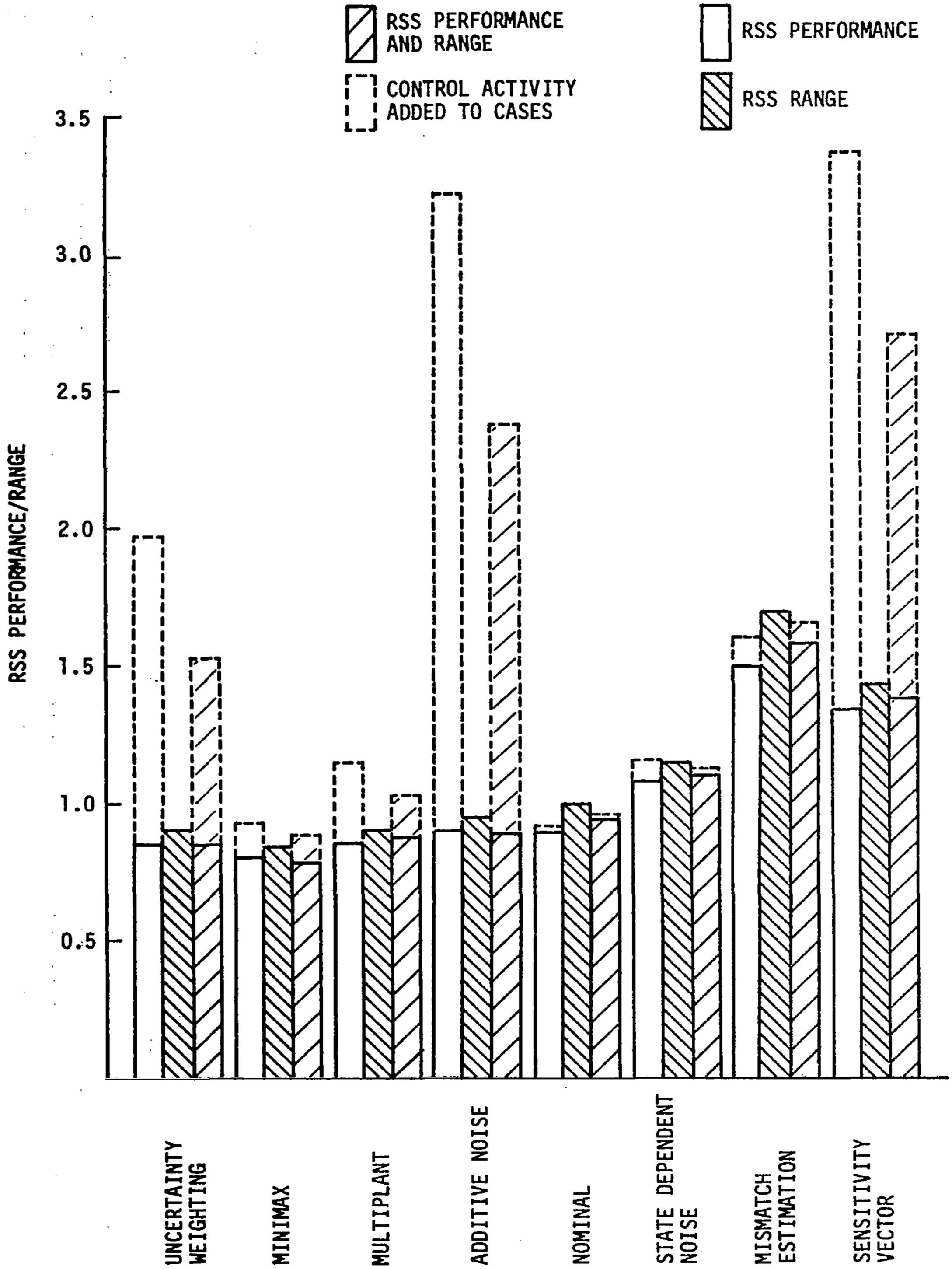


Figure 22. Normalized Performance and Range for Case 4R for Six Evaluation Conditions

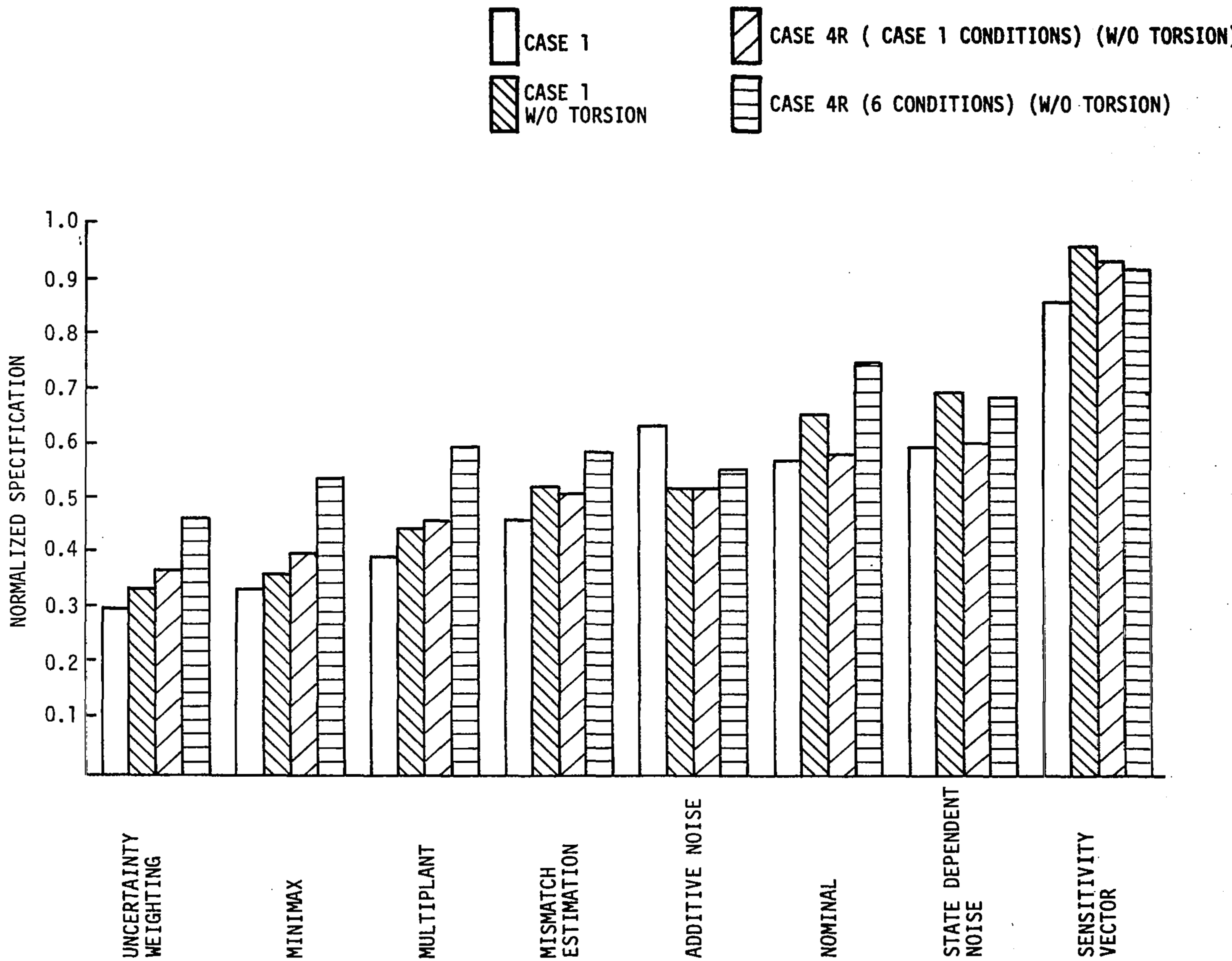


Figure 23. Normalized Spec Violations for the Controllers

Overall Comparison

As a final means of comparison, the controllers were ranked with respect to each of the overall sensitivity measures with 1 indicating least sensitivity. The rankings are given in Table 33. Although there is some variation in relative ranking, it is clear that the minimax and uncertainty weighting controllers are least sensitive. They are less sensitive than the nominal controller according to every ranking. Also, it is clear that the sensitivity vector controller is the most sensitive and is always more sensitive than the nominal. The additive noise and multiplant controllers generally rate less sensitive than the nominal. However, they each are rated more sensitive than the nominal in at least one ranking. The additive noise controller could get a worse rating because of its high surface activity and because of the fact that it is the only controller which violates the RMS torsion spec in Case 1.

The major conclusion to be drawn from the comparison data is that the minimax and uncertainty weighting controllers were significantly less sensitive than the others and that the sensitivity vector controller was actually much more sensitive than the nominal controller.

TABLE 33. RANKING OF INSENSITIVE CONTROLLERS WITH RESPECT TO THE OVERALL SENSITIVITY MEASURES

Controller	Overall relative scoring			Overall performance/range			Overall specification violation			
		Case 4R			Case 4R		Case 1	Case 4R		Sum
	Case 1	3 Cond.	6 Cond.	Case 1	3 Cond.	6 Cond.	w/o torsion	3 Cond.	6 Cond.	
Minimax	1	1	1	1	1	1	2	2	2	12
Uncertainty weighting	3	3	2	2	3	2	1	1	1	18
Additive noise	2	1	3	3	5	4	4	4	3	29
Multiplant	6	5	4	4	2	3	3	3	5	35
Mismatch estimation	4	4	5	7	7	8	5	4	4	48
Nominal	4	6	6	5	4	5	6	6	7	49
State dependent noise	7	7	7	6	6	6	7	7	6	59
Sensitivity vector	8	8	8	8	8	7	8	8	8	71

SECTION VIII

NEW CONCEPTS FOR INSENSITIVE CONTROL

Four new concepts, developed by Professor Kleinman and Professor Porter, are described in this section. Synthesis of controllers based on these concepts was not attempted due to the preliminary nature of the conceptual development. Professor Kleinman's mismatch estimation technique and Professor Porter's uncertainty weighting technique were developed to the stage where synthesis was appropriate as described in Section VI. In this section Professor Porter's concept developments are summarized in the following subsections: The Finite Dimensional Inverse Concept, Sensitivity Reduction Subject to Terminal Equivalence, Interrelations Between Terminal Equivalence, Model Following, and Observers. The final subsection, Sensitivity Design for Maximum Difficulty, is a summary of Professor Kleinman's concept development.

THE FINITE DIMENSIONAL INVERSE CONCEPT

Many parameter sensitivity problems for systems under optimal control are finite dimensional in nature. This provides an opportunity for the design of sensitivity reducing compensators which circumvent, in part, the construction of inverse systems.

Consider the linear system modeled by the equation

$$\dot{x} = F(p)x + Gu, \quad x(0) = x_o \tag{149}$$

where F and G are matrices, and x, u, and p are vectors. Suppose that p_o denotes the nominal value of the parameter vector, p. Further, suppose that $u_o(t)$ is the optimal control for the nominal system and $x_o(t)$ is the corresponding state, i.e.,

$$\dot{x}_o = F(p_o)x_o + Gu_o, \quad x_o(0) = x_o \tag{150}$$

Defining δx as the difference between the state of system (149) and the nominal state, $\delta x \triangleq x - x_o$, we have

$$\dot{\delta x} = F(p)\delta x + [F(p) - F(p_o)]x_o + G\delta u, \quad \delta x(0) = 0 \tag{151}$$

where $\delta u \triangleq u - u_o$. We are interested in a method for choosing δu to compensate for the variation from the nominal plant.

We now add a few explicit assumptions.

A.1 The system of Equation (150) is under minimum energy optimal control.

A.2 The parameter vector, p, and hence the perturbation, $F(p) - F(p_o)$, is constant.

The import of A.1 is that the input u(t) is basically determined by x_o. For example, with minimum energy state transfer from x_o at $t = 0$ to x_f at $t = t_f$,

$$u_o(t) = G^T \Phi^T (t_f, t) \Xi (t_f, 0)^{-1} [x_f - \Phi(t_f, 0) x_o] \tag{152}$$

where

$$\Phi(t, \tau) = e^{F(p_o)(t-\tau)} \text{ and } \Xi(t, \tau) = \int_\tau^t \Phi(t, \sigma) G G^T \Phi^T (\tau, \sigma) d\sigma$$

Equation (152) may be viewed as placing the input set in linear correspondence with R^n, the state set. From this observation we modify A.1 to the following form:

A'.1 The control is of the form $u = \sum_{i=1}^{n} \alpha_i u_i$ where α_i are scalars and u_i are known functions.

Using A'.1 and A.2 leads to the final form of the explicit assumptions:

A. The term, $[F(p) - F(p_o)]x_o$, in Equation (151) may be written as

$$[F(p) - F(p_o)]x_o(t) = \sum_{i=1}^{N} \gamma_i a_i(t) \tag{153}$$

where the γ_i are scalars and the $a_i(t)$ are known functions of time. Moreover the $a_i(t)$ may be taken to be linearly independent. From Equations (151) and (153), we conclude that

$$\delta x(t) = \int_o^t \Phi(t, s) \sum_{i=1}^{N} \gamma_i a_i(s) ds + \int_o^t \Phi(t, s) G \delta u(s) ds \tag{154}$$

or

$$\delta x(t) = \sum_{i=1}^{N} \gamma_i b_i(t) + (T\delta u)(t) \tag{154'}$$

where

$$b_i(t) = \int_0^t \Phi(t,s)\, a_i(s)\, ds \tag{155}$$

and T is the integral operator with kernel $\Phi(t, s)\, G$; i.e.,

$$(T\delta u)(t) = \int_0^t \Phi(t,s)\, G\, \delta u(s)\, ds \tag{156}$$

If δu is zero, the system is operating in an open-loop manner and the open-loop state response is (from Equation (154'))

$$\delta x(t)_{OL} = \sum_{i=1}^{N} \gamma_i b_i(t) \tag{157}$$

The sensitivity ratio for a system with control, δu, is then

$$\begin{aligned} \$ &= \|\delta x(t)_{CL}\| / \|\delta x(t)_{OL}\| \\ &= \left\| \sum_{i=1}^{N} \gamma_i b_i(t) + (T\delta u)(t) \right\| \Big/ \left\| \sum_{i=1}^{N} \gamma_i b_i(t) \right\| \end{aligned} \tag{158}$$

Our goal for sensitivity reduction is to find δu's which give $\$ < 1$. Proceeding toward this goal, let us define L as the linear subspace spanned by the set $\{b_i(t)\}$, and let P_L denote the projection operator from the space of all absolutely continuous n-dimensional functions onto L. If we could take the control to be given by

$$\delta u = H\delta x \tag{159}$$

where H is a causal operator with the property that

$$TH = \left(1 - \frac{1}{\varepsilon}\right) P_L \tag{160}$$

then this control would yield a system with sensitivity ratio

$$\$ = \varepsilon \tag{161}$$

Choosing ε such that $0 < \varepsilon < 1$, we would achieve reduced sensitivity. Equation (161) may be established from (154') as follows. With δu given by (159) and H satisfying (160), we have

$$\begin{aligned}
\delta x(t)_{CL} &= \sum_{i=1}^{N} \gamma_i b_i(t) + (T\delta u)(t) \\
&= \delta x(t)_{OL} + (TH\delta x)(t)_{CL} \\
&= \delta x(t)_{OL} + (1 - \tfrac{1}{\varepsilon})(P_L\ \delta x)(t)_{CL} \\
&= \delta x(t)_{OL} + (1 - \tfrac{1}{\varepsilon})\ \delta x(t)_{CL}
\end{aligned}$$

Thus,

$$\delta x(t)_{CL} = \varepsilon\, \delta x(t)_{OL}$$

Hence,

$$\begin{aligned}
\$ &= \| \delta x(t)_{CL} \| / \| \delta x(t)_{OL} \| \\
&= \| \varepsilon\, \delta x(t)_{OL} \| / \| \delta x(t)_{OL} \| = \varepsilon
\end{aligned}$$

The heart of the problem of synthesizing such controls is to develop methods for computing the operator H in a realizable form. The realizability of such an operator is assured for many systems as the following development indicates.

Suppose that the set $\{v_i\}$ of controls is found such that $b_i = Tv_i$. It might be found, for instance, by building (mathematically) a T^{-1} and recording the outputs v_i of T^{-1} corresponding to the inputs b_i. Suppose that P^t, $t > 0$ is the truncation map

$$(P^t x)(\beta) = \begin{cases} x(\beta) & \beta \le t \\ 0 & \beta > t \end{cases}$$

We want

$$P^t v_i = P^t \hat{H} b_i = P^t \hat{H}(P^t b_i) \qquad t \geq 0$$

for $i = 1, \ldots, N$ and

$$\hat{H} P^t v = 0$$

whenever

$$P^t v \in \text{span}\,\{P^t b_1, \ldots, P^t b_N\}, \; t \geq 0$$

Then we would take $H = (1 - \frac{1}{\varepsilon})\,\hat{H}$.

As a specific example, consider the problem on the finite time interval [0, 1] of finding $\hat{H}$ with the properties that $\hat{H}$ is causal and $\hat{H}b_i = v_i$ for $i = 1, 2, \ldots, N$. A necessary condition for the solution to exist is that, if for any $t \in [0, 1]$,

$$P^t b_i = P^t b_j$$

then

$$P^t v_i = P^t v_j$$

Reasonable assumptions that could be made concerning the set of functions b_i are:

Assumption 1: The functions $P^t b_i$, $i = 1, 2, \ldots, N$ are distinct for any $t > 0$.

Assumption 2: In addition to being distinct, the functions $P^t b_i$, $i = 1, 2, \ldots, N$ are linearly independent for any $t > 0$.

An example of a set of b_i's which satisfy the second assumption is:

$$b_{i+1}(t) = t^i, \qquad i = 0, 1, \ldots$$

For this general problem we have the known results:[47, 48]

1. A solution exists whenever necessary condition holds.

2. A fixed order Volterra solution exists whenever Assumption 1 holds.

3. A linear solution exists whenever Assumption 2 holds.

4. An explicit realization is available for results 1 through 3.

5. Realization in differential equation form is possible.

When Assumption 2 holds, the linear solution, $\hat{H}$, may be determined as follows. Let us define a normalized projection, η_j^t, of b_j as

$$\eta_j^t = ||P^t b_j||^{-1} P^t b_j$$

Then η_j^t are linearly independent for $t > 0$ and $||\eta_j^t|| = 1$ for all $t > 0$ and $j = 1, 2, \ldots$. Let N(t) be the Grammian matrix

$$N(t) = [< \eta_i^t, \eta_j^t >]$$

The linear independence of the set of functions $\{\eta_i^t\}$ implies that N(t) is nonsingular. Let the inverse of N(t) be denoted by

$$N^{-1}(t) = [\alpha_{ij}(t)]$$

Now define

$$(\eta_i^t)^+ = \sum_j \alpha_{ij}(t)\eta_j^t, \quad i = 1, 2, \ldots$$

Then performing another normalization, let

$$B_j^t = ||P^t b_j|| (\eta_j^t)^+, \quad j = 1, 2, \ldots$$

The set of functions B_j^t have the desirable properties that

$$< B_j^t, b_i > = \delta_{ij}$$

$$< B_j^t, P^\beta u > = < B_j^t, u > \text{ if } \beta \geq t$$

This leads to the desired linear solution:

$$[\hat{H}u](t) = \int_o^t w(t, \beta)\, u(\beta)\, d\beta$$

where

$$w(t, \beta) = \Sigma\, v_i(t)\, B_i^t(\beta)$$

SENSITIVITY REDUCTION SUBJECT TO TERMINAL EQUIVALENCE

In this discussion, we will be concerned with the relation between open-loop and closed-loop control of a plant which has uncertain parameters. Sensitivity reduction is measured in terms of the ratio of closed-loop to open-loop perturbations caused by perturbations in the parameters from their nominal values. The open-loop and closed-loop systems are constrained to be terminally equivalent for the case of nominal parameter values. Here, terminal equivalence refers to equivalence of certain input-output relations for the open-loop and closed-loop systems.

The open-loop system is shown schematically in Figure 24. The plant is represented by an operator P. Operators Q and R represent open-loop compensators. The input-output description of the open-loop system is

$$S_{OL} : x = PQu + (I - PQR)\eta \tag{162}$$

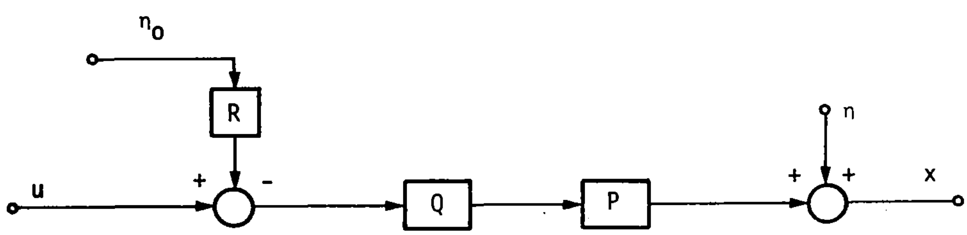


Figure 24. Schematic Diagram of Open-Loop System

The closed-loop system is shown schematically in Figure 25. The feedback operator is represented by H, and G denotes a compensator. These operators, G and H, are the design choices. The input-output description of the closed-loop system is

$$S_{CL} : x = (I + PGH)^{-1}\,(PGu + \eta) \tag{163}$$

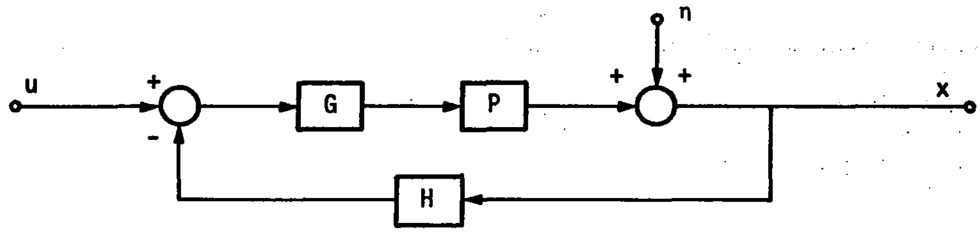


Figure 25. Schematic Diagram of Closed-Loop System

Three types of terminal equivalence of the open-loop and closed-loop systems can be imposed. Type 1 equivalence requires the transformations from u to x for the two systems to be identical for all u with $\eta = 0$. Type 2 requires the transformations from η to x to be identical for all η with u = 0. Finally, Type 3 requires the transformations from all pairs (u, η) to x to be identical. These types of equivalence impose constraints on the pairs of operators (Q, R) and (G, H). For example, Type 1 equivalence requires

$$x_{OL} = PQu \equiv x_{CL} = (I + PGH)^{-1} PGu \tag{164}$$

for all u. This implies

$$PQ = (I + PGH)^{-1} PG$$

which is satisfied if

$$Q = (I + GHP)^{-1} G \tag{165}$$

or

$$G = Q(I - HPQ)^{-1} \tag{165'}$$

Thus, the first equation specifies Q in terms of G and H, and the second specifies G in terms of Q and H.

Type 2 equivalence yields the constraining relation

$$QR = (I + GHP)^{-1} GH = GH(I + PGH)^{-1} \tag{166}$$

or

$$GH = QR\,(I - PQR)^{-1} = (I - QRP)^{-1}\,QR \tag{166'}$$

Finally, Type 3 equivalence yields the relations

$$Q = (I + GHP)^{-1}\,G, \quad R = H \tag{167}$$

for (Q, R) in terms of (G, H) or

$$G = Q\,(I - RPQ)^{-1}, \quad H = R \tag{167'}$$

for (G, H) in terms of (Q, R).

These types of equivalence will be imposed on the open-loop and closed-loop systems for the nominal values of the plant in the definitions of sensitivity reduction. The sensitivity index which will serve as a measure of sensitivity reduction will be defined as the "ratio" of closed-loop to open-loop output perturbations. For this purpose let P_o denote the nominal plant, η_o denote the nominal open-loop input, and x_o denote the nominal open-loop response. Let δP, $\delta\eta$, and δx denote the perturbations from the nominals. Thus, $P = P_o + \delta P$, $\eta = \eta_o + \delta\eta$, and $x = x_o + \delta x$. Then for the open-loop system we have

$$
\begin{aligned}
S_{OL}:\quad x &= PQ(u - R\eta_o) + \eta \\
x_o &= P_oQ\,(u - R\eta_o) + \eta_o \\
\delta x &= \delta PQ\,(u - R\eta_o) + \delta\eta \qquad \text{(to first order)}
\end{aligned}
\tag{168}
$$

The corresponding closed-loop system is

$$
\begin{aligned}
S_{CL}:\quad (I + PGH)\,x &= PGu + \eta \\
(I + P_oGH)\,x_o &= P_oGu + \eta_o \\
(I + P_oGH)\,\delta x &= \delta PGu + \delta\eta - \delta PGHx_o \quad \text{(to first order)} \\
&= \delta PG\,[u - H\,(I + P_oGH)^{-1}\,(P_oGu + \eta_o)] + \delta\eta \\
&= \delta PG\,(I + HP_oG)^{-1}\,(u - H\eta_o) + \delta\eta
\end{aligned}
\tag{169}
$$

Now suppose we impose Type 1 equivalence and make the further assumption for simplicity that $Q = I$. Then

$$\delta x_{OL} = \delta Pu + \delta \eta \tag{170}$$

$$\delta x_{CL} = (I + P_o GH)^{-1} [\delta PG (I + HP_o G)^{-1} u + \delta \eta] \tag{171}$$

But Type 1 equivalence implies that Equations (165) and (165') hold. Equation (165') with $Q = I$ gives $G = (I - HP_o)^{-1}$ which yields the identities $I + P_o GH = (I - P_o H)^{-1}$ and $G(I + HP_o G)^{-1} = I$. Therefore, using these identities and Equations (170) and (171), we have

$$\delta x_{CL} = (I + P_o GH)^{-1} \delta x_{OL} = (I - P_o H) \delta x_{OL} \tag{172}$$

Thus defining the sensitivity index, $\$$, to be the "ratio" of δx_{CL} to δx_{OL}, we have

$$\$ = (I + P_o GH)^{-1} = I - P_o H \tag{173}$$

We would also consider Type 2 equivalence. In this case, $u_o = 0$. Thus

$$\delta x_{OL} = - \delta P \eta_o + \delta \eta \tag{174}$$

$$\delta x_{CL} = (I + P_o QR)^{-1} [\delta PG(I + HP_o G)^{-1} (-H\eta_o) + \delta \eta] \tag{175}$$

But $G(I + HP_o G)^{-1} = (I + GHP_o)^{-1} G$ so that

$$\begin{aligned} \delta x_{CL} &= (I + P_o GH)^{-1} [\delta P (I + GHP_o)^{-1} GH(-\eta_o) + \delta \eta] \\ &= (I + P_o GH)^{-1} [-\delta PQR\eta_o + \delta \eta] \\ &= (I + P_o GH)^{-1} \delta x_{OL} + (I - P_o QR) \delta x_{OL} \end{aligned} \tag{176}$$

since $(I + P_o GH)^{-1} = I - P_o QR$ by virtue of Equation (165'). Again defining the sensitivity index, $\$$, to be the "ratio" of δx_{CL} to δx_{OL}, we find

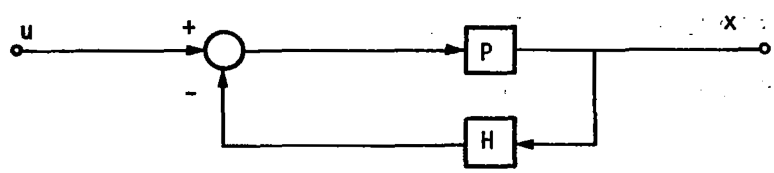
$$\$ = (I + P_o GH)^{-1} = I - P_o QR \tag{177}$$

These two examples motivate the definition of the generalized sensitivity operator as

$$\$ = (I + P_o GH)^{-1} \tag{178}$$

This sensitivity index is consistent with the original Bode type of index. Recall that Bode's index for the system shown in Figure 26 is

$$S_P^T = \frac{dT/T}{dP/P} = \frac{d[\ell nT]}{d[\ell nP]}$$



$\frac{x}{u} = T + P/(1 + PH)$

Figure 26. Single Input-Single Output Plant with Feedback

For this system $dT = \frac{\partial T}{\partial P} dP/(1 + PH)^2$ so that

$$S_P^T = \frac{dP}{(1+PH)^2} \left(\frac{P}{1 + PH}\right)^{-1} \left(\frac{dP}{P}\right)^{-1} = (1 + PH)^{-1} \tag{179}$$

With this background let us proceed to develop general conditions which are necessary and/or sufficient to assure sensitivity reduction as measured by the generalized sensitivity index (178). We will formulate the conditions as conditions on linear operators acting on Hilbert spaces. We may consider the space of n-tuples of Lebesque square integrable functions on $(0, \infty)$ as an example Hilbert space $\underline{H}$ with inner produce denoted by $<, >$. Thus $x(t) = (x_1(t), x_2(t), \ldots , x_n(t))$ is an element of $\underline{H}$ if

$$\int_o^\infty \sum_{i=1}^n x_i(t)\, \bar{x}_i(t)\, dt < \infty$$

The inner product of elements x and y is

$$< x, y > = \int_0^\infty \sum_{i=1}^n x_i(t)\, \bar{y}_i(t)\, dt$$

The norm of x is denoted by $||x||$, where

$$|| x ||^2 = \int_0^\infty \sum_{i=1}^n x_i(t)\, \bar{x}_i(t)\, dt = \sum_{i=1}^n \int_0^\infty | x_i(t) |^2 dt = < x, x >$$

We may consider an operator T as a mapping from one Hilbert space, $\underline{H}_1$, into another, $\underline{H}_2$. In case $\underline{H}_1 = \underline{H}_2 = \underline{H}$ we have $T : \underline{H} \rightarrow \underline{H}$. The operator is linear if $T(ax + by) = aTx + bTy$. We say that T is positive denoted by $T > 0$ (nonnegative denoted by $\overset{\circ}{T} \geq 0$) if and only if

$$< x, Tx > > 0 \ (\geq 0) \text{ for all } x \neq 0$$

The adjoint operator, T^*, of the operator T is defined as the operator which satisfies

$$< x, Ty > = < T^*x, y > \text{ for all } x, y \epsilon H$$

The norm of the operator T is defined as

$$|| T || = \sup \{ || T x || / || x || \}$$

where the supremum is taken over all $x \neq 0$ belonging to $\underline{H}$. The operator T is a contraction if and only if $||T|| \leq 1$.

Now let us consider a plant, P, to be the linear operator mapping inputs $u \epsilon L_2^m (0, \infty) = \underline{H}$ into outputs $y \epsilon \underline{H}$ defined by the set of equations:

$$y(t) = M(t)\, x(t)$$

$$P : \dot{x}(t) = F(t)\, x(t) + G(t)\, u(t) \tag{180}$$

$$x(0) = 0$$

We will consider H to be a feedback operator mapping y into u. This is shown schematically in Figure 27. In this case the sensitivity index is

$$\$ = (I + PH)^{-1} \tag{181}$$

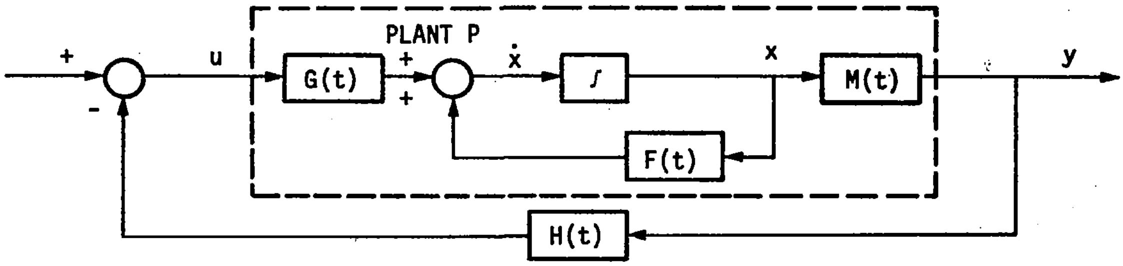


Figure 27. Schematic Diagram of Plant, P, with Feedback H

i.e., $\delta y_{CL} = \$ \, \delta y_{OL}$. For sensitivity reduction we want $\$$ to be a contraction so that

$$\| \delta y_{CL} \| \leq \| \$ \| \, \| \delta y_{OL} \| \leq \| \delta y_{OL} \| \tag{182}$$

Actually, we want somewhat more than (182). We generally want

$$\int_0^t \sum_{i=1}^m |(\delta y_{CL})_i(\tau)|^2 \, d\tau \leq \int_0^t \sum_{i=1}^m |(\delta y_{OL})_i(\tau)|^2 \, d\tau \tag{183}$$

for all $t \geq 0$ instead of (182) which only states that

$$\int_0^\infty \sum_{i=1}^m |(\delta y_{CL})_i(t)|^2 \, dt \leq \int_0^\infty \sum_{i=1}^m |(\delta y_{OL})_i(t)|^2 \, dt$$

This requires $\$$ to be a causal contraction. A simple scalar example to illustrate the need for this causal requirement is provided by taking $\$$ to be the time advancing contraction

$$\$: x(t) = \frac{1}{2} x(t+1)$$

In this case, $\$$ is a contraction with $\| \$ \| = 1/2$. But if we take $y = x$ and choose

$$\delta y_{OL}(t) = \begin{cases} 0 & 0 < t \leq 1 \\ 1 & 1 < t \leq 2 \end{cases}$$

we obtain

$$\int_0^t |\delta y_{OL}(\tau)|^2 \, d\tau = 0 \text{ for } 0 \le t \le 1$$

but

$$\int_0^t |\delta y_{CL}(\tau)|^2 \, d\tau > 0 \text{ for } 0 < t \le 1$$

In stating our result for the system shown in Figure 27, it is convenient to have in hand the transition matrix, ψ, for the closed-loop system

$$\dot{\psi}(t, \tau) = [F(t) - G(t)\, H(t)\, M(t)]\, \psi(t, \tau), \quad \psi(\tau, \tau) = I \tag{184}$$

Another useful quantity is the integral operator, W, whose kernel is given by

$$w(t, \tau) = [M(t) - H^T(t)\, G^T(t)\, K(t)]\, \psi(t, \tau)\, G(\tau)\, H(\tau), \quad t \ge \tau \tag{185}$$

Here K is the unique self-adjoint solution of

$$-\dot{K}(t) = K(t)\,[F(t) - G(t)\, H(t)\, M(t)] + [F(t) - G(t)\, H(t)\, M(t)]^T K(t) + M^T(t)\, M(t)$$
$$K(t_o) = \int_{t_o}^{t_f} \psi^T(\tau, t_o)\, M^T(\tau)\, M(\tau)\, \psi(\tau, t_o)\, d\tau \tag{186}$$

Finally we identify the kernel

$$h(t, \tau) = M(t)\, \psi(t, \tau)\, G(t)\, M(t) \tag{187}$$

and let $\tilde{H}$ denote its associated operator. The operator $\tilde{H}$ may be considered as mapping γ into y via the integral equation

$$y(t) = \int_0^t h(t, \tau)\gamma(\tau)\, d\tau \tag{188}$$

with γ viewed as an input introduced in the feedback path. The sensitivity operator may be viewed as the mapping into y of an input, $\tilde{\gamma}$, introduced additively to the output.

These two situations are depicted in Figures 28a and 28b. Comparing these schematic diagrams we see that the operators $\$$ and $\tilde{H}$ are related by the equation

$$\$ = I - \tilde{H} \tag{189}$$

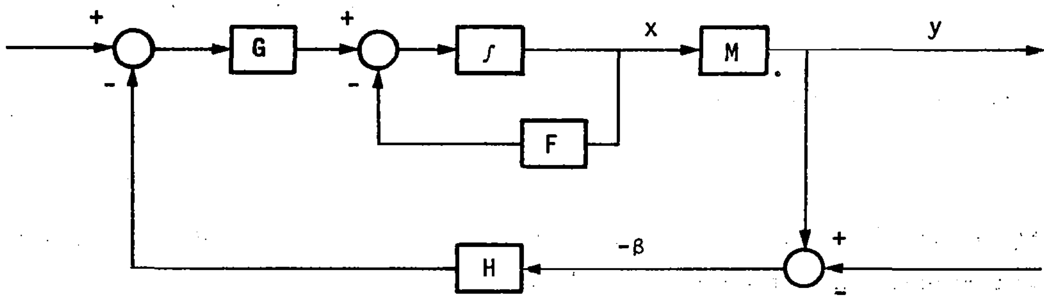


Figure 28a. Schematic of Operator $\tilde{H} : \gamma \to y$

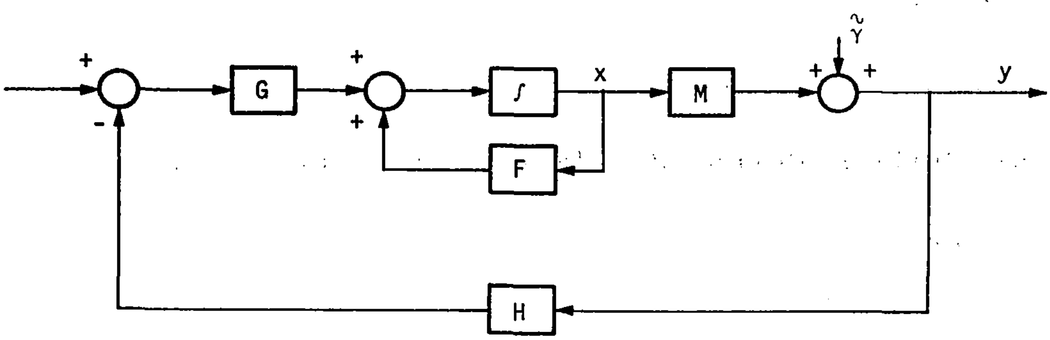


Figure 28b. Schematic of Operator $\$: \tilde{\gamma} \to y$

This result may be derived analytically as follows. The operator $\tilde{H}$ may be expressed as:

$$\tilde{H}: y(t) = M(t)x(t)$$

$$\dot{x}(t) = F(t)x(t) + G(t)H(t)\,[\gamma(t) - y(t)],\ x(0) = 0$$

$$= [F(t) - G(t)H(t)M(t)]\,x(t) + G(t)H(t)\gamma(t)$$

The operator $\$$ may be expressed as:

$$\$: \; y(t) = M(t)x(t) + \tilde{\gamma}(t)$$

$$\dot{x}(t) = F(t)x(t) - G(t)H(t)y(t), \; x(0) = 0$$

$$= [F(t) - G(t)H(t)M(t)]\, x(t) - G(t)H(t)\, \tilde{\gamma}(t)$$

The operator I-$\tilde{H}$ may be expressed as:

$$I\text{-}\tilde{H}: \; y(t) = \gamma(t) - M(t)x(t) = \gamma(t) + M(t)[-x(t)]$$

$$[-\dot{x}(t)] = [F(t) - G(t)H(t)M(t)]\,[-x(t)] - G(t)H(t)\gamma(t)$$

which is the same as the expression for the operator $\$$.

Clearly, $\$$ is causal if and only if $\tilde{H}$ is causal, a property built into this latter operator. Let us note that (asterisk denotes adjoint)

$$||\delta y_{OL}||^2 - ||\delta y_{CL}||^2 = ||\delta y_{OL}||^2 - ||\$\delta y_{OL}||^2$$

$$= \langle \delta y_{OL}, \delta y_{OL} \rangle - \langle \$\delta y_{OL}, \$\delta y_{OL} \rangle$$

$$= \langle \delta y_{OL}, \delta y_{OL} \rangle - \langle \delta y_{OL}, \$^*\$\, \delta y_{OL} \rangle$$

$$= \langle \delta y_{OL}, I - \$^*\$\, \delta y_{OL} \rangle$$

Hence $||\$|| \leq 1$ if and only if $I - \$^*\$ \geq 0$. Using the relation (189), we obtain the result that $||\$|| \leq 1$ is equivalent to

$$\tilde{H} + \tilde{H}^* - \tilde{H}^*\tilde{H} \geq 0 \tag{190}$$

let us now introduce the following definition. An integral map $W : L_2^m \to L_2^m$ is positive real if its kernel $W(t, \tau)$ satisfies

(1) $W(t, \tau)$ is real

(2) $W(t, \tau) = 0, \; \tau > t$

(3) W is bounded

(4) $W + W^* \geq 0$

Now consider the kernel

$$W(t, \tau) = \begin{cases} M(t)\, \Phi(t, \tau)\, G(\tau) & t \geq \tau \\ 0 & t < \tau \end{cases} \tag{191}$$

where $\dot{\Phi}(t, \tau) = F(t)\, \Phi(t, \tau), \quad \Phi(\tau, \tau) = I.$

The associated operator W may be viewed as the mapping from u to y defined by the system

$$S : \dot{x}(t) = F(t)x(t) + G(t)u(t), \qquad x(0) = 0$$
$$y(t) = M(t)x(t)$$

For such an operator W, we have the following result.

Theorem 1: Let W be bounded with kernel W given by (191). If there exists a continuous, positive, self-adjoint matrix Q and a continuous m x n matrix L such that

(1) $-\dot{Q} = QF + F^*Q + L^*L$ (time suppressed)

(2) $Q(0)\ G(0) = M^*(0)$

(3) $||A||$ is finite where $(Az)(t) \triangleq \int_0^t L(t)\Phi(t, \tau)G(\tau)z(\tau)\, d\tau$

then W is positive real.

Proof: Consider

$$\frac{d}{d\tau}\{\Phi^*(\tau, t)Q(\tau)\Phi(\tau, s)\} = \Phi^*(\tau, t)\, \{F^*(\tau)Q(\tau) + \dot{Q}(\tau) + Q(\tau)F(\tau)\}\, \Phi(\tau, s)$$
$$= -\Phi^*(\tau, t)L^*(\tau)L(\tau)\Phi(\tau, s)$$

We have also the identity (see later remark)

$$(A^*Az)(t) = \int_0^t \int_t^b G^*(t)\Phi^*(\tau, t)L^*(\tau)L(\tau)\Phi(\tau, s)G(s)z(s)\, d\tau ds$$
$$+ \int_t^b \int_s^b G^*(t)\Phi^*(\tau, t)L^*(\tau)L(\tau)\Phi(\tau, s)G(s)z(s)d\tau ds \tag{192}$$

Using the first identity, we have

$$(A^*Az)(t) = -\int_o^t \int_t^b G^*(t) \frac{d}{d\tau} \{\Phi^*(\tau, t)\, Q(\tau)\Phi(\tau, s)\} G(s)z(s)d\tau ds$$

$$-\int_t^b \int_s^b G^*(t) \frac{d}{d\tau} \{\Phi^*(\tau, t)Q(\tau)\Phi(\tau, s)\}\; G(s)z(s)d\tau ds$$

Using the Fundamental Theorem of Calculus, we have

$$(A^*Az)(t) = \int_o^t G^*(t)\; \{Q(t)\Phi(t, s) - \Phi^*(b, t)Q(b)\Phi(b, s)\}\; G(s)z(s)ds$$

$$+ \int_t^b G^*(t)\; \{\Phi^*(s, t)Q(s) - \Phi^*(b, t)Q(b)\Phi(b, s)\}\, G(s)z(s)ds$$

Using causality of Φ, we have

$$(A^*Az)(t) = \int_o^b G^*(t)\; [Q(t)\Phi(t, s) + \Phi^*(s, t)Q(s)]\; G(s)z(s)ds$$

$$- \int_o^b G^*(t)\Phi^*(b,t)Q(b)\Phi(b, s)G(s)z(s)ds$$

$$= ([W + W^*]z)(t) - \int_o^b G^*(t)\Phi^*(b, t)Q(b)\Phi(b, s)\; G(s)z(s)ds$$

Since A^*A is positive and bounded, $W+W^* \geq 0$ and the theorem is proved.

<u>Remark</u>: With A as in condition 3, we compute

$$< y,\; Az > = \int_o^b \; [y(\beta),\; \int_o^\beta \; L(\beta)\Phi(\beta, \tau)G(\tau)z(\tau)d\tau]d\beta$$

$$= \int_o^b \int_o^\beta [z(\tau),\; G^*(\tau)\Phi^*(\beta, \tau)\; L^*(\beta)y(\beta)]\, d\tau d\beta$$

$$= \int_o^b \int_\tau^b [z(\tau),\; G^*(\tau)\Phi^*(\beta, \tau)L^*(\beta)y(\beta)]\; d\beta d\tau$$

$$= \langle A^* y, z \rangle$$

$$\Rightarrow (A^* y)(\tau) = \int_{\tau}^{b} G^*(\tau)\Phi^*(\beta, \tau)L^*(\beta)y(\beta)d\beta$$

By direct substitution, we have

$$(A^*Az)(t) = \int_{t}^{b} G^*(t)\Phi^*(\beta, t)L^*(\beta)\int_{o}^{\beta} L(\beta)\Phi(\beta, \tau)G(\tau)z(\tau)d\tau$$

Using the identity

$$\int_{t}^{b}\int_{o}^{\tau} f(t, \tau, s)dsd\tau = \int_{o}^{t}\int_{t}^{b} f(t, \tau, s)d\tau ds + \int_{t}^{b}\int_{s}^{b} f(t, \tau, s)d\tau ds$$

Equation (192) results.

Now using the analogy between $\tilde{H}$ and A and the form of Equation (192), we have

$$(\tilde{H}^*\tilde{H}x)(t) = \int_{o}^{t}\int_{t}^{b} H^*(t)G^*(t)\Psi^*(\tau, t)M^*(\tau)M(\tau)\Psi(\tau, s)G(s)M(s)x(s)d\tau ds$$

$$+ \int_{t}^{b}\int_{s}^{b} H^*(t)G^*(t)\Psi^*(\tau, t)M^*(\tau)M(\tau)\Psi(\tau, s)G(s)H(s)x(s)d\tau ds$$

Then it can easily be verified that

$$\Psi^*(\tau, t)M^*(\tau)M(\tau)\Psi(\tau, s) = -\frac{d}{d\tau}\{\Psi^*(\tau, t)K(\tau)\Psi(\tau, s)\}$$

We substitute this expression into $\tilde{H}^*\tilde{H}$, recalling that $\Psi(a, b) = 0$ for $b > a$:

$$(\tilde{H}^*\tilde{H}x)(t) = \int_{o}^{b} H^*(t)G^*(t)[K(t)\Psi(t, s) + \Psi^*(s, t)K(s)]G(s)H(s)x(s)ds$$

$$- \int_{o}^{b} H^*(t)G^*(t)\Psi^*(b, t)K(b)\Psi(b, s)G(s)H(s)x(s)ds$$

Therefore the kernel of $\tilde{H}^*\tilde{H}$ is given by

$$H^*(t)G^*(t)[K(t)\Phi(t,\tau) + \Phi^*(\tau,t)K(\tau)]G(\tau)H(\tau)$$
$$- H^*(t)G^*(t)\Phi^*(b,t)K(b)\Phi(b,\tau)G(\tau)H(\tau)$$

Recall that the kernal of $\tilde{H}+\tilde{H}^*$ is given by

$$M(t)\Phi(t,\tau)G(\tau)H(\tau) + H^*(t)G^*(t)\Phi^*(\tau,t)M(\tau)$$

and hence it is immediate that the kernel of $\tilde{H}+\tilde{H}^* - \tilde{H}^*\tilde{H}$ is computed by

$$[M(t) - H^*(t)G^*(t)K(t)]\Phi(t,\tau)G(\tau)H(\tau) + H^*(t)G^*(t)\Phi^*(\tau,t)\,[M^*(\tau)-K(\tau)G(\tau)M(\tau)]$$
$$+ H^*(t)G^*(t)\Phi^*(t_f,t)\Phi(t_f,\tau)G(\tau)H(\tau)$$

Letting W be the operator whose kernel is

$$W(t,\tau) = [M(t) - H^*(t)G^*(t)K(t)]\Phi(t,\tau)G(\tau)H(\tau)$$

and letting π be the operator whose kernel is

$$\pi(t,\tau) = H^*(t)G^*(t)\Phi^*(t_f,t)K(t_f)\Phi(t_f,\tau)\ G(\tau)H(\tau)$$

we note that $\pi \geq 0$ and that

$$I - \mathcal{S}^*\mathcal{S} = \tilde{H}^* + \tilde{H} - \tilde{H}^*\tilde{H} = W^* + W + \pi \tag{193}$$

This leads to the following sufficiency theorem:

<u>Theorem 2:</u> If $W + W^* \geq 0$, then sensitivity is reduced. A necessary condition for $W + W^* \geq 0$ is that

$$M(t)G(t)H(t) + H^T(t)G^T(t)M^T(t) \geq\ 2H^T(t)G^T(t)K(t)G(t)H(t)$$

where K is given in Equation (186).

A sufficient condition for positivity of $W + W^*$ is obtained by expressing Theorem 1 in terms of the closed-loop system as follows:

Theorem 3: A sufficient condition for positivity of $W + W^*$ is the existence of a continuous positive, self-adjoint matrix Q and a continuous matrix L such that

(1) $-\dot{Q} = Q[F - GHM] + [F - GHM]^T Q + L^T L$ (times suppressed)

(2) $[Q + K] GH = M^T$ (defines $Q(t_o)$)

(3) $(Ax)(t) = \int_o^t L(t)\phi(t, \tau)G(\tau)H(\tau)x(\tau)d\tau$ is a bounded map

Condition (3) is automatically satisfied on all finite intervals, and choosing H(0) arbitrarily in (2), we see that Q(0) is arbitrary in (1).

Example: Consider the scalar case with F, G, and M denoted by f, g, and m. Moreover let g and m be constants (without loss of generality) and suppose g, m, and $h > 0$ while $\beta \geq f(t) \geq f > 0$. Then from (184)

$$\phi(t, \tau) = \exp\{-\int_\tau^t [f(s) - ghm]\, ds\}\quad t \geq \tau$$

In Equation (186) we have

$$-\dot{k}(t) = -2[f(t) - ghm]\, k(t) + m^2 \tag{194}$$

which gives

$$k(t) = \phi^2(t_o, t)\int_t^\infty m^2\, \phi^2(\tau, t_o)\, d\tau$$

from which we deduce

$$0 \leq k(t) \leq m^2/2\,(\beta - ghm) < -m/gh$$

Defining

$$q(t) = m/gh - k(t)$$

and substituting in Equation (194) yields

$$\dot{q} = -2[f(t) - ghm]\,[m/gh - q(t)] + m^2$$

which may be rewritten as

$$-\dot{q} = -2[f(t) - ghm]q(t) + m^2 - \frac{2f(t)m}{gh}$$

Consequently choosing

$$\ell(t) = [m^2 - \frac{2f(t)m}{gh}]^{1/2}$$

Theorem 3 is satisfied.

The Use of $\$^{-1}$

Taking a different approach we may obtain results similar in spirit with the above. With $\$ = [I + PH]^{-1}$ and manipulation, it follows that

$$||\$|| \leq 1 \Longleftrightarrow (\$*)^{-1}\$^{-1} - I \geq 0 \Longleftrightarrow PH + H^*P^* + H^*P^* PH \geq 0$$

Since $H*P*PH \geq 0$, a sufficient condition is that $\$$ be causal and $PH+H*P* \geq 0$. This line of thought culminates in Theorem 4.

Theorem 4: A sufficient condition for $||\$|| \leq 1$ is the existence of positive self-adjoint Q and continuous L satisfying

(1) $-\dot{Q} = QF + F^TQ + L^TL$

(2) $QGH = M^T$ (defines Q(0) also)

(3) $(Ax)(t) = \int_0^t L(t)\, \psi(t,\tau)G(\tau)H(\tau)x(\tau)d\tau$ is bounded

Remark: Notice that with $G = M = I$ we choose $H = Q^{-1}$, and it follows easily that $\dot{H} = FH + HF^T + HL^T LH$ which suggests a cost function being optimized. If $G = I$, then $H = Q^{-1}M^T$ and progress also is in hand. However, if $M = I$, then a "rank" check on condition (2) reveals an immediate difficulty if $G \neq I$. Conclusion: It is more important to move toward $G = I$ than $M = I$ in the design assumptions.

In Theorem 4, the term H*P*PH is not used; hence the ensuring sufficient condition is possibly too severe. One can use a factoring process to improve this situation. The conditions, however, become more complicated to state.

Let V denote the integral operator with kernel

$$v(t, \tau) = [M(t) + H^T(t)G^T(t)K(t)]\psi(t, \tau)G(\tau)H(\tau)$$

where K is the unique self-adjoint solution of

$$-\dot{K}(t) = K(t)F(t) + F^T(t)K(t) + M^T(t)M(t)$$

$$K(t_o) = \int_{t_o}^{t_f} \psi^T(\tau, t_o)M^T(\tau)M(\tau)\psi(\tau, t_o)d\tau$$

Then we have

$$PH + H^*P^* + H^*P^*PH \geq 0 \Longleftrightarrow V+V^* \geq 0$$

Rather than continue the present line of development we shift to the stationary case.

Stationary Results

Let s and ω denote the Laplace and Fourier variables, respectively. Our input-output spaces become $L_2^n(-\infty, \infty)$. Let (using Equation (184))

$$\psi(s) = [sI - (F - GHM)]^{-1} \tag{195}$$

Then using Equations (187) and (189),

$$\$(s) = I - M\psi(s)GH \tag{196}$$

A stationary form of Theorem 2 is the following:

Theorem 2': If $\text{Re}\sigma(F - GHM) \leq \epsilon < 0$, then a necessary and sufficient condition for $W + W^* \geq 0$ is that

$$W(s) = (M - H^TG^TK)\,\psi(s)\,GH$$

be positive real where K is the unique positive, self-adjoint solution of

$$K(F - GHM) + (F - GHM)^TK = -M^TM$$

Necessary conditions for W to be positive real are

(1) $MGH = H^T G^T M^T$

(2) $MGH \geq M^T G^T KGH \geq 0$

The stationary form of Theorem 3 is the following:

<u>Theorem 3':</u> If Re $\{\sigma(F - GHM)\} \leq \epsilon < 0$, then a sufficient condition for W(s) to be positive real is that positive, self-adjoint Q and L exist such that

(1) $Q(F - GHM) + (F - GHM)^T Q = L^T L$

(2) $(Q + K)GH = M^T$

<u>Example:</u>

$$F = \frac{1}{115} \begin{bmatrix} 60 & 218 \\ -85 & -213 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

We choose rank (G) = rank (M) so that H can be invertable (see condition (2), Theorem 3'). Consider the compensation

$$H = \frac{6}{115} \begin{bmatrix} -29 & 5 \\ 17 & 5 \end{bmatrix}$$

Note that

$$MGH = H^T G^T M^T = \frac{6}{115} \begin{bmatrix} 17 & 5 \\ 5 & 15 \end{bmatrix} \geq 0$$

$$\sigma(F - GHM) = \{-1, -2\}$$

Moreover

$$K = \begin{bmatrix} 7/12 & 1/2 \\ 1/2 & 2/3 \end{bmatrix}$$

Hence

$$W(s) = \frac{3}{(115)^2(s+1)(s+2)} \begin{bmatrix} -18 & 166 \\ 130 & 130 \end{bmatrix} \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix} \begin{bmatrix} -12 & 10 \\ 17 & 5 \end{bmatrix}$$

and hence

$$(W + W^*)(j\omega) = \frac{(6/115)^2}{(2-\omega^2)^2 + 9\omega^2} \begin{bmatrix} 676 + 1181\,\omega^2 & -180 + 415\,\omega^2 + 920j\omega \\ -180 + 415\,\omega^2 - 920j\omega & 1300 + 325\omega^2 \end{bmatrix}$$

which is a positive matrix for all ω. Theorem 3' could also have been used with

$$Q = \frac{1}{6} \begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Example: Suppose MG is square and nonsingular. Let H be chosen as $H = (MG)^{-1} N$ where $N = N^* > 0$. The necessary conditions for positive $W + W^*$ are then met (taking N not too large).

Let $\eta(M)$ denote the null space of M and let $\bar{P}$ denote the orthogonal projection on $\eta(M)$. Then M has a right inverse M^{-R} with the properties $MM^{-R} = I$ and $M^{-R}M = I - \bar{P}$. Let $\hat{N} = M^{-R}NM$; the n $M\hat{N} = NM$ and

$$F - GHM = F - G(MG)^{-1} M\hat{N} = F - (I - \bar{P})\hat{N} = F - \hat{N}$$

We can then rearrange Theorem 3' to read ($\hat{K} = K + Q$)

(1) $\hat{K}(F - \hat{N}) + (F - \hat{N})^T\hat{K} = -(L^TL + M^TM)$

(2) $\hat{K}\hat{N} = M^TM$

(3) $\operatorname{Re} \sigma(F - \hat{N}) \leq \epsilon < 0$

A sufficient condition for sensitivity reduction is the existence of a solution triplet $(\hat{K}, L, \hat{N})$.

Remark: It was noted earlier that the assumption $G = I$ does more for the analysis than the assumption $M = I$. The primary reason for this, from a physical point of view, is that $M = I$ gives more information to compensate with; however it also lets "more" of the internal perturbations show up in the output. The assumption $G = I$ gives better control with no detrimental effects.

L_2-Positive Realness and Stability

Consider PH : $M(t)\Phi(t,\tau)G(\tau)H(\tau)$, and suppose our opening theorem holds, i.e., $Q(t) > 0$. Then, we have

Proposition: The closed-loop system is stable.

Proof: With $V[x,t] = [x(t), Q(t)x(t)]$ and the identities of Theorem 1, it is easily verified that

$$\dot{V}[x,t] = -\,||L(t)\psi(t,t_o)||^2 - 2\,||M(t)\psi(t,t_o)x(t_o)||^2 \leq 0$$

where

$$\dot{\psi}(t,t_o) = [F(t) - G(t)H(t)M(t)]\psi(t,t_o)$$

$$\dot{x}(t) = [F(t) - G(t)H(t)M(t)]x(t)$$

Corollary: If either $\{F - GHM, L\}$ or $\{F - GHM, M\}$ is observable and if $||Q(t)||$ is finite, then the closed-loop system is asymptotically stable.

INTERRELATIONS BETWEEN TERMINAL EQUIVALENCE, MODEL FOLLOWING, AND OBSERVERS[48]

Many of the insensitive control concepts are closely related. This discussion illustrates the type of comparison of concepts that is possible. The concept of terminal equivalence was introduced earlier in this section. Let us consider the single-loop feedback system with two compensators shown in Figure 29. The plant is represented by P. The mapping from u to y is

$$y = (I + PGH)^{-1}\,PGu \tag{197}$$

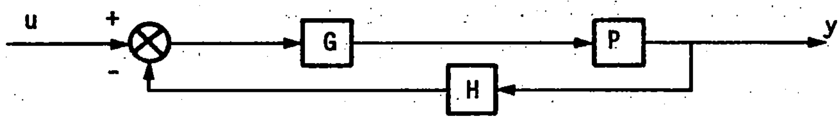


Figure 29. Single-Loop Feedback System with Two Compensators

If the nominal desired closed-loop plant is denoted by $\hat{P}$, and the compensators, G and H, are constrained to satisfy

$$G = (I - H\hat{P})^{-1} \tag{198}$$

the system in Figure 29 is terminally equivalent to the system shown in Figure 30 where the mapping from u to y is $y = \hat{P}u$. That is, if P in Equation (197) is equal to $\hat{P}$, then the mapping from u to y is the same for the two systems shown in Figures 29 and 30. Thus, we will refer to the system in Figure 29 as being in terminally equivalent form if Equation (198) holds.

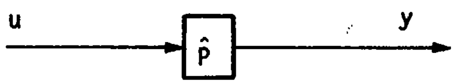


Figure 30. Desired Closed-Loop Plant

The system shown in Figure 31 will be referred to as a model-following form.

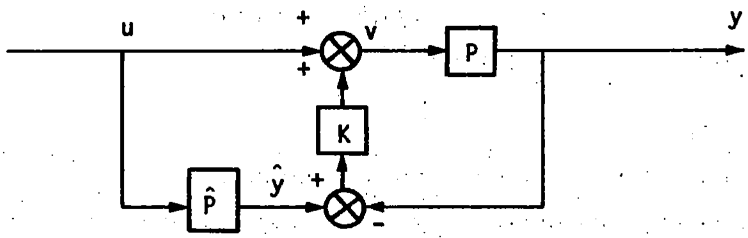


Figure 31. System in Model-Following Form

The model-following form and terminally equivalent form are related as follows:

A system is realizable in terminally equivalent form if and only if it is realizable in model-following form.

Using block diagram manipulation, we may transform Figure 31 into Figure 32 where I is the identity operator. Figure 32 is of the form of Figure 29 and is in terminally equivalent form if Equation (198) is satisfied with $G = I + K\hat{P}$ and $H = (I + K\hat{P})^{-1}K$; i.e.,

$$I + K\hat{P} = (I - (I+K\hat{P})^{-1} K\hat{P})^{-1} \tag{199}$$

This is readily verified by multiplying both sides of Equation (199) on the right by $I - (I+K\hat{P})^{-1} K\hat{P}$.

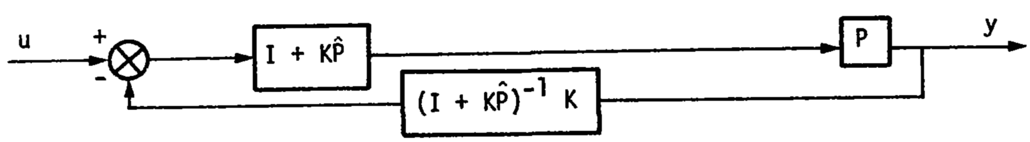


Figure 32. Equivalent Representation of Figure 6

To prove the converse, Figure 29 is equivalent to Figure 32 if we set $G = I+K\hat{P}$ and $H = (H+K\hat{P})^{-1} K$. Then Equation (199) implies that Equation (198) holds. The block diagram manipulation from Figure 31 to Figure 32 is reversible, which yields our desired result.

For simplicity the above manipulations ignore some technicalities on existence of inverses, stability of various forms, etc. What seems to be suggested, however, it the following:

Theorem 5: If a compensated system satisfies the following conditions, then it is realizable in either a terminally equivalent or model-following form:

(1) At nominal parameter values the compensated and uncompensated systems agree.

(2) It has good behavior (i.e., it is causal, bounded, internally stable).

A potential application of the model-following concept is to use a reduced order system as the model to reduce sensitivity to neglected and unknown dynamics. The plant, P, may be represented as

$$\begin{aligned} P:\ \dot{x}(t) &= F(t)x(t) + G(t)u(t) \\ y(t) &= M(t)x(t) \end{aligned} \tag{200}$$

The reduced order model is taken to be the plant

$$\hat{P}: \quad \dot{\hat{x}}(t) = \hat{F}(t)\hat{x}(t) + \hat{G}(t)\hat{u}(t)$$
$$\hat{y}(t) = \hat{M}(t)\hat{x}(t) \tag{201}$$

If $\hat{P}$ is, in some sense, a good approximation of P, then one can take the point of view that $\hat{P}$ is the nominal plant. Thus, the uncertainties and disturbances of P and the difference $P - \hat{P}$ become the undesirables for $\hat{P}$.

Two nice things could result from a sensitivity design based on $\hat{P}$ as nominal. First the complexity of compensations calculations, etc., would be reduced. Secondly, since $\hat{P} - P$ is now a disturbance, the natural tendency of P to look like $\hat{P}$ would be further enhanced.

The model-following form of the control is then

$$u = -K(y-\hat{y}) + \hat{u} \tag{202}$$

where $\hat{u}$ is the optimal control for the system (201). Equation (202) incorporates terminal equivalence, and the resulting sensitivity operator is $\$ = (I + \hat{P}K)^{-1}$. Thus the theorems dealing with the choice of K given in the discussion of terminal equivalence can be invoked.

The model-following form of the control system may be related to observer theory[49] as follows. Consider the plant P to be given in the form of Equation (200) and the nominal plant to be in the form of Equation (201) where now, however, we will assume the nominal plant is not of reduced order. We introduce a third system $\bar{P}$ given by

$$\bar{P}: \quad \dot{\bar{x}} = \bar{F}\bar{x} + \bar{G}\bar{u}$$
$$\bar{y} = \bar{M}\bar{x} \tag{203}$$

The model-following form of Figure 31 may be expressed as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} F-GKM & GK\hat{M} \\ 0 & \hat{F} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} G \\ \hat{G} \end{bmatrix} \hat{u} \tag{204}$$

The system $\bar{P}$ is called an *observer for* $\hat{P}$ *provided that* $\hat{P}$ and $\bar{P}$ are coupled so that $\bar{x} - \hat{x} \rightarrow 0$ as $t \rightarrow \infty$. One such coupling is obtained by setting $\bar{F} = \hat{F}-\hat{G}K\hat{M}$ and $\bar{G}\bar{u} = \hat{G}K\hat{y} + \hat{G}\hat{u}$. This yields the coupled system

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} \hat{F}-\hat{G}\hat{K}\hat{M} & \hat{G}\hat{K}\hat{M} \\ 0 & \hat{F} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} \hat{G} \\ \hat{G} \end{bmatrix} \hat{u} \tag{205}$$

for which $e = \bar{x} - \hat{x}$ satisfies

$$\dot{e} = (\hat{F} - \hat{G}\hat{K}\hat{M})\, e \tag{206}$$

which will be asymptotically stable if the nominal system is observable and controllable and K is chosen properly.

The similarity between Equations (204) and (205) is apparent which leads to the results:

<u>Theorem 6:</u> $(P, \hat{P})$ is in model-following form if and only if $(\hat{P} - \bar{P})$ is in observer form.

Theorems 5 and 6 provide complete correspondence between the terminally equivalent, model-following, and observer forms.

SENSITIVITY DESIGN FOR MAXIMUM DIFFICULTY

We assume that the system is represented by

$$\dot{x} = F(p)x + G_1 u + G_2 \eta, \; x(0) = x_o \tag{207}$$

where x is the state vector, p is a vector of uncertain parameters, u is the control vector, and η is a vector of white-noise inputs. The object of insensitive design is to determine an (optimal) feedback control that renders the closed-loop responses insensitive to variations in the uncertain parameters. In this development we assume that all states can be measured. Further, it is assumed that for the nominal system $(p = p_o)$ a quadratic cost functional

$$J = E\{x^T Q x + u^T R u\} \tag{208}$$

has been found such that the closed-loop nominal system responses meet or exceed specifications, where the nominal controller is

$$u = K^* x \tag{209}$$

which is optimal with respect to (207) and (208) for $p = p_o$. The problem may then be viewed as modifying the controller to account for sensitivity requirements. This problem may be approached in two ways:

- Structural modification of the controller, and
- Algebraic modification of the gains, K*.

The mismatch estimation and control concept, described in Section VI, and model following schemes exemplify the first approach. The basic idea of these concepts is to have a feedback loop in parallel with the nominal feedback controller so that the perturbed plant is forced to look like the nominal assumed plant. If there are no parameter variations, then one indeed has the optimal control. However, trying to bring the perturbed open-loop (or closed-loop) dynamics back to their nominal values is not necessarily the best thing to do! It may well be that in the perturbed situation the nominal control is entirely satisfactory and that model-following feedbacks even degrade performance.

An example of the second approach is to follow a "worst case" design in which an insensitive control $u = K_o x$ is computed a priori subject to various assumptions on the parameter variations. Thus, these variations are included directly into the design. The result will generally be improved response over the parameter range but somewhat degraded response over the situation where the actual parameters are known (or have no variations). Examples of concepts derived from this approach include additive and multiplicative noise, eigenvalue sensitivity, the multiplant concept, and the minimax concept.

The concept of sensitivity design for maximum difficulty is based on the idea that a system that performs well at a worst case will perform satisfactorily for all other cases corresponding to admissible parameter values. Finding the worst case is a matter of interpretation. The minimax concept uses this approach with the worst case defined on the basis of closed-loop responses and may require a large amount of computation. The maximum difficulty concept is aimed at finding the worst case based on some property of the open-loop system such as stability, controllability, etc., that influences closed-loop performance.

For convenience in developing measures of system difficulty, let us consider the deterministic problem. Minimize

$$J = \int_0^\infty (y^T Q y + u^T R u)\,dt \tag{210}$$

with

$$\dot{x} = F(p)x + G(p)u, \quad x(0) = x_o \tag{211}$$

$$y = H(p)x \tag{212}$$

The solution to this problem is well-known and the minimum is

$$J* = x_o^T P(p)\, x_o \tag{213}$$

with P(p) denoting the Riccati matrix which satisfies

$$O = PF + F^TP + H^TQH - PGR^{-1}G^TP \tag{214}$$

where the dependence on p is suppressed. The worst case might be defined as the value of p for which P(p) is the largest in some sense.

One of the major problems, however, is that a scalar performance criterion must be used to maximize P. Three are common:

1. Trace (P)
2. det (P)
3. The minimum eigenvalue of P

Also, we have not really kept to the goal of working with the open-loop system properties. All properties (i.e., stability, observability, and controllability) are involved. To get a rough "feel" we know heuristically that P increases as

1. F(p) becomes more unstable
2. $H^T(p)QH(p)$ becomes more positive definite
3. $G(p)R^{-1}G^T(p)$ becomes less positive definite

What we need is a measure that neatly wraps up all three!

First Proposed Metric

Foregoing use of P directly, we can make use of the almost inequalities

$$[W(0,T) + M^{-1}(0,T)]^{-1} \le P \le M(0,t) + W^{-1}(0,T) \tag{215}$$

This is not strictly correct but is a reasonable approximation, where

$$M(0, T) = \int_0^T e^{F^T \tau} H^T Q H e^{F\tau} \, d\tau \tag{216}$$

$$W(0, T) = \int_0^T e^{-F\tau} G R^{-1} G^T e^{-F^T \tau} \, d\tau \tag{217}$$

are the open-loop observability and controllability matrices, respectively. T is any time > 0. The inequality is derived by using the special control that drives the state of the system to zero at time T using minimum energy. The equation shows that to maximize P we should pick the parameter point p* that

1. Minimizes controllability, $W(0, t)$
2. Maximizes observability, $M(0, T)$

These results have intuitive appeal. The controllability part is clear. The observability part is clear if one considers that if the outputs show up strongly then so will the effects of parameter variations. However, there are difficulties with the above W and M measures arising from the nature of the inequality derivation that assumed $\underline{x}(T) = 0$. It is common for not all states to be controllable, as in noise shaping states. Thus, the system may be only stabilizable and W^{-1} will not exist. A possible solution is to use the generalized inverse so that

$$(W + M^{\dagger})^{\dagger} \leq P \leq M + W^{\dagger} \tag{218}$$

Then we pick p to either minimize $(W + M^{\dagger})$ or maximize $(M + W^{\dagger})$. The first measure seems preferable since it increases the lower bound.* Also, a scalar metric such as TR or det must be used. So,

<u>Sensitivity Problem 1:</u> Find p* to minimize

$$J_s = \mathrm{tr}\,(W + M^{\dagger}) \tag{219}$$

---

* Also, the matrix M is rarely singular in optimization problems since this would mean non-zero states did not affect $y = Hx$.

The solution p* should be very close to that needed to maximize P directly. The only arbitrary quantity here is the time T, and different choices may result in different minimizing p*. However, this is not a major drawback for linear time-invariant systems.

Second Proposed Metric

The metric J_s^o is not really a single metric but is composed of both controllability and observability. It would still be nice if both items could be rolled into one. Since the basic inequality was derived from driving $x(T) \to 0$, a logical consideration would be to drive $y(T) \to 0$ also with minimum energy. This will bring up the output controllability matrix,

$$H \int_o^T e^{-F\tau} GR^{-1}G^T e^{-F^T\tau} d\tau \, H^T = W_y(0, T) \quad (220)$$

Also, there is another consideration. While the above discussions are able to include parametric variations in the output H matrix, it may not be of interest to have low sensitivity in all of $y = Hx$ but rather in some output subset

$$y_s = \tilde{H}x \quad (221)$$

Taking all these ideas into account, the following optimization problem is posed:

Find the control to minimize

$$J = \int_o^T (y^TQy + u^TRu)\, dt \quad (210)$$

while meeting the terminal condition $\tilde{H}x(T) = 0$.

The above takes into account the design goals via J, and the desensitive design via the terminal condition. If we are interested only in the desensitive design part, set $Q = 0$ so that minimum energy output control is of concern. Once the minimum J is found, it will be necessary to maximize with respect to the parameters p.

Solution to the Optimization Problem

Hamilton-Jacobi theory is used to solve the above posed problem. Appending the terminal constraint to J via a Lagrange multiplier gives

$$J_\nu = x^T(T)\tilde{H}^T\nu + 1/2 \int_o^T (y^TQy + u^TRu)\, dt \qquad (222)$$

The Hamiltonian for this problem is

$$\mathcal{H}(\eta, x, u) = \dot{x}^T \eta + 1/2\, [x^TH^TQHx + u^TRu] \qquad (223)$$

Let S(x, t) denote the cost associated with the initial condition pair (x, t); that is, S(x, t) is given by Equation (210) with the lower limit on the integral replaced by t and the initial condition in Equation (211) replaced by x(t) = x. Then the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \mathcal{H}\left(\frac{\partial S}{\partial x}, x, u\right) = 0 \qquad (224)$$

with boundary condition

$$S(x, T) = x^T \tilde{H}^T\nu \qquad (225)$$

The control is found from $\partial\mathcal{H}/\partial u = 0$ which yields

$$u = -R^{-1}G^T \frac{\partial S}{\partial x} \qquad (226)$$

Using Equation (226) to eliminate u from Equation (224) gives

$$\frac{\partial S}{\partial t} + x^TF \frac{\partial S}{\partial x} + 1/2\, x^TH^TQHx - 1/2 \left(\frac{\partial S}{\partial x}\right)^T GR^{-1}G^T \frac{\partial S}{\partial x} = 0 \qquad (227)$$

Assuming that S(x, t) has the form

$$S(x, t) = 1/2\, x^TPx + x^Tq(t) + r(t) \qquad (228)$$

and performing the necessary differentiations and substituting in Equation (227) and using the boundary condition of Equation (225) yields

$$-\dot{P} = PF + F^TP + H^TQH - PGR^{-1}G^TP; \; P(T) = 0 \qquad (229)$$

$$\dot{q} = -(F-GR^{-1}G^TP)^Tq; \;\; q(T) = \tilde{H}^T\nu \qquad (230)$$

$$\dot{r} = 1/2\, q^TGR^{-1}G^Tq; \;\; r(T) = 0 \qquad (231)$$

The multiplier, ν, must be determined such that the terminal condition, $\tilde{H}x(T) = 0$, is satisfied. Now, $x(t)$ is given by

$$\dot{x} = (F - GR^{-1}G^T P)x - GR^{-1}G^T q, \quad x(0) = x_o \tag{232}$$

Let $\Phi(t, \tau)$ denote the fundamental matrix for Equation (232). Then

$$x(T) = \Phi(T, 0)x_o - \int_o^T \Phi(T, \tau)GR^{-1}G^T q(\tau)\, d\tau \tag{233}$$

and from Equation (230) we have

$$q(\tau) = \Phi^T(T, \tau)q(T) = \Phi^T(T, \tau)\tilde{H}^T \nu \tag{234}$$

Thus,

$$x(T) = \Phi(T, 0)x_o - \int_o^T \Phi(T, \tau)GR^{-1}G^T \Phi^T(T, \tau)d\tau\, \tilde{H}^T \nu \tag{235}$$

and $\tilde{H}x(T) = 0$ if

$$\nu = [\tilde{H} \int_o^T \Phi(T, \tau)GR^{-1}G^T \Phi^T(T, \tau)d\tau\, \tilde{H}^T]^{-1}\, \tilde{H}\, \Phi(T, 0)\, x_o \tag{236}$$

To simplify notation, let $\hat{H} = \tilde{H}\, \Phi(T, 0)$ and

$$\hat{W} = \int_o^T \Phi(0, \tau)\, GR^{-1}G^T \Phi^T(0, \tau)\, d\tau \tag{237}$$

Thus, $\hat{W}$ denotes the closed-loop controllability matrix. The minimum of Equation (222) is given by

$$J^* = S(x_o, 0) = 1/2\, x_o^T P x_o + x_o^T \hat{H}^T (\hat{H}\hat{W}\hat{H}^T)^{-1} \hat{H}x_o + r(0) \tag{238}$$

with

$$r(0) = -1/2\, x_o^T \hat{H}^T (\hat{H}\hat{W}\hat{H}^T)^{-1} \hat{H}x_o \tag{239}$$

Substituting (239) into (238) yields

$$J^* = 1/2\, x_o^T \,[P + \hat{H}^T (\hat{H}\hat{W}\hat{H}^T)^{-1} \hat{H}]x_o \tag{240}$$

The cost may be viewed as consisting of two terms, the first associated with the control task and the second associated with sensitivity.

We will choose the parameters p* to maximize the matrix $\hat{H}^T (\hat{H}\hat{W}\hat{H}^T)^{-1} \hat{H}$. At this point we must decide whether to consider open-loop versus closed-loop sensitivity. If open-loop Q = 0 and P = 0, the problem becomes simpler since $\hat{W} = W$ = usual controllability matrix. The cost functional J* is then the minimum energy to drive $\tilde{H}x(T) = 0$. Choosing the open-loop case yields:

<u>Sensitivity Problem 2:</u> Find p* to maximize

$$J_s = \mathrm{tr}\, [\hat{H}^T (\hat{H}\hat{W}\hat{H}^T)^{-1} \hat{H}]$$

where $\hat{H} = \tilde{H}\Phi(T, 0)$ and T is arbitrary.

There are many interesting interpretations of this sensitivity metric. Note that both output and control measures are included (one can set $\tilde{H} = H$ if all output sensitivities are of concern). If $\tilde{H}^{-1}$ exists, then the problem reduces to full state controllability since the only way to get $\tilde{H}x = 0$ is to get $x(T) = 0$. Also, there are many similarities to the metric proposed under problem 1.

SECTION IX

CONCLUSION

The results of the study of synthesis techniques for the design of insensitive aircraft control systems indicate that reduced sensitivity to model uncertainties can be achieved. Specifically, when sensitivity reduction was equated with performance improvement, two synthesis techniques produced controllers which demonstrated a significant improvement in performance over that of a nominal controller designed with no regard to sensitivity considerations. These techniques were the minimax synthesis technique and the uncertainty weighting synthesis technique. The former technique was formulated based on existing minimax theory, and while it performed well on the design example, it has the potential for severe computational requirements. The latter technique was based on a new concept developed in the study which was termed the uncertainty weighting concept. The controller designed with this technique equaled the performance of the minimax controller. In addition, the uncertainty weighting synthesis procedure is straightforward and suffers none of the computational burden of the minimax concept.

On the negative side, study results indicate that the sensitivity vector augmentation synthesis technique is not well suited for use on a controller design problem of the scale of the C-5A example. In all performance evaluations, the sensitivity vector augmentation controller could not compete with the other controllers. It must be emphasized that the results presented here do not preclude use of the sensitivity vector augmentation concept on other control problems which require sensitivity reduction. In this study, however, the range of the uncertain parameters that were investigated, the number of approximations that were necessary for the technique to be workable, and the requirement for sensitivity state feedback were major deterrents to a successful formulation.

The other techniques (additive noise, multiplant, state dependent noise, mismatch estimation, and re-residualization) produced controllers whose performance grouped rather closely about the performance of the nominal. The limited performance improvement that was demonstrated for these techniques compromises the additional design effort that was required. The additive noise technique did demonstrate better performance but at the expense of an order of magnitude increase in control requirements. The multiplant controller produced results very close to the nominal. The state dependent noise controller generally behaved poorer than the nominal; however, it is felt that some tuning may improve performance. The controller designed with the newly developed mismatch estimation synthesis technique did not score very high on the overall evaluation but did demonstrate significantly improved performance at some evaluation conditions. Since this was a new concept, it is felt that further refinements are required before making final judgment. The controller designed with the re-residualized reduced order model virtually duplicated the results of the residualization based nominal controller. It is felt that the relatively weak coupling between the system modes that were eliminated and those that were retained did not fully exercise the re-residualization concept. The results presented in Appendix E, however, indicate that re-residualization can maintain high order system characteristics in the reduced order model that residualization cannot.

To summarize the major conclusion of the study, the uncertainty weighting and minimax syntheses techniques can improve performance with respect to variations from the design condition. The latter technique has potential computational problems. The sensitivity vector augmentation technique was not applicable to the C-5A design example. The remaining techniques do not offer much improvement over nominal techniques in their current formulation.

These conclusions have relied upon a solid basis for the final comparative evaluations. This basis was constructed by 1) determining qualitative and quantitative design criteria which specified sensitivity in terms of performance, 2) defining a broad category of design model variations which consisted of model parameter uncertainties and both known and assumed unknown neglected dynamics, 3) selecting realistic model parameter uncertainties (dynamic pressure, structural frequency and damping, M_W), known neglected dynamics (structural modes), and assumed unknown neglected dynamics (unsteady aerodynamics), 4) determining the range of uncertain parameter variation through experimental search for design specification violation conditions, and 5) defining a practical set of insensitive controller evaluation criteria, both qualitative and quantitative.

Extending from these conclusions are the following recommendations:

- The minimax or uncertainty weighting concept should be used for current applications with uncertainty weighting being preferred because of the potential computational load required for minimax design.

- Further developments of design methodology for the uncertainty weighting, mismatch estimation, and state dependent noise concepts should be investigated before a final judgment is made.

- Of the new concepts for which controllers were not synthesized, the finite dimensional inverse, maximum difficulty, and dual Lyapunov equation concepts should be further developed and resulting designs compared with the existing synthesis techniques.

- The real world constraint of limited measurements could compromise if not negate the sensitivity reduction obtained with the techniques described in this report. Investigation of the effects of these constraints is recommended for future study.

APPENDIX A

DESIGN AND EVALUATION MODELS

APPENDIX A

DESIGN AND EVALUATION MODELS

All models are formulated in the following form:

$$\dot{x} = Fx + G_1 u + G_2 \eta \qquad \text{(A-1)}$$

$$r = Hx + Du \qquad \text{(A-2)}$$

Six cases were used for design and/or evaluation. These have been labeled

Case 1
Case 2
Case 3R
Case 3T
Case 4R
Case 4T

Case 1 was described in Section IV. The system states and responses were given in Tables 2 and 3. Table 34 contains the F, G_1, G_2, H_1, and D matrices for Case 1, and Table 35 presents eigenvalues and statistical response data.

The remaining cases are defined in Table 36. The F, G_1, G_2, H, and D matrices and accompanying eigenvalue and statistical response data are given in Tables 37 through 46.

(Note: The data presented in the tables containing the F, G_1, G_2, H, and D matrices are in computer card image form. Only the non-zero elements of each matrix are shown. There are five matrix elements per card with the row index specified in the first two columns, the column index specified in the next two columns, and the value of the matrix element given in exponential format in the next 12 columns.)

F-MATRIX FOR CASE 1

1 1	-.66218E+00	1 2	.32776E+01	1 3	-.28017E-01	1 4	-.69419E-02	1 5	-.23237E-01
1 6	.14600E-01	1 7	-.24692E-01	1 8	.50997E-01	1 9	.89406E-01	110	-.41943E-02
111	-.36092E-01	112	-.98817E-02	113	-.78981E-02	114	.14250E-01	115	-.15659E-01
116	-.30474E-01	117	.22346E+00	118	-.55979E+00	119	-.58786E+00	120	-.15561E+01
121	.40758E+01	122	-.13389E+01	123	.20001E+01	124	.86617E+01	125	-.19745E+00
126	-.23875E+01	127	-.45817E+00	128	-.39201E+00	129	.19497E+01	130	-.14466E+01
131	-.32445E+01	132	.32607E+02	133	-.24690E+03	134	-.20139E+03	135	-.40941E+02
136	-.48934E+00	137	-.67639E+01	138	-.98990E+00	143	-.53777E+00	144	-.61617E-01
145	.31474E+00	146	-.44937E+00	147	-.13379E+01	148	.35371E+01	149	-.74565E+00
150	.69614E+00	151	.64374E+01	152	.58554E-01	153	-.91028E+00	154	.15883E+00
155	.53838E-01	156	.19618E+01	157	-.10967E+01	158	-.27490E+01	159	.27778E+02
160	-.56393E-01	161	-.82787E-01	162	-.48040E+00	163	.90061E-01	164	.69993E+00
165	.65671E+00	166	.15334E+00	167	-.29326E+00	168	-.19208E-01	169	-.40373E-01
170	-.23787E+00	171	-.12419E+00	172	-.59290E-01	173	-.10445E+00	174	.10298E+00
175	.18357E+00	176	.30302E+00	177	-.24690E+03	178	-.20139E+03	179	-.40941E+02
2 1	-.56837E+00	2 2	-.12293E+01	2 3	.58290E-01	2 4	-.46882E-01	2 5	-.34790E+00
2 6	-.35559E+00	2 7	-.12779E+00	2 8	.27254E+00	2 9	.27873E-01	210	-.13122E+00
211	-.22584E+00	212	-.15208E+00	213	-.13356E+00	214	-.11210E+00	215	-.10421E+00
216	-.77966E-01	217	.35119E+00	218	.54826E-01	219	-.16428E+01	220	-.10445E+02
221	-.68234E+01	222	-.51239E+01	223	.98465E+01	224	.69654E+01	225	-.72091E+01
226	-.13368E+02	227	-.77423E+01	228	-.64829E+01	229	-.35922E+01	230	-.58873E+01
231	-.60006E+01	232	.53301E+02	233	-.60551E+03	234	-.26235E+04	235	-.56095E+03
236	.27253E+01	237	-.80421E+00	238	-.12530E+02	243	-.75308E-01	244	-.49520E-01
245	.12353E+01	246	-.45015E+00	247	-.18200E+01	248	.35175E+01	249	-.78860E+00
250	-.24959E+00	251	.54178E+01	252	-.18738E+01	253	-.16688E+01	254	.33011E+00
255	.31507E+00	256	.27096E+01	257	-.19621E+01	258	-.42261E+01	259	.42222E+02
260	-.71946E+00	261	-.10568E+01	262	-.61363E+01	263	.11562E+01	264	.89852E+01
265	.84574E+01	266	.19869E+01	267	-.38185E+01	268	-.25341E+00	269	-.47473E+00
270	-.29223E+01	271	-.14905E+01	272	-.66956E+00	273	-.12448E+01	274	.13838E+01
275	.23839E+01	276	.38615E+01	277	-.60551E+03	278	-.26235E+04	279	-.56095E+03
3 1	-.12620E+01	3 2	.16571E+00	3 3	-.93166E+00	3 4	-.19642E-01	3 5	-.11204E+00
3 6	.48658E+00	3 7	.25263E-01	3 8	-.25497E+00	3 9	.17433E+00	310	-.30890E-01
311	.28651E-01	312	.10389E+00	313	.96035E-01	314	.23093E+00	315	-.89069E-01
316	-.23731E+00	317	.22748E+01	318	-.29214E+02	319	-.35241E+01	320	-.14694E+02
321	.37541E+02	322	-.52607E+01	323	-.63746E+01	324	.44558E+02	325	-.10078E+02
326	-.73334E+01	327	.74485E+01	328	.67119E+01	329	.27593E+02	330	-.17349E+02
331	-.38705E+02	332	.35564E+03	333	-.35548E+04	334	.14284E+04	335	.30829E+03
336	.57304E+00	337	-.20517E+02	338	.68753E+01	343	-.17476E+01	344	-.40282E+00
345	.78138E+01	346	-.31825E+01	347	-.11241E+02	348	.25805E+02	349	-.59674E+01
350	.25667E+00	351	.41193E+02	352	-.10396E+02	353	-.12264E+02	354	.26922E+01
355	.23846E+01	356	.20189E+02	357	-.15651E+02	358	-.33071E+02	359	.30806E+03
360	.39401E+00	361	.57874E+00	362	.33603E+01	363	-.63313E+00	364	-.49205E+01
365	-.46322E+01	366	-.10885E+01	367	.20915E+01	368	.13883E+00	369	.25937E+00
370	.15979E+01	371	.81217E+00	372	.36274E+00	373	.67734E+00	374	-.76274E+00
375	-.13093E+01	376	-.21167E+01	377	-.35548E+04	378	.14284E+04	379	.30829E+03
4 1	.35519E-01	4 2	-.95920E-01	4 3	.14365E-01	4 4	-.46616E+00	4 5	-.70439E-01
4 6	-.60690E-01	4 7	-.86751E-02	4 8	-.12895E-02	4 9	-.29856E-01	410	-.57857E-02
411	-.21051E-01	412	-.22688E-01	413	-.11339E-01	414	-.22035E-01	415	-.34194E-01
416	-.21614E-02	417	-.14305E-01	418	.26115E+00	419	-.12387E+03	420	-.93701E+00
421	-.16216E+01	422	-.17073E+00	423	.15038E+01	424	-.43895E+01	425	-.69265E+00
426	-.11279E+01	427	-.16325E+01	428	-.26963E-01	429	-.81533E+00	430	-.21706E+01
431	.57163E+00	432	.34758E+00	433	-.13414E+03	434	-.30139E+03	435	-.77226E+02
436	-.20616E+00	437	.16060E+01	438	-.13990E+01	443	.12039E+00	444	.11129E-01
445	.16683E+00	446	.12855E+00	447	.48804E+00	448	-.85502E+00	449	.13866E+00
450	-.18298E+00	451	-.16680E+01	452	-.18694E+00	453	-.95956E-01	454	.72823E-01

F MATRIX FOR CASE 1 (CONTINUED)

455	.10621E+00	456	-.10533E+00	457	-.29904E+00	458	-.30804E+00	459	-.76569E+00
460	-.80774E-01	461	-.11887E+00	462	-.69134E+00	463	.13219E+00	464	.10271E+01
465	.97637E+00	466	.23352E+00	467	-.45484E+00	468	-.31302E-01	469	-.39107E-01
470	-.28755E+00	471	-.13125E+00	472	-.41060E-01	473	-.10522E+00	474	.18499E+00
475	.28665E+00	476	.43471E+00	477	-.13414E+03	478	-.30139E+03	479	-.77226E+02
5 1	-.76870E-03	5 2	-.91786E+00	5 3	.63949E-01	5 4	-.76045E-01	5 5	-.13099E+01
5 6	-.43621E+00	5 7	-.43833E-01	5 8	-.42346E-01	5 9	-.44043E+00	510	-.54702E-01
511	-.26611E+00	512	-.15270E+00	513	-.14680E+00	514	-.16376E+00	515	-.18148E+00
516	-.18341E+00	517	.44391E-02	518	.14951E+01	519	-.72542E+00	520	-.19351E+03
521	-.16261E+02	522	-.11716E+01	523	-.25174E+01	524	-.21711E+02	525	-.63347E+01
526	-.17814E+02	527	-.67761E+01	528	-.57960E+01	529	-.61759E+01	530	-.95811E+01
531	-.13363E+02	532	.50916E+02	533	-.14545E+04	534	-.26722E+04	535	-.70943E+03
536	-.22475E+01	537	.10995E+02	538	-.12582E+02	543	.85901E+00	544	.52165E-01
545	.28433E+01	546	.89611E+00	547	.40636E+01	548	-.51444E+01	549	.48272E+00
550	-.13436E+01	551	-.10909E+02	552	-.24888E+01	553	-.32955E+01	554	.13575E+01
555	.16760E+01	556	.24355E+01	557	-.65613E+01	558	-.99129E+01	559	.43848E+02
560	-.72639E+00	561	-.10692E+01	562	-.62184E+01	563	.11903E+01	564	.92505E+01
565	.88003E+01	566	.21080E+01	567	-.41105E+01	568	-.28371E+00	569	-.33899E+00
570	-.25444E+01	571	-.11373E+01	572	-.32583E+00	573	-.90225E+00	574	.17083E+01
575	.26103E+01	576	.39200E+01	577	-.14545E+04	578	-.26722E+04	579	-.70943E+03
6 1	-.11317E+01	6 2	-.10485E+01	6 3	.94093E-01	6 4	-.41706E-01	6 5	-.29513E+00
6 6	-.11228E+01	6 7	-.18509E+00	6 8	.46229E+00	6 9	.28063E+00	610	-.13900E+00
611	-.17667E+00	612	-.21490E+00	613	-.15173E+00	614	-.12465E+00	615	-.18933E+00
616	-.83926E-01	617	.22999E+00	618	-.49739E-01	619	-.68533E+00	620	-.96007E+01
621	-.24366E+03	622	-.73565E+01	623	.17716E+02	624	.12927E+02	625	-.78022E+01
626	-.11319E+02	627	-.12021E+02	628	-.58092E+01	629	-.47687E+01	630	-.71866E+01
631	.97311E+00	632	.33548E+02	633	.78907E+03	634	-.26279E+04	635	-.75360E+03
636	-.99210E+00	637	-.40611E+01	638	-.12268E+02	643	-.29818E+00	644	-.56881E-01
645	-.47038E-02	646	-.78646E+00	647	-.34870E+01	648	.54284E+01	649	-.99924E+00
650	.48857E-01	651	.91696E+01	652	-.15350E+01	653	-.46002E+00	654	-.16803E+00
655	-.30948E+00	656	.18071E+01	657	.52544E+00	658	-.43857E+00	659	.26164E+02
660	-.71047E+00	661	-.10468E+01	662	-.60919E+01	663	.11740E+01	664	.91233E+01
665	.87176E+01	666	.21048E+01	667	-.41281E+01	668	-.28922E+00	669	-.27276E+00
670	-.23188E+01	671	-.95951E+00	672	-.17302E+00	673	-.73488E+00	674	.17997E+01
675	.26373E+01	676	.38422E+01	677	.78907E+03	678	-.26279E+04	679	-.75360E+03
7 6	-.11610E+00	7 7	-.86288E+00	7 8	.13017E+00	7 9	.56848E-01	710	-.35799E-01
7 1	-.21536E+00	7 2	-.25291E+00	7 3	.63700E-01	7 4	-.21769E-01	7 5	-.55068E-01
711	-.63619E-01	712	-.68431E-02	713	-.91757E-01	714	-.16280E-01	715	-.11195E+00
716	.91292E-01	717	.13750E-01	718	.25469E+00	719	-.72510E+00	720	-.14228E+01
721	-.33588E+01	722	-.30664E+03	723	.62918E+01	724	.48704E+01	725	-.24640E+01
726	-.44727E+01	727	.13600E+01	728	-.58688E+01	729	-.35973E+00	730	-.71001E+01
731	.11813E+02	732	-.67247E+01	733	-.10291E+02	734	-.67998E+03	735	-.21418E+03
736	-.32477E+00	737	-.70521E-01	738	-.31534E+01	743	-.31705E-01	744	-.19425E-02
745	-.35171E+00	746	-.10075E+00	747	-.60307E+00	748	.42189E+00	749	.10880E-01
750	.35842E-01	751	.98065E+00	752	.18442E+00	753	.60773E+00	754	-.23862E+00
755	-.26513E+00	756	-.58289E+00	757	.11683E+01	758	.18811E+01	759	-.93515E+01
760	-.18312E+00	761	-.27011E+00	762	-.15733E+01	763	.30579E+00	764	.23764E+01
765	.22834E+01	766	.55667E+00	767	-.10996E+01	768	-.78438E-01	769	-.50768E-01
770	-.54098E+00	771	-.19579E+00	772	.45796E-02	773	-.13976E+00	774	.50770E+00
775	.70844E+00	776	.99344E+00	777	-.10291E+02	778	-.67998E+03	779	-.21418E+03
8 1	.48110E+00	8 2	.45053E+00	8 3	-.11352E+00	8 4	.19362E-01	8 5	.45570E-01
8 6	.20063E+00	8 7	.10432E+00	8 8	-.11603E+01	8 9	-.24073E+00	810	.78782E-01
811	.53088E-01	812	.78267E-01	813	.12550E+00	814	.95678E-01	815	-.50959E-01
816	.70074E-01	817	-.91262E-01	818	-.15696E+00	819	.19545E+01	820	.17357E+01
821	.47678E+01	822	.49136E+01	823	-.35292E+03	824	-.15028E+02	825	.36451E+01
826	.42262E+01	827	.32839E+01	828	.67126E+01	829	.64133E+01	830	-.98954E+01

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F MATRIX FOR CASE 1 (CONTINUED)
 831  .53454E+00 832  .17075E+02 833 -.42360E+03 834  .11629E+04 835  .38364E+03
 836  .43958E+00 837  .16028E+01 838  .53777E+01 843  .17391E+00 844  .16852E-01
 845  .10778E+01 846  .46646E+00 847  .26915E+01 848 -.26075E+01 849  .20692E+00
 850  .33896E+00 851 -.43578E+01 852  .37718E+00 853 -.14613E+01 854  .67474E+00
 855  .80593E+00 856  .11674E+01 857 -.34184E+01 858 -.52679E+01 859  .19132E+02
 860  .31272E+00 861  .46157E+00 862  .26898E+01 863 -.52507E+00 864 -.40806E+01
 865 -.39319E+01 866 -.96332E+00 867  .19096E+01 868  .13742E+00 869  .69164E-01
 870  .87300E+00 871  .28821E+00 872 -.51887E-01 873  .19390E+00 874 -.90643E+00
 875 -.12358E+01 876 -.16996E+01 877 -.42360E+03 878  .11629E+04 879  .38364E+03
 9 1 -.37983E+00 9 2 -.55269E-01 9 3 -.10552E+00 9 4 -.64061E-02 9 5 -.50959E-01
 9 6  .96177E-01 9 7 -.25539E-02 9 8 -.55047E-01 9 9 -.87550E+00 910  .84688E-02
 911 -.61592E-01 912  .39394E-01 913  .33352E-02 914  .32776E-01 915 -.22464E-01
 916 -.89065E-01 917  .15490E+00 918 -.10754E+01 919 -.15547E+01 920 -.23622E+01
 921  .47058E+01 922  .11709E+00 923 -.34107E+01 924 -.39258E+03 925  .77745E-01
 926 -.42830E+01 927  .31505E+01 928 -.69322E+00 929  .44623E+01 930 -.20029E+01
 931 -.10128E+02 932  .54795E+02 933 -.77082E+02 934  .69060E+02 935  .24520E+02
 936  .39132E+00 937 -.51706E+01 938  .31886E+00 943 -.35286E+00 944 -.65435E-01
 945  .12989E+01 946 -.43857E+00 947 -.89740E+00 948  .37853E+01 949 -.10200E+01
 950  .94695E+00 951  .70772E+01 952 -.61166E+00 953 -.22436E+01 954  .58798E+00
 955  .56027E+00 956  .36780E+01 957 -.35140E+01 958 -.70468E+01 959  .54932E+02
 960  .18522E-01 961  .27367E-01 962  .15960E+00 963 -.31387E-01 964 -.24394E+00
 965 -.23615E+00 966 -.58329E-01 967  .11631E+00 968  .84882E-02 969  .23369E-02
 970  .46598E-01 971  .12422E-01 972 -.75196E-02 973  .69900E-02 974 -.57656E-01
 975 -.75813E-01 976 -.10096E+00 977 -.77082E+02 978  .69060E+02 979  .24520E+02
10 1 -.37519E+0010 2 -.19915E+0010 3  .78391E-0110 4  .58402E-0310 5  .66927E-02
10 6 -.57961E-0110 7 -.25543E-0110 8  .90318E-0110 9  .47146E-011010 -.11075E+01
1011 -.77572E-011012 -.52878E-011013 -.41357E-011014 -.43910E-011015 -.15384E-01
1016 -.81760E-021017 -.76703E-011018  .36342E+001019 -.46182E-011020  .40640E+00
1021 -.19296E+011022 -.17498E+011023  .35783E+011024  .25782E+011025 -.74025E+03
1026 -.23150E+011027 -.26347E+011028 -.17795E+011029 -.30680E+011030  .12924E+01
1031  .86341E+001032 -.17573E+021033 -.74070E+021034 -.42369E+031035 -.22855E+03
1036  .53623E+001037 -.36219E+011038 -.20487E+011043 -.24705E+001044 -.59292E-02
1045 -.73806E+001046 -.14252E-011047  .13266E+001048 -.13893E+001049  .12878E+00
1050  .66985E+001051  .60701E+001052  .21655E+011053  .77811E+001054 -.22775E+00
1055 -.28622E+001056 -.94056E+001057  .11736E+011058  .20492E+011059 -.16363E+02
1060 -.12003E+001061 -.17847E+001062 -.10446E+011063  .21390E+001064  .16640E+01
1065  .16520E+011066  .42564E+001067 -.87325E+001068 -.68071E-011069  .52032E-01
1070 -.10371E+001071  .10431E+001072  .22740E+001073  .13546E+001074  .53793E+00
1075  .60181E+001076  .67237E+001077 -.74070E+021078 -.42369E+031079 -.22855E+03
11 1 -.20075E+0111 2 -.19147E+0111 3  .45368E+0011 4 -.72765E-0111 5 -.57468E+00
11 6 -.66399E+0011 7 -.17396E+0011 8  .38344E+0011 9 -.18370E+001110 -.94098E-01
1111 -.23055E+011112 -.62163E+001113 -.50184E+001114 -.57649E+001115 -.74054E+00
1116 -.58281E+001117 -.13913E+011118  .38616E+011119  .97200E+001120 -.53094E+01
1121 -.21074E+021122 -.11679E+021123  .27641E+021124 -.13670E+021125  .62737E+01
1126 -.11151E+041127 -.30055E+021128 -.16156E+021129 -.26872E+021130 -.23799E+02
1131 -.13880E+021132 -.10372E+031133 -.10275E+041134 -.35972E+041135 -.31896E+04
1136  .23869E+011137 -.12360E+021138 -.20415E+021143 -.10411E+011144  .13579E-01
1145 -.20017E+011146  .10538E+011147  .84077E+011148 -.78455E+011149  .11618E+01
1150  .77291E+011151 -.23052E+011152  .12281E+021153  .40057E+011154 -.54081E+00
1155 -.46459E+001156 -.44276E+011157  .13030E+011158  .30710E+011159 -.77927E+02
1160 -.11959E+011161 -.17934E+011162 -.10537E+021163  .22632E+011164  .17638E+02
1165  .18010E+021166  .48485E+011167 -.10229E+021168 -.84549E+001169  .13892E+01
1170  .15757E+011171  .35199E+011172  .46819E+011173  .37943E+011174  .76284E+01
1175  .75469E+011176  .70061E+011177 -.10275E+041178 -.35972E+041179 -.31896E+04
12 1 -.24694E+0112 2 -.37095E+0112 3  .55723E+0012 4 -.17836E+0012 5 -.14560E+01
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F MATRIX FOR CASE 1 (CONTINUED)
12 6 -.16376E+0112 7 -.65109E+0012 8  .13833E+0112 9  .94993E-011210 -.64967E+00
1211 -.15420E+011212 -.29582E+011213 -.15915E+011214 -.14920E+011215 -.11792E+01
1216 -.68848E+001217  .56209E+001218  .30512E+011219 -.10787E+021220 -.26684E+02
1221 -.53151E+021222  .15942E+021223  .14840E+021224  .34064E+021225 -.30431E+02
1226 -.78141E+021227 -.14307E+041228 -.75811E+021229 -.50402E+021230 -.36113E+02
1231 -.14196E+021232  .52383E+021233  .50725E+031234 -.50985E+041235 -.75441E+04
1236  .67629E+001237 -.14909E+011238 -.41524E+021243 -.12032E+001244 -.10825E-01
1245  .32479E+001246 -.10294E+001247 -.18559E+011248  .19591E+011249 -.16048E+00
1250 -.40101E+011251 -.38515E+011252 -.37792E+011253 -.26821E+011254  .73088E+00
1255  .60145E+001256  .20410E+011257 -.22179E+011258 -.33302E+011259  .34028E+02
1260 -.24128E+011261 -.36380E+011262 -.21384E+021263  .46995E+011264  .36721E+02
1265  .38009E+021266  .10446E+021267 -.22310E+021268 -.18896E+011269  .38182E+01
1270  .63729E+011271  .10351E+021272  .12694E+021273  .10945E+021274  .18637E+02
1275  .17554E+021276  .14812E+021277  .50725E+031278 -.50985E+041279 -.75441E+04
13 1 -.17448E+0113 2 -.36254E+0113 3  .46329E+0013 4 -.17596E+0013 5 -.15314E+01
13 6 -.20034E+0113 7 -.55334E+0013 8  .15209E+0113 9  .37598E+001310 -.72919E+00
1311 -.13917E+011312 -.16930E+011313 -.30940E+011314 -.15821E+011315 -.13130E+01
1316 -.82367E+001317  .80444E+001318  .29161E+011319  .30446E+011320 -.36387E+02
1321 -.26717E+021322 -.43714E+021323  .56387E+021324 -.30321E+021325 -.31614E+02
1326 -.51969E+021327 -.83134E+021328 -.15393E+041329 -.56930E+021330 -.49877E+02
1331 -.29688E+021332  .74957E+021333  .26129E+031334 -.43554E+041335 -.78570E+04
1336 -.12348E+011337  .98173E+011338 -.42470E+021343  .81535E+001344  .18343E-01
1345  .10908E+011346 -.40820E+001347 -.50085E+011348  .32450E+011349 -.19190E+00
1350 -.67498E+011351 -.42881E+011352 -.81635E+011353 -.28955E+011354  .50919E+00
1355  .49564E+001356  .20753E+011357 -.95702E+001358 -.12630E+011359  .39155E+02
1360 -.24591E+011361 -.37111E+011362 -.21801E+021363  .48001E+011364  .37539E+02
1365  .38918E+021366  .10720E+021367 -.22929E+021368 -.19470E+011369  .40303E+01
1370  .69914E+011371  .11130E+021372  .13549E+021373  .11769E+021374  .19671E+02
1375  .18404E+021376  .15309E+021377  .26129E+031378 -.43554E+041379 -.78570E+04
14 1 -.13266E+0114 2 -.11113E+0114 3 -.95060E-0114 4 -.84651E-0114 5 -.44431E+00
14 6 -.35689E+0014 7 -.18637E+0014 8  .34980E+0014 9  .14949E+001410 -.21786E+00
1411 -.45392E+001412 -.43391E+001413 -.43766E+001414 -.19754E+011415 -.42109E+00
1416 -.27609E+001417  .96969E+001418 -.17827E+011419 -.23750E+011420 -.15745E+02
1421  .43094E+011422 -.19124E+011423  .13931E+021424  .19740E+021425 -.13529E+02
1426 -.23170E+021427 -.13850E+021428 -.14601E+021429 -.15774E+041430 -.26704E+02
1436  .68405E+001437 -.92678E+011438 -.10553E+021443 -.71194E+001444 -.15055E+00
1445  .24810E+011446 -.13995E+011447 -.55400E+011448  .11488E+021449 -.25022E+01
1450 -.11311E+011451  .16066E+021452 -.51046E+011453 -.56653E+011454  .12560E+01
1431 -.14400E+021432  .15494E+031433  .10260E+041434 -.89608E+031435 -.21997E+04
1455  .10690E+011456  .86714E+011457 -.66909E+011458 -.13886E+021459  .13177E+03
1460 -.61105E+001461 -.92632E+001462 -.54520E+011463  .12284E+011464  .96153E+01
1465  .10090E+021466  .28284E+011467 -.61113E+011468 -.52914E+001469  .12387E+01
1470  .24502E+011471  .34465E+011472  .40307E+011473  .35973E+011474  .55142E+01
1475  .50027E+011476  .38918E+011477  .10260E+041478 -.89608E+031479 -.21997E+04
15 1  .19477E-0115 2 -.58489E+0015 3  .20441E+0015 4 -.46303E-0115 5 -.30807E+00
15 6 -.41732E+0015 7 -.97993E-0115 8  .35500E+0015 9 -.81434E-011510 -.62737E-01
1511 -.38106E+001512 -.27815E+001513 -.28703E+001514 -.32015E+001515 -.20359E+01
1516 -.46163E+001517 -.98539E+001518  .18724E+011519 -.26872E+011520  .30788E+00
1521 -.19265E+021522 -.14966E+021523 -.66485E+011524  .55132E+011525  .17121E+01
1526 -.12497E+021527 -.10826E+021528 -.12680E+021529 -.21024E+021530 -.17384E+04
1531 -.25208E+021532 -.91848E+021533 -.47353E+031534 -.44453E+031535 -.15745E+04
1536 -.80869E+001537  .68469E+011538 -.83214E+011543  .55391E+001544  .72406E-01
1545 -.80653E+001546  .59725E+001547  .31210E+011548 -.55321E+011549  .94920E+00
1550  .27588E+011551 -.36625E+011552  .28172E+011553  .37746E+011554 -.93512E+00
1555 -.76684E+001556 -.43268E+011557  .38812E+011558  .71134E+011559 -.65753E+02
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F MATRIX FOR CASE 1 (CONTINUED)
1560 -.47796E+001561 -.72246E+001562 -.42357E+011563  .93298E+001564  .73130E+01
1565  .75919E+011566  .20961E+011567 -.44901E+011568 -.38233E+001569  .81808E+00
1570  .15042E+011571  .23565E+011572  .28441E+011573  .25008E+011574  .40730E+01
1575  .37750E+011576  .30718E+011577 -.47353E+031578 -.44453E+031579 -.15745E+04
16 1  .53424E+0016 2 -.22955E+0016 3  .24430E+0016 4 -.20947E-0116 5 -.11441E+00
16 6 -.36665E+0016 7 -.13219E+0016 8  .20594E+0016 9 -.69189E-011610 -.14963E-01
1611 -.20857E+001612 -.14532E+001613 -.15596E+001614 -.16790E+001615 -.35235E+00
1616 -.19016E+011617 -.12405E+011618  .26336E+011619  .20826E+011620  .22394E+01
1621 -.14040E+021622  .13016E+021623  .19706E+021624 -.15890E+021625  .31093E+00
1626 -.33716E+011627 -.60378E+011628 -.80368E+011629 -.99897E+011630 -.18921E+02
1631 -.18401E+041632 -.14065E+031633 -.84061E+031634 -.19840E+031635 -.87244E+03
1636 -.91625E+001637  .10848E+021638 -.50395E+011643  .88225E+001644  .13403E+00
1645 -.18605E+011646  .10974E+011647  .46843E+011648 -.96992E+011649  .19350E+01
1650  .24789E+011651 -.10393E+021652  .40085E+011653  .59839E+011654 -.14709E+01
1655 -.12493E+011656 -.78533E+011657  .69372E+011658  .13368E+021659 -.11887E+03
1660 -.28830E+001661 -.43464E+001662 -.25412E+011663  .54929E+001664  .43083E+01
1665  .44299E+011666  .12063E+011667 -.25629E+011668 -.21477E+001669  .41572E+00
1670  .68797E+001671  .12312E+011672  .15371E+011673  .13283E+011674  .23069E+01
1675  .21836E+011676  .18533E+011677 -.84061E+031678 -.19840E+031679 -.87244E+03
17 1 -.92163E+0017 2 -.18082E+0017 3 -.21466E+0017 4 -.17658E-0117 5 -.10524E+00
17 6  .11402E+0017 7 -.26968E-0117 8 -.14242E-0117 9  .10208E+001710 -.10156E-01
1711 -.64258E-011712  .86108E-021713  .76977E-021714  .67624E-011715 -.71534E-01
1716 -.12792E+001717 -.12395E+011718 -.24700E+011719 -.16375E+011720 -.66277E+01
1721  .13586E+021722 -.29476E+011723  .12621E+011724  .20395E+021725 -.28539E+01
1726 -.62156E+011727  .99626E+001728  .10153E+011729  .96046E+011730 -.81471E+01
1731 -.15726E+021732 -.24167E+041733  .82144E+031734 -.45825E+011735 -.19786E+02
1736 -.28069E-011737 -.10869E+021738 -.23249E+001743 -.94076E+001744 -.16570E+00
1745  .22936E+011746 -.14001E+011747 -.45551E+011748  .11168E+021749 -.25524E+01
1750  .11751E+011751  .19090E+021752 -.26063E+011753 -.46454E+011754  .10039E+01
1755  .82667E+001756  .81824E+011757 -.61330E+011758 -.13351E+021759  .12242E+03
1760 -.13194E-011761 -.19581E-011762 -.11306E+001763  .21893E-011764  .17192E+00
1765  .16590E+001766  .40845E-011767 -.81272E-011768 -.58962E-021769 -.76836E-03
1770 -.25848E-011771  .44948E-021772  .20871E-011773  .10691E-011774  .61627E-01
1775  .71216E-011776  .81945E-011777  .82144E+031778 -.45825E+011779 -.19786E+02
18 3  .10000E+01
19 4  .10000E+01
20 5  .10000E+01
21 6  .10000E+01
22 7  .10000E+01
23 8  .10000E+01
24 9  .10000E+01
2510  .10000E+01
2611  .10000E+01
2712  .10000E+01
2813  .10000E+01
2914  .10000E+01
3015  .10000E+01
3116  .10000E+01
3217  .10000E+01
3333 -.60000E+01
3434 -.75000E+01
3535 -.75000E+01
3636 -.22185E+023642  .22185E+02
3737 -.85492E+013740  .85492E+01
3836 -.50960E+013839  .10000E+01
```

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F MATRIX FOR CASE 1 (CONTINUED)
3936   .90891E+023938 -.38953E+023939 -.10192E+02
4040 -.10983E+024042  .10983E+02
4141 -.49524E+004142 -.61315E-01
4241   .10000E+01
43 1   .33109E+0043 2 -.16388E+0143 3  .14008E-0143 4  .34710E-0243 5  .11618E-01
43 6 -.73002E-0243 7  .12346E-0143 8 -.25498E-0143 9 -.44703E-014310  .20971E-02
4311   .18046E-014312  .49408E-024313  .39491E-024314 -.71249E-024315  .78295E-02
4316   .15237E-014317 -.11173E+004318  .27989E+004319  .29393E+004320  .77806E+00
4321 -.20379E+014322  .66943E+004323 -.10001E+014324 -.43308E+014325  .98724E-01
4326   .11937E+014327  .22908E+004328  .19600E+004329 -.97486E+004330  .72330E+00
4331   .16222E+014332 -.16303E+024333  .12345E+034334  .10070E+034335  .20471E+02
4336   .24467E+004337  .33820E+014338  .49495E+004343 -.10714E+024344  .30808E-01
4345 -.15737E+004346  .22468E+004347  .66895E+004348 -.17686E+014349  .37282E+00
4350 -.34807E+004351 -.32187E+014352 -.29277E-014353  .45514E+004354 -.79414E-01
4355 -.26919E-014356 -.98089E+004357  .54836E+004358  .13745E+014359 -.13889E+02
4360   .28197E-014361  .41394E-014362  .24020E+004363 -.45030E-014364 -.34996E+00
4365 -.32835E+004366 -.76668E-014367  .14663E+004368  .96040E-024369  .20186E-01
4370   .11894E+004371  .62094E-014372  .29645E-014373  .52224E-014374 -.51491E-01
4375 -.91785E-014376 -.15151E+004377  .12345F+034378  .10070E+034379  .20471E+02
44 1   .28419E+0044 2  .61463E+0044 3 -.29145E-0144 4  .23441E-0144 5  .17395E+00
44 6   .17779E+0044 7  .63894E-0144 8 -.13627E+0044 9 -.13936E-014410  .65609E-01
4411   .11292E+004412  .76041E-014413  .66782E-014414  .56051E-014415  .52103E-01
4416   .38983E-014417 -.17560E+004418 -.27413E-014419  .82138E+004420  .52225E+01
4421   .34117E+014422  .25620E+014423 -.49233E+014424 -.34827E+014425  .36045E+01
4426   .66839E+014427  .38711E+014428  .32415E+014429  .17961E+014430  .29436E+01
4431   .30003E+014432 -.26650E+024433  .30276E+034434  .13117E+044435  .28048E+03
4436 -.13626E+014437  .40210E+004438  .62650E+014443  .37654E-014444 -.10959E+02
4445 -.61766E+004446  .22507E+004447  .91001E+004448 -.17588E+014449  .39430E+00
4450   .12480E+004451 -.27089E+014452  .93688E+004453  .83438E+004454 -.16505E+00
4455 -.15753E+004456 -.13548E+014457  .98103E+004458  .21130E+014459 -.21111E+02
4460   .35973E+004461  .52839E+004462  .30681E+014463 -.57811E+004464 -.44926E+01
4465 -.42287E+014466 -.99347E+004467  .19093E+014468  .12671E+004469  .23737E+00
4470   .14611E+014471  .74523E+004472  .33478E+004473  .62240E+004474 -.69189E+00
4475 -.11919E+014476 -.19308E+014477  .30276E+034478  .13117E+044479  .28048E+03
45 3 -.50000E+004545 -.10983E+02
46 4 -.50000E+004646 -.10983E+02
47 5 -.50000E+004747 -.10983E+02
48 6 -.50000E+004848 -.10983E+02
49 7 -.50000E+004949 -.10983E+02
50 8 -.50000E+005050 -.10983E+02
51 9 -.50000E+005151 -.10983E+02
5210 -.50000E+005252 -.10983E+02
5311 -.50000E+005353 -.10983E+02
5412 -.50000E+005454 -.10983E+02
5513 -.50000E+005555 -.10983E+02
5614 -.50000E+005656 -.10983E+02
5715 -.50000E+005757 -.10983E+02
5816 -.50000E+005858 -.10983E+02
5917 -.50000E+005959 -.10983E+02
60 1   .33109E+0060 2 -.16388E+0160 3  .14008E-0160 4  .34710E-0260 5  .11618E-01
60 6 -.73002E-0260 7  .12346E-0160 8 -.25498E-0160 9 -.44703E-016010  .20971E-02
6011   .18046E-016012  .49408E-026013  .39491E-026014 -.71249E-026015  .78295E-02
6016   .15237E-016017 -.11173E+006018  .27989E+006019  .29393E+006020  .77806E+00
6021 -.20379E+016022  .66943E+006023 -.10001E+016024 -.43308E+016025  .98724E-01
6026   .11937E+016027  .22908E+006028  .19600E+006029 -.97486E+006030  .72330E+00
```

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F MATRIX FOR CASE 1 (CONTINUED)
6031  .16222E+016032 -.16303E+026033  .12345E+036034  .10070E+036035  .20471E+02
6036  .24467E+006037  .33820E+016038  .49495E+006043  .26889E+006044  .30808E-01
6045 -.15737E+006046  .22468E+006047  .66895E+006048 -.17686E+016049  .37282E+00
6050 -.34807E+006051 -.32187E+016052 -.29277E-016053  .45514E+006054 -.79414E-01
6055 -.26919E-016056 -.98089E+006057  .54836E+006058  .13745E+016059 -.13889E+02
6060 -.22157E+026061  .41394E-016062  .24020E+006063 -.45030E-016064 -.34996E+00
6065 -.32835E+006066 -.76668E-016067  .14663E+006068  .96040E-026069  .20186E-01
6070  .11894E+006071  .62094E-016072  .29645E-016073  .52224E-016074 -.51491E-01
6075 -.91785E-016076 -.15151E+006077  .12345E+036078  .10070E+036079  .20471E+02
61 1  .28419E+0061 2  .61463E+0061 3 -.29145E-0161 4  .23441E-0161 5  .17395E+00
61 6  .17779E+0061 7  .63894E-0161 8 -.13627E+0061 9 -.13936E-016110  .65609E-01
6111  .11292E+006112  .76041E-016113  .66782E-016114  .56051E-016115  .52103E-01
6116  .38983E-016117 -.17560E+006118 -.27413E-016119  .82138E+006120  .52225E+01
6121  .34117E+016122  .25620E+016123 -.49233E+016124 -.34827E+016125  .36045E+01
6126  .66839E+016127  .38711E+016128  .32415E+016129  .17961E+016130  .29436E+01
6131  .30003E+016132 -.26650E+026133  .30276E+036134  .13117E+046135  .28048E+03
5136 -.13626E+016137  .40210E+006138  .62650E+016143  .37654E-016144  .24760E-01
6145 -.61766E+006146  .22507E+006147  .91001E+006148 -.17588E+016149  .39430E+00
6150  .12480E+006151 -.27089E+016152  .93688E+006153  .83438E+006154 -.16505E+00
6155 -.15753E+006156 -.13548E+016157  .98103E+006158  .21130E+016159 -.21111E+02
6160  .35973E+006161 -.21657E+026162  .30681E+016163 -.57811E+006164 -.44926E+01
6165 -.42287E+016166 -.99347E+006167  .19093E+016168  .12671E+006169  .23737E+00
6170  .14611E+016171  .74523E+006172  .33478E+006173  .62240E+006174 -.69189E+00
6175 -.11919E+016176 -.19308E+016177  .30276E+036178  .13117E+046179  .28048E+03
62 3 -.50000E+006262 -.22185E+02
63 4 -.50000E+006363 -.22185E+02
64 5 -.50000E+006464 -.22185E+02
65 6 -.50000E+006565 -.22185E+02
66 7 -.50000E+006666 -.22185E+02
67 8 -.50000E+006767 -.22185E+02
68 9 -.50000E+006868 -.22185E+02
6910 -.50000E+006969 -.22185E+02
7011 -.50000E+007070 -.22185E+02
7112 -.50000E+007171 -.22185E+02
7213 -.50000E+007272 -.22185E+02
7314 -.50000E+007373 -.22185E+02
7415 -.50000E+007474 -.22185E+02
7516 -.50000E+007575 -.22185E+02
7617 -.50000E+007676 -.22185E+02
7733  .30000E+017777 -.10983E+02
7834  .37500E+017878 -.22185E+02
7935  .37500E+017979 -.22185E+02


G1-MATRIX FOR CASE 1

33 1  .60000E 0134 2  .75000E 01


G2-MATRIX FOR CASE 1

41 1 -.30360E 0042 1  .86190E 00
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H-MATRIX FOR CASE 1

1 1	-.46710E+04	1 2	-.33999E+03	1 3	.11376E+05	1 4	-.10751E+04	1 5	-.10405E+05
1 6	.55658E+04	1 7	.38226E+04	1 8	-.92675E+04	1 9	.15864E+04	110	.56826E+04
111	.33517E+04	112	-.11905E+04	113	-.14716E+04	114	-.16984E+04	115	.29672E+04
116	.39830E+04	117	.11692E+05	118	.12620E+07	119	-.27540E+06	120	-.33662E+07
121	.19521E+07	122	.16782E+07	123	-.43864E+07	124	.73394E+06	125	.35311E+07
126	.30311E+07	127	-.11240E+07	128	-.11306E+07	129	-.15274E+07	130	.23935E+07
131	.42056E+07	132	.12556E+08	133	-.14144E+07	134	.42460E+06	135	-.41530E+06
136	.10650E+04	137	-.53974E+05	138	-.44260E+04	143	-.88378E+04	144	.76136E+06
145	-.89139E+04	146	.26964E+04	147	-.18863E+05	148	.73781E+05	149	-.16472E+05
150	.72396E+05	151	.11983E+06	152	.50978E+05	153	-.23429E+04	154	-.13261E+05
155	.14508E+05	156	.14077E+05	157	-.35668E+05	158	.15379E+05	159	-.32088E+06
160	-.24412E+03	161	-.36153E+03	162	-.20801E+04	163	.39832E+03	164	.31760E+04
165	.30845E+04	166	.76841E+03	167	-.15086E+04	168	-.11054E+03	169	.21955E+02
170	-.31097E+03	171	.34156E+03	172	.64254E+03	173	.47579E+03	174	.14581E+04
175	.15702E+04	176	.16393E+04	177	-.14144E+07	178	.42460E+06	179	-.41530E+06
2 1	-.86102E+04	2 2	-.29804E+03	2 3	.41358E+03	2 4	-.91562E+03	2 5	-.10198E+04
2 6	.47160E+04	2 7	-.75835E+03	2 8	.34602E+04	2 9	.95815E+04	210	.47989E+04
211	-.83995E+03	212	-.13651E+02	213	-.54810E+02	214	.65389E+03	215	.27201E+03
216	-.59538E+03	217	.34453E+04	218	-.58771E+05	219	-.24519E+06	220	-.52211E+06
221	.15353E+07	222	-.38944E+06	223	.16019E+07	224	.44838E+07	225	.30578E+07
226	-.39053E+06	227	.21336E+05	228	-.25863E+05	229	.69313E+06	230	-.21120E+06
231	-.94568E+06	232	.46212E+07	233	-.21198E+07	234	.32960E+05	235	-.26307E+05
236	.20473E+04	237	-.10503E+06	238	-.47021E+03	243	-.10762E+05	244	.19480E+05
245	-.14873E+05	246	.24395E+04	247	.36783E+04	248	.19437E+05	249	-.28138E+04
250	.42324E+05	251	.43344E+05	252	.53808E+05	253	.12331E+05	254	-.84319E+04
255	.16238E+04	256	-.81883E+04	257	-.47726E+03	258	.36264E+05	259	-.33536E+06
260	-.25324E+02	261	-.37338E+02	262	-.21491E+03	263	.39535E+02	264	.31332E+03
265	.29203E+03	266	.67149E+02	267	-.13113E+03	268	-.87359E+01	269	-.13711E+02
270	-.78539E+02	271	-.11714E+02	272	.24346E+02	273	.21997E+01	274	.11077E+03
275	.13437E+03	276	.16489E+03	277	-.21198E+07	278	.32960E+05	279	-.26307E+05
3 1	.52393E+03	3 2	.93350E+02	3 3	.85424E+04	3 4	-.42479E+03	3 5	-.46136E+04
3 6	.41100E+04	3 7	.25838E+04	3 8	-.52743E+04	3 9	-.14002E+04	310	-.64333E+02
311	-.10987E+03	312	-.30653E+03	313	-.22620E+03	314	-.47990E+03	315	.19195E+04
316	.15359E+04	317	.31187E+04	318	.10001E+07	319	-.10815E+06	320	-.14847E+07
321	.14485E+07	322	.11440E+07	323	-.24666E+07	324	-.60274E+06	325	-.94317E+05
326	-.39443E+05	327	-.45073E+06	328	-.75681E+05	329	-.44955E+06	330	.16760E+07
331	.18975E+07	332	.26194E+07	333	-.10291E+06	334	.10081E+06	335	-.42412E+03
336	.30895E+02	337	.49688E+04	338	.18374E+04	343	-.14008E+04	344	.46130E+06
345	.17441E+04	346	.35234E+04	347	-.23045E+05	348	.55947E+05	349	-.12408E+05
350	.47160E+05	351	.67932E+05	352	-.23435E+05	353	.48005E+04	354	-.13034E+05
355	.11200E+05	356	.13537E+05	357	-.26725E+05	358	.15047E+05	359	-.10508E+06
360	.10484E+03	361	.15436E+03	362	.88142E+03	363	-.15347E+03	364	-.11879E+04
365	-.10418E+04	366	-.21333E+03	367	.38901E+03	368	.18158E+02	369	.15720E+03
370	.65380E+03	371	.40284E+03	372	.23896E+03	373	.33872E+03	374	-.10933E+03
375	-.31181E+03	376	-.60824E+03	377	-.10291E+06	378	.10081E+06	379	-.42412E+03
4 1	-.99994E+03	4 2	.96088E+02	4 3	-.38659E+02	4 4	-.96419E+03	4 5	-.13792E+04
4 6	.40642E+04	4 7	-.90475E+03	4 8	.36825E+04	4 9	.91867E+04	410	.44513E+04
411	-.51475E+03	412	.25702E+02	413	-.89011E+02	414	.55229E+03	415	.18977E+03
416	-.56835E+03	417	.32929E+04	418	-.58505E+05	419	-.25187E+06	420	-.56242E+06
421	.15131E+07	422	-.37629E+06	423	.15326E+07	424	.43898E+07	425	.29169E+07
426	-.39738E+06	427	-.21787E+05	428	-.21933E+05	429	.58681E+06	430	-.90507E+03
431	-.55499E+06	432	.45207E+07	433	-.17330E+07	434	.10935E+06	435	-.22459E+05
436	.13863E+03	437	-.12468E+05	439	.47957E+03	443	-.29505E+04	444	.13082E+05
445	-.72576E+04	446	.16392E+04	447	-.59501E+04	448	.21709E+05	449	-.21506E+04
450	.34872E+05	451	.42990E+05	452	.46902E+05	453	.88593E+04	454	-.91932E+04

H MATRIX FOR CASE 1 (CONTINUED)

455	.22939E+04	456	-.62827E+04	457	.14942E+04	458	.38179E+05	459	-.27516E+06
460	.28206E+02	461	.40810E+02	462	.23303E+03	463	-.38337E+02	464	-.29626E+03
465	-.25210E+03	466	-.47945E+02	467	.71679E+02	468	.12957E+01	469	.61952E+02
470	.24340E+03	471	.17716E+03	472	.14123E+03	473	.16285E+03	474	.50387E+02
475	-.24515E+02	476	-.14109E+03	477	-.17390E+07	478	.10935E+06	479	-.22459E+05
5 1	.11047E+04	5 2	.93032E+02	5 3	.55441E+04	5 4	.24661E+02	5 5	.85374E+03
5 6	.98392E+03	5 7	-.10366E+04	5 8	-.90088E+03	5 9	-.13917E+04	510	-.26489E+03
511	-.14244E+04	512	.48363E+03	513	.21700E+03	514	.54278E+03	515	-.84323E+03
516	-.69240E+03	517	-.22374E+04	518	.64873E+06	519	.16472E+05	520	.36237E+06
521	.31944E+06	522	-.44925E+06	523	-.32028E+06	524	-.53826E+06	525	-.11830E+06
526	-.11934E+07	527	.33294E+06	528	.31857E+06	529	.57286E+06	530	-.10198E+07
531	-.88453E+06	532	-.30774E+07	533	.14502E+06	534	-.11724E+06	535	.90101E+05
536	.34228E+03	537	.12549E+05	538	.50000E+03	543	.62580E+03	544	.21850E+06
545	.22251E+04	546	.49008E+04	547	-.51497E+04	548	.16788E+05	549	-.21512E+05
550	.13277E+05	551	-.48850E+04	552	-.28268E+05	553	.76069E+04	554	-.11242E+05
555	.93408E+04	556	-.29454E+03	557	-.23552E+04	558	.13545E+04	559	-.57026E+05
560	.26398E+02	561	.40796E+02	562	.23564E+03	563	-.50793E+02	564	-.39896E+03
565	-.39872E+03	566	-.10693E+03	567	.23967E+03	568	.19843E+02	569	-.45077E+02
570	-.97438E+02	571	-.15432E+03	572	-.20017E+03	573	-.17292E+03	574	-.28072E+03
575	-.26051E+03	576	-.20205E+03	577	.14502E+06	578	-.11724E+06	579	.90101E+05
6 1	-.68182E+02	6 2	.56242E+02	6 3	-.26137E+03	6 4	-.92587E+03	6 5	-.10061E+04
6 6	.28011E+04	6 7	-.16043E+04	6 8	.15380E+04	6 9	.54196E+04	610	-.33927E+04
611	.59362E+02	612	.12623E+03	613	-.14087E+03	614	.62433E+03	615	-.19265E+03
616	-.51697E+03	617	.47463E+04	618	-.53753E+05	619	-.24748E+06	620	-.40897E+06
621	.10800E+07	622	-.64709E+06	623	.55760E+06	624	.25443E+07	625	-.22988E+07
626	-.13221E+05	627	.69734E+05	628	-.59495E+05	629	.61750E+06	630	-.28299E+06
631	-.62617E+06	632	.55968E+07	633	-.92437E+06	634	.14868E+05	635	.22823E+05
636	-.24773E+03	637	-.79323E+03	638	.31682E+03	643	-.82759E+03	644	-.86858E+04
645	-.23162E+04	646	-.26017E+03	647	-.10750E+05	648	.20808E+05	649	-.12552E+05
650	.11138E+05	651	.30920E+05	652	-.17198E+05	653	.92885E+04	654	-.81346E+04
655	.26062E+04	656	-.31198E+04	657	.76499E+04	658	.14798E+05	659	-.13180E+06
660	.17828E+02	661	.26439E+02	662	.15164E+03	663	-.29034E+02	664	-.22753E+03
665	-.21843E+03	666	-.53568E+02	667	.10393E+03	668	.72766E+01	669	.43244E+01
670	.44918E+02	671	.38756E+01	672	-.17041E+02	673	-.54214E+01	674	-.72475E+02
675	-.89158E+02	676	-.10829E+03	677	-.92437E+06	678	.14868E+05	679	.22823E+05
7 1	-.12248E+04	7 2	-.22167E+03	7 3	.32852E+04	7 4	.16866E+03	7 5	.31314E+04
7 6	-.13581E+04	7 7	-.18389E+03	7 8	.20779E+04	7 9	.25470E+04	710	.41739E+02
711	-.51962E+03	712	.16119E+03	713	.36844E+03	714	.24976E+03	715	-.12770E+04
716	-.16279E+04	717	-.20002E+04	718	.38864E+06	719	.50672E+05	720	.10476E+07
721	-.53054E+06	722	-.46980E+05	723	.87873E+06	724	.11607E+07	725	.40645E+04
726	-.38095E+06	727	.15782E+06	728	.39216E+06	729	.22357E+06	730	-.13394E+07
731	-.17388E+07	732	-.32170E+07	733	-.10129E+06	734	-.18734E+05	735	-.64253E+05
736	-.15372E+03	737	-.13481E+05	738	-.15250E+04	743	-.12022E+04	744	.10345E+06
745	-.41785E+03	746	-.96021E+03	747	-.29952E+04	748	.68456E+04	749	-.10703E+04
750	.23162E+04	751	.12627E+05	752	.82133E+04	753	-.14439E+04	754	.12595E+03
755	-.33459E+03	756	.28695E+04	757	-.52004E+03	758	-.29463E+04	759	.45997E+05
760	-.86953E+02	761	-.12769E+03	762	-.73173E+03	763	.13300E+03	764	.10467E+04
765	.97745E+03	766	.22511E+03	767	-.41695E+03	768	-.26142E+02	769	-.70009E+02
770	-.35766E+03	771	-.13382E+03	772	-.24336E+02	773	-.86342E+02	774	.26529E+03
775	.37604E+03	776	.52998E+03	777	-.10129E+06	778	-.18734E+05	779	-.64253E+05
8 1	-.13272E+03	8 2	-.27330E+02	8 3	-.12862E+03	8 4	-.97640E+02	8 5	.37214E+02
8 6	.33717E+03	8 7	-.20650E+03	8 8	.44970E+03	8 9	.13421E+04	810	-.43832E+03
811	.39010E+01	812	.13694E+02	813	-.49166E+02	814	.51588E+03	815	-.13858E+03
816	-.61310E+03	817	.11628E+05	818	-.76596E+04	819	-.27607E+05	820	-.49078E+05
821	.15404E+06	822	-.56204E+05	823	.55930E+05	824	.45044E+06	825	-.19703E+06

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H MATRIX FOR CASE 1 (CONTINUED)
 826 -.15255E+06 827  .37496E+05 828  .74024E+04 829  .51861E+06 830 -.28648E+06
 831 -.80805E+06 832  .13561E+08 833 -.29188E+06 834 -.69061E+04 935  .24827E+05
 836 -.18698E+03 837 -.16823E+04 838  .39389E+03 843 -.11111E+03 844 -.14865E+04
 845  .13841E+03 846 -.12458E+04 847 -.76851E+04 848  .90630E+04 849 -.16752E+04
 850 -.33386E+04 851  .11282E+05 852 -.10587E+05 853  .91962E+03 854 -.94849E+03
 855 -.13688E+04 856  .95737E+03 857  .54061E+04 858  .72069E+04 859  .21363E+05
 860  .22212E+02 861  .32875E+02 862  .18872E+03 863 -.35271E+02 864 -.27769E+03
 865 -.26217E+03 866 -.62286E+02 867  .12021E+03 868  .81057E+01 869  .96985E+01
 870  .66613E+02 871  .11184E+02 872 -.17685E+02 873 -.69369E+00 874 -.90147E+02
 875 -.11234E+03 876 -.13960E+03 877 -.29188E+06 878 -.69061E+04 879  .24827E+05
 9 1  .53648E+03 9 2  .96039E+02 9 3  .16623E+04 9 4  .14510E+03 9 5  .23225E+04
 9 6 -.10967E+04 9 7 -.31730E+03 9 8  .23422E+04 9 9  .24682E+04 910 -.30565E+03
 911  .72749E+03 912 -.16232E+03 913 -.17541E+03 914 -.13615E+03 915  .39420E+03
 916  .41367E+03 917  .87559E+03 918  .18379E+06 919  .37819E+05 920  .78952E+06
 921 -.40974E+06 922 -.14473E+06 923  .11113E+07 924  .13009E+07 925 -.23283E+06
 926  .61005E+06 927 -.15451E+06 928 -.16954E+06 929 -.10259E+06 930  .41620E+06
 931  .41980E+06 932  .17982E+07 933 -.20505E+06 934  .15347E+05 935 -.20083E+05
 936 -.12918E+03 937  .66299E+04 938 -.91784E+02 943  .55474E+03 944  .40728E+05
 945 -.12260E+04 946  .70934E+03 947  .29529E+04 948 -.64211E+04 949  .15635E+04
 950 -.43498E+03 951 -.10032E+05 952  .36046E+04 953  .10268E+04 954 -.37907E+02
 955  .42608E+03 956 -.33247E+04 957 -.14450E+03 958  .23299E+04 959 -.62061E+05
 960 -.50770E+01 961 -.76268E+01 962 -.43946E+02 963  .96448E+01 964  .76840E+02
 965  .84858E+02 966  .24806E+02 967 -.49599E+02 968 -.43419E+01 969  .10997E+02
 970  .24387E+02 971  .39558E+02 972  .42342E+02 973  .42393E+02 974  .58591E+02
 975  .52585E+02 976  .37634E+02 977 -.20505E+06 978  .15347E+05 979 -.20083E+05
10 1  .22409E+0310 2 -.39454E+0210 3 -.28404E+0310 4 -.81995E+0210 5 -.39832E+03
10 6  .25627E+0310 7  .15819E+0210 8 -.29934E+0310 9  .24790E+021010 -.11869E+03
1011 -.18024E+031012  .49374E+021013  .36300E+021014  .35954E+031015 -.27140E+03
1016 -.60451E+031017  .74963E+041018 -.58586E+041019 -.16761E+051020 -.46131E+05
1021  .81652E+051022 -.16192E+051023 -.26972E+051024  .13755E+061025 -.75201E+05
1026 -.14822E+061027  .39533E+051028  .33321E+051029  .34997E+061030 -.27831E+06
1031 -.64008E+061032  .86299E+071033  .15034E+071034 -.65201E+041035  .12976E+05
1036 -.15469E+031037  .27320E+041038  .15902E+031043  .26917E+031044 -.78984E+03
1045  .37949E+041046 -.27606E+031047 -.14769E+041048  .37635E+041049 -.98714E+03
1050 -.29913E+041051  .55142E+031052 -.48208E+041053 -.66726E+041054  .19340E+04
1055  .20440E+041056  .76238E+041057 -.94182E+041058 -.16743E+051059  .12533E+06
1060  .89320E+011061  .13279E+021062  .76306E+021063 -.14583E+021064 -.11496E+03
1065 -.11026E+031066 -.26990E+021067  .53094E+021068  .37760E+011069  .10556E+01
1070  .18203E+021071 -.40224E+011072 -.15385E+021073 -.87100E+011074 -.44220E+02
1075 -.50861E+021076 -.57519E+021077  .15034E+071078 -.65201E+041079  .12976E+05
11 1  .18029E+0611 2  .46752E+0611 3  .12355E+0711 4 -.25858E+0611 5 -.32155E+07
11 6  .20524E+0711 7  .17302E+0711 8 -.45153E+0711 9  .67208E+061110  .35494E+07
1111  .31135E+071112 -.10556E+071113 -.10831E+071114 -.14848E+071115  .24434E+07
1116  .42152E+071117  .12579E+081118 -.37126E+061119  .69573E+061120  .57756E+07
1121  .18498E+071122  .61815E+061123 -.34805E+061124 -.21784E+071125 -.14958E+07
1126  .14034E+071127  .47073E+071128  .47459E+071129  .43920E+071130 -.30712E+07
1131 -.55310E+071132 -.46670E+081133  .21613E+091134  .10114E+101135  .22155E+09
1136 -.10144E+071137  .25063E+061138  .48553E+071139 -.44260E+041140 -.46143E+06
1147  .70430E+061148 -.17186E+071149  .38926E+061150 -.60615E+061151 -.25397E+07
1142  .23627E+051143  .82332E+051144 -.83504E+071145 -.32287E+061146  .82752E+05
1152  .12173E+061153  .58165E+061154  .25410E+051155 -.28311E+061156 -.96834E+06
1157  .10570E+071158  .11801E+071159 -.94037E+071160  .28414E+061161  .41747E+06
1162  .24237E+071163 -.45712E+061164 -.35544E+071165 -.33492E+071166 -.78851E+06
1167  .15171E+071168  .10110E+061169  .18093E+061170  .11313E+071171  .56232E+06
1172  .23773E+061173  .46419E+061174 -.57562E+061175 -.96338E+061176 -.15336E+07
```

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H MATRIX FOR CASE 1 (CONTINUED)
1177  .22742E+091178  .10036E+101179  .22921E+09
12 1 -.42178E+0412 2 -.13275E+0412 3 -.54122E+0512 4 -.24566E+0612 5 -.52092E+06
12 6  .15267E+0712 7 -.38663E+0612 8  .15755E+0712 9  .44550E+071210  .30266E+07
1211 -.39400E+061212  .27131E+051213 -.25134E+051214  .69799E+061215 -.21117E+06
1216 -.96309E+061217  .47853E+071218 -.36305E+051219  .11001E+061220  .21534E+06
1221 -.93274E+061222  .25116E+061223 -.12910E+071224 -.36901E+071225 -.35197E+07
1226  .97286E+061227  .10607E+061228  .15429E+061229 -.89556E+061230 -.49143E+06
1231  .10108E+071232 -.79649E+071233  .19240E+081234  .23901E+081235  .56439E+07
1236 -.63895E+051237  .78785E+061238  .11513E+061239 -.47021E+031240 -.89792E+06
1242  .45419E+051243  .11038E+061244 -.21510E+061245  .17505E+061246 -.36565E+05
1247 -.69754E+051248 -.13489E+061249  .12670E+051250 -.45276E+061251 -.33951E+06
1252 -.59927E+061253 -.16827E+061254  .10003E+061255 -.12022E+051256  .14710E+06
1257 -.38945E+051258 -.49336E+061259  .45224E+071260  .71701E+041261  .10541E+05
1262  .61189E+051263 -.11557E+051264 -.89956E+051265 -.84855E+051266 -.20011E+05
1267  .38664E+051268  .25960E+041269  .42854E+041270  .27468E+051271  .12918E+05
1272  .46168E+041273  .10381E+051274 -.16073E+051275 -.25480E+051276 -.39217E+05
1277  .36163E+081278  .23293E+081279  .61289E+07
13 1  .11232E+0613 2  .28645E+0613 3  .97816E+0613 4 -.99003E+0513 5 -.13888E+07
13 6  .15034E+0713 7  .11766E+0713 8 -.25468E+0713 9 -.63582E+061310 -.53145E+05
1311  .10282E+051312 -.40828E+061313 -.49941E+051314 -.42782E+061315  .17090E+07
1316  .19029E+071317  .26038E+071318 -.26833E+061319  .38781E+061320  .31337E+07
1321  .93956E+061322  .28151E+061323 -.36266E+061324 -.38091E+061325  .16128E+07
1326  .31469E+071327  .22536E+071328  .18522E+071329  .17974E+071330 -.20935E+07
1331 -.17474E+071332 -.17522E+081333  .12140E+091334  .61253E+091335  .13065E+09
1336 -.63518E+061337 -.10951E+061338  .29240E+071339  .18374E+041340  .42479E+05
1342  .68541E+031343  .11258E+051344 -.50595E+071345 -.25709E+061346  .27026E+05
1347  .52989E+061348 -.11419E+071349  .25827E+061350 -.43875E+061351 -.15007E+07
1352  .61280E+061353  .25775E+061354  .76430E+051355 -.19156E+061356 -.61498E+06
1357  .67245E+061358  .61029E+061359 -.62159E+071360  .16555E+061361  .24315E+06
1362  .14122E+071363 -.26636E+061364 -.20700E+071365 -.19500E+071366 -.45876E+06
1367  .88204E+061368  .58691E+051369  .10751E+061370  .66807E+061371  .33961E+06
1372  .15172E+061373  .28374E+061374 -.31963E+061375 -.54870E+061376 -.88734E+06
1377  .12223E+091378  .61067E+091379  .13066E+09
14 1 -.73497E+0414 2  .70668E+0414 3 -.57282E+0514 4 -.25200E+0614 5 -.55079E+06
14 6  .15030E+0714 7 -.37405E+0614 8  .15106E+0714 9  .43617E+071410  .28892E+07
1411 -.40062E+061412 -.16311E+051413 -.21995E+051414  .59037E+061415 -.20413E+04
1416 -.57356E+061417  .46554E+071418 -.23498E+051419  .11205E+061420  .25749E+06
1421 -.80321E+061422  .28272E+061423 -.13287E+071424 -.35121E+071425 -.32720E+07
1426  .57812E+061427  .22805E+051428  .18245E+061429 -.75600E+061430 -.36345E+06
1431  .95343E+061432 -.75452E+071433  .16087E+081434  .14971E+081435  .35491E+07
1436 -.16674E+051437 -.21735E+051438  .73510E+051439  .47957E+031440 -.10659E+06
1442  .30755E+041443  .23099E+051444 -.14496E+061445  .91168E+051446 -.28419E+05
1447  .35432E+051448 -.15500E+061449  .44083E+041450 -.36707E+061451 -.32269E+06
1452 -.51880E+061453 -.12975E+061454  .10818E+061455 -.19802E+051456  .12765E+06
1457 -.60596E+051458 -.51595E+061459  .38758E+071460  .35946E+041461  .52963E+04
1462  .30852E+051463 -.59591E+041464 -.46348E+051465 -.44331E+051466 -.10713E+05
1467  .21115E+051468  .14913E+041469  .12416E+041470  .11248E+051471  .43589E+04
1472  .35928E+031473  .32499E+041474 -.96285E+041475 -.13699E+051476 -.19557E+05
1477  .29970E+081478  .12956E+081479  .39631E+07
15 1  .56089E+0515 2  .13645E+0615 3  .63607E+0615 4  .18974E+0515 5  .40128E+06
15 6  .35099E+0615 7 -.42383E+0615 8 -.35655E+0615 9 -.53621E+061510 -.90299E+05
1511 -.11700E+071512  .35453E+061513  .32819E+061514  .58518E+061515 -.10057E+07
1516 -.87601E+061517 -.30668E+071518 -.16811E+061519  .15343E+061520  .87733E+06
1521  .69217E+061522  .85873E+061523 -.80154E+061524  .42352E+051525  .88005E+06
1526  .29515E+071527  .19691E+061528  .39389E+061529 -.32558E+061530  .20372E+07
1531  .17579E+071532  .20298E+071533  .47290E+081534  .29117E+091535  .61109E+08
1536 -.31011E+061537 -.10126E+061538  .13861E+071539  .50000E+031540  .10728E+06
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H MATRIX FOR CASE 1 (CONTINUED)
1542   .75935E+041543 -.56409E+041544 -.23966E+071545 -.11411E+061546 -.22912E+05
1547   .17871E+061548 -.41954E+061549  .28976E+061550 -.14125E+061551 -.33511E+06
1552   .43037E+061553  .22638E+051554  .10383E+061555 -.12245E+061556 -.17737E+06
1557   .15269E+061558  .26500E+061559 -.21990E+071560  .79000E+051561  .11598E+06
1562   .67347E+061563 -.12669E+061564 -.98439E+061565 -.92569E+061566 -.21702E+06
1567   .41609E+061568  .27483E+051569  .54091E+051570  .32715E+061571  .16996E+06
1572   .80138E+051573  .14319E+061574 -.14530E+061575 -.25684E+061576 -.42245E+06
1577   .46132E+081578  .29333E+091579  .59448E+08
16 1 -.10956E+0516 2 -.61730E+0416 3 -.54322E+0516 4 -.24724E+0616 5 -.40523E+06
16 6   .10672E+0716 7 -.64038E+0616 8  .55196E+0616 9  .25254E+071610 -.22873E+07
1611 -.19535E+051612  .72926E+051613 -.61067E+051614  .61809E+061615 -.28754E+06
1616 -.63407E+061617  .56608E+071618 -.16264E+051619  .91369E+051620  .75880E+05
1621 -.58430E+061622  .45370E+061623 -.48587E+061624 -.19593E+071625  .24560E+07
1626 -.18049E+061627 -.20190E+061628  .18990E+061629 -.91362E+061630  .24108E+06
1631   .79647E+061632 -.10705E+081633  .89911E+071634 -.12342E+081635 -.28717E+07
1636   .18015E+051637 -.97347E+051638 -.59441E+051639  .31682E+031640 -.67815E+04
1642 -.54959E+041643  .80366E+031644  .93776E+051645  .50978E+051646 -.11204E+05
1647   .70927E+051648 -.11560E+061649  .11185E+061650 -.11431E+061651 -.15151E+06
1652   .15463E+061653 -.15144E+061654  .10009E+061655 -.19463E+051656  .11894E+06
1657 -.14840E+061658 -.30095E+061659  .27207E+071660 -.38073E+041661 -.56003E+04
1662 -.32479E+051663  .61441E+041664  .47797E+051665  .45156E+051666  .10692E+05
1667 -.20616E+051668 -.13853E+041669 -.22284E+041670 -.14490E+051671 -.67980E+04
1672 -.24469E+041673 -.54270E+041674  .85083E+041675  .13520E+051676  .20772E+05
1677   .16370E+081678 -.12616E+081679 -.32924E+07
17 1   .27944E+0517 2  .62053E+0517 3  .38248E+0617 4  .53277E+0517 5  .10627E+07
17 6 -.51343E+0617 7 -.39220E+0517 8  .85935E+0617 9  .11493E+071710  .66098E+04
1711 -.36757E+061712  .16582E+061713  .39898E+061714  .22821E+061715 -.13317E+07
1716 -.17310E+071717 -.32441E+071718 -.99420E+051719  .53596E+051720 -.11032E+06
1721   .79103E+061722  .31833E+061723 -.13215E+071724 -.13538E+071725  .28882E+06
1726   .12046E+071727  .20832E+061728 -.18581E+061729 -.97689E+051730  .24529E+07
1731   .31657E+071732  .39823E+071733  .14757E+081734  .13896E+091735  .30528E+08
1736 -.13061E+061737  .12656E+061738  .66769E+061739 -.15250E+041740 -.11525E+06
1742 -.34103E+041743  .14637E+051744 -.11349E+071745 -.17911E+051746  .26659E+05
1747   .10174E+061748 -.18548E+061749  .33644E+051750 -.30557E+051751 -.33984E+06
1752 -.49819E+051753  .33871E+051754  .14854E+031755  .61036E+041756 -.82391E+05
1757   .12159F+051758  .74707E+051759 -.12916F+071760  .40266E+051761  .59152E+05
1762   .34325E+061763 -.64606E+051764 -.50240E+061765 -.47293E+061766 -.11110E+06
1767   .21331E+061768  .14148E+051769  .26491E+051770  .16250E+061771  .81207E+05
1772   .35030E+051773  .67042E+051774 -.80825E+051775 -.13623E+061776 -.21780E+06
1777   .15565E+081778  .13930E+091779  .31712E+08
18 1 -.12334E+0518 2 -.36149E+0418 3 -.10604E+0518 4 -.27187E+0518 5 -.46791E+05
18 6   .15070E+0618 7 -.55554E+0518 8  .57035E+0518 9  .44485E+061810 -.19152E+06
1811 -.15404E+061812  .37927E+051813  .82405E+041814  .51806E+061815 -.28979E+06
1816 -.81213E+061817  .13538E+081818 -.29469E+051819 -.11309E+051820 -.10310E+06
1821   .89419E+051822  .22313E+051823 -.14465E+061824 -.27520E+061825  .28076E+06
1826 -.97769E+051827 -.98802E+041828  .82083E+051829 -.68872E+061830  .13530E+06
1831   .92725E+061832 -.27834E+081833  .11549E+081834 -.21914E+071835 -.78586E+06
1836   .49531E+041837 -.12799E+061838 -.12099E+051839  .39389E+031840 -.14382E+05
1842 -.41482E+041843 -.10891E+051844  .14157E+051845  .30168E+051846 -.45146E+04
1847   .24821E+051848  .47232E+051849 -.15263E+051850  .48991E+051851  .12123E+06
1852   .78678E+051853 -.75023E+051854  .24852E+051855  .27203E+051856  .10026E+06
1857 -.14522E+061858 -.26280E+061859  .14265E+071860 -.11870E+041861 -.17518E+04
1862 -.10124E+051863  .19162E+041864  .14985E+051865  .14204E+051866  .33893E+04
1867 -.65790E+041868 -.44937E+031869 -.53449E+031870 -.38596E+041871 -.12391E+04
1872   .19354E+031873 -.73305E+031874  .37839E+041875  .51147E+041876  .69190E+04
```

```
H MATRIX FOR CASE 1 (CONTINUED)
1877  .13879E+081878 -.20641E+071879 -.12435E+07
19 1  .97123E+0419 2  .25620E+0519 3  .18133E+0619 4  .3818 E+0519 5  8792025 06
19 6 -.39762E+0619 7 -.14229E+0619 8  .11020E+0719 9  .13014E+071910 -.23130E+06
1911  .61231E+061912 -.15086E+061913 -.16642E+061914 -.97980E+051915  .41678E+06
1916  .41816E+061917  .18225E+071918 -.47759E+051919  .99710E+041920 -.24941E+06
1921  .45129E+061922  .20344E+061923 -.10602E+071924 -.11359E+071925  .37399E+06
1926 -.56763E+061927  .40314E+061928  .41687E+061929  .34640E+061930 -.65017E+06
1931 -.78889E+061932 -.25447E+071933  .66182E+061934  .54132E+081935  .11792E+08
1936 -.52265E+051937 -.65455E+051938  .25932E+061939 -.91784E+021940  .56680E+05
1942 -.28659E+041943 -.66752E+041944 -.44700E+061945  .13087E+051946 -.55909E+03
1947  .10216E+051948  .23609E+051949 -.10960E+051950  .20912E+051951  .50169E+05
1952 -.12194E+051953 -.10116E+051954  .38981E+041955 -.15997E+031956  .28738E+05
1957 -.16808E+051958 -.47656E+051959  .55185E+061960  .15002E+051961  .22042E+05
1962  .12798E+061963 -.24158E+051964 -.18778E+061965 -.17709E+061966 -.41742E+05
1967  .80304E+051968  .53601E+041969  .94800E+041970  .59631E+051971  .29672E+05
1972  .12628E+051973  .24527E+051974 -.30215E+051975 -.50694E+051976 -.80797E+05
1977  .22987E+071978  .53849E+081979  .12162E+08
20 1 -.78385E+0420 2 -.16362E+0420 3 -.93750E+0420 4 -.16715E+0520 5 -.45811E+05
20 6  .80430E+0520 7 -.15993E+0520 8 -.25006E+0520 9  .13847E+062010 -.72904E+05
2011 -.14506E+062012  .38452E+052013  .32191E+052014  .34603E+062015 -.27344E+06
2016 -.63136E+062017  .85592E+072018 -.14076E+052019 -.42376E+042020  .15981E+05
2021  .47650E+052022 -.33207E+052023  .11823E+062024  .16851E+062025  .56952E+05
2026  .14608E+062027 -.64837E+052028 -.51334E+052029 -.49218E+062030  .42081E+06
2031  .10107E+072032 -.18020E+082033  .45764E+072034 -.10821E+072035 -.39737E+06
2036  .42021E+042037 -.11483E+062038 -.59784E+042039  .15902E+032040  .23356E+05
2042 -.34318E+042043 -.10773E+052044  .73258E+042045 -.24951E+052046 -.90128E+04
2047 -.26346E+052048  .54114E+052049 -.10049E+052050  .37477E+052051  .14916E+06
2052  .28225E+052053  .35149E+052054 -.13359E+052055 -.16289E+052056 -.17658E+05
2057  .56689E+052058  .80773E+052059 -.38826E+062060 -.54120E+032061 -.79998E+03
2062 -.46283E+042063  .88496E+032064  .69198E+042065  .66040E+042066  .15959E+04
2067 -.31235E+042068 -.21830E+032069 -.17472E+032070 -.15634E+042071 -.38580E+03
2072  .25621E+032073 -.16318E+032074  .18727E+042075  .24303E+042076  .31636E+04
24 6  .10000E+01
22 4  .10000E+01
2077 -.74252E+072078 -.96194E+062079 -.63659E+06
21 3  .10000E+01
23 5  .10000E+01
26 8  .10000E+01
25 7  .10000E+01
27 9  .10000E+01
2810  .10000E+01
2911  .10000E+01
3012  .10000E+01
3113  .10000E+01
3214  .10000E+01
3315  .10000E+01
3416  .10000E+01
```

```
H MATRIX FOR CASE 1 (CONTINUED)
3517  .10000E+01
3618  .10000E+01
3719  .10000E+01
3820  .10000E+01
3921  .10000E+01
4022  .10000E+01
4123  .10000E+01
4224  .10000E+01
4325  .10000E+01
4426  .10000E+01
4527  .10000E+01
4628  .10000E+01
4729  .10000F+01
4830  .10000E+01
4931  .10000E+01
5032  .10000E+01
5133 -.60000E+01
5234 -.75000E+01
5333  .10000E+01
5434  .10000E+01
55 1  .10000E+01
56 2  .10000E+01
```

```
D-MATRIX FOR CASE 1

11 1 -.84864E+0711 2  .31845E+07
12 1 -.12719E+0812 2  .24720E+06
13 1 -.61746E+0613 2  .75607E+06
14 1 -.10434E+0814 2  .82012E+06
15 1  .87012E+0615 2 -.87930E+06
16 1 -.55462E+0716 2  .11151E+06
17 1 -.60774E+0617 2 -.14050E+06
18 1 -.17513E+0718 2 -.51796E+05
19 1 -.12303E+0719 2  .11510E+06
20 1  .90204E+0720 2 -.48901E+05
51 1  .60000E 0152 2  .75000E 01
```

TABLE 35. EIGENVALUES AND RMS RESPONSES FOR CASE 1

OPEN LOOP

EIGENVALUES

	REAL	IMAG	FREQ	DAMP
Kussner	-.24680489	0.00000000	.24680489	-1.00000000
	-.24843511	0.00000000	.24843511	-1.00000000
Aileron	-6.00000000	0.00000000	6.00000000	-1.00000000
Kussner/Wagner	-10.98300000	0.00000000	10.98300000	-1.00000000
	-22.18500000	0.00000000	22.18500000	-1.00000000
Kussner	-5.09600000	3.60330182	6.24123385	-.81650522
Kussner/Wagner	-8.54920000	0.00000000	8.54920000	-1.00000000
	-10.98300000	0.00000000	10.98300000	-1.00000000
Elevator	-7.50000000	0.00000000	7.50000000	-1.00000000
Kussner/Wagner	-22.18500000	0.00000000	22.18500000	-1.00000000
Elevator	-7.50000000	0.00000000	7.50000000	-1.00000000
Wagner	-22.18500000	0.00000000	22.18500000	-1.00000000
	-22.18500004	0.00000000	22.18500004	-1.00000000
	-22.18499954	0.00000000	22.18499954	-1.00000000
	-22.18499954	0.00000000	22.18499954	-1.00000000
	-22.18500006	0.00000000	22.18500006	-1.00000000
	-22.18500006	0.00000000	22.18500006	-1.00000000
	-22.18499932	0.00000000	22.18499932	-1.00000000
	-22.18499932	0.00000000	22.18499932	-1.00000000
	-22.18500352	0.00000000	22.18500352	-1.00000000
	-22.18500352	0.00000000	22.18500352	-1.00000000
	-22.18499766	0.00000000	22.18499766	-1.00000000
	-22.18499766	0.00000000	22.18499766	-1.00000000
	-22.18498832	0.00000000	22.18498832	-1.00000000
	-10.98299999	0.00000000	10.98299999	-1.00000000
	-10.98299999	0.00000000	10.98299999	-1.00000000
	-10.98299938	0.00000000	10.98299938	-1.00000000
	-10.98299592	0.00000000	10.98299592	-1.00000000
	-10.98299592	0.00000000	10.98299592	-1.00000000
	-10.98297826	0.00000000	10.98297826	-1.00000000
	-10.98297826	0.00000000	10.98297826	-1.00000000
	-22.18516865	0.00000000	22.18516865	-1.00000000
	-22.18516865	0.00000000	22.18516865	-1.00000000
	-10.98353111	.00013084	10.98353111	-1.00000000
	-10.98195856	.00074758	10.98195858	-1.00000000
	-10.97703485	0.00000000	10.97703485	-1.00000000
	-22.18163848	0.00000000	22.18163848	-1.00000000
	-10.93147807	.00788580	10.93148092	-.99999974
	-10.88522896	0.00000000	10.88522896	-1.00000000
	-10.74226749	0.00000000	10.74226749	-1.00000000
	-21.87218932	0.00000000	21.87218932	-1.00000000
	-10.16535534	0.00000000	10.16535534	-1.00000000
	-21.39156456	0.00000000	21.39156456	-1.00000000
Short Period	-.86710311	1.33027733	1.58792493	-.54606052
Bending Modes 1-15	-.67787290	5.53703588	5.57837593	-.12151797
	-.23390114	11.12613802	11.12859636	-.02101803
	-.66381152	13.88935949	13.90521315	-.04773832
	-.71189325	15.69722948	15.71336391	-.04530496
	-.43016385	17.48300516	17.48829639	-.02459724
	-.61462065	18.77419694	18.78425482	-.03271999
	-.50636749	19.89619669	19.90263930	-.02544223
	-.55891810	27.21771559	27.22345370	-.02053076
	-.99205830	33.31684647	33.33161319	-.02976328
	-.79858336	37.35654713	37.36508195	-.02137245
	-.83990987	39.39941788	39.40836939	-.02131298
	-2.49913278	40.34719457	40.42451947	-.06182220
	-1.11727885	41.55773336	41.57274966	-.02687527
	-1.10415915	42.89188820	42.90609794	-.02573432
	-.76662165	49.76614620	49.77205055	-.01540265

TABLE 35. Concluded

1	.10121382E+07
2	.16302733E+06
3	.69469100E+06
4	.88558057E+05
5	.39317435E+06
6	.48690990E+05
7	.23265728E+06
8	.55777699E+05
9	.10340485E+06
10	.31300667E+05
11	.24137047E+07
12	.47154683E+06
13	.17695985E+07
14	.37390724E+06
15	.11089511E+07
16	.25167444E+06
17	.71808161E+06
18	.13691056E+06
19	.37008233E+06
20	.81342190E+05
21	.18462290E+01
22	.69030339E-01
23	.19773704E+00
24	.12922158E+00
25	.23080328E-01
26	.37000113E-01
27	.36920803E-01
28	.14791733E-01
29	.46697515E-01
30	.43904216E-01
31	.34796844E-01
32	.18248172E-01
33	.78772318E-02
34	.10927018E-01
35	.78947763E-02
36	.68132548E+00
37	.80861673E-02
38	.22704271E-01
39	.38493616E-01
40	.55158298E-02
41	.11107994E-01
42	.91198926E-02
43	.45301163E-02
44	.15590481E-01
45	.12701031E-01
46	.77672093E-02
47	.66490324E-02
48	.14119698E-02
49	.32484778E-02
50	.33709014E-02
51	.31937363E-14
52	.22627982E-13
53	.53228939E-15
54	.30170643E-14
55	.15881054E+02
56	.24569286E+01

1	.15881054E+02
2	.24569286E+01
3	.18462290E+01
4	.69030339E-01
5	.19773704E+00
6	.12922158E+00
7	.23080328E-01
8	.37000113E-01
9	.36920803E-01
10	.14791733E-01
11	.46697515E-01
12	.43904216E-01
13	.34796844E-01
14	.18248172E-01
15	.78772318E-02
16	.10927018E-01
17	.78947763E-02
18	.68132548E+00
19	.80861673E-02
20	.22704271E-01
21	.38493616E-01
22	.55158298E-02
23	.11107994E-01
24	.91198926E-02
25	.45301163E-02
26	.15590481E-01
27	.12701031E-01
28	.77672093E-02
29	.66490324E-02
30	.14119698E-02
31	.32484778E-02
32	.33709014E-02
33	.53228939E-15
34	.30170643E-14
35	.23647580E-14
36	.99178200E+00
37	.97183710E+00
38	.97979891E+00
39	.55751508E+01
40	.98351309E+00
41	.37288852E+00
42	.10000642E+01
43	.46190556E+00
44	.17852216E+00
45	.77064958E-01
46	.22502400E-02
47	.59778695E-02
48	.45303568E-02
49	.75114084E-03
50	.12768364E-02
51	.10536364E-02
52	.46060422E-03
53	.16470572E-02
54	.15477549E-02
55	.11710190E-02
56	.60616702E-03
57	.22011686E-03
58	.31019115E-03
59	.27933908E-03
60	.23008115E+00
61	.91467413E-01
62	.40632342E-01
63	.13987475E-02
64	.38647145E-02
65	.26310788E-02
66	.45221500E-03
67	.73822350E-03
68	.67941286E-03
69	.27106400E-03
70	.91206133E-03
71	.86177223E-03
72	.67033689E-03
73	.33596179E-03
74	.13600091E-03
75	.18527349E-03
76	.15293583E-03
77	.12870008E-14
78	.59315791E-15
79	.14820492E-14

TABLE 36. STATES, RESPONSES, AND CONTROLS FOR CASES 2, 3, AND 4

Case 2 42 States 56 Responses 2 Controls			Case 3 6 Modes Truncated, Residualized 24 States 38 Responses 2 Controls			Case 4 2 Modes (1 & 3) Truncated, Residualized 16 States 30 Responses 2 Controls		
States	Responses	Controls	States	Responses	Controls	States	Responses	Controls
1 w	1-10 (B_i, T_i)	δ_{ac}	1 w	1-10 (B_i, T_i)	δ_{ac}	1 w	1-10 (B_i, T_i)	δ_{ac}
2 q/n_2	$i = 1, 5$	$\delta_{e_{i_c}}$	2 q/n_2	$i = 1, 5$	$\delta_{e_{i_c}}$	2 q/n_2	$i = 1, 5$	$\delta_{e_{i_c}}$
3-17 $\dot{\eta}_i$	11-20 $(\dot{B}_i, \dot{T}_i)$		3-8 $\dot{\eta}_1 - \dot{\eta}_6$	11-20 $(\dot{B}_i, \dot{T}_i)$		3 $\dot{\eta}_1$	11-20 $(\dot{B}_i, \dot{T}_i)$	
18-32 η_i	$i = 1, 5$		9-14 $\eta_1 - \eta_6$	$i = 1, 5$		4 $\dot{\eta}_3$	$i = 1, 5$	
33 δ_a	21-35 $\dot{\eta}_i$		15 δ_a	21-26 $\dot{\eta}_1 - \dot{\eta}_6$		5 η_1	21 $\dot{\eta}_1$	
34 δ_{e_i}	36-50 η_i		16 δ_{e_i}	27-32 $\eta_1 - \eta_6$		6 η_3	22 $\dot{\eta}_3$	
35 δ_{e_o}	51 $\dot{\delta}_a$		17 δ_{e_o}	33 $\dot{\delta}_a$		7 δ_a	23 η_1	
36-41 $p_1 - p_6$	52 $\dot{\delta}_{e_i}$		18-23 $p_1 - p_6$	34 $\dot{\delta}_{e_i}$		8 δ_{e_i}	24 η_3	
42 w_g	53 δ_a		24 w_g	35 δ_a		9 δ_{e_o}	25 $\dot{\delta}_a$	
	54 δ_{e_i}			36 δ_{e_i}		10-15 $p_1 - p_6$	26 $\dot{\delta}_{e_i}$	
	55 w			37 w		16 w_g	27 δ_a	
	56 q/n_2			38 q/n_2			28 δ_{e_i}	
							29 w	
							30 q/n_2	

TABLE 37. COEFFICIENT MATRICES FOR CASE 2

F-MATRIX FOR CASE 2

1 1	-.66218E+00	1 2	.32776E+01	1 3	-.28017E-01	1 4	-.69419E-02	1 5	-.23237E-01
1 6	.14600E-01	1 7	-.24692E-01	1 8	.50997E-01	1 9	.89406E-01	110	-.41943E-02
111	-.36092E-01	112	-.98817E-02	113	-.78981E-02	114	.14250E-01	115	-.15659E-01
116	-.30474E-01	117	.22346E+00	118	-.55979E+00	119	-.58786E+00	120	-.15561E+01
121	.40758E+01	122	-.13389E+01	123	.20001E+01	124	.86617E+01	125	-.19745E+00
126	-.23875E+01	127	-.45817E+00	128	-.39201E+00	129	.19497E+01	130	-.14466E+01
131	-.32445E+01	132	.32607E+02	133	-.24690E+03	134	-.20139E+03	135	-.40941E+02
136	-.48934E+00	137	-.67639E+01	138	-.98990E+00				
2 1	-.56837E+00	2 2	-.12293E+01	2 3	.58290E-01	2 4	-.46882E-01	2 5	-.34790E+00
2 6	-.35559E+00	2 7	-.12779E+00	2 8	.27254E+00	2 9	.27873E-01	210	-.13122E+00
211	-.22584E+00	212	-.15208E+00	213	-.13356E+00	214	-.11210E+00	215	-.10421E+00
216	-.77966E-01	217	.35119E+00	218	.54826E-01	219	-.16428E+01	220	-.10445E+02
221	-.68234E+01	222	-.51239E+01	223	.98465E+01	224	.69654E+01	225	-.72091E+01
226	-.13368E+02	227	-.77423E+01	228	-.64829E+01	229	-.35922E+01	230	-.58873E+01
231	-.60006E+01	232	.53301E+02	233	-.60551E+03	234	-.26235E+04	235	-.56095E+03
236	.27253E+01	237	-.80421E+00	238	-.12530E+02				
3 1	-.12620E+01	3 2	.16571E+00	3 3	-.93166E+00	3 4	-.19642E-01	3 5	-.11204E+00
3 6	.48658E+00	3 7	.25263E-01	3 8	-.25497E+00	3 9	.17433E+00	310	-.30890E-01
311	.28651E-01	312	.10389E+00	313	.96035E-01	314	.23093E+00	315	-.89069E-01
316	-.23731E+00	317	.22748E+01	318	-.29214E+02	319	-.35241E+01	320	-.14694E+02
321	.37541E+02	322	-.52607E+01	323	-.63746E+01	324	.44558E+02	325	-.10078E+02
326	-.73334E+01	327	.74485E+01	328	.67119E+01	329	.27593E+02	330	-.17349E+02
331	-.38705E+02	332	.35564E+03	333	-.35548E+04	334	.14284E+04	335	.30829E+03
336	.57304E+00	337	-.20517E+02	338	.68753E+01				
4 1	.35519E-01	4 2	-.95920E-01	4 3	.14365E-01	4 4	-.46616E+00	4 5	-.70439E-01
4 6	-.60690E-01	4 7	-.86751E-02	4 8	-.12895E-02	4 9	-.29856E-01	410	-.57857E-02
411	-.21051E-01	412	-.22688E-01	413	-.11339E-01	414	-.22035E-01	415	-.34194E-01
416	-.21614E-02	417	-.14305E-01	418	.26115E+00	419	-.12387E+03	420	-.93701E+00
421	-.16216E+01	422	-.17073E+00	423	.15038E+01	424	-.43895E+01	425	-.69265E+00
426	-.11279E+01	427	-.16325E+01	428	-.26963E-01	429	-.81533E+00	430	-.21706E+01
431	.57163E+00	432	.34758E+00	433	-.13414E+03	434	-.30139E+03	435	-.77226E+02
436	-.20616E+00	437	.16060E+01	438	-.13990E+01				
5 1	-.76870E-03	5 2	-.91786E+00	5 3	.63949E-01	5 4	-.76045E-01	5 5	-.13099E+01
5 6	-.43621E+00	5 7	-.43833E-01	5 8	-.42346E-01	5 9	-.44043E+00	510	-.54702E-01
511	-.26611E+00	512	-.15270E+00	513	-.14680E+00	514	-.16376E+00	515	-.18148E+00
516	-.18341E+00	517	.44391E-02	518	.14951E+01	519	-.72542E+00	520	-.19351E+03
521	-.16261E+02	522	-.11716E+01	523	-.25174E+01	524	-.21711E+02	525	-.63347E+01
526	-.17814E+02	527	-.67761E+01	528	-.57960E+01	529	-.61759E+01	530	-.95811E+01
531	-.13363E+02	532	.50916E+02	533	-.14545E+04	534	.26722E+04	535	-.70943E+03
536	-.22475E+01	537	.10995E+02	538	-.12582E+02				
6 1	-.11317E+01	6 2	-.10485E+01	6 3	.94093E-01	6 4	-.41706E-01	6 5	-.29513E+00
6 6	-.11228E+01	6 7	-.18509E+00	6 8	.46229E+00	6 9	.28063E+00	610	-.13900E+00
611	-.17667E+00	612	-.21490E+00	613	-.15173E+00	614	-.12465E+00	615	-.18933E+00
616	-.83926E-01	617	.22999E+00	618	-.49739E-01	619	-.68533E+00	620	-.96007E+01
621	-.24366E+03	622	-.73565E+01	623	.17716E+02	624	.12927E+02	625	-.78022E+01
626	-.11319E+02	627	-.12021E+02	628	-.58092E+01	629	-.47687E+01	630	-.71866E+01
631	.97311E+00	632	.33548E+02	633	.78907E+03	634	-.26279E+04	635	-.75360E+03
636	-.99210E+00	637	-.40611E+01	638	-.12268E+02				
7 1	-.21536E+00	7 2	-.25291E+00	7 3	.63700E-01	7 4	-.21769E-01	7 5	-.55068E-01
7 6	-.11610E+00	7 7	-.86288E+00	7 8	.13017E+00	7 9	.56848E-01	710	-.35799E-01
711	-.63619E-01	712	-.68431E-02	713	-.91757E-01	714	-.16280E-01	715	-.11195E+00
716	.91292E-01	717	.13750E-01	718	.25469E+00	719	-.72510E+00	720	-.14228E+01
721	-.33588E+01	722	-.30664E+03	723	.62918E+01	724	.48704E+01	725	-.24640E+01
726	-.44727E+01	727	.13600E+01	728	-.58688E+01	729	-.35973E+00	730	-.71001E+01
731	.11813E+02	732	-.67247E+01	733	-.10291E+02	734	-.67998E+03	735	-.21418E+03

```
F MATRIX FOR CASE 2 (CONTINUED)
 736 -.32477E+00 737 -.70521E-01 738 -.31534E+01
 8 1  .48110E+00 8 2  .45053E+00 8 3 -.11352E+00 8 4  .19362E-01 8 5  .45570E-01
 8 6  .20063E+00 8 7  .10432E+00 8 8 -.11603E+01 8 9 -.24073E+00 810  .78782E-01
 811  .53088E-01 812  .78267E-01 813  .12550E+00 814  .95678E-01 815 -.50959E-01
 816  .70074E-01 817 -.91262E-01 818 -.15696E+00 819  .19545E+01 820  .17357E+01
 821  .47678E+01 822  .49136E+01 823 -.35292E+03 824 -.15028E+02 825  .36451E+01
 826  .42262E+01 827  .32839E+01 828  .67126E+01 829  .64133E+01 830 -.98954E+01
 831  .53454E+00 832  .17075E+02 833 -.42360E+03 834  .11629E+04 835  .38364E+03
 836  .43958E+00 837  .16028E+01 838  .53777E+01
 9 1 -.37983E+00 9 2 -.55269E-01 9 3 -.10552E+00 9 4 -.64061E-02 9 5 -.50959E-01
 9 6  .96177E-01 9 7 -.25539E-02 9 8 -.55047E-01 9 9 -.87550E+00 910  .84688E-02
 911 -.61592E-01 912  .39394E-01 913  .33352E-02 914  .32776E-01 915 -.22464E-01
 916 -.89065E-01 917  .15490E+00 918 -.10754E+01 919 -.15547E+01 920 -.23622E+01
 921  .47058E+01 922  .11709E+00 923 -.34107E+01 924 -.39258E+03 925  .77745E-01
 926 -.42830E+01 927  .31505E+01 928 -.69322E+00 929  .44623E+01 930 -.20029E+01
 931 -.10128E+02 932  .54795E+02 933 -.77082E+02 934  .69060E+02 935  .24520E+02
 936  .39132E+00 937 -.51706E+01 938  .31886E+00
10 1 -.37519E+0010 2 -.19915E+0010 3  .78391E-0110 4  .58402E-0310 5  .66927E-02
10 6 -.57961E-0110 7 -.25543E-0110 8  .90318E-0110 9  .47146E-011010 -.11075E+01
1011 -.77572E-011012 -.52878E-011013 -.41357E-011014 -.43910E-011015 -.15384E-01
1016 -.81760E-021017 -.76703E-011018  .36342E+001019 -.46182E-011020  .40640E+00
1021 -.19296E+011022 -.17498E+011023  .35783E+011024  .25782E+011025 -.74025E+03
1026 -.23150E+011027 -.26347E+011028 -.17795E+011029 -.30680E+011030  .12924E+01
1031  .86341E+001032 -.17573E+021033 -.74070E+021034 -.42369E+031035 -.22855E+03
1036  .53623E+001037 -.36219E+011038 -.20487E+01
11 1 -.20075E+0111 2 -.19147E+0111 3  .45368E+0011 4 -.72765E-0111 5 -.57468E+00
11 6 -.66399E+0011 7 -.17396E+0011 8  .38344E+0011 9 -.18370E+001110 -.94098E-01
1111 -.23055E+011112 -.62163E+001113 -.50184E+001114 -.57649E+001115 -.74054E+00
1116 -.58281E+001117 -.13913E+011118  .38616E+011119  .97200E+001120 -.53094E+01
1121 -.21074E+021122 -.11679E+021123  .27641E+021124 -.13670E+021125  .62737E+01
1126 -.11151E+041127 -.30055E+021128 -.16156E+021129 -.26872E+021130 -.23799E+02
1131 -.13880E+021132 -.10372E+031133 -.10275E+041134 -.35972E+041135 -.31896E+04
1136  .23869E+011137 -.12360E+021138 -.20415E+02
12 1 -.24694E+0112 2 -.37095E+0112 3  .55723E+0012 4 -.17836E+0012 5 -.14560E+01
12 6 -.16376E+0112 7 -.65109E+0012 8  .13833E+0112 9  .94993E-011210 -.64967E+00
1211 -.15420E+011212 -.29582E+011213 -.15915E+011214 -.14920E+011215 -.11792E+01
1216 -.68848E+001217  .56209E+001218  .30512E+011219 -.10787E+021220 -.26684E+02
1221 -.53151E+021222  .15942E+021223  .14840E+021224  .34064E+021225 -.30431E+02
1226 -.78141E+021227 -.14307E+041228 -.75811E+021229 -.50402E+021230 -.36113E+02
1231 -.14196E+021232  .52383E+021233  .50725E+03123' -.50985E+041235 -.75441E+04
1236  .67629E+001237 -.14909E+011238 -.41524E 02
13 1 -.17448E+0113 2 -.36254E+0113 3  .46329E+0013 4 -.17596E+0013 5 -.15314E+01
13 6 -.20034E+0113 7 -.55334E+0013 8  .15209E+0113 9  .37598E+001310 -.72919E+00
1311 -.13917E+011312 -.16930E+011313 -.30940E+011314 -.15821E+011315 -.13130E+01
1316 -.82367E+001317  .80444E+001318  .29161E+011319  .30446E+011320 -.36387E+02
1321 -.26717E+021322 -.43714E+021323  .56387E+021324 -.30321E+021325 -.31614E+02
1326 -.51969E+021327 -.83134E+021328 -.15393F+041329 -.56930E+021330 -.49877E+02
1331 -.29688E+021332  .74957E+021333  .26129E+031334 -.43554E+041335 -.78570E+04
1336 -.12348E+011337  .98173E+011338 -.42470E+02
14 1 -.13266E+0114 2 -.11113E+0114 3 -.95060E-0114 4 -.84651E-0114 5 -.44431E+00
14 6 -.35689E+0014 7 -.18637E+0014 8  .34980E+0014 9  .14949E+001410 -.21786E+00
1411 -.45392E+001412 -.43391E+001413 -.43766E+001414 -.19754E-011415 -.42109E+00
1416 -.27609E+001417  .96969E+001418 -.17827E+011419 -.23750E+011420 -.15745E+02
1421  .43094E+011422 -.19124E+011423  .13931E+021424  .19740E+021425 -.13529E+02
1426 -.23170E+021427 -.13850E+021428 -.14601E+021429 -.15774E+041430 -.26704E+02
```

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F MATRIX FOR CASE 2 (CONTINUED)
1431 -.14400E+021432  .15494E+031433  .10260E+041434 -.89608E+031435 -.21997E+04
1436  .68405E+001437 -.92678E+011438 -.10553E+02
15 1  .19477E-0115 2 -.58489E+0015 3  .20441E+0015 4 -.46303E-0115 5 -.30807E+00
15 6 -.41732E+0015 7 -.97993E-0115 8  .35500E+0015 9 -.81434E-011510 -.62737E-01
1511 -.38106E+001512 -.27815E+001513 -.28703E+001514 -.32015E+001515 -.20359E+01
1516 -.46163E+001517 -.98539E+001518  .18724E+011519 -.26872E+011520  .30788E+00
1521 -.19265E+021522 -.14966E+021523 -.66485E+011524  .55132E+011525  .17121E+01
1526 -.12497E+021527 -.10826E+021528 -.12680E+021529 -.21024E+021530 -.17384E+04
1531 -.25208E+021532 -.91848E+021533 -.47353E+031534 -.44453E+031535 -.15745E+04
1536 -.80869E+001537  .68469E+011538 -.83214E+01
16 1  .53424E+0016 2 -.22955E+0016 3  .24430E+0016 4 -.20947E-0116 5 -.11441E+00
16 6 -.36665E+0016 7 -.13219E+0016 8  .20594E+0016 9 -.69189E-011610 -.14963E-01
1611 -.20857E+001612 -.14532E+001613 -.15595E+001614 -.16790E+001615 -.35235E+00
1616 -.19016E+011617 -.12405E+011618  .26335E+011619  .20826E+011620  .22394E+01
1621 -.14040E+021622  .13016E+021623  .19706E+021624 -.15890E+021625  .31093E+00
1626 -.33716E+011627 -.60378E+011628 -.80368E+011629 -.99897E+011630 -.18921E+02
1631 -.18401E+041632 -.14065E+031633 -.84061E+031634 -.19840E+031635 -.87244E+03
1636 -.91625E+001637  .10848E+021638 -.50395E+01
17 1 -.92163E+0017 2 -.18082E+0017 3 -.21465E+0017 4 -.17658E-0117 5 -.10524E+00
17 6  .11402E+0017 7 -.26968E-0117 8 -.14242E-0117 9  .10208E+001710 -.10156E-01
1711 -.64258E-011712  .86108E-021713  .76977E-021714  .67624E-011715 -.71534E-01
1716 -.12792E+001717 -.12395E+011718 -.24700E+011719 -.16375E+011720 -.66277E+01
1721  .13586E+021722 -.29476E+011723  .12621E+011724  .20395E+021725 -.28539E+01
1726 -.62156E+011727  .99626E+001728  .10153E+011729  .96046E+011730 -.81471E+01
1731 -.15726E+021732 -.24167E+041733  .82144E+031734 -.45825E+011735 -.19786E+02
1736 -.28069E-011737 -.10869E+021738 -.23249E+00
18 3  .10000E+01
19 4  .10000E+01
20 5  .10000E+01
21 6  .10000E+01
22 7  .10000E+01
23 8  .10000E+01
24 9  .10000E+01
2510  .10000E+01
2611  .10000E+01
2712  .10000E+01
2813  .10000E+01
2914  .10000E+01
3015  .10000E+01
3116  .10000E+01
3217  .10000E+01
3333 -.60000E+01
3434 -.75000E+01
3535 -.75000E+01
3636 -.22185E+023642  .22185E+02
3737 -.85492E+013740  .85492E+01
3836 -.50960E+013839  .10000E+01
3936  .90891E+023938 -.38953E+023939 -.10192E+02
4040 -.10983E+024042  .10983E+02
4141 -.49524E+004142 -.61315E-01
4241  .10000E+01


G1-MATRIX FOR CASE 2

33 1  .60000E+01
34 2  .75000E+01
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G2-MATRIX FOR CASE 2

41 1 -.30360E+00
42 1  .86190E+00


H-MATRIX FOR CASE 2

1 1 -.46710E+04 1 2 -.33999E+03 1 3  .11376E+05 1 4 -.10751E+04 1 5 -.10405E+05
1 6  .55658E+04 1 7  .38226E+04 1 8 -.92675E+04 1 9  .15864E+04 110  .56826E+04
111  .33517E+04 112 -.11905E+04 113 -.14716E+04 114 -.16984E+04 115  .29672E+04
116  .39830E+04 117  .11692E+05 118  .12620E+07 119 -.27540E+06 120 -.33662E+07
121  .19521E+07 122  .16782E+07 123 -.43864E+07 124  .73394E+06 125  .35311E+07
126  .30311E+07 127 -.11240E+07 128 -.11306E+07 129 -.15274E+07 130  .23935E+07
131  .42056E+07 132  .12556E+08 133 -.14144E+07 134  .42460E+06 135 -.41530E+06
136  .10650E+04 137 -.53974E+05 138 -.44260E+04
2 1 -.86102E+04 2 2 -.29804E+03 2 3  .41358E+03 2 4 -.91562E+03 2 5 -.10198E+04
2 6  .47160E+04 2 7 -.75835E+03 2 8  .34602E+04 2 9  .95815E+04 210  .47989E+04
211 -.83995E+03 212 -.13651E+02 213 -.54810E+02 214  .65389E+03 215  .27201E+03
216 -.59538E+03 217  .34453E+04 218 -.58771E+05 219 -.24519E+06 220 -.52211E+06
221  .15353E+07 222 -.38944E+06 223  .16019E+07 224  .44838E+07 225  .30578E+07
226 -.39053E+06 227  .21336E+05 228 -.25863E+05 229  .69313E+06 230 -.21120E+06
231 -.94568E+06 232  .46212E+07 233 -.21198E+07 234  .32960E+05 235 -.26307E+05
236  .20473E+04 237 -.10503E+06 238 -.47021E+03
3 1  .52393E+03 3 2  .93350E+02 3 3  .85424E+04 3 4 -.42479E+03 3 5 -.46136E+04
3 6  .41100E+04 3 7  .25838E+04 3 8 -.52743E+04 3 9 -.14002E+04 310 -.64333E+02
311 -.10987E+03 312 -.30653E+03 313 -.22620E+03 314 -.47990E+03 315  .19195E+04
316  .15359E+04 317  .31187E+04 318  .10001E+07 319 -.10815E+06 320 -.14847E+07
321  .14485E+07 322  .11440E+07 323 -.24666E+07 324 -.60274E+06 325 -.94317E+05
326 -.39443E+05 327 -.45073E+06 328 -.75681E+05 329 -.44955E+06 330  .16760E+07
331  .18975E+07 332  .26194E+07 333 -.10291E+06 334  .10081E+06 335 -.42412E+03
336  .30895E+02 337  .49688E+04 338  .18374E+04
4 1 -.99994E+03 4 2  .96088E+02 4 3 -.38659E+02 4 4 -.96419E+03 4 5 -.13792E+04
4 6  .40642E+04 4 7 -.90475E+03 4 8  .36825E+04 4 9  .91867E+04 410  .44513E+04
411 -.51475E+03 412  .25702E+02 413 -.89011E+02 414  .55229E+03 415  .18977E+03
416 -.56835E+03 417  .32929E+04 418 -.58505E+05 419 -.25187E+06 420 -.56242E+06
421  .15131E+07 422 -.37629E+06 423  .15326E+07 424  .43898E+07 425  .29169E+07
426 -.39738E+06 427 -.21787E+05 428 -.21933E+05 429  .58681E+06 430 -.90507E+03
431 -.55499E+06 432  .45207E+07 433 -.17390E+07 434  .10935E+06 435 -.22459E+05
436  .13863E+03 437 -.12468E+05 438  .47957E+03
5 1  .11047E+04 5 2  .93032E+02 5 3  .55441E+04 5 4  .24661E+02 5 5  .85374E+03
5 6  .98392E+03 5 7 -.10366E+04 5 8 -.90088E+03 5 9 -.13917E+04 510 -.26489E+03
511 -.14244E+04 512  .48363E+03 513  .21700E+03 514  .54278E+03 515 -.84323E+03
516 -.69240E+03 517 -.22374E+04 518  .64873E+06 519  .16472E+05 520  .36237E+06
521  .31944E+06 522 -.44925E+06 523 -.32028E+06 524 -.53826E+06 525 -.11830E+06
526 -.11934E+07 527  .33294E+06 528  .31857E+06 529  .57286E+06 530 -.10198E+07
531 -.88453E+06 532 -.30774E+07 533  .14502E+06 534 -.11724E+06 535  .90101E+05
536  .34228E+03 537  .12549E+05 538  .50000E+03
6 1 -.68182E+02 6 2  .56242E+02 6 3 -.26137E+03 6 4 -.92587E+03 6 5 -.10061E+04
6 6  .28011E+04 6 7 -.16043E+04 6 8  .15380E+04 6 9  .54196E+04 610 -.33927E+04
611  .59362E+02 612  .12623E+03 613 -.14087E+03 614  .62433E+03 615 -.19265E+03
616 -.51697E+03 617  .47463E+04 618 -.53753E+05 619 -.24748E+06 620 -.40897E+06
621  .10800E+07 622 -.64709E+06 623  .55760E+06 624  .25443E+07 625 -.22988E+07
626 -.13221E+05 627  .69734E+05 628 -.59495E+05 629  .61750E+06 630 -.28299E+06
631 -.62617E+06 632  .55968E+07 633 -.92437E+06 634  .14868E+05 635  .22823E+05
636 -.24773E+03 637 -.79323E+03 638  .31682E+03
```

```
H MATRIX FOR CASE 2 (CONTINUED)
 7 1 -.12248E+04 7 2 -.22167E+03 7 3  .32852E+04 7 4  .16866E+03 7 5  .31314E+04
 7 6 -.13581E+04 7 7 -.18389E+03 7 8  .20779E+04 7 9  .25470E+04 710  .41739E+02
 711 -.51962E+03 712  .16119E+03 713  .36844E+03 714  .24976E+03 715 -.12770E+04
 716 -.16279E+04 717 -.20002E+04 718  .38864E+06 719  .50672E+05 720  .10476E+07
 721 -.53054E+06 722 -.46980E+05 723  .87873E+06 724  .11607E+07 725  .40645E+04
 726 -.38095E+06 727  .15782E+06 728  .39216E+06 729  .22357E+06 730 -.13394E+07
 731 -.17388E+07 732 -.32170E+07 733 -.10129E+06 734 -.18734E+05 735 -.64253E+05
 736 -.15372E+03 737 -.13481E+05 738 -.15250E+04
 8 1 -.13272E+03 8 2 -.27330E+02 8 3 -.12862E+03 8 4 -.97640E+02 8 5  .37214E+02
 8 6  .33717E+03 8 7 -.20650E+03 8 8  .44970E+03 8 9  .13421E+04 810 -.43832E+03
 811  .39010E+01 812  .13694E+02 813 -.49166E+02 814  .51588E+03 815 -.13858E+03
 816 -.61310E+03 817  .11628E+05 818 -.76596E+04 819 -.27607E+05 820 -.49078E+05
 821  .15404E+06 822 -.56204E+05 823  .55930E+05 824  .45044E+06 825 -.19703E+06
 826 -.15255E+06 827  .37496E+05 828  .74824E+04 829  .51861E+06 830 -.28648E+06
 831 -.80805E+06 832  .13561E+08 833 -.29188E+06 834 -.69061E+04 835  .24827E+05
 836 -.18698E+03 837 -.16823E+04 838  .39389E+03
 9 1  .53648E+03 9 2  .96039E+02 9 3  .16623E+04 9 4  .14510E+03 9 5  .23225E+04
 9 6 -.10967E+04 9 7 -.31730E+03 9 8  .23422E+04 9 9  .24682E+04 910 -.30565E+03
 911  .72749E+03 912 -.16232E+03 913 -.17541E+03 914 -.13615E+03 915  .39420E+03
 916  .41367E+03 917  .87559E+03 918  .18379E+06 919  .37819E+05 920  .78952E+06
 921 -.40974E+06 922 -.14473E+06 923  .11113E+07 924  .13009E+07 925 -.23283E+06
 926  .61005E+06 927 -.15451E+06 928 -.16954E+06 929 -.10259E+06 930  .41620E+06
 931  .41980E+06 932  .17982E+07 933 -.20505E+06 934  .15347E+05 935 -.20083E+05
 936 -.12918E+03 937  .66299E+04 938 -.91784E+02
10 1  .22409E+0310 2 -.39454E+0210 3 -.28404E+0310 4 -.81995E+0210 5 -.39832E+03
10 6  .25627E+0310 7  .15819E+0210 8 -.29934E+0310 9  .24790E+021010 -.11869E+03
1011 -.18024E+031012  .49374E+021013  .36300E+021014  .35954E+031015 -.27140E+03
1016 -.60451E+031017  .74963E+041018 -.58586E+041019 -.16761E+051020 -.46131E+05
1021  .81652E+051022 -.16192E+051023 -.26972E+051024  .13755E+061025 -.75201E+05
1026 -.14822E+061027  .39533E+051028  .33321E+051029  .34997E+061030 -.27831E+06
1031 -.64008E+061032  .86299E+071033  .15034E+071034 -.65201E+041035  .12976E+05
1036 -.15469E+031037  .27320E+041038  .15902E+03
11 1  .18029E+0611 2  .46752E+0611 3  .12355E+0711 4 -.25858E+0611 5 -.32155E+07
11 6  .20524E+0711 7  .17302E+0711 8 -.45153E+0711 9  .67208E+061110  .35494E+07
1111  .31135E+071112 -.10556E+071113 -.10831E+071114 -.14848E+071115  .24434E+07
1116  .42152E+071117  .12579E+081118 -.37126E+061119  .69573E+061120  .57756E+07
1121  .18498E+071122  .61815E+061123 -.34805E+061124 -.21784E+071125 -.14958E+07
1126  .14034E+071127  .47073E+071128  .47459E+071129  .43920E+071130 -.30712E+07
1131 -.55310E+071132 -.46670E+081133  .21613E+091134  .10114E+101135  .22155E+09
1136 -.10144E+071137  .25063E+061138  .48553E+071139 -.44260E+041140 -.46143E+06
1142  .23627E+05
12 1 -.42178E+0412 2 -.13275E+0412 3 -.54122E+0512 4 -.24566E+0612 5 -.52092E+06
12 6  .15267E+0712 7 -.38663E+0612 8  .15755E+0712 9  .44550E+071210  .30266E+07
1211 -.39400E+061212  .27131E+051213 -.25134E+051214  .69799E+061215 -.21117E+06
1216 -.96309E+061217  .47853E+071218 -.36305E+051219  .11001E+061220  .21534E+06
1221 -.93274E+061222  .25116E+061223 -.12910E+071224 -.36901E+071225 -.35197E+07
1226  .97286E+061227  .10607E+061228  .15429E+061229 -.89556E+061230 -.49143E+06
1231  .10108E+071232 -.79649E+071233  .19240E+081234  .23901E+081235  .56439E+07
1236 -.63895E+051237  .78785E+061238  .11513E+061239 -.47021E+031240 -.89792E+06
1242  .45419E+05
13 1  .11232E+0613 2  .28645E+0613 3  .97816E+0613 4 -.99003E+0513 5 -.13888E+07
13 6  .15034E+0713 7  .11766E+0713 8 -.25468E+0713 9 -.63582E+061310 -.53145E+05
1311  .10282E+051312 -.40828E+061313 -.49941E+051314 -.42782E+061315  .17090E+07
1316  .19029E+071317  .26038E+071318 -.26833E+061319  .38781E+061320  .31337E+07
1321  .93956E+061322  .28151E+061323 -.36265E+061324 -.38091E+061325  .16128E+07
```

```
H MATRIX FOR CASE 2 (CONTINUED)
1326  .31469E+071327  .22536E+071328  .18522E+071329  .17974E+071330 -.20935E+07
1331 -.17474E+071332 -.17522E+081333  .12140E+091334  .61253E+091335  .13065E+09
1336 -.63518E+061337 -.10951E+061338  .29240E+071339  .18374E+041340  .42479E+05
1342  .68541E+03
14 1 -.73497E+0414 2  .70668E+0414 3 -.57282E+0514 4 -.25200E+0614 5 -.55679E+06
14 6  .15030E+0714 7 -.37405E+0614 8  .15106E+0714 9  .43617E+071410  .28892E+07
1411 -.40062E+061412 -.16311E+051413 -.21995E+051414  .59037E+061415 -.20413E+04
1416 -.57356E+061417  .46554E+071418 -.23498E+051419  .11205E+061420  .25749E+06
1421 -.80321E+061422  .28272E+061423 -.13287E+071424 -.35121E+071425 -.32720E+07
1426  .57812E+061427  .22805E+051428  .18245E+061429 -.75600E+061430 -.36345E+06
1431  .95343E+061432 -.75452E+071433  .16087E+081434  .14971E+081435  .35491E+07
1436 -.16674E+051437 -.21735E+051438  .73510E+051439  .47957E+031440 -.10659E+06
1442  .30755E+04
15 1  .56089E+0515 2  .13645E+0615 3  .63607E+0615 4  .18974E+0515 5  .40128E+06
15 6  .35099E+0615 7 -.42383E+0615 8 -.35655E+0615 9 -.53621E+061510 -.90299E+05
1511 -.11700E+071512  .35453E+061513  .32819E+061514  .58518E+061515 -.10057E+07
1516 -.87601E+061517 -.30668E+071518 -.16811E+061519  .15343E+061520  .87733E+06
1521  .69217E+061522  .85873E+061523 -.80154E+061524  .42352E+051525  .88005E+06
1526  .29515E+071527  .19691E+061528  .39389E+061529 -.32558E+061530  .20372E+07
1531  .17579E+071532  .20298E+071533  .47290E+081534  .29117E+091535  .61109E+08
1536 -.31011E+061537 -.10126E+061538  .13861E+071539  .50000E+031540  .10728E+06
1542  .75935E+04
16 1 -.10956E+0516 2 -.61730E+0416 3 -.54322E+0516 4 -.24724E+0616 5 -.40523E+06
16 6  .10672E+0716 7 -.64038E+0616 8  .55196E+0616 9  .25254E+071610 -.22873E+07
1611 -.19535E+051612  .72926E+051613 -.61067E+051614  .61809E+061615 -.28754E+06
1616 -.63407E+061617  .56608E+071618 -.16264E+051619  .91369E+051620  .75880E+05
1621 -.58430E+061622  .45370E+061623 -.48587E+061624 -.19593E+071625  .24560E+07
1626 -.18049E+061627 -.20190E+061628  .18990E+061629 -.91362E+061630  .24108E+06
1631  .79647E+061632 -.10705E+081633  .89911E+071634 -.12342E+081635 -.28717E+07
1636  .18015E+051637 -.97347E+051638 -.59441E+051639  .31682E+031640 -.67815E+04
1642 -.54959E+04
17 1  .27944E+0517 2  .62053E+0517 3  .38248E+0617 4  .53277E+0517 5  .10627E+07
17 6 -.51343E+0617 7 -.39220E+0517 8  .85935E+0617 9  .11493E+071710  .66098E+04
1711 -.36757E+061712  .16582E+061713  .39898E+061714  .22821E+061715 -.13317E+07
1716 -.17310E+071717 -.32441E+071718 -.99420E+051719  .53596E+051720 -.11032E+06
1721  .79103E+061722  .31833E+061723 -.13215E+071724 -.13538E+071725  .28882E+06
1726  .12046E+071727  .20832E+061728 -.18581E+061729 -.97689E+051730  .24529E+07
1731  .31657E+071732  .39823E+071733  .14757E+081734  .13896E+091735  .30528E+08
1736 -.13061E+061737  .12656E+061738  .66769E+061739 -.15250E+041740 -.11525E+06
1742 -.34103E+04
18 1 -.12334E+0518 2 -.36149E+0418 3 -.10604E+0518 4 -.27187E+0518 5 -.46791E+05
18 6  .15070E+0618 7 -.55554E+0518 8  .57035E+0518 9  .44485E+061810 -.19152E+06
1811 -.15404E+061812  .37927E+051813  .82405E+041814  .51806E+061815 -.28979E+06
1816 -.81213E+061817  .13538E+081818 -.29469E+051819 -.11309E+051820 -.10310E+06
1821  .89419E+051822  .22313E+051823 -.14465E+061824 -.27520E+061825  .28076E+06
1826 -.97769E+051827 -.98802E+041828  .82083E+051829 -.68872E+061830  .13530E+06
1831  .92725E+061832 -.27834E+081833  .11549E+081834 -.21914E+071835 -.78586E+06
1836  .49531E+041837 -.12799E+061838 -.12099E+051839  .39389E+031840 -.14382E+05
1842 -.41482E+04
19 1  .97123E+0419 2  .25620E+0519 3  .18133E+0619 4  .38189E+0519 5  .79202E+06
19 6 -.39762E+0619 7 -.14229E+0619 8  .11020E+0719 9  .13014E+071910 -.23130E+06
1911  .61231E+061912 -.15086E+061913 -.16642E+061914 -.97980E+051915  .41678E+06
1916  .41816E+061917  .18225E+071918 -.47759E+051919  .99710E+041920 -.24941E+06
1921  .45129E+061922  .20344E+061923 -.10602E+071924 -.11359E+071925  .37399E+06
```

TABLE 37. Concluded

```
H MATRIX FOR CASE 2 (CONTINUED)
1926 -.56763E+061927  .40314E+061928  .41687E+061929  .34640E+061930 -.65017E+06
1931 -.78889E+061932 -.25447E+071933  .66182E+061934  .54132E+081935  .11792E+08
1936 -.52265E+051937 -.65455E+051938  .25932E+061939 -.91784E+021940  .56680E+05
1942 -.28659E+04
20 1 -.78385E+0420 2 -.16362E+0420 3 -.93750E+0420 4 -.16715E+0520 5 -.45811E+05
20 6  .80430E+0520 7 -.15993E+0520 8 -.25006E+0520 9  .13847E+062010 -.72904E+05
2011 -.14506E+062012  .38452E+052013  .32191E+052014  .34603E+062015 -.27344E+06
2016 -.63136E+062017  .85592E+072018 -.14076E+052019 -.42376E+042020  .15981E+05
2021  .47650E+052022 -.33207E+052023  .11823E+062024  .16851E+062025  .56952E+05
2026  .14608E+062027 -.64837E+052028 -.51334E+052029 -.49218E+062030  .42081E+06
2031  .10107E+072032 -.18020E+082033  .45764E+072034 -.10821E+072035 -.39737E+06
2036  .42021E+042037 -.11483E+062038 -.59784E+042039  .15902E+032040  .23356E+05
2042 -.34318E+04
21 3  .10000E+01
22 4  .10000E+01
23 5  .10000E+01
24 6  .10000E+01
25 7  .10000E+01
26 8  .10000E+01
27 9  .10000E+01
2810  .10000E+01
2911  .10000E+01
3012  .10000E+01
3113  .10000E+01
3214  .10000E+01
3315  .10000E+01
3416  .10000E+01
3517  .10000E+01
3618  .10000E+01
3719  .10000E+01
3820  .10000E+01
3921  .10000E+01
4022  .10000E+01
4123  .10000E+01
4224  .10000E+01
4325  .10000E+01
4426  .10000E+01
4527  .10000E+01
4628  .10000E+01
4729  .10000E+01
4830  .10000E+01
4931  .10000E+01
5032  .10000E+01
5133 -.60000E+01
5234 -.75000E+01
5333  .10000E+01
5434  .10000E+01
55 1  .10000E+01
56 2  .10000E+01
57 2 -.22748E-02
```

```
D-MATRIX FOR CASE 2

11 1 -.84864E+0711 2  .31845E+07
12 1 -.12719E+0812 2  .24720E+06
13 1 -.61746E+0613 2  .75607E+06
14 1 -.10434E+0814 2  .82012E+06
15 1  .87012E+0615 2 -.87930E+06
16 1 -.55462E+0716 2  .11151E+06
17 1 -.60774E+0617 2 -.14050E+06
18 1 -.17513E+0718 2 -.51796E+05
19 1 -.12303E+0719 2  .11510E+06
20 1  .90204E+0720 2 -.48901E+05
51 1  .60000E 0152 2  .75000E 01
57 2  .75000E 01
```

TABLE 38. EIGENVALUES AND RMS RESPONSES FOR CASE 2

OPEN LOOP

EIGENVALUES

REAL	IMAG	FREQ	DAMP
-.24680489	0.00000000	.24680489	-1.00000000
-.24843511	0.00000000	.24843511	-1.00000000
-10.98300000	0.00000000	10.98300000	-1.00000000
-8.54920000	0.00000000	8.54920000	-1.00000000
-5.09600000	3.60330182	6.24123385	-.81650522
-22.18500000	0.00000000	22.18500000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-6.00000000	0.00000000	6.00000000	-1.00000000
-.87958028	1.27414073	1.54825581	-.56811043
-.51114709	5.45910641	5.48298406	-.09322425
-.23153375	11.12377051	11.12617985	-.02080982
-.56235202	13.79095081	13.80241153	-.04074303
-.58926994	15.59071845	15.60185056	-.03776923
-.42543603	17.48034181	17.48551817	-.02433076
-.57814367	18.75374268	18.76265212	-.03081354
-.46764186	19.82490739	19.83042213	-.02358204
-.55183151	27.19946735	27.20506465	-.02028415
-.98417727	33.27741024	33.29196055	-.02956201
-.80458111	37.35352141	37.36218560	-.02153464
-.81469289	39.36582774	39.37425704	-.02069100
-2.41964243	40.11805440	40.19095617	-.06020365
-1.11127619	41.55086856	41.56572642	-.02673540
-1.02606573	42.78253654	42.79483898	-.02397639
-.66176822	49.22033625	49.22478479	-.01344380

TABLE 38. Concluded

States

R.M.S. RESPONSES

1	.15749265E+02
2	.24047537E+01
3	.21375944E+01
4	.71228885E-01
5	.21399880E+00
6	.12941657E+00
7	.21727066E-01
8	.35101296E-01
9	.38867383E-01
10	.14528055E-01
11	.45516509E-01
12	.42391354E-01
13	.33773654E-01
14	.18483173E-01
15	.81967769E-02
16	.11651000E-01
17	.81831300E-02
18	.70211777E+00
19	.82045150E-02
20	.23440668E-01
21	.37561075E-01
22	.53221984E-02
23	.10707587E-01
24	.90147168E-02
25	.44006460E-02
26	.15131846E-01
27	.12323695E-01
28	.75592331E-02
29	.64967921E-02
30	.14155077E-02
31	.32029698E-02
32	.33047234E-02
33	.15460685E-14
34	.29099085E-15
35	.30474335E-14
36	.99178200E+00
37	.97183710E+00
38	.97979891E+00
39	.55751508E+01
40	.98351309E+00
41	.37288852E+00
42	.10000642E+01

Responses

R.M.S. RESPONSES

1	.11221924E+07
2	.16183051E+06
3	.77227746E+06
4	.89494276E+05
5	.43855596E+06
6	.49681893E+05
7	.25746560E+06
8	.54714230E+05
9	.11495801E+06
10	.30820437E+05
11	.35579708E+07
12	.48785318E+06
13	.24948393E+07
14	.40364431E+06
15	.14816548E+07
16	.27421917E+06
17	.90853222E+06
18	.14260969E+06
19	.45453683E+06
20	.86699336E+05
21	.21375944E+01
22	.71228885E-01
23	.21399880E+00
24	.12941657E+00
25	.21727066E-01
26	.35101296E-01
27	.38867383E-01
28	.14528055E-01
29	.45516509E-01
30	.42391354E-01
31	.33773654E-01
32	.18483173E-01
33	.81967769E-02
34	.11651000E-01
35	.81831300E-02
36	.70211777E+00
37	.82045150E-02
38	.23440668E-01
39	.37561075E-01
40	.53221984E-02
41	.10707587E-01
42	.90147168E-02
43	.44006460E-02
44	.15131846E-01
45	.12323695E-01
46	.75592331E-02
47	.64967921E-02
48	.14155077E-02
49	.32029698E-02
50	.33047234E-02
51	.92764110E-14
52	.21824314E-14
53	.15460685E-14
54	.29099085E-15
55	.15749265E+02
56	.24047537E+01

TABLE 39. COEFFICIENT MATRICES FOR CASE 3R

```
F-MATRIX FOR CASE 3R
 1 1 -.68156E+00 1 2  .32791E+01 1 3 -.35615E-01 1 4 -.71438E-02 1 5 -.24133E-01
 1 6  .21332E-01 1 7 -.26404E-01 1 8  .47797E-01 1 9 -.63975E+00 110 -.65316E+00
 111 -.17082E+01 112  .44958E+01 113 -.13601E+01 114  .18468E+01 115 -.23106E+03
 116 -.19002E+03 117 -.30147E+02 118 -.48273E+00 119 -.70523E+01 120 -.91890E+00
 2 1 -.55140E+00 2 2 -.11746E+01 2 3  .39579E-01 2 4 -.44709E-01 2 5 -.32990E+00
 2 6 -.32436E+00 2 7 -.11995E+00 2 8  .25113E+00 2 9 -.10850E+00 210 -.16757E+01
 211 -.10304E+02 212 -.57006E+01 213 -.49434E+01 214  .91053E+01 215 -.57617E+03
 216 -.25322E+04 217 -.44466E+03 218  .27053E+01 219 -.10493E+01 220 -.11854E+02
 3 1 -.14863E+01 3 2  .10534E+00 3 3 -.98741E+00 3 4 -.24874E-01 3 5 -.14662E+00
 3 6  .51128E+00 3 7  .17325E-01 3 8 -.25652E+00 3 9 -.29851E+02 310 -.40858E+01
 311 -.16621E+02 312  .40631E+02 313 -.58418E+01 314 -.66903E+01 315 -.33801E+04
 316  .14194F+04 317  .25982E+03 318  .62979E+00 319 -.23132E+02 320  .66833E+01
 4 1  .45522E-01 4 2 -.88309E-01 4 3  .14416E-01 4 4 -.46574E+00 4 5 -.67265E-01
 4 6 -.58908E-01 4 7 -.76114E-02 4 8 -.30630E-02 4 9  .26632E+00 410 -.12384E+03
 411 -.86920E+00 412 -.15823E+01 413 -.15692E+00 414  .15112E+01 415 -.13323E+03
 416 -.29227E+03 417 -.63867E+02 418 -.21372E+00 419  .16827E+01 420 -.13256E+01
 5 1  .51060E-01 5 2 -.85213E+00 5 3  .51344E-01 5 4 -.72981E-01 5 5 -.12857E+01
 5 6 -.40920E+00 5 7 -.34829E-01 5 8 -.61413E-01 5 9  .13958E+01 510 -.64745E+00
 511 -.19318E+03 512 -.15404E+02 513 -.97897E+00 514 -.31528E+01 515 -.14152E+04
 516 -.25753E+04 517 -.58207E+03 518 -.23009E+01 519  .11163E+02 520 -.11851E+02
 6 1 -.11068E+01 6 2 -.98721E+00 6 3  .75927E-01 6 4 -.39164E-01 6 5 -.27445E+00
 6 6 -.10894E+01 6 7 -.17602E+00 6 8  .43783E+00 6 9 -.19816E+00 610 -.67868E+00
 611 -.93687E+01 612 -.24252E+03 613 -.71834E+01 614  .17025E+02 615  .80354E+03
 616 -.25292E+04 617 -.62504E+03 618 -.10079E+01 619 -.42601E+01 620 -.11528E+02
 7 1 -.20164E+00 7 2 -.23495E+00 7 3  .60625E-01 7 4 -.21022E-01 7 5 -.48689E-01
 7 6 -.10753E+00 7 7 -.86125E+00 7 8  .12357E+00 7 9  .23384E+00 710 -.74504E+00
 711 -.12963E+01 712 -.32269E+01 713 -.30625E+03 714  .60832E+01 715 -.13080E+02
 716 -.65296F+03 717 -.17943E+03 718 -.32804E+00 719 -.42523E-01 720 -.29517E+01
 8 1  .46358E+00 8 2  .42046E+00 8 3 -.10741E+00 8 4  .18093E-01 8 5  .35634E-01
 8 6  .18446E+00 8 7  .99742E-01 8 8 -.11481E+01 8 9 -.11571E+00 810  .20064E+01
 811  .14987E+01 812  .45063E+01 813  .47710E+01 814 -.35230E+03 815 -.41074E+03
 816  .11165E+04 817  .32318E+03 818  .43848E+00 819  .16346E+01 820  .50304E+01
 9 3  .10000E+01
10 4  .10000E+01
11 5  .10000E+01
12 6  .10000E+01
13 7  .10000E+01
14 8  .10000E+01
1515 -.60000E+01
1616 -.75000E+01
1717 -.75000E+01
1818 -.22185E+021824  .22185E+02
1919 -.85492E+011922  .85492E+01
2018 -.50960E+012021  .10000E+01
2118  .90891E+022120 -.38953E+022121 -.10192E+02
2222 -.10983E+022224  .10983E+02
2323 -.49524E+002324 -.61315E-01
2423  .10000E+01


G1-MATRIX FOR CASE 3R

15 1  .60000E+01
16 2  .75000E+01
```

```
G2-MATRIX FOR CASE 3R

23 1 -.30360E+00
24 1  .86190E+00

H-MATRIX FOR CASE 3R

 1 1 -.11402E+05 1 2 -.20980E+04 1 3  .11892E+05 1 4 -.11222E+04 1 5 -.10486E+05
 1 6  .59635E+04 1 7  .37766E+04 1 8 -.95956E+04 1 9  .12660E+07 110 -.27397E+06
 111 -.33460E+07 112  .19604E+07 113  .16535E+07 114 -.43256E+07 115 -.49243E+07
 116 -.42718E+07 117 -.37233E+06 118  .74621E+04 119 -.13161E+06 120 -.18320E+05
 2 1 -.16815E+05 2 2 -.17551E+04 2 3 -.13706E+04 2 4 -.10226E+04 2 5 -.17113E+04
 2 6  .62092E+04 2 7 -.89423E+03 2 8  .30213E+04 2 9 -.79580E+05 210 -.27011E+06
 211 -.56988E+06 212  .16328E+07 213 -.40178E+06 214  .15634E+07 215  .35508E+05
 216  .23209E+06 217  .14186E+06 218  .87754E+04 219 -.21133E+06 220  .11202E+04
 3 1  .21601E+04 3 2  .94467E+03 3 3  .87658E+04 3 4 -.40629E+03 3 5 -.43716E+04
 3 6  .40248E+04 3 7  .26270E+04 3 8 -.52791E+04 3 9  .10033E+07 310 -.10366E+06
 311 -.14698E+07 312  .14385E+07 313  .11376E+07 314 -.24584E+07 315 -.98587E+06
 316  .15608E+07 317  .10258E+07 318 -.27862E+04 319  .24007E+05 320  .68258E+04
 4 1 -.86859E+04 4 2 -.12267E+04 4 3 -.17298E+04 4 4 -.10673E+04 4 5 -.20389E+04
 4 6  .54799E+04 4 7 -.10409E+04 4 8  .32638E+04 4 9 -.78122E+05 410 -.27563E+06
 411 -.60684E+06 412  .16049E+07 413 -.38764E+06 414  .14963E+07 415  .73225E+05
 416  .49587E+06 417  .18605E+06 418  .62943E+04 419 -.11212E+06 420  .22936E+04
 5 1  .32461E+04 5 2  .86847E+03 5 3  .54010E+04 5 4  .57899E+02 5 5  .10606E+04
 5 6  .90921E+03 5 7 -.10337E+04 5 8 -.75766E+03 5 9  .64725E+06 510  .17219E+05
 511  .35796E+06 512  .32090E+06 513 -.43627E+06 514 -.33237E+06 515  .16249E+07
 516  .16435E+07 517  .60190E+06 518 -.18108E+04 519  .35857E+05 520  .71056E+04
 6 1 -.45678E+04 6 2 -.45726E+03 6 3 -.19342E+04 6 4 -.10373E+04 6 5 -.17389E+04
 6 6  .39764E+04 6 7 -.16366E+04 6 8  .87787E+03 6 9 -.70613E+05 610 -.26417E+06
 611 -.45087E+06 612  .11613E+07 613 -.64807E+06 614  .52377E+06 615  .18458E+07
 616  .15519E+07 617  .46543E+06 618  .13897E+04 619 -.61029E+05 620  .69635E+04
 7 1 -.20237E+04 7 2 -.36966E+03 7 3  .28416E+04 7 4  .17184E+03 7 5  .30157E+04
 7 6 -.10691E+04 7 7 -.14644E+03 7 8  .19386E+04 7 9  .38385E+06 710  .46899E+05
 711  .10328E+07 712 -.50920E+06 713 -.49705E+05 714  .86234E+06 715  .54150E+06
 716  .14635E+06 717 -.55748E+05 718  .15095E+04 719 -.26246E+05 720  .63792E+03
 8 1 -.63651E+04 8 2 -.10989E+04 8 3 -.17894E+04 8 4 -.21584E+03 8 5 -.65703E+03
 8 6  .13185E+04 8 7 -.34017E+03 8 8  .21615E+03 8 9 -.26382E+05 810 -.41223E+05
 811 -.98042E+05 812  .25316E+06 813 -.74333E+05 814  .49803E+05 815  .54116E+07
 816  .46564E+06 817  .15154E+06 818  .52326E+03 819 -.78329E+05 820  .20544E+04
 9 1 -.17620E+04 9 2 -.51343E+03 9 3  .13572E+04 9 4  .99645E+02 9 5  .20198E+04
 9 6 -.77769E+03 9 7 -.33906E+03 9 8  .21110E+04 9 9  .18069E+06 910  .32667E+05
 911  .78195E+06 912 -.39323E+06 913 -.14985E+06 914  .11084E+07 915 -.78084E+06
 916 -.63165E+06 917 -.32589E+06 918  .19434E+04 919 -.21225E+05 920 -.27122E+04
10 1 -.36447E+0410 2 -.70178E+0310 3 -.13305E+0410 4 -.15386E+0310 5 -.82027E+03
10 6  .85787E+0310 7 -.67775E+0210 8 -.42362E+0310 9 -.17731E+051010 -.24964E+05
1011 -.77475E+051012  .14476E+061013 -.28098E+051014 -.30134E+051015  .53083E+07
1016  .29360E+061017  .10060E+061018  .17237E+031019 -.44805E+051020  .12945E+04
11 1  .18137E+0611 2  .44677E+0611 3  .12423E+0711 4 -.25941E+0611 5 -.32230E+07
11 6  .20391E+0711 7  .17271E+0711 8 -.45062E+0711 9 -.31213E+061110  .70640E+06
1111  .56752E+071112  .13907E+071113  .57191E+061114 -.12992E+061115  .20835E+09
1116  .97908E+091117  .17283E+091118 -.10109E+071119  .44184E+061120  .45933E+07
1121 -.44260E+041122 -.46143E+061124  .23627E+05
12 1  .35457E+0412 2 -.80794E+0312 3 -.52043E+0512 4 -.24557E+0612 5 -.52054E+06
12 6  .15249E+0712 7 -.38657E+0612 8  .15761E+0712 9 -.12380E+051210  .13555E+06
1211  .26164E+061212 -.10491E+071213  .26751E+061214 -.12415E+071215  .15421E+08
1216  .21769E+081217  .37171E+071218 -.68760E+051219  .89422E+061220  .10307E+06
```

```
H-MATRIX FOR CASE 3R (CONTINUED)
1221 -.47021E+031222 -.89792E+061224  .45419E+05
13 1  .10591E+0613 2  .27273E+0613 3  .98197E+0613 4 -.99521E+0513 5 -.13933E+07
13 6  .14964E+0713 7  .11748E+0713 8 -.25420E+0713 9 -.23688E+061310  .39108E+06
1311  .30743E+071312  .70998E+061313  .24726E+061314 -.19728E+061315  .11607E+09
1316  .58954E+091317  .10272E+091318 -.62575E+061319 -.84798E+051320  .27637E+07
1321  .18374E+041322  .42479E+051324  .68541E+03
14 1  .58499E+0314 2  .81114E+0414 3 -.55453E+0514 4 -.25191E+0614 5 -.55627E+06
14 6  .15014E+0714 7 -.37394E+0614 8  .15111E+0714 9 -.20644E+041410  .13629E+06
1411  .30236E+061412 -.90600E+061413  .29851E+061414 -.12885E+071415  .12812E+08
1416  .14075E+081417  .26117E+071418 -.22062E+051419  .82742E+051420  .67924E+05
1421  .47957E+031422 -.10659E+061424  .30755E+04
15 1  .50178E+0515 2  .12954E+0615 3  .63783E+0615 4  .18678E+0515 5  .39898E+06
15 6  .34803E+0615 7 -.42467E+0615 8 -.35456E+0615 9 -.15329E+061510  .15447E+06
1511  .85731E+061512  .59861E+061513  .81152E+061514 -.70459E+061515  .43514E+08
1516  .27958E+091517  .48408E+081518 -.30539E+061519 -.12251E+061520  .13076E+07
1521  .50000E+031522  .10728E+061524  .75935E+04
16 1 -.44169E+0416 2 -.48777E+0416 3 -.52366E+0516 4 -.24707E+0616 5 -.40418E+06
16 6  .10660E+0716 7 -.64024E+0616 8  .55249E+0616 9  .43519E+041610  .11119E+06
1611  .13188E+061612 -.68436E+061613  .45904E+061614 -.45983E+061615  .42389E+07
1616 -.13054E+081617 -.24351E+071618  .16309E+051619 -.16289E+051620 -.60716E+05
1621  .31682E+031622 -.67815E+041624 -.54959E+04
17 1  .26792E+0517 2  .58958E+0517 3  .38376E+0617 4  .53115E+0517 5  .10616E+07
17 6 -.51536E+0617 7 -.39769E+0517 8  .86068E+0617 9 -.87665E+051710  .56207E+05
1711 -.10800E+061712  .71745E+061713  .31097E+061714 -.12614E+071715  .12831E+08
1716  .13398E+091717  .23903E+081718 -.13172E+061719  .14222E+061720  .62683E+06
1721 -.15250E+041722 -.11525E+061724 -.34103E+04
18 1 -.26000E+0318 2 -.12429E+0418 3 -.77001E+0418 4 -.26952E+0518 5 -.45414E+05
18 6  .14907E+0618 7 -.55256E+0518 8  .57347E+0518 9  .36788E+041810  .12306E+05
1811 -.14240E+051812 -.89106E+051813  .60356E+051814 -.14714E+061815  .97702E+06
1816 -.21942E+071817 -.47868E+061818  .38551E+041819  .16805E+051820 -.10397E+05
1821  .39389E 031822 -.14382E 051824 -.41482E 04
19 1  .11014E+0519 2  .24985E+0519 3  .18174E+0619 4  .38177E+0519 5  .79184E+06
19 6 -.39844E+0619 7 -.14245E+0619 8  .11027E+0719 9 -.44438E+051910  .13201E+05
1911 -.25386E+061912  .42578E+061913  .20404E+061914 -.10480E+071915  .14664E+07
1916  .53135E+081917  .96484E+071918 -.53703E+051919 -.41029E+051920  .24873E+06
1921 -.91784E+021922  .56680E+051924 -.28659E+04
20 1 -.27781E+0320 2 -.19670E+0320 3 -.74720E+0420 4 -.16573E+0520 5 -.44975E+05
20 6  .79401E+0520 7 -.15808E+0520 8 -.24856E+0520 9  .77964E+042010  .10146E+05
2011  .75129E+052012 -.70432E+052013 -.78915E+042014  .11628E+062015 -.28674E+07
2016 -.12211E+072017 -.30281E+062018  .38844E+042019 -.23918E+052020 -.57892E+04
2021  .15902E+032022  .23356E+052024 -.34318E+04
21 3  .10000E+01
22 4  .10000E+01
23 5  .10000E+01
24 6  .10000E+01
25 7  .10000E+01
26 8  .10000E+01
27 9  .10000F+01
2810  .10000E+01
2911  .10000E+01
3012  .10000E+01
3113  .10000E+01
3214  .10000E+01
3315 -.60000E+01

H-MATRIX FOR CASE 3R (CONTINUED)
3416 -.75000E+01
3515  .10000E+01
3616  .10000E+01
37 1  .10000E+01
38 2  .10000E+01
40 2 -.22748E-02
```

```
D-MATRIX FOR CASE 3R

11 1 -.84864E+0711 2  .31845E+07
12 1 -.12719E+0812 2  .24720E+06
13 1 -.61746E+0613 2  .75607E+06
14 1 -.10434E+0814 2  .82012E+06
15 1  .87012E+0615 2 -.87930E+06
16 1 -.55462E+0716 2  .11151E+06
17 1 -.60774E+0617 2 -.14050E+06
18 1 -.17513E+0718 2 -.51796E+05
19 1 -.12303E+0719 2  .11510E+06
20 1  .90204E+0720 2 -.48901E+05
33 1  .60000E 01
34 2  .75000E 0140 2  .75000E 01
```

TABLE 40. EIGENVALUES AND RMS RESPONSES FOR CASE 3R

EIGENVALUES

REAL	IMAG	FREQ	DAMP
-.24680489	0.00000000	.24680489	-1.00000000
-.24843511	0.00000000	.24843511	-1.00000000
-10.98300000	0.00000000	10.98300000	-1.00000000
-8.54920000	0.00000000	8.54920000	-1.00000000
-5.09600000	3.60330182	6.24123385	-.81650522
-22.18500000	0.00000000	22.18500000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-6.00000000	0.00000000	6.00000000	-1.00000000
-.89639730	1.28832477	1.56949317	-.57113807
-.47291589	5.39825762	5.41893299	-.08727104
-.23197946	11.12583909	11.12825817	-.02084598
-.58051611	13.80188476	13.81408780	-.04202349
-.62879051	15.63094804	15.64359019	-.04019477
-.42916431	17.49286539	17.49812909	-.02452630
-.63282642	18.79828981	18.80893854	-.03364498

Responses

R.M.S. RESPONSES

1	.97691330E+06
2	.95948058E+05
3	.71717221E+06
4	.33235365E+05
5	.41364008E+06
6	.22612479E+05
7	.22832789E+06
8	.37925111E+04
9	.88437194E+05
10	.28281492E+04
11	.32640649E+07
12	.33403210E+06
13	.23570807E+07
14	.27606995E+06
15	.14024411E+07
16	.20932398E+06
17	.84463933E+06
18	.10995822E+06
19	.41152790E+06
20	.68150133E+05
21	.19583924E+01
22	.67654134E-01
23	.20722379E+00
24	.13572309E+00
25	.23649271E-01
26	.35310963E-01
27	.63519868E+00
28	.77335982E-02
29	.22888029E-01
30	.38568559E-01
31	.56414905E-02
32	.11094970E-01
33	.48164338E-14
34	.13204196E-13
35	.80273897E-15
36	.17605594E-14
37	.15841310E+02
38	.24460384E+01

States

R.M.S. RESPONSES

1	.15841310E+02
2	.24460384E+01
3	.19583924E+01
4	.67654134E-01
5	.20722379E+00
6	.13572309E+00
7	.23649271E-01
8	.35310963E-01
9	.63519868E+00
10	.77335982E-02
11	.22888029E-01
12	.38568559E-01
13	.56414905E-02
14	.11094970E-01
15	.80273897E-15
16	.17605594E-14
17	.32509803E-14
18	.99178200E+00
19	.97183710E+00
20	.97979891E+00
21	.55751508E+01
22	.98351309E+00
23	.37288852E+00
24	.10000642E+01

TABLE 41. COEFFICIENT MATRICES FOR CASE 3T

```
F-MATRIX FOR CASE 3T

 1 1 -.66218E+00 1 2  .32776E+01 1 3 -.28017E-01 1 4 -.69419E-07 1 5 -.23237E-01
 1 6  .14600E-01 1 7 -.24692E-01 1 8  .50997E-01 1 9 -.55979E+00 110 -.58786E+00
 111 -.15561E+01 112  .40758E+01 113 -.13389E+01 114  .20001E+01 115 -.24690E+03
 116 -.20139E+03 117 -.40941E+02 118 -.48934E+00 119 -.67639E+01 120 -.98990E+00
 2 1 -.56837E+00 2 2 -.12293E+01 2 3  .582[illegible] 2 4 -.46882E-01 2 5 -.34790E+00
 2 6 -.35559E+00 2 7 -.12779E+00 2 8  .27254E+00 2 9  .54826E-01 210 -.16428E+01
 211 -.10445E+02 212 -.68234E+01 213 -.51239E+01 214  .98465E+01 215 -.60551E+03
 216 -.26235E+04 217 -.56095E+03 218  .27253E+01 219 -.80421E+00 220 -.12530E+02
 3 1 -.12620E+01 3 2  .16571E+00 3 3 -.93166E+00 3 4 -.19642E-01 3 5 -.11204E+00
 3 6  .48658E+00 3 7  .25263E-01 3 8 -.25497E+00 3 9 -.29214E+02 310 -.35241E+01
 311 -.14694E+02 312  .37541E+02 313 -.52607E+01 314 -.63746E+01 315 -.35548E+04
 316  .14284E+04 317  .30829E+03 318  .57304E+00 319 -.20517E+02 320  .68753E+01
 4 1  .35519E-01 4 2 -.95920E-01 4 3  .14365E-01 4 4 -.46616E+00 4 5 -.70439E-01
 4 6 -.60690E-01 4 7 -.86751E-02 4 8 -.12895E-02 4 9  .26115E+00 410 -.12387E+03
 411 -.93701E+00 412 -.16216E+01 413 -.17073E+00 414  .15038E+01 415 -.13414E+03
 416 -.30139E+03 417 -.77226E+02 418 -.20616E+00 419  .16060E+01 420 -.13990E+01
 5 1 -.76870E-03 5 2 -.91786E+00 5 3  .63949E-01 5 4 -.76045E-01 5 5 -.13099E+01
 5 6 -.43621E+00 5 7 -.43833E-01 5 8 -.42346E-01 5 9  .14951E+01 510 -.72542E+00
 511 -.19351E+03 512 -.16261E+02 513 -.11716E+01 514 -.25174E+01 515 -.14545E+04
 516 -.26722E+04 517 -.70943E+03 518 -.22475E+01 519  .10995E+02 520 -.12582E+02
 6 1 -.11317E+01 6 2 -.10485E+01 6 3  .94093E-01 6 4 -.41706E-01 6 5 -.29513E+00
 6 6 -.11228E+01 6 7 -.18509E+00 6 8  .46229E+00 6 9 -.49739E-01 610 -.68533E+00
 611 -.96007E+01 612 -.24366E+03 613 -.73565E+01 614  .17716E+02 615  .78907E+03
 616 -.26279E+04 617 -.75360E+03 618 -.99210E+00 619 -.40611E+01 620 -.12268E+02
 7 1 -.21536E+00 7 2 -.25291E+00 7 3  .63700E-01 7 4 -.21769E-01 7 5 -.55068E-01
 7 6 -.11610E+00 7 7 -.86288E+00 7 8  .13017E+00 7 9  .25469E+00 710 -.72510E+00
 711 -.14228E+01 712 -.33588E+01 713 -.30664E+03 714  .62918E+01 715 -.10291E+02
 716 -.67998E+03 717 -.21418E+03 718 -.32477E+00 719 -.70521E-01 720 -.31534E+01
 8 1  .48110E+00 8 2  .45053E+00 8 3 -.11352E+00 8 4  .19362E-01 8 5  .45570E-01
 8 6  .20063E+00 8 7  .10432E+00 8 8 -.11603E+01 8 9 -.15696E+00 810  .19545E+01
 811  .17357E+01 812  .47678E+01 813  .49136E+01 814 -.35292E+03 815 -.42360E+03
 816  .11629E+04 817  .38364E+03 818  .43958E+00 819  .16028E+01 820  .53777E+01
 9 3  .10000E+01
 10 4  .10000E+01
 11 5  .10000E+01
 12 6  .10000E+01
 13 7  .10000E+01
 14 8  .10000E+01
 1515 -.60000E+01
 1616 -.75000E+01
 1717 -.75000E+01
 1818 -.22185E+021824  .22185E+02
 1919 -.85492E+011922  .85492E+01
 2018 -.50960E+012021  .10000E+01
 2118  .90891E+022120 -.38953E+022121 -.10192E+02
 2222 -.10983E+022224  .10983E+02
 2323 -.49524E+002324 -.61315E-01
 2423  .10000E+01


GI-MATRIX FOR CASE 3T

 15 1  .60000E+01
 16 2  .75000E+01
```

G2-MATRIX FOR CASE 3T

```
23 1 -.30360E+00
24 1  .86190E+00
```

H MATRIX FOR CASE 3T

```
 1 1 -.46710E+04 1 2 -.33999E+03 1 3  .11376E+05 1 4 -.10751E+04 1 5 -.10405E+05
 1 6  .55658E+04 1 7  .38226E+04 1 8 -.92675E+04 1 9  .12620E+07 110 -.27540E+06
 111 -.33662E+07 112  .19521E+07 113  .16782E+07 114 -.43864E+07 115 -.14144E+07
 116  .42460E+06 117 -.41530E+06 118  .10650E+04 119 -.53974E+05 120 -.44260E+04
 2 1 -.86102E+04 2 2 -.29804E+03 2 3  .41358E+03 2 4 -.91562E+03 2 5 -.10198E+04
 2 6  .47160E+04 2 7 -.75835E+03 2 8  .34602E+04 2 9 -.58771E+05 210 -.24519E+06
 211 -.52211E+06 212  .15353E+07 213 -.38944E+06 214  .16019E+07 215 -.21198E+07
 216  .32960E+05 217 -.26307E+05 218  .20473E+04 219 -.10503E+06 220 -.47021E+03
 3 1  .52393E+03 3 2  .93350E+02 3 3  .85424E+04 3 4 -.42479E+03 3 5 -.46136E+04
 3 6  .41100E+04 3 7  .25838E+04 3 8 -.52743E+04 3 9  .10001E+07 310 -.10815E+06
 311 -.14847E+07 312  .14485E+07 313  .11440E+07 314 -.24666E+07 315 -.10291E+06
 316  .10081E+06 317 -.42412E+03 318  .30895E+02 319  .49688E+04 320  .18374E+04
 4 1 -.99994E+03 4 2  .96088E+02 4 3 -.38659E+02 4 4 -.96419E+03 4 5 -.13792E+04
 4 6  .40642E+04 4 7 -.90475E+03 4 8  .36825E+04 4 9 -.58505E+05 410 -.25187E+06
 411 -.56242E+06 412  .15131E+07 413 -.37629E+06 414  .15326E+07 415 -.17390E+07
 416  .10935E+06 417 -.22459E+05 418  .13863E+03 419 -.12468E+05 420  .47957E+03
 5 1  .11047E+04 5 2  .93032E+02 5 3  .55441E+04 5 4  .24661E+02 5 5  .85374E+03
 5 6  .98392E+03 5 7 -.10366E+04 5 8 -.90088E+03 5 9  .64873E+06 510  .16472E+05
 511  .36237E+06 512  .31944E+06 513 -.44925E+06 514 -.32028E+06 515  .14502E+06
 516 -.11724E+06 517  .90101E+05 518  .34228E+03 519  .12549E+05 520  .50000E+03
 6 1 -.68182E+02 6 2  .56242E+02 6 3 -.26137E+03 6 4 -.92587E+03 6 5 -.10061E+04
 6 6  .28011E+04 6 7 -.16043E+04 6 8  .15380E+04 6 9 -.53753E+05 610 -.24748E+06
 611 -.40897E+06 612  .10800E+07 613 -.64709E+06 614  .55760E+06 615 -.92437E+06
 616  .14868E+05 617  .22823E+05 618 -.24773E+03 619 -.79323E+03 620  .31682E+03
 7 1 -.12248E+04 7 2 -.22167E+03 7 3  .32852E+04 7 4  .16866E+03 7 5  .31314E+04
 7 6 -.13581E+04 7 7 -.18389E+03 7 8  .20779E+04 7 9  .38864E+06 710  .50672E+05
 711  .10476E+07 712 -.53054E+06 713 -.46980E+05 714  .87873E+06 715 -.10129E+06
 716 -.18734E+05 717 -.64253E+05 718 -.15372E+03 719 -.13481E+05 720 -.15250E+04
 8 1 -.13272E+03 8 2 -.27330E+02 8 3 -.12862E+03 8 4 -.97640E+02 8 5  .37214E+02
 8 6  .33717E+03 8 7 -.20650E+03 8 8  .44970E+03 8 9 -.76596E+04 810 -.27607E+05
 811 -.49078E+05 812  .15404E+06 813 -.56204E+05 814  .55930E+05 815 -.29188E+06
 816 -.69061E+04 817  .24827E+05 818 -.18698E+03 819 -.16823E+04 820  .39389E+03
 9 1  .53648E+03 9 2  .96039E+02 9 3  .16623E+04 9 4  .14510E+03 9 5  .23225E+04
 9 6 -.10967E+04 9 7 -.31730E+03 9 8  .23422E+04 9 9  .18379E+06 910  .37819E+05
 911  .78952E+06 912 -.40974E+06 913 -.14473E+06 914  .11113E+07 915 -.20505E+06
 916  .15347E+05 917 -.20083E+05 918 -.12918E+03 919  .66299E+04 920 -.91784E+02
10 1  .22409E+0310 2 -.39454E+0210 3 -.28404E+0310 4 -.81995E+0210 5 -.39832E+03
10 6  .25627E+0310 7  .15819E+0210 8 -.29934E+0310 9 -.58586E+041010 -.16761E+05
1011 -.46131E+051012  .81652E+051013 -.16192E+051014 -.26972E+051015  .15034E+07
1016 -.65201E+041017  .12976E+051018 -.15469E+031019  .27320E+041020  .15902E+03
11 1  .18029E+0611 2  .46752E+0611 3  .12355E+0711 4 -.25858E+0611 5 -.32155E+07
11 6  .20524E+0711 7  .17302E+0711 8 -.45153E+0711 9 -.37126E+061110  .69573E+06
1111  .57756E+071112  .18498E+071113  .61815E+061114 -.34805E+061115  .21613E+09
1116  .10114E+101117  .22155E+091118 -.10144E+071119  .25063E+061120  .48553E+07
1121 -.44260E+041122 -.46143E+061124  .23627E+05
12 1 -.42178E+0412 2 -.13275E+0412 3 -.54122E+0512 4 -.24566E+0612 5 -.52092E+06
12 6  .15267E+0712 7 -.38663E+0612 8  .15755E+0712 9 -.36305E+051210  .11001E+06
1211  .21533E+061212 -.93274E+061213  .25116E+061214 -.12910E+071215  .19240E+08
```

```
H-MATRIX FOR CASE 3T (CONTINUED)
1216  .23901E+081217  .56439E+071218 -.63895E+051219  .78785E+061220  .11513E+06
1221 -.47021E+031222 -.89792E+061224  .45419E+05
13 1  .11232E+0613 2  .28645E+0613 3  .97816E+0613 4 -.99003E+0513 5 -.13888E+07
13 6  .15034E+0713 7  .11766E+0713 8 -.25468E+0713 9 -.26833E+061310  .38781E+06
1311  .31337E+071312  .93956E+061313  .28151E+061314 -.36266E+061315  .12140E+09
1316  .61253E+091317  .13065E+091318 -.63519E+061319 -.10951E+061320  .29240E+07
1321  .18374E+041322  .42479E+051324  .68541E+03
14 1 -.73497E+0414 2  .70668E+0414 3 -.57282E+0514 4 -.25200E+0614 5 -.55679E+06
14 6  .15030E+0714 7 -.37405E+0614 8  .15106E+0714 9 -.23498E+051410  .11205E+06
1411  .25749E+061412 -.80321E+061413  .28272E+061414 -.13287E+071415  .16087E+08
1416  .14971E+081417  .35491E+071418 -.16674E+051419 -.21735E+051420  .73510E+05
1421  .47957E+031422 -.10659E+061424  .30755E+04
15 1  .56089E+0515 2  .13645E+0615 3  .63607E+0615 4  .18974E+0515 5  .40128E+06
15 6  .35099E+0615 7 -.42383E+0615 8 -.35655E+0615 9 -.16811E+061510  .15343E+06
1511  .87733E+061512  .69217E+061513  .85873E+061514 -.80154E+061515  .47290E+08
1516  .29117E+091517  .61109E+081518 -.31011E+061519 -.10126E+061520  .13861E+07
1521  .50000E+031522  .10728E+061524  .75935E+04
16 1 -.10956E+0516 2 -.61730E+0416 3 -.54322E+0516 4 -.24724E+0616 5 -.40523E+06
16 6  .10672E+0716 7 -.64038E+0616 8  .55196E+0616 9 -.16264E+051610  .91369E+05
1611  .75880E+051612 -.58430E+061613  .45370E+061614 -.48587E+061615  .89911E+07
1616 -.12342E+081617 -.28717E+071618  .18015E+051619 -.97347E+051620 -.59441E+05
1621  .31682E+031622 -.67815E+041624 -.54959E+04
17 1  .27944E+0517 2  .62053E+0517 3  .38248E+0617 4  .53277E+0517 5  .10627E+07
17 6 -.51343E+0617 7 -.39220E+0517 8  .85935E+0617 9 -.99420E+051710  .53596E+05
1711 -.11032E+061712  .79103E+061713  .31833E+061714 -.13215E+071715  .14757E+08
1716  .13896E+091717  .30528E+081718 -.13061E+061719  .12656E+061720  .66769E+06
1721 -.15250E+041722 -.11525E+061724 -.34103E+04
18 1 -.12334E+0518 2 -.36149E+0418 3 -.10604E+0518 4 -.27187E+0518 5 -.46791E+05
18 6  .15070E+0618 7 -.55554E+0518 8  .57035E+0518 9 -.29469E+051810 -.11309E+05
1811 -.10310E+061812  .89419E+051813  .22313E+051814 -.14465E+061815  .11549E+08
1816 -.21914E+071817 -.78586E+061818  .49531E+041819 -.12799E+061820 -.12099E+05
1821  .39389E+031822 -.14382E+051824 -.41482E+04
19 1  .97123E+0419 2  .25620E+0519 3  .18133E+0619 4  .38189E+0519 5  .79202E+06
19 6 -.39762E+0619 7 -.14229E+0619 8  .11020E+0719 9 -.47759E+051910  .99710E+04
1911 -.24941E+061912  .45129E+061913  .20344E+061914 -.10602E+071915  .66182E+06
1916  .54132E+081917  .11792E+081918 -.52265E+051919 -.65455E+051920  .25932E+06
1921 -.91784E+021922  .56680E+051924 -.28659E+04
20 1 -.78385E+0420 2 -.16362E+0420 3 -.93750E+0420 4 -.16715E+0520 5 -.45811E+05
20 6  .80430E+0520 7 -.15993E+0520 8 -.25006E+0520 9 -.14076E+052010 -.42376E+04
2011  .15981E+052012  .47650E+052013 -.33207E+052014  .11823E+062015  .45764E+07
2016 -.10821E+072017 -.39737E+062018  .42021E+042019 -.11483E+062020 -.59784E+04
2021  .15902E+032022  .23356E+052024 -.34318E+04
21 3  .10000E+01
22 4  .10000E+01
23 5  .10000E+01
24 6  .10000E+01
25 7  .10000E+01
26 8  .10000E+01
27 9  .10000E+01
2810  .10000E+01
2911  .10000E+01
3012  .10000E+01
3113  .10000E+01
3214  .10000E+01
3315 -.20000E 01

H-MATRIX FOR CASE 3T (CONTINUED)
3416 -.75000E+01
3515  .10000E+01
3616  .10000E+01
37 1  .10000E+01
38 2  .10000E+01
39 2  .06110E 003915 -.24690E 033916 -.20139E 03
40 2 -.22748E-02
```

```
D MATRIX FOR CASE 3T

11 1 -.84864E+0711 2  .31845E+07
12 1 -.12719E+0812 2  .24720E+06
13 1 -.61746E+0613 2  .75607E+06
14 1 -.10434E+0814 2  .82012E+06
15 1  .87012E+0615 2 -.87930E+06
16 1 -.55462E+0716 2  .11151E+06
17 1 -.60774E+0617 2 -.14050E+06
18 1 -.17513E+0718 2 -.51796E+05
19 1 -.12303E+0719 2  .11510E+06
20 1  .90204E+0720 2 -.48901E+05
33 1  .60000E 01
34 2  .75000E 01
40 1  .00000E 0140 2  .75000E 01
```

TABLE 42. EIGENVALUES AND RMS RESPONSES FOR CASE 3T

EIGENVALUES

REAL	IMAG	FREQ	DAMP
-.24680489	0.00000000	.24680489	-1.00000000
-.24843511	0.00000000	.24843511	-1.00000000
-10.98300000	0.00000000	10.98300000	-1.00000000
-8.54920000	0.00000000	8.54920000	-1.00000000
-5.09600000	3.60330182	6.24123385	-.81650522
-22.18500000	0.00000000	22.18500000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-6.00000000	0.00000000	6.00000000	-1.00000000
-.88003874	1.27317857	1.54772473	-.56860159
-.50790698	5.45874459	5.48232269	-.09264449
-.23175692	11.12423163	11.12664553	-.02082900
-.57035683	13.79429153	13.80620212	-.04152893
-.60493468	15.59460920	15.60633788	-.03876212
-.42821137	17.48254868	17.48779213	-.02448630
-.62071352	18.78156474	18.79181895	-.03303105

States

R.M.S. RESPONSES

1	.15747433E+02
2	.24026060E+01
3	.21337662E+01
4	.69787019E-01
5	.20710092E+00
6	.12959656E+00
7	.22042549E-01
8	.32871672E-01
9	.70145739E+00
10	.80971864E-02
11	.23061739E-01
12	.37550928E-01
13	.53251662E-02
14	.10686758E-01
15	.21144986E-14
16	.72863426E-14
17	.10386349E-13
18	.99178200E+00
19	.97183710E+00
20	.97979891E+00
21	.55751508E+01
22	.98351309E+00
23	.37288852E+00
24	.10000642E+01

Responses

R.M.S. RESPONSES

1	.11209333E+07
2	.16080005E+06
3	.77160745E+06
4	.88160057E+05
5	.43819204E+06
6	.48810738E+05
7	.25718039E+06
8	.54495741E+05
9	.11474430E+06
10	.30694623E+05
11	.34517775E+07
12	.32302646E+06
13	.25001679E+07
14	.26995730E+06
15	.15015515E+07
16	.19977758E+06
17	.90853657E+06
18	.26653026E+05
19	.44151039E+06
20	.22142522E+05
21	.21337662E+01
22	.69787019E-01
23	.20710092E+00
24	.12959656E+00
25	.22042549E-01
26	.32871672E-01
27	.70145739E+00
28	.80971864E-02
29	.23061739E-01
30	.37550928E-01
31	.53251662E-02
32	.10686758E-01
33	.12686991E-13
34	.54647569E-13
35	.21144986E-14
36	.72863426E-14
37	.15747433E+02
38	.24026060E+01

TABLE 43. COEFFICIENT MATRICES FOR CASE 4R

```
F-MATRIX FOR CASE 4R

 1 1 -.69879E+00 1 2  .32647E+01 1 3 -.35206E-01 1 4 -.28477E-01 1 5 -.64668E+00
 1 6 -.18640E+01 1 7 -.21772E+03 1 8 -.22603E+03 1 9 -.38719E+02 110 -.49629E+00
 111 -.71309E+01 112 -.10835E+01
 2 1 -.51240E+00 2 2 -.11372E+01 2 3  .34118E-01 2 4 -.32117E+00 2 5 -.11381E+00
 2 6 -.10023E+02 2 7 -.60258E+03 2 8 -.24336E+04 2 9 -.41898E+03 210  .27467E+01
 211 -.93350E+00 212 -.11407E+02
 3 1 -.16743E+01 3 2 -.56727E-01 3 3 -.97534E+00 3 4 -.18968E+00 3 5 -.29898E+02
 3 6 -.18151E+02 3 7 -.32357E+04 3 8  .10072E+04 3 9  .15754E+03 310  .47046E+00
 311 -.23922E+02 312  .48042E+01
 4 1  .11610E+00 4 2 -.79345E+00 4 3  .47618E-01 4 4 -.12682E+01 4 5  .14080E+01
 4 6 -.19259E+03 4 7 -.14607E+04 4 8 -.24250E+04 4 9 -.54550E+03 410 -.22403E+01
 411  .11406E+02 412 -.11165E+02
 5 3  .10000E+01
 6 4  .10000E+01
 7 7 -.60000E+01
 8 8 -.75000E+01
 9 9 -.75000E+01
1010 -.22185E+021016  .22185E+02
1111 -.85492E+011114  .85492E+01
1210 -.50960E+011213  .10000E+01
1310  .90891E+021312 -.38953E+021313 -.10192E+02
1414 -.10983E+021416  .10983E+02
1515 -.49524E+001516 -.61315E-01
1615  .10000E+01


G1-MATRIX FOR CASE 4R

 7 1  .60000E+01
 8 2  .75000E+01


G2-MATRIX FOR CASE 4R

15 1 -.30360E+00
16 1  .86190E+00


H-MATRIX FOR CASE 4R

 1 1 -.26658E+05 1 2 -.15746E+05 1 3  .14025E+05 1 4 -.13130E+05 1 5  .12663E+07
 1 6 -.34413E+07 1 7  .64745E+07 1 8 -.39797E+08 1 9 -.98219E+07 110 -.67189E+04
 111 -.18860E+06 112 -.17944E+06
 2 1 -.22021E+05 2 2 -.60129E+04 2 3 -.14744E+04 2 4 -.32119E+04 2 5 -.82381E+05
 2 6 -.62332E+06 2 7  .38999E+07 2 8 -.10278E+08 2 9 -.22243E+07 210  .48588E+04
 211 -.23632E+06 212 -.47110E+05
 3 1 -.80508E+04 3 2 -.82720E+04 3 3  .10114E+05 3 4 -.62936E+04 3 5  .10035E+07
 3 6 -.15374E+07 3 7  .64654E+07 3 8 -.22425E+08 3 9 -.52868E+07 310 -.12441E+05
 311 -.13252E+05 312 -.10203E+06
```

```
H MATRIX FOR CASE 4R (CONTINUED)
 4 1 -.13862E+05 4 2 -.54558E+04 4 3 -.18196E+04 4 4 -.35129E+04 4 5 -.90879E+05
 4 6 -.65948E+06 4 7  .39280E+07 4 8 -.99475E+07 4 9 -.21741E+07 410  .24070E+04
 411 -.13700E+06 412 -.45621E+05
 5 1  .16623E+04 5 2 -.48570E+03 5 3  .55115E+04 5 4  .72781E+03 5 5  .64679E+06
 5 6  .34598E+06 5 7  .30635E+07 5 8 -.17947E+07 5 9 -.26291E+06 510 -.30877E+04
 511  .29008E+05 512 -.85479E+04
 6 1 -.82262E+04 6 2 -.34491E+04 6 3 -.19089E+04 6 4 -.27641E+04 6 5 -.72850E+05
 6 6 -.48923E+06 6 7  .53767E+07 6 8 -.68195E+07 6 9 -.15022E+07 610 -.16018E+04
 611 -.82481E+05 612 -.31361E+05
 7 1  .13779E+04 7 2  .26297E+04 7 3  .24345E+04 7 4  .36391E+04 7 5  .38408E+06
 7 6  .10553E+07 7 7 -.21328E+07 7 8  .78988E+07 7 9  .19761E+07 710  .45473E+04
 711 -.12917E+05 712  .35836E+05
 8 1 -.74062E+04 8 2 -.19685E+04 8 3 -.17503E+04 8 4 -.90406E+03 8 5 -.26763E+05
 8 6 -.10700E+06 8 7  .62292E+07 8 8 -.17195E+07 8 9 -.37769E+06 810 -.29833E+03
 811 -.83061E+05 812 -.79252E+04
 9 1  .14835E+04 9 2  .23831E+04 9 3  .88956E+03 9 4  .25554E+04 9 5  .18064E+06
 9 6  .80141E+06 9 7 -.33323E+07 9 8  .69091E+07 9 9  .16925E+07 910  .49246E+04
 911 -.90009E+04 912  .31471E+05
10 1 -.43225E+0410 2 -.12744E+0410 3 -.12881E+0410 4 -.96766E+0310 5 -.[illegible]7922E+05
10 6 -.82855E+0510 7  .58423E+0710 8 -.11569E+0710 9 -.26046E+061010 -.[illegible]7810E+03
1011 -.47793E+051012 -.53174E+04
11 1  .17498E+0611 2  .44024E+0611 3  .12429E+0711 4 -.32250E+0711 5 -.31134E+06
11 6  .56148E+0711 7  .21215E+0911 8  .96190E+0911 9  .16860E+091110 -.10184E+07
1111  .42705E+061112  .45151E+071113 -.44260E+041114 -.46143E+061116  .23627E+05
12 1  .65708E+0412 2  .16779E+0412 3 -.51908E+0512 4 -.51958E+0612 5 -.10580E+05
12 6  .29532E+0612 7  .13249E+0812 8  .27875E+0812 9  .50426E+071210 -.66467E+05
1211  .90869E+061212  .13115E+061213 -.47021E+031214 -.89792E+061216  .45419E+05
13 1  .10252E+0613 2  .26926E+0613 3  .98233E+0613 4 -.13943E+0713 5 -.23640E+06
13 6  .30430E+0713 7  .11813E+0913 8  .58039E+0913 9  .10045E+091310 -.62974E+06
1311 -.92516E+051312  .27221E+071313  .18374E+041314  .42479E+051316  .68541E+03
14 1  .28952E+0414 2  .99540E+0414 3 -.55259E+0514 4 -.55548E+0614 5 -.34852E+03
14 6  .33027E+0614 7  .11152E+0814 8  .18525E+0814 9  .35215E+071410 -.20434E+05
1411  .94531E+051412  .88462E+051413  .47957E+031414 -.10659E+061416  .30755E+04
15 1  .46216E+0515 2  .12570E+0615 3  .63839E+0615 4  .39805E+0615 5 -.15263E+06
15 6  .82779E+0615 7  .45998E+0815 8  .26946E+0915 9  .45784E+081510 -.3097[illegible]E+06
1511 -.13381E+061512  .12616E+071513  .50000E+031514  .10728E+061516  .75[illegible]
16 1 -.21619E+0416 2 -.30750E+0416 3 -.52326E+0516 4 -.40357E+0616 5  .56808E+04
16 6  .15392E+0616 7  .23723E+0716 8 -.86284E+0716 9 -.14286E+071610  .17892E+05
1611 -.49373E+041612 -.40416E+051613  .31682E+031614 -.67815E+041616 -.54959E+04
17 1  .21854E+0517 2  .54432E+0517 3  .38441E+0617 4  .10607E+0717 5 -.87530E+05
17 6 -.14161E+0617 7  .16490E+0817 8  .12217E+0917 9  .20811E+081710 -.13650E+06
1711  .12495E+061712  .57326E+061713 -.15250E+041714 -.11525E+061716 -.34103E+04
18 1 -.77073E+0218 2 -.11058E+0418 3 -.76690E+0418 4 -.45342E+0518 5  .38753E+04
18 6 -.11679E+0518 7  .83518E+0618 8 -.18761E+0718 9 -.42332E+061810  .39620E+04
1811  .17860E+051812 -.89166E+041813  .39389E+031814 -.14382E+051816 -.41482E+04
19 1  .76872E+0419 2  .21960E+0519 3  .18221E+0619 4  .79125E+0619 5 -.44291E+05
19 6 -.27485E+0619 7  .39978E+0719 8  .45238E+0819 9  .75505E+071910 -.56875E+05
1911 -.52904E+051912  .21292E+061913 -.91784E+021914  .56680E+051916 -.28659E+04
20 1  .19115E+0320 2  .21313E+0320 3 -.75272E+0420 4 -.44890E+0520 5  .78366E+04
20 6  .78214E+0520 7 -.32380E+0720 8 -.16377E+0620 9 -.25568E+052010  .42960E+04
2011 -.22037E+052012 -.98840E+032013  .15902E+032014  .23356E+052016 -.34318E+04
21 3  .10000E+01
22 4  .10000E+01
23 5  .10000E+01
24 6  .10000E+01
25 7 -.60000E+01
26 8 -.75000E+01
27 7  .10000E+01
```

```
H-MATRIX FOR CASE 4R (CONTINUED)
28 8  .10000E+01
29 1  .10000E+01
30 2  .10000E+01
32 2 -.22748E-02
```

```
D-MATRIX FOR CASE 4R

11 1 -.84864E+0711 2  .31845E+07
12 1 -.12719E+0812 2  .24720E+06
13 1 -.61746E+0613 2  .75607E+06
14 1 -.10434E+0814 2  .82012E+06
15 1  .87012E+0615 2 -.87930E+06
16 1 -.55462E+0716 2  .11151E+06
17 1 -.60774E+0617 2 -.14050E+06
18 1 -.17513E+0718 2 -.51796E+05
19 1 -.12303E+0719 2  .11510E+06
20 1  .90204E+0720 2 -.48901E+05
32 2  .75000E+01
26 2  .75000E+01
25 1  .60000E+01
```

TABLE 44. EIGENVALUES AND RMS RESPONSES FOR CASE 4R

EIGENVALUES REAL	IMAG	FREQ	DAMP
-.24680489	0.00000000	.24680489	-1.00000000
-.24843511	0.00000000	.24843511	-1.00000000
-10.98300000	0.00000000	10.98300000	-1.00000000
-8.54920000	0.00000000	8.54920000	-1.00000000
-5.09600000	3.60330182	6.24123385	-.81650522
-22.18500000	0.00000000	22.18500000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-6.00000000	0.00000000	6.00000000	-1.00000000
-.91300458	1.33658410	1.61865204	-.56405241
-.47781095	5.39333882	5.41446276	-.08824716
-.67570447	13.87828693	13.89472651	-.04863028

States

R.M.S. RESPONSES

1	.15805554E+02
2	.24996092E+01
3	.17714140E+01
4	.20777196E+00
5	.57109424E+00
6	.22436704E-01
7	.79153434E-14
8	.68686857E-14
9	.11051351E-13
10	.99178200E+00
11	.97183710E+00
12	.97979891E+00
13	.55751508E+01
14	.98351309E+00
15	.37288852E+00
16	.10000642E+01

Responses

R.M.S. RESPONSES

1	.77017052E+06
2	.60395885E+05
3	.57100123E+06
4	.28680908E+05
5	.36105986E+06
6	.30804690E+05
7	.23096387E+06
8	.35132370E+04
9	.10087294E+06
10	.52078202E+04
11	.28386422E+07
12	.22757708E+06
13	.20406015E+07
14	.18361243E+06
15	.12788421E+07
16	.15983672E+06
17	.81839401E+06
18	.10174884E+06
19	.40601504E+06
20	.66246680E+05
21	.17714140E+01
22	.20777196E+00
23	.57109424E+00
24	.22436704E-01
25	.47492060E-13
26	.51515143E-13
27	.79153434E-14
28	.68686857E-14
29	.15805554E+02
30	.24996092E+01

TABLE 45. COEFFICIENT MATRICES FOR CASE 4T

F-MATRIX FOR CASE 4T

```
 1 1 -.66218E+00 1 2  .32776E+01 1 3 -.28017E-01 1 4 -.23237E-01 1 5 -.55979E+00
 1 6 -.15561E+01 1 7 -.24690E+03 1 8 -.20139E+03 1 9 -.40941E+02 110 -.48934E+00
 111 -.67639E+01 112 -.98990E+00
 2 1 -.56837E+00 2 2 -.12293E+01 2 3  .58290E-01 2 4 -.34790E+00 2 5  .54826E-01
 2 6 -.10445E+02 2 7 -.60551E+03 2 8 -.26235E+04 2 9 -.56095E+03 210  .27253E+01
 211 -.80421E+00 212 -.12530E+02
 3 1 -.12620E+01 3 2  .16571E+00 3 3 -.93166E+00 3 4 -.11204E+00 3 5 -.29214E+02
 3 6 -.14694E+02 3 7 -.35548E+04 3 8  .14284E+04 3 9  .30829E+03 310  .57304E+00
 311 -.20517E+02 312  .68753E+01
 4 1 -.76870E-03 4 2 -.91786E+00 4 3  .63949E-01 4 4 -.13099E+01 4 5  .14951E+01
 4 6 -.19351E+03 4 7 -.14545E+04 4 8 -.26722E+04 4 9 -.70943E+03 410 -.22475E+01
 411  .10995E+02 412 -.12582E+02
 5 3  .10000E+01
 6 4  .10000E+01
 7 7 -.60000E+01
 8 8 -.75000E+01
 9 9 -.75000E+01
1010 -.22185E+021016  .22185E+02
1111 -.85492E+011114  .85492E+01
1210 -.50960E+011213  .10000E+01
1310  .90891E+021312 -.38953E+021313 -.10192E+02
1414 -.10983E+021416  .10983E+02
1515 -.49524E+001516 -.61315E-01
1615  .10000E+01
```

G1-MATRIX FOR CASE 4T

```
 7 1  .60000E+01
 8 2  .75000E+01
```

G2-MATRIX FOR CASE 4T

```
15 1 -.30360E+00
16 1  .86190E+00
```

H-MATRIX FOR CASE 4T

```
 1 1 -.46710E+04 1 2 -.33999E+03 1 3  .11376E+05 1 4 -.10405E+05 1 5  .12020E+07
 1 6 -.33662E+07 1 7 -.14144E+07 1 8  .42460E+06 1 9 -.41530E+06 110  .10650E+04
 111 -.53974E+05 112 -.44260E+04
 2 1 -.86102E+04 2 2 -.29804E+03 2 3  .41558E+03 2 4  .10198E+04 2 5 -.58771E+05
 2 6 -.52211E+06 2 7 -.21198E+07 2 8  .32960E+05 2 9 -.26307E+05 210  .20473E+04
 211 -.10503E+06 212 -.47021E+03
 3 1  .52393E+03 3 2  .93350E+02 3 3  .85424E+04 3 4 -.46136E+04 3 5  .10001E+07
 3 6 -.14847E+07 3 7 -.10291E+06 3 8  .10081E+06 3 9 -.42412E+03 310  .30895E+02
 311  .49688E+04 312  .18374E+04
 4 1 -.99994E+03 4 2  .96088E+02 4 3 -.38659E+02 4 4 -.13792E+04 4 5 -.58505E+05
 4 6 -.56242E+06 4 7 -.17390E+07 4 8  .10935E+06 4 9 -.22459E+05 410  .13863E+03
 411 -.12468E+05 412  .47957E+03
 5 1  .11047E+04 5 2  .93032E+02 5 3  .55441E+04 5 4  .85374E+03 5 5  .64873E+06
```

```
H-MATRIX FOR CASE 4T (CONTINUED)
 5 6   .36237E+06 5 7   .14502E+06 5 8 -.11724E+06 5 9   .90101E+05 510   .34228E+03
 511   .12549E+05 512   .50000E+03
 5 1 -.68182E+02 6 2   .56242E+02 6 3 -.26137E+03 6 4 -.10061E+04 6 5 -.53753E+05
 ; 6 -.40897E+06 6 7 -.92437E+06 6 8   .14868E+05 6 9   .22823E+05 610 -.24773E+03
 611 -.79323E+03 612  -31682E+03
 7 1 -.12248E+04 7 2 -.22167E+03 7 3   .32852E+04 7 4   .31314E+04 7 5   .38864E+06
 7 6   .10476E+07 7 7 -.10129E+06 7 8 -.18734E+05 7 9 -.64253E+05 710 -.15372E+03
 711 -.13481E+05 712 -.15250E+04
 8 1 -.13272E+03 8 2 -.27330E+02 8 3 -.12862E+03 8 4   .37214E+02 8 5 -.76596E+04
 8 6 -.49078E+05 8 7 -.29188E+06 8 8 -.69061E+04 8 9   .24827E+05 810 -.18698E+03
 811 -.16823E+04 812   .39389E+03
 9 1   .53648E+03 9 2   .96039E+02 9 3   .16623E+04 9 4   .23225E+04 y 5   .18379E+06
 9 6   .78952E+06 9 7 -.20505E+06 9 8   .15347E+05 9 9 -.20083E+05 910 -.12918E+03
 911   .66299E+04 912 -.91784E+02
10 1   .22409E+0310 2 -.39454E+0210 3 -.28404E+0310 4 -.39832E+0310 5 -.58586E+04
10 6 -.46131E+0510 7   .15034E+0710 8 -.65201E+0410 9   .12976E+051010 -.15469E+03
1011   .27320E+041012   .15902E+03
11 1   .18029E+0611 2   .46752E+0611 3   .12355E+0711 4 -.32155E+0711 5 -.37126E+06
11 6   .57756E+0711 7   .21613E+0911 8   .10114E+1011 9   .22155E+091110 -.10144E+07
1111   .25063E+061112   .48553E+071113 -.44260E+041114 -.46143E+061116   .23627E+05
12 1 -.42178E+0412 2 -.13275E+0412 3 -.54122E+0512 4 -.52092E+0612 5 -.36305E+05
12 6   .21534E+0612 7   .19240E+0812 8   .23901E+0812 9   .56439E+071210 -.63895E+05
1211   .78785E+061212   .11513E+061213 -.47021E+031214 -.89792E+061216   .45419E+05
13 1   .11232E+0613 2   .28645E+0613 3   .97816E+0613 4 -.13888E+0713 5 -.26833E+06
13 6   .31337E+0713 7   .12140E+0913 8   .61253E+0913 9   .13065E+091310 -.63518E+06
1311 -.10951E+061312   .29240E+071313   .18374E+041314   .42479E+051316   .68541E+03
14 1 -.73497E+0414 2   .70668E+0414 3 -.57282E+0514 4 -.55679E+0614 5 -.23498E+05
14 6   .25749E+0614 7   .16087E+0814 8   .14971E+0814 9   .35491E+071410 -.16674E+05
1411 -.21735E+051412   .73510E+051413   .47957E+031414 -.10659E+061416   .30755E+04
15 1   .56089E+0515 2   .13645E+0615 3   .63607E+0615 4   .40128E+0615 5 -.16811E+06
15 6   .87733E+0615 7   .47290E+0815 8   .29117E+0915 9   .61109E+081510 -.31011E+06
1511 -.10126E+061512   .13861E+071513   .50000E+031514   .10728E+061516   .75935E+04
16 1 -.10956E+0516 2 -.61730E+0416 3 -.54322E+0516 4 -.40523E+0616 5 -.16264E+05
16 6   .75880E+0516 7   .89911E+0716 8 -.12342E+0816 9 -.28717E+071610   .18015E+05
1611 -.97347E+051612 -.59441E+051613   .31682E+031614 -.67815E+041616 -.54959E+04
17 1   .27944E+0517 2   .62053E+0517 3   .38248E+0617 4   .10627E+0717 5 -.99420E+05
17 6 -.11032E+0617 7  .14757E+0817 8   .13896E+0917 9   .30528E+081710 -.13061E+06
1711   .12656E+061712   .66769E+061713 -.15250E+041714 -.11525E+061716 -.34103E+04
18 1 -.12334E+0518 2 -.36149E+0418 3 -.10604E+0518 4 -.46791E+0518 5 -.29469E+05
18 6 -.10310E+0618 7   .11549E+0818 8 -.21914E+0718 9 -.78586E+061810   .49531E+04
19 1   .97123E+0419 2   .25620E+0519 3   .18133E+0619 4   .79202E+0619 5 -.47759E+05
1811 -.12799E+061812 -.12099E+051813   .39389E+031814 -.14382E+051816 -.41482E+04
19 6 -.24941E+0619 7   .66182E+0619 8   .54132E+0819 9   .11792E+081910 -.52265E+05
1911 -.65455E+051912   .25932E+061913 -.91784E+021914   .56680E+051916 -.28659E+04
20 1 -.78385E+0420 2 -.16362E+0420 3 -.93750E+0420 4 -.45811E+0520 5 -.14076E+05
20 6   .15981E+0520 7   .45764E+0720 8 -.10821E+0720 9 -.39737E+062010   .42021E+04
2011 -.11483E+062012 -.59784E+042013   .15902E+032014   .23356E+052016 -.34318E+04
21 3   .10000E+01
22 4   .10000E+01
23 5   .10000E+01
24 6   .10000E+01
25 7 -.60000E+01
26 8 -.75000E+01
27 7   .10000E+01
28 8   .10000E+01
```

```
H-MATRIX FOR CASE 4T (CONTINUED)
29 1   .10000E+01
30 2   .10000E+01
32 2 -.22748E-02
```

```
D-MATRIX FOR CASE 4T

11 1 -.84864E+0711 2   .31845E+07
12 1 -.12719E+0812 2   .24720E+06
13 1 -.61746E+0613 2   .75607E+06
14 1 -.10434E+0814 2   .82012E+06
15 1   .87012E+0615 2 -.87930E+06
16 1 -.55462E+0716 2   .11151E+06
17 1 -.60774E+0617 2 -.14050E+06
18 1 -.17513E+0718 2 -.51796E+05
19 1 -.12303E+0719 2   .11510E+06
20 1   .90204E+0720 2 -.48901E+05
25 1   .60000E+01
26 2   .75000E+01
32 2   .75000E+01
```

TABLE 46. EIGENVALUES AND RMS RESPONSES FOR CASE 4T

OPEN LOOP

EIGENVALUES

REAL	IMAG	FREQ	DAMP
-.24680489	0.00000000	.24680489	-1.00000000
-.24843511	0.00000000	.24843511	-1.00000000
-10.98300000	0.00000000	10.98300000	-1.00000000
-8.54920000	0.00000000	8.54920000	-1.00000000
-5.09600000	3.60330182	6.24123385	-.81650522
-22.18500000	0.00000000	22.18500000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-7.50000000	0.00000000	7.50000000	-1.00000000
-6.00000000	0.00000000	6.00000000	-1.00000000
-.88215376	1.27062167	1.54682730	-.57029881
-.50773730	5.45751506	5.48108272	-.09263449
-.64986528	13.84743358	13.86267440	-.04687878

States

R.M.S. RESPONSES

1	.15740274E+02
2	.23962136E+01
3	.21072839E+01
4	.20585036E+00
5	.69758088E+00
6	.22877193E-01
7	.54273281E-14
8	.13807059E-13
9	.19833243E-13
10	.99178200E+00
11	.97183710E+00
12	.97979891E+00
13	.55751508E+01
14	.98351309E+00
15	.37288852E+00
16	.10000642E+01

Responses

R.M.S. RESPONSES

1	.11139601E+07
2	.15921548E+06
3	.76623691E+06
4	.86150862E+05
5	.43539257E+06
6	.47057712E+05
7	.25592687E+06
8	.54251833E+05
9	.11421182E+06
10	.30540298E+05
11	.31940719E+07
12	.21960789E+06
13	.23267321E+07
14	.18487121E+06
15	.14660200E+07
16	.15349598E+06
17	.92810167E+06
18	.20455724E+05
19	.45712662E+06
20	.22068265E+05
21	.21072839E+01
22	.20585036E+00
23	.69758088E+00
24	.22877193E-01
25	.32563969E-13
26	.10355295E-12
27	.54273281E-14
28	.13807059E-13
29	.15740274E+02
30	.23962136E+01

APPENDIX B

MODELING PARAMETER UNCERTAINTIES

APPENDIX B

MODELING PARAMETER UNCERTAINTIES

There are many ways to incorporate parameter uncertainties in the model. One could impose an independent variation on each uncertain element of the coefficient matrices in the equations of motion and response equations. Such an approach has two major disadvantages: the number of components in the parameter vector is large, and specific combinations of such variations could violate physical constraints. To overcome these disadvantages, we introduced the physically "independent" parameters: dynamic pressure, structural damping, and structural stiffness. Variations in dynamic pressure cause variations in essentially all of the uncertain coefficients. Variations in structural damping and stiffness cause variations in certain subsets of the coefficients. To permit the assessment of a variation in a single coefficient, we included the coefficient, M_w, corresponding to an uncertainty in C_{m_α} in the vector of uncertain parameters. Relative variations from the nominal values were chosen as the actual components of the parameter vector. Thus this vector has the form

$$p = (\bar{q}_f,\ \omega_f,\ M_{wf},\ \zeta_f)^T \tag{B-1}$$

with the nominal value for each component of p being unity.

The state space representation of the "complete model" of the C-5A with explicit dependence on these parameters is given by

$$\dot{x} = F(p)x + G_1(p)u + G_2(p)\eta \tag{B-2}$$

$$r = H(p)x + D(p)u \tag{B-3}$$

The matrix F(p) may be written as

$$F(p) = F_o + \bar{q}_o(\bar{q}_f)\, F_{\bar{q}} + (\omega_f)^2 \sum_{i=1}^{15} \omega_{io}{}^2 F_{\omega_i 2} + \zeta_o(\zeta_f)(\omega_f) \sum_{i=1}^{15} \omega_{io} F_{\zeta\omega_i}$$
$$+ \bar{q}_o(\bar{q}_f) M_{w_o}(M_{wf} - 1) F_{M_w} \tag{B-4}$$

where F_o includes actuator, gust, and Wagner dynamics and pure integrations. The matrix, $G_1(p)$, is actually independent of p since it depends only on the actuator models. Thus

$$G_1(p) \equiv G_1(p_o) \triangleq G_1 \tag{B-5}$$

It was assumed that the gust model did not vary so that G_2 is also independent of p

$$G_2(p) \equiv G_2(p_o) \triangleq G_2 \tag{B-6}$$

The matrices H(p) and D(p) should be expressible in forms similar to that used for F(p) in Equation (B-4). However, it was not possible to properly isolate the effects of M_w variations in the responses from the data available. Therefore, H(p) was expressed as

$$H(p) = H_o + \bar{q}_o(\bar{q}_f)H_{\bar{q}} + (\omega_f)^2 \sum_{i=1}^{15} \omega_{io}{}^2 H_{\omega_i 2} + \zeta_o(\zeta_f)(\omega_f) \sum_{i=1}^{15} \omega_{io} H_{\zeta\omega_i} \tag{B-7}$$

and D(p) was similarly expressed as

$$D(p) = D_o + \bar{q}_o(\bar{q}_f)\, D_{\bar{q}} + (\omega_f)^2 \sum_{i=1}^{15} \omega_{io}{}^2\, D_{\omega_i 2} + \zeta_o(\zeta_f)(\omega_f) \sum_{i=1}^{15} \omega_{io} D_{\zeta\omega_i} \tag{B-8}$$

The manner in which the matrices F_o, $F_{\bar{q}}$, $F_{\omega_i 2}$, $F_{\zeta\omega_i}$, etc., were determined will now be described. The matrix, F, may be represented in the partitioned form

$$F = \left[\begin{array}{cccccc} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & F_{24} & F_{25} & F_{26} \\ 0 & F_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{55} & 0 \\ \hline \multicolumn{6}{c}{F_w} \end{array}\right] \tag{B-9}$$

corresponding to partitioning the x vector into $x^T = [(x^1)^T, (x^2)^T, (x^3)^T, (x^4)^T, (x^5)^T, (x^6)^T]$ where $x^1 = (w, q/n_2)^T$, $x^2 = (\dot{\eta}_1, \dot{\eta}_2, \ldots, \dot{\eta}_{15})^T$, $x^3 = (\eta_1, \eta_2, \ldots, \eta_{15})^T$, $x^4 = (\delta_a, \delta e_i, \delta e_o)^T$, x^5 = (gust states), and x^6 = (Wagner states). The matrix, F_o, is

$$F_o = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{55} & 0 \\ \hline & & & F_w & & \end{array}\right] \tag{B-10}$$

where F_{32} is the 15 x 15 identity matrix corresponding to integration $(\eta_i = \int \dot{\eta}_i)$

F_{44} is the 3 x 3 diagonal actuator dynamics matrix

F_{55} is the 7 x 7 gust dynamics matrix

F_w is the 37 x 79 Wagner dynamics matrix

The only matrices in (B-9) that depend on structural damping and frequency are F_{22} and F_{23}. The matrix, F_{22}, may be written as

$$F_{22} = (F_{22})_{aero} + (F_{22})_{structure} \tag{B-11}$$

where

$$(F_{22})_{structure} = -2\zeta \operatorname{diag}(\omega_i) \tag{B-12}$$

Similarly, the matrix, F_{23}, may be written as

$$F_{23} = (F_{23})_{aero} + (F_{23})_{structure} \tag{B-13}$$

where

$$(F_{23})_{structure} = -\operatorname{diag}(\omega_i^2) \tag{B-14}$$

The structural data consists of the values of $\zeta_o = 0.02$ and the model frequencies, ω_{io}, listed in Table 47.

TABLE 47. MODAL FREQUENCIES (radians/second)

i	ω_{io}	i	ω_{io}	i	ω_{io}
1	4.7225	6	18.723	11	38.792
2	11.146	7	20.116	12	39.671
3	13.575	8	27.339	13	41.897
4	15.564	9	32.980	14	43.230
5	17.749	10	37.425	15	50.568

From Equation (B-12) we have $F_{\zeta\omega_i}$ as the matrix with a "-1" in the i^{th} diagonal element of "the F_{22} block" and with all other elements zero. Similarly, from (B-14) $F_{\omega_i}2$ is the matrix with "-1" in the i^{th} diagonal element of "the F_{23} block" and zero elsewhere. The matrix $\bar{q}F_{\bar{q}}$ corresponds to the remainder of $F - F_o - F_{structure}$, i.e.,

$$\bar{q}F_{\bar{q}} = \left[\begin{array}{cccccc} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & (F_{22})_{aero} & (F_{23})_{aero} & F_{24} & F_{25} & F_{26} \\ \hdashline \multicolumn{6}{c}{0} \end{array}\right] \qquad \text{(B-15)}$$

The matrix F_{M_w} is the matrix with a "+1" in the first row and second column and zeros elsewhere.

The only responses which were included in the performance index that depend on the uncertain parameters are the bending and torsion moments at the wing root and their derivatives. In Reference 35 the bending and torsion moments, denoted by BM, where given as

$$BM = I\ddot{q} + N_i\dot{q} + N_e q + \text{(gust penetration moments)} + \text{(control surface moments)} \qquad \text{(B-16)}$$

where q is a vector of generalized coordinates, I is an inertia matrix, and N_i and N_e are aerodynamic load coefficient matrices. The equations of motion, written in terms of the generalized coordinate vector, are:

$$M\ddot{q} + A_o\dot{q} + Kq - \left(\begin{matrix}\text{generalized} \\ \text{aerodynamic forces}\end{matrix}\right) = \left(\begin{matrix}\text{generalized} \\ \text{control forces}\end{matrix}\right) \qquad \text{(B-17)}$$

The generalized vector, q, may be partitioned into two sub-vectors q^1 and q^2 with q^1 denoting rigid body coordinates and q^2 the vector of flexure mode displacements, i.e.,

$$q^2 = (\eta_1, \eta_2, \ldots, \eta_{15})^T \qquad \text{(B-18)}$$

The corresponding partitioning of Equation (B-17) has the form

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 I_{15} \end{bmatrix} \begin{bmatrix} \ddot{q}^1 \\ \ddot{q}^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -M_2(F_{22})_{\text{structure}} \end{bmatrix} \begin{bmatrix} \dot{q}^1 \\ \dot{q}^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -M_2(F_{23})_{\text{structure}} \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}$$

$$= \Sigma \text{ (partitioned generalized forces)} \qquad \text{(B-19)}$$

where I_{15} is the 15 x 15 identity matrix, M_2 is a diagonal matrix, and $(F_{22})_{\text{structure}}$ and $(F_{23})_{\text{structure}}$ are given in Equations (B-12) and (B-14), respectively. With I in Equation (B-16) partitioned appropriately into $[I^1, I^2]$, it is possible to rewrite Equation (B-16) using (B-19) as

$$BM = I^2 \left[(F_{22})_{\text{structure}} \dot{q}^2 + (F_{23})_{\text{structure}} q^2\right] + \text{(aerodynamic moments)} \qquad \text{(B-20)}$$

with the aerodynamic moments being proportional to $\bar{q}$.

The matrix I^2 is given in Table 48 for the case in which BM is the vector

$$BM = \begin{bmatrix} B_1 \triangleq \text{bending moment at the wing root} \\ T_1 \triangleq \text{torsion moment at the wing root} \end{bmatrix} \qquad \text{(B-21)}$$

The response vector, BM, may be written in the form

$$BM = \underline{H}x \qquad \text{(B-22)}$$

TABLE 48. I^2 MATRIX

```
ROW   1
  -.56448E+05  .22275E+04  .18360E+05 -.80742E+04 -.53074E+04  .12540E+05 -.18274E+04 -.46399E+04
  -.28034E+04  .80097E+03  .74888E+03  .96566E+03 -.13630E+04 -.22474E+04 -.49429E+04
ROW   2
   .28913E+04  .20000E+04  .30008E+04 -.63192E+04  .12468E+04 -.45179E+04 -.11045E+05 -.40468E+04
   .36353E+03 -.18577E+02  .15368E+02 -.44941E+03  .13077E+03  .52363E+03 -.19210E+04
```

since it is independent of actual commands. The dependence of $\underline{H}$ on the parameter vector p may be expressed as

$$\underline{H} = \bar{q}_o(q_f)\underline{H}_{\bar{q}} + (\omega_f)^2 \sum_{i=1}^{15} \omega_{i_o}^{\ 2} \underline{H}_{\omega_i^2} + \zeta_o(\zeta_f)(\omega_f) \sum_{i=1}^{15} \omega_{io}\underline{H}_{\zeta\omega_i} \tag{B-23}$$

where

$$\underline{H}_{\bar{q}} = (\bar{q}_o)^{-1} [\underline{H} - \sum_{i=1}^{15} \omega_{io}^{\ 2} \underline{H}_{\omega_i^2} - \zeta_o \sum_{i=1}^{15} \omega_{io} \underline{H}_{\zeta\omega_i}] \tag{B-24}$$

$$\zeta_o \sum_{i=1}^{15} \omega_{io} \underline{H}_{\zeta\omega_i} = [0 \quad I^2(F_{22})_{structure} \quad 0 \quad 0 \quad 0 \quad 0] \tag{B-25}$$

$$\sum_{i=1}^{15} \omega_{io}^{\ 2} \underline{H}_{\omega_i^2} = [0 \quad 0 \quad I^2(F_{23})_{structure} \quad 0 \quad 0 \quad 0] \tag{B-26}$$

and the partitioning in (B-25) and (B-26) is consistent with the partitioning indicated in (B-9). The decomposition into parameter dependent components of the H and D matrices associated with the derivative of BM may be derived using the decomposition of $\underline{H}$ and F and the equation

$$\frac{d}{dt} (BM) = \underline{H} (Fx + G_1u + G_2\eta) \tag{B-27}$$

The coefficient matrices for the remaining responses of interest (surface displacements, surface rates, and control follower response) are independent of parameter variations.

APPENDIX C

RESIDUALIZED PARTIALS

APPENDIX C

RESIDUALIZED PARTIALS

In computing the sensitivity partials for use in the sensitivity vector augmentation design technique, it was discovered that residualization computations and partial derivative computations are not commutative processes. In other words, the partial derivatives that are computed on a higher order system and then modified through residualization are not equal to the partial derivatives computed on a residualized system. Consider our standard plant representation given by

$$\dot{x} = F(p)x + G_1 u + G_2 \eta \tag{C-1}$$

The sensitivity vector augmentation approach, as discussed in Section V, defines sensitivity state dynamics given by

$$\dot{\sigma} = \frac{\partial \dot{x}}{\partial p} = \left.\frac{\partial F(p)}{\partial p}\right|_{p_o} x + F(p_o)\sigma \tag{C-2}$$

These sensitivity state dynamics are then augmented to Equation (C-1) to form a sensitivity state vector design model, or

$$\dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} F(p_o) & 0 \\ \left.\frac{\partial F(p)}{\partial p}\right|_{p_o} & F(p_o) \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \eta \tag{C-3}$$

To illustrate the difference discussed above, let us first compute the partials on the 42nd order Case 2 model and residualize the partials to Case 4R. Equation (C-3) may be represented by

$$\begin{bmatrix} \dot{x}_1^{\,2} \\ \dot{x}_2^{\,2} \\ \dot{\sigma}_1^{\,2} \\ \dot{\sigma}_2^{\,2} \end{bmatrix} = \begin{bmatrix} F_{11}(p_o) & F_{12}(p_o) & 0 & 0 \\ F_{21}(p_o) & F_{22}(p_o) & 0 & 0 \\ \left.\frac{\partial F_{11}(p)}{\partial p}\right|_{p_o} & \left.\frac{\partial F_{12}(p)}{\partial p}\right|_{p_o} & F_{11}(p_o) & F_{12}(p_o) \\ \left.\frac{\partial F_{21}(p)}{\partial p}\right|_{p_o} & \left.\frac{\partial F_{22}(p)}{\partial p}\right|_{p_o} & F_{21}(p_o) & F_{22}(p_o) \end{bmatrix} \begin{bmatrix} x_1^{\,2} \\ x_2^{\,2} \\ \sigma_1^{\,2} \\ \sigma_2^{\,2} \end{bmatrix} \tag{C-4}$$

+ control and noise drivers

where the superscript 2 refers to the Case 2 model states and all partials are evaluated at p_o. Let us consider the case where, for demonstration purposes, we wish to eliminate $x_2^{\,2}$ and $\sigma_2^{\,2}$ which are scalars. Let us assume also that p is a scalar. If we compute the partial derivatives first and then residualize, we obtain a sensitivity vector augmentation model given by

$$\begin{bmatrix} \dot{x}_1^{\,2} \\ \dot{\sigma}_1 \end{bmatrix} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix} \begin{bmatrix} x_1^{\,2} \\ \sigma_1 \end{bmatrix} + \text{control and noise drivers} \tag{C-5}$$

where

$$\tilde{F}_{11} = F_{11}(p) - F_{12}(p)\,F_{21}(p)/f_{22}(p) + F_{12}(p)\,\frac{\partial f_{22}(p)}{\partial p}\,\frac{\partial F_{21}(p)}{\partial p}\,/f_{22}(p)\,f_{22}(p_o)$$

$$\tilde{F}_{12} = F_{12}(p)\,F_{21}(p_o)\,\frac{\partial f_{22}(p)}{\partial p}\,/\,f_{22}(p)\,f_{22}(p_o)$$

$$\tilde{F}_{21} = \frac{\partial F_{11}(p)}{\partial p} - \frac{\partial F_{12}(p)}{\partial p}\,F_{21}(p)/f_{22}(p)$$

$$+ \left[\frac{\partial F_{12}(p)}{\partial p}\,\frac{\partial f_{22}(p)}{\partial p}\,/\,f_{22}(p)\,f_{22}(p_o) - F_{12}(p_o)/f_{22}(p_o)\right]\frac{\partial F_{12}(p)}{\partial p}$$

$$\tilde{F}_{22} = F_{11}(p_o) + \left[\frac{\partial F_{12}(p)}{\partial p}\,\frac{\partial f_{22}(p)}{\partial p}\,/\,f_{22}(p)\,f_{22}(p_o) - F_{12}(p_o)/f_{11}(p_o)\right]F_{21}(p_o)$$

This may be contrasted to calculating the partials after residualizing. For example, the residualized form of Equation (C-4) with the sensitivity states eliminated is:

$$\dot{x}_1^2 = [F_{11}(p) - F_{12}(p)\, f_{22}^{-1}(p)\, F_{21}(p)]x_1 + \text{control and noise drivers}$$

Computing partial derivatives based on this residualized form results in the following sensitivity vector augmentation design model:

$$\begin{bmatrix} \dot{x}_1^2 \\ \dot{\sigma}_1^2 \end{bmatrix} = \begin{bmatrix} F_{11}(p) - F_{12}(p)\, f_{22}^{-1}(p)\, F_{21}(p) & 0 \\ \hat{F}_{21} & F_{11}(p_o) - F_{12}(p_o) f_{22}^{-1}(p_o) F_{21}(p_o) \end{bmatrix} \begin{bmatrix} x_1 \\ \sigma_1 \end{bmatrix}$$

$$+ \text{ control and noise drivers} \qquad \text{(C-6)}$$

where

$$\hat{F}_{21} = \frac{\partial F_{11}}{\partial p} - \frac{\partial F_{12}(p)}{\partial p} F_{21}(p)/f_{22}(p) - F_{12}(p) \frac{\partial F_{21}(p)}{\partial p} / f_{22}(p)$$

$$- [f_{22}(p) \frac{\partial F_{12}(p)}{\partial p} F_{21}(p) + f_{22}(p)\, F_{12}(p) \frac{\partial F_{21}(p)}{\partial p}]/f_{22}(p)^2$$

As can be seen, each element of the F matrix in Equation (C-5) contains elements that are functions of the partials of x_2 matrix elements that do not appear in Equation (C-6).

No attempt was made to evaluate the effects of the two procedures on the sensitivity vector augmentation controller performance. It is stated here purely as an observation.

The partials that were used in the actual design were truncated partials; that is,

$$\frac{\partial F_{21}}{\partial p} = \frac{\partial F_{12}}{\partial p} = \frac{\partial F_{22}}{\partial p} = 0$$

This produces a sensitivity vector augmentation design model given by

$$\begin{bmatrix} \dot{x}_1^2 \\ \dot{\sigma}_1^2 \end{bmatrix} = \begin{bmatrix} F_{11}(p) - F_{12}(p)\, F_{22}^{-1}(p)\, F_{21}(p) & 0 \\ \frac{\partial F_{11}(p)}{\partial p} & F_{11}(p_o) \end{bmatrix} \begin{bmatrix} x_1^2 \\ \sigma_1^2 \end{bmatrix} \qquad \text{(C-7)}$$

$$+ \text{ control and noise drivers}$$

APPENDIX D

RESPONSE RATE CALCULATIONS

APPENDIX D

RESPONSE RATE CALCULATIONS

In controlling the level of response of specific elements of this response vector, it is sometimes advantageous to introduce rates of change of those elements as additional response states. Control is then exercised through manipulation of weights on the response elements and their rate of change. This procedure was used in control of the RMS bending and torsion moment response to gust input. Consider the standard Case 1 model representation

$$\dot{x}^1 = Fx^1 + G_1u + G_2\eta \tag{D-1}$$

$$r^1 = Hx^1 + Du \tag{D-2}$$

This is the Case 1 model structure given in Appendix A. The response vector may be partitioned as follows:

$$r = \begin{bmatrix} r_1^1 \\ r_2^1 \\ r_3^1 \end{bmatrix} = \begin{bmatrix} r_1^1 \\ \dot{r}_1^1 \\ r_3^1 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} x^1 + \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} u \tag{D-3}$$

where

r_1^1 = bending, torsion moment response

$r_2^1 = \dot{r}_1^1$ = rate of change of bending, torsion moment response

r_3^1 = dynamic states and state derivatives

The original form of the data did not include the r_2^1 data. These response terms were computed by differentiating

$$r_1^1 = H_1x^1 + D_1u \tag{D-4}$$

or

$$\dot{r}_1^1 = H_1\dot{x}^1 \qquad \text{(D-5)}$$

where

$\dot{u} = 0$

Substituting (D-1) into (D-5), we obtain

$$\begin{aligned}\dot{r}_1^1 &= H_1Fx^1 + H_1G_1u \qquad \text{(D-6)}\\ &= H_2x^1 + D_2u\end{aligned}$$

where the noise driver term was neglected.

Variations similar to those discussed in Appendix C were observed in the computations of response rate terms. For example, if one were to construct response rate terms using Case 2 instead of Case 1, elements of the H_2 matrix would be significantly different, some even changing signs. This can be seen from representing (D-6) by

$$\dot{r}_1^1 = [H_{11}, H_{12}] \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} + [H_{11}, H_{12}] \begin{bmatrix} G_1 \\ 0 \end{bmatrix} u$$

$$\begin{aligned}&= [H_{11}F_{11} + H_{12}F_{11}]x_1^1 + [H_{11}F_{12} + H_{12}F_{22}]x_2^1 \qquad \text{(D-7)}\\ &+ H_{11}G_{1_1} u\end{aligned}$$

Using Case 2, the resulting expression is

$$\dot{r}_1^2 = H_{11}F_{11}x_1^1 + H_{11}G_{1_1} u \qquad \text{(D-8)}$$

As can be seen, the matrix coefficients of x_1^1 in (D-7) and (D-8) differ by $H_{12}F_{11}$ which does have non-zero elements.

As was the case with the computation of the partial derivatives, the computation of the response rates is not commutative with the residualization process. Rewriting Equation (D-8) as

$$\dot{r}_1^2 = \tilde{H}_2^2 x^2 + D^2 u \qquad \text{(D-9)}$$

or

$$\dot{r}_1^2 = [\tilde{H}_{21}^2 \; \tilde{H}_{22}^2] \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} + D^2 u$$

where

x_1^2 = states to be retained

x_2^2 = states to be eliminated

and

$$\tilde{H}_{21}^2 = H_{11}^2 F_{11}^2 + H_{12}^2 F_{21}^2$$

$$\tilde{H}_{22}^2 = H_{11}^2 F_{12}^2 + H_{12}^2 F_{22}^2$$

$$D^2 = H_{11}^2 G_1$$

Eliminating states x_2^2 through residualization, the equation reads

$$\dot{r}_1^3 = [H_{11}^2 F_{11}^2 + H_{12}^2 F_{21}^2 - [H_{11}^2 F_{12}^2 + H_{12}^2 F_{22}^2] (F_{22}^2)^{-1} F_{21}^2] x_1^2 + D^2 u$$

$$= [H_{11}^2 F_{11}^2 - H_{11}^2 F_{12}^2 (F_{22}^2)^{-1} F_{21}^2] x_1^2 + D^2 u \qquad \text{(D-10)}$$

$$= \hat{H}_2^3 x_1^2 + \hat{D}^3 u$$

However, beginning with

$$r_1^2 = H_1^2 x^2 + D^2 u \qquad \text{(D-11)}$$

we residualize to get

$$r_1^3 = [H_{11}^{\;2} - H_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2}]\, x_1^2 + D^2 u \tag{D-12}$$

If the response rate terms are now computed using the derivative of Equation (D-12), we obtain

$$\dot{r}_1^3 = [H_{11}^{\;2} - H_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2}]\,[F_{11}^{\;2} - F_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}]\, x_1^2$$

$$+ [H_{11}^{\;2} - H_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2}]\, G_1 u \tag{D-13}$$

$$= \tilde{H}_2^{\;3}\, x_1^{\;2} + \tilde{D}^3 u$$

Now

$$\tilde{H}_2^{\;3} - \hat{H}_2^{\;3} = -H_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2} F_{11}^{\;2} + H_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2} F_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2}$$

$$= -H_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2}\,[F_{11}^{\;2} - F_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2}] \tag{D-14}$$

$$\neq 0$$

and

$$\tilde{D}^3 = \hat{D}^3 = -H_{12}^{\;2}(F_{22}^{\;2})^{-1} F_{21}^{\;2}\, G_1 \tag{D-15}$$

There was no attempt made to determine if any one procedure was better in some sense than another. It is noted here only to recognize that differences exist and must be considered. For this study, after we recognized that differences did exist due to computational procedures, we maintained consistent models for all designs.

APPENDIX E

MODEL REDUCTION TECHNIQUE COMPARISON

APPENDIX E

MODEL REDUCTION TECHNIQUE COMPARISON

Three techniques--truncation, residualization, and re-residualization--have been described in the main body of this report. All three techniques can be used to reduce the order of an unwieldy model. The latter two techniques retain the characteristics of the eliminated dynamics in some sense. This appendix presents additional comparitive results on model reduction techniques. An additional technique, singular perturbation, is also described and included in the comparison evaluations.

SINGULAR PERTURBATION

The singular perturbation method assumes a plant representation given by

$$\dot{x}_1 = F_1 x_1 + F_2 x_2 + G_1 u \tag{E-1}$$

$$\epsilon \dot{x}_2 = F_3 x_1 + F_4 x_2 \tag{E-2}$$

where, again,

x_1 = states to be retained

x_2 = states to be eliminated

We may note in passing that the residualization method may be derived by assuming $\epsilon = 0$ in Equation (E-2). But the perturbation approach consists of writing the solutions as power series in ϵ. To this end, suppose

$$x_1 = x_1^o + \epsilon x_1^1 + \frac{1}{2}\epsilon^2 x_1^2 + \dots \tag{E-3}$$

$$x_2 = x_2^o + \epsilon x_1^1 + \frac{1}{2}\epsilon^2 x_2^2 + \dots \tag{E-4}$$

These expressions may be substituted into (E-1) and (E-2). This gives

$$\dot{x}_1^0 + \epsilon \dot{x}_1^1 + \dots = F_1(x_1^0 + \epsilon x_1^1 + \dots) + F_2(x_2^0 + \epsilon x_2^1 + \dots) + G_1 u \tag{E-5}$$

$$\epsilon \dot{x}_2^0 + \epsilon^2 \dot{x}_2^1 + \dots = F_3(x_1^0 + \epsilon x_1^1 + \dots) + F_4(x_2^0 + \epsilon x_2^1 + \dots) \tag{E-6}$$

Equating coefficients of like powers of ϵ gives

$$\dot{x}_1^0 = F_1 x_1^0 + F_2 x_2^0 + G_1 u \tag{E-7}$$

$$0 = F_3 x_1^0 + F_4 x_2^0 \tag{E-8}$$

$$\dot{x}_1^1 = F_1 x_1^1 + F_2 x_2^1 \tag{E-9}$$

$$\dot{x}_1^0 = F_3 x_1^1 + F_4 x_2^1 \tag{E-10}$$

Thus from (E-8), $x_2^0 = -F_4^{-1} F_3 x_1^0$, and from (E-7) this yields

$$\dot{x}_1^0 = (F_1 - F_2 F_4^{-1} F_3)\, x_1^0 + G_1 u \tag{E-11}$$

We may solve (E-10) for x_2^1 in terms of x_1^1 and $\dot{x}_2^0$ and substitute this expression for x_2^1 in (E-9) to obtain

$$\dot{x}_1^1 = (F_1 - F_2 F_4^{-1} F_3)\, x_1^1 + F_2 F_4^{-1} \dot{x}_2^0 \tag{E-12}$$

But $\dot{x}_2^0 = -F_4^{-1} F_3 \dot{x}_1^0$ from (E-8), and using (E-11) we have

$$\dot{x}_2^0 = -F_4^{-1} F_3 (F_1 - F_2 F_4^{-1} F_3)\, x_1^0 - F_4^{-1} F_3 G_1 u \tag{E-13}$$

Substituting for $\dot{x}_2^0$ from (E-13) into (E-12) yields

$$\dot{x}_1^1 = (F_1 - F_2 F_4^{-1} F_3)\, x_1^1 - F_2 F_4^{-2} F_3 (F_1 - F_2 F_4^{-1} F_3)\, x_1^0 - F_2 F_4^{-2} F_3 G_1 u \tag{E-14}$$

Now if we set $y = x_1^0 + \epsilon x_1^1$ and perform the required algebra, we finally have

$$\dot{y} = (I - \epsilon F_2 F_4^{-2} F_3)\, [(F_1 - F_2 F_4^{-1} F_3) y + G_1 u] + \epsilon^2 F_2 F_4^{-2} F_3 (F_1 - F_2 F_4^{-1} F_3)\, x_1^1 \tag{E-15}$$

Assuming the last term is negligible (which may or may not be a valid assumption) leads to the reduced order model

$$\dot{y} = (I - \epsilon F_2 F_4^{-2} F_3)\, [(F_1 - F_2 F_4^{-1} F_3) y + G_1 u]$$

Apparently one can derive an infinite number of methods to derive reduced order models. We have stopped with these four and examined them in the light of a very simple example.

Examples

We will treat a very simple system with numerical values chosen somewhat arbitrarily but intended to be typical of what may occur in an authentic design problem. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -3 & 0 \\ -0.5 & a & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} u \qquad \text{(E-16)}$$

A loose interpretation of this system as typical of an authentic system is to consider x_1 as corresponding to a low frequency mode, x_3 as corresponding to a high frequency mode, with x_2 a surface position driven by a first order actuator. Viewed in this manner the relative magnitudes of the numerical coefficients are realistic. Consistent with the physical interpretation, we will assume as our goal a reduced order model for the low frequency mode and actuator. Thus we group the first two components of x into the sub-vector x_1 and take the last component of x to be the sub-vector x_2. This partitioning yields

$$F_1 = \begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad G_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$F_3 = [-0.5 \quad a] \qquad F_4 = -10$$

The truncated method yields

$$\dot{x}_T = \begin{bmatrix} -1 & +1 \\ 0 & -3 \end{bmatrix} x_T + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u \qquad \text{(E-17)}$$

The residualization method yields

$$\dot{x}_R = \begin{bmatrix} -1.1 & 1+(a/5) \\ 0 & -3 \end{bmatrix} x_R + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u \qquad \text{(E-18)}$$

The re-residualization method yields

$$\dot{x}_{RR} = \begin{bmatrix} \frac{-110}{99} & \frac{100+26a}{99} \\ 0 & -3 \end{bmatrix} x_{RR} + \begin{bmatrix} \frac{-6a}{99} \\ 3 \end{bmatrix} u \qquad \text{(E-19)}$$

The singular perturbation method with $\epsilon = 1$ yields

$$\dot{x}_p \begin{bmatrix} -1.111 & 1.01+0.262a \\ 0 & -3 \end{bmatrix} x_p + \begin{bmatrix} -0.06a \\ 3 \end{bmatrix} u \qquad \text{(E-20)}$$

Now, the question arises as to how these reduced order models compare, and more basically, what should be the basis for the comparison. We will not attempt to answer the second question but will make a comparison based on two criteria. The first comparison will be with respect to controllability. The second will be with respect to closed-loop eigenvalues of the "complete" system corresponding to designs with the reduced order models aimed at specified closed-loop eigenvalues.

The determinant of the controllability matrix for the system (E-16) is $27[0.5 + 9a + 2a^2]$ which is zero at a = -4.444 and -0.056. For system (E-17) the corresponding determinant is -9. The determinant of the controllability matrix for the residualized system (E-18) is $-9[5 + a]/5$ which is zero at a = -5. The determinant for system (E-19) is $-(900 + 200a)/99$ which is zero at a = -4.5. For system (E-20), the determinant is $-[9.09+2.018a]$ which is zero at a = -4.505. Thus the latter two systems more accurately reflect the controllability of the original system.

Another possible check on the validity of the reduced order models is to specify the closed-loop eigenvalues for these models in order to define a controller. Then with these controllers, find the closed-loop eigenvalues of the complete system and compare the difference. This was done for the two cases a = -4 and a = -5. The closed-loop eigenvalues specified for the lower order models were -2 and -3 in each case. The results are given in Table 49. The models based on re-residualization and perturbations give very similar results which are significantly better than those for the truncation and residualization models.

TABLE 49. CLOSED-LOOP EIGENVALUE COMPARISON

Value of a	Reduced order model	K_1	K_2	Closed-loop eigenvalues of complete system
-4	Truncated	-2/3	-1/3	-1.18, -4.25, -9.57
	Residualized	-2.85	-0.3	-1.36, -5.42, -8.11
	Re-Residualized	-4.99	+0.11	-1.68, -5.25, -6.77
	Perturbation	-4.95	+0.10	-1.68, -5.14, -6.88
-5	Truncated	-2/3	-1/3	-1.11, -4.01, -9.88
	Residualized	No solution		- - -
	Re-Residualized	4.99	-0.80	$-2.06 \pm 0.77j$, -12.28
	Perturbation	5.04	-0.80	$-2.05 \pm 0.79j$, -12.30

APPENDIX F

MINIMAX THEOREMS

APPENDIX F

MINIMAX THEOREMS

Two theorems are presented to support the procedure used in the design of the minimax controller. The first theorem is an existence theorem providing a sufficiency condition for optimal insensitive controllers locally, that is, for sufficiently small variations. This theorem also implies that such controllers are optimal controllers corresponding to boundary points of admissible parameter variation sets. The second theorem states a necessary condition for an optimal controller corresponding to a point in the boundary of the domain of admissible parameter variations to be an optimal insensitive controller. Here, the expression optimal insensitive controller refers to a controller which is optimal for some admissible value of the parameters which minimizes the maximum of the performance over the range of admissible parameter values.

- Theorem 1: Consider the system $\dot{x} = F(p)x + G(p)u$, $x(0) = x_o$ and associated performance functional

  $$J(u, p) = \int_o^\infty (Hx+Du)^T Q(Hx+Du)dt$$

  where p is a vector of parameters. Let $J^*(p) = \min J(u, p)$. Suppose p_o is a point with the property that $|\nabla_p J^*(p_o)|$ and $\nabla_{pp} J^*(p_o) > 0$. Then there exists an $\epsilon > 0$ such that the control $u^*(p_o)$ which minimizes $J(u, p_o)$ also minimizes the maximum of $J(u, p)$ with respect to p in an ϵ-ball, with p_o on the shell (boundary) of the ball.

  Proof: For any $\epsilon > 0$, let $B_\epsilon(p_o)$ denote the ϵ-ball with center at $p_o - \epsilon\nabla_p J^*(p_o)$, i.e.,

  $$B_\epsilon = \{p{:}p = p_o - \epsilon\nabla_p J^*(p_o) + \epsilon\eta, \ |\eta| \leq |\nabla_p J^*(p_o)|\}$$

  Also, define $M(u; p_o, \epsilon)$ to be $\max\limits_{p\epsilon B_\epsilon(p_o)} J(u, p)$. Then

  $$M(u; p_o, \epsilon) \geq J(u, p_o) \geq J^*(p_o)$$

For $p \epsilon B_\varepsilon(p_o)$ we may express $J(u^*(p_o), p)$ as

$$J[u^*(p_o), p] = J[u^*(p_o), p_o] + \nabla_p J[u^*(p_o), p]\Big|_{p=p_o}^{T} (p-p_o) +$$

$$1/2(p-p_o)^T \left.\frac{\partial^2 J[u^*(p_o), p]}{\partial p \partial p^T}\right|_{p=p_o} (p-p_o) + o(\varepsilon)^2$$

$$\equiv J^*(p_o) + g^T (p-p_o) + 1/2(p-p_o)^T H(p-p_o) + o(\varepsilon^2), \ H > 0$$

Note that $g \equiv \nabla_p J[u^*(p_o), p]\Big|_{p=p_o} = \nabla_p J^*(p)\Big|_{p=p_o}$

For $p \epsilon B_\varepsilon$ and ε to be sufficiently small, the only possibilities for extreme points of $J[u^*(p_o), p]$ are

1. Approximately $p_o - H^{-1}g$ if this point lies within B_ε, or

2. Points on the shell of B_ε.

The point near $p_o - H^{-1}g$ is a minimizing point. Therefore, maximizing points lie on the boundary of B_ε.

The problem of extremizing $J[u^*(p_o), p]$ subject to $p = p_o - \varepsilon g + \varepsilon \eta$ with $|\eta| = |g|$ may be treated with a Lagrange multiplier as minimizing

$$H = J_o + g^T (p-p_o) + 1/2(p-p_o)^T H(p-p_o) + o(\varepsilon^2) + \lambda(\eta^T \eta - g^T g)$$

This yields $0 = \varepsilon g + \lambda \eta^T \varepsilon^2 H(g-\eta) + 0\ (\varepsilon^2)$ and $|\eta| = |g|$ as necessary conditions. For ε small these conditions imply that

$$\lambda = \pm \varepsilon\ [1 + o(1)] \text{ and } \eta = \pm g[1 + o(1)]$$

The bottom signs yield the maximum and the top signs yield the minimum. The exact solution for the bottom signs is $\lambda = -\varepsilon$, $\eta = g$ which describes the point p_o.

Thus, on B_ϵ

$$J[u^*(p_o), p] \le J[u^*(p_o), p_o] = J^*(p_o)$$

Hence,

$$M[u^*(p_o); p_o, \epsilon] = J^*(p_o) \le M(u; p_o, \epsilon)$$

which was to be proved.

- Theorem 2: Consider $J[u^*(p_o), p] = J^*(p_o) + g^T (p-p_o) + o(|p-p_o|)$ with $g = \nabla_p J^*(p)|_{p=p_o}$. Let Ω denote a closed convex set with nonempty interior in the parameter space. Suppose p_o is a point in the boundary of Ω with the property that

$$J[u^*(p_o), p_o] = \max_{p \in \Omega} J[u^*(p_o), p]$$

Then g must be an external normal to Ω at p_o.

Proof: Assume that g is not an external normal to Ω at p_o; i.e., there exists a p_1 in Ω such that $(p_1-p_o)^T g = c_1 > 0$. Since Ω is convex, $p(\lambda) = p_o + \lambda(p_1-p_o)$ lies in Ω for $0 \le \lambda \le 1$ and $[p(\lambda)-p_o]^T g = \lambda(p_1-p_o)^T g = \lambda c_1$. Thus,

$$J[u^*(p_o), p(\lambda)] = J^*(p_o) + [p(\lambda) - p_o]^T g + o[|p(\lambda) - p_o|]$$

$$= J^*(p_o) + \lambda c_1 + o(\lambda) > J^*(p_o)$$

for λ to be sufficiently small. This contradicts the hypothesis that p_o has the property that $J[u^*(p_o), p_o] = \max_{p \in \Omega} J[u^*(p_o), p]$.

APPENDIX G

DUAL LYAPUNOV CONCEPT

APPENDIX G

DUAL LYAPUNOV CONCEPT

This appendix presents results derived by Professor D. L. Russell in his role as a consultant to Honeywell. The work reported is part of Honeywell's Independent Research Program on insensitive control.

Several different definitions for insensitivity of a linear control system are introduced, and relationships between these definitions are established. It is found that a certain degree of agreement between a priori distinct notions of insensitivity can be expected. The idea of maneuverability is introduced, and the extent to which it can be considered to be an attribute complementary to insensitivity is explored. Finally, numerical schemes for implementing the design techniques suggested by the theoretical developments are examined.

INTRODUCTION

Our purpose in this report is to explore several different possible definitions for insensitivity of a linear control system and to establish relationships, where possible, between these definitions. We shall see that a certain degree of agreement between a priori distinct notions of insensitivity can be expected. Finally we introduce the idea of maneuverability and explore the extent to which it can be considered to be an attribute which is complementary to insensitivity.

Let us assume that we begin with a linear control system

$$\dot{x} = F(p)x + G(p)u, \; x(0) = x_o, \qquad x \epsilon R^n, \quad u \epsilon R^m \tag{G-1}$$

and determine, via familiar techniques, a linear feedback law

$$u = Kx \tag{G-2}$$

such that the nominal closed-loop system

$$\dot{x} = (F(p_0) + G(p_o)K)x = Sx, \; x(0) = x_o \tag{G-3}$$

For $p \epsilon B_\epsilon(p_o)$ we may express $J(u^*(p_o), p)$ as

$$J[u^*(p_o), p] = J[u^*(p_o), p_o] + \nabla_p J[u^*(p_o), p] \Big|_{p=p_o}^{T} (p-p_o) +$$

$$1/2(p-p_o)^T \left. \frac{\partial^2 J[u^*(p_o), p]}{\partial p \partial p^T} \right|_{p=p_o} (p-p_o)+o(\epsilon)^2$$

$$\equiv J^*(p_o) + g^T (p-p_o) + 1/2(p-p_o)^T H(p-p_o) + o(\epsilon^2), \; H > 0$$

Note that $g \equiv \nabla_p J[u^*(p_o), p] \Big|_{p=p_o} = \nabla_p J^*(p) \Big|_{p=p_o}$

For $p \epsilon B_\epsilon$ and ϵ to be sufficiently small, the only possibilities for extreme points of $J[u^*(p_o), p]$ are

1. Approximately $p_o - H^{-1} g$ if this point lies within B_ϵ, or

2. Points on the shell of B_ϵ.

The point near $p_o - H^{-1} g$ is a minimizing point. Therefore, maximizing points lie on the boundary of B_ϵ.

The problem of extremizing $J[u^*(p_o), p]$ subject to $p = p_o - \epsilon g + \epsilon \eta$ with $|\eta| = |g|$ may be treated with a Lagrange multiplier as minimizing

$$H = J_o + g^T (p-p_o) + 1/2(p-p_o)^T H(p-p_o) + o(\epsilon^2) + \lambda(\eta^T \eta - g^T g)$$

This yields $0 = \epsilon g + \lambda \eta^T \epsilon^2 H(g-\eta) + 0\,(\epsilon^2)$ and $|\eta| = |g|$ as necessary conditions. For ϵ small these conditions imply that

$$\lambda = \pm \epsilon\, [1 + o(1)] \text{ and } \eta = \pm g[1 + o(1)]$$

The bottom signs yield the maximum and the top signs yield the minimum. The exact solution for the bottom signs is $\lambda = -\epsilon$, $\eta = g$ which describes the point p_o.

Thus, on B_ϵ

$$J[u^*(p_o), p] \le J[u^*(p_o), p_o] = J^*(p_o)$$

Hence,

$$M[u^*(p_o); p_o, \epsilon] = J^*(p_o) \le M(u; p_o, \epsilon)$$

which was to be proved.

- Theorem 2: Consider $J[u^*(p_o), p] = J^*(p_o) + g^T (p-p_o) + o(|p-p_o|)$ with $g = \nabla_p J^*(p)|_{p=p_o}$. Let Ω denote a closed convex set with nonempty interior in the parameter space. Suppose p_o is a point in the boundary of Ω with the property that

$$J[u^*(p_o), p_o] = \max_{p \in \Omega} J[u^*(p_o), p]$$

Then g must be an external normal to Ω at p_o.

Proof: Assume that g is not an external normal to Ω at p_o; i.e., there exists a p_1 in Ω such that $(p_1-p_o)^T g = c_1 > 0$. Since Ω is convex, $p(\lambda) = p_o + \lambda(p_1-p_o)$ lies in Ω for $0 \le \lambda \le 1$ and $[p(\lambda)-p_o]^T g = \lambda(p_1-p_o)^T g = \lambda c_1$. Thus,

$$J[u^*(p_o), p(\lambda)] = J^*(p_o) + [p(\lambda) - p_o]^T g + o[|p(\lambda) - p_o|]$$

$$= J^*(p_o) + \lambda c_1 + o(\lambda) > J^*(p_o)$$

for λ to be sufficiently small. This contradicts the hypothesis that p_o has the property that $J[u^*(p_o), p_o] = \max_{p \in \Omega} J[u^*(p_o), p]$.

APPENDIX G

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INTRODUCTION

Our purpose in this report is to explore several different possible definitions for insensitivity of a linear control system and to establish relationships, where possible, between these definitions. We shall see that a certain degree of agreement between a priori distinct notions of insensitivity can be expected. Finally we introduce the idea of maneuverability and explore the extent to which it can be considered to be an attribute which is complementary to insensitivity.

Let us assume that we begin with a linear control system

$$\dot{x} = F(p)x + G(p)u, \ x(0) = x_o, \quad x \epsilon R^n, \quad u \epsilon R^m \tag{G-1}$$

and determine, via familiar techniques, a linear feedback law

$$u = Kx \tag{G-2}$$

such that the nominal closed-loop system

$$\dot{x} = (F(p_0) + G(p_o)K)x = Sx, \ x(0) = x_o \tag{G-3}$$

is asymptotically stable; i.e., $S = F(p_o) + G(p_o)K$ is a stability matrix.

One of the types of insensitivity which we shall discuss is insensitivity to parameter variations. If the feedback law (G-2) is employed in (G-1), we obtain

$$\dot{x} = (F(p_o) + [F(p) - F(p_o)] + G(p_o)K + [G(p) - G(p_o)]K)x = (S + \delta S)x \qquad \text{(G-4)}$$

where

$$\delta S = F(p) - F(p_o) + [G(p) - G(p_o)]K \qquad \text{(G-5)}$$

We shall use the term "insensitivity with respect to parameter variations" (IPV) to refer to the study of the perseverence of the stability character of the matrix $S+\delta S$ for perturbations δS about S.

Another aspect of insensitivity has to do with the manner in which the system (G-3) is affected by external disturbances. Specifically we shall suppose (G-3) to be replaced by

$$\dot{x} = Sx+v \qquad \text{(G-6)}$$

where v is a "white noise" disturbance with covariance

$$\text{cov}\,(v(t),\ v(s)) = \delta(t-s)V$$

The term "insensitivity to external disturbances" (IED) will be used to refer to the study of the relationship of the covariance X(t) of x(t) to the covariance matrix V.

Finally, one might suppose the two systems (G-3) and (G-4) to be augmented by a command vector:

$$\dot{x} = Sx + Cw \qquad \text{(G-7)}$$

$$\dot{x} = (S+\delta S)x + Cw \qquad \text{(G-8)}$$

In both cases we consider the problem of steering from

$$x(0) = 0 \qquad \text{(G-9)}$$

to

$$x(T) = x_1 \tag{G-10}$$

by means of an appropriate command input w. The study of the manner in which w must change as the system matrix passes from S to S+δS will be referred to as "insensitivity with respect to command requirements," or ICR for short.

In connection with the system (G-7) and specified boundary conditions (G-9) and (G-10), one may also ask what relationship exists between the norm and location of the target vector x_1 and the norm in, say, $L^2[0, T]$ of the control w required to achieve the transfer from (G-9) to (G-10). This is what we refer to as "maneuverability."

INSENSITIVITY WITH RESPECT TO PARAMETER VARIATIONS

We consider the nominal closed-loop system (G-3), i.e., $\dot{x} = Sx$, and the perturbed system (G-4), i.e., $\dot{x} = (S+\delta S)x$. If S (=A+BK) is a stability matrix, the familiar theorem of Lyaponov[38] guarantees that the matrix equation†

$$SZ+ZS^* + L = 0 \tag{G-11}$$

has a unique positive definite symmetric solution Z = Z(L) for each positive definite symmetric matrix L. In fact, we have

$$Z = \int_o^\infty e^{St} Le^{S^*t} dt$$

We will attempt to extract information about IPV from Equation (G-11).

Multiplying Equation (G-11) on the right and on the left by Z^{-1} one has

$$S^*Z^{-1} + Z^{-1}S + Z^{-1} LZ^{-1} = 0$$

For a perturbed matrix S+δS one then has

$$(S+\delta S)^*Z^{-1} + Z^{-1}(S+\delta S) + Z^{-1} LZ^{-1} - Z^{-1}\delta S - \delta S^*Z^{-1} = 0 \tag{G-12}$$

† In this appendix superscript asterisks will be used to denote transpose operation.

Citing Lyapunov's theorem again we see that $S+\delta S$ remains a stability matrix if

$$Z^{-1} LZ^{-1} - Z^{-1}\delta S - \delta S^*Z^{-1} > 0 \tag{G-13}$$

Moreover, $S+\delta S$ has purely imaginary eigenvalues if $Z^{-1}LZ^{-1} - Z^{-1}\delta S - \delta S^*Z^{-1} = 0$ and becomes completely unstable if $Z^{-1}LZ^{-1} - Z^{-1}\delta S - \delta S^*Z^{-1} < 0$.

Proposition 1: The inequality (G-13) is valid if

$$\delta S^*L^{-1}\delta S < 1/4\, Z^{-1}LZ^{-1}$$

and this is the best estimate of its kind.

Proof: Let $L^{1/2}$ denote the positive symmetric square root of L and let $L^{-1/2} = (L^{1/2})^{-1}$. We rewrite (G-13) in the form

$$Z^{-1}LZ^{-1} - Z^{-1}L^{1/2}L^{-1/2}\delta S - \delta S^*L^{-1/2}L^{1/2}Z^{-1} > 0$$

and note that

$$0 \le (\tfrac{1}{\sqrt{2}} Z^{-1} L^{1/2} \pm \sqrt{2}\delta S^*L^{-1/2})(\tfrac{1}{\sqrt{2}} Z^{-1} L^{1/2} \pm \sqrt{2}\delta S^*L^{-1/2})^*$$

$$= 1/2Z^{-1} LZ^{-1} \pm Z^{-1} L^{1/2} L^{-1/2}\delta S \pm \delta S^*L^{-1/2} L^{1/2} Z^{-1} + 2\delta S^*L^{-1}\delta S$$

so that

$$\pm (Z^{-1} \delta S + \delta S^*Z^{-1}) \le 1/2Z^{-1} LZ^{-1} + 2\delta S^*L^{-1}\delta S$$

and hence

$$Z^{-1} LZ^{-1} - Z^{-1}\delta S - \delta S^*Z^{-1} \ge 1/2Z^{-1} LZ^{-1} - 2\delta S^*L^{-1} \delta S \tag{G-14}$$

The inequality (G-13) is true, therefore, if the right-hand side of (G-14) is positive, i.e.,

$$\delta S^*L^{-1} \delta S < 1/4\, Z^{-1} LZ^{-1} \tag{G-15}$$

That this is the best estimate of its type follows from the fact that if we take

$$\delta S = \mu LZ^{-1} \tag{G-16}$$

which violates (G-13) for $\mu \geq 1/2$, then (G-12) becomes

$$(S+\delta S)^*Z^{-1} + Z^{-1}(S+\delta S) + (1 - 2\mu)Z^{-1}LZ^{-1} = 0$$

showing that, for the particular δS in (G-16), $S+\delta S$ has purely imaginary eigenvalues when (G-15) becomes equality and becomes completely unstable when < becomes > in (G-15), i.e., for $\mu > 1/2$.

Remarks

The estimate (G-15) does not appear to be independent of the matrix L. It may, therefore, be true that some choices by L yield better estimates than others for various purposes.

If we take

$$\delta S = \lambda I$$

(G-15) becomes

$$|\lambda|^2 I < 1/4\, L^{1/2} Z^{-1} LZ^{-1} L^{1/2}$$

or

$$|\lambda|\, I < 1/2\, L^{1/2} Z^{-1} L^{1/2}$$

showing that $S + \lambda I$ remains a stability matrix if $|\lambda| < \mu_1 \equiv 1/2 \times$ (smallest eigenvalue of $L^{1/2} Z^{-1} L^{1/2}$). This means that all eigenvalues of S lie to the left of the line $\mathrm{Re}(\lambda) = -\mu_1$. This result is similar to one obtained by Wonham.[39]

INSENSITIVITY WITH RESPECT TO EXTERNAL DISTURBANCES

We consider the system (G-6) with white noise disturbance. Letting $\tilde{X}(t)$ be the covariance matrix for the vector x(t), we find that

$$\dot{\tilde{X}} = S\tilde{X} + \tilde{X}S^* + V$$

where, as indicated following (G-6), V is the covaraince matrix for V. Since S is a stability matrix, for any value of $\tilde{X}(0)$ we have

$$\lim_{t\to\infty} \tilde{X}(t) = X$$

where X satisfies

$$SX + XS^* + V = 0$$

A natural way in which to assess the insensitivity of $\dot{x} = Sx + v$ to the white noise disturbance v would seem to be afforded by comparison of X and V. Let us define the index of insensitivity with respect to v to be the largest positive number μ_o for which

$$\mu_o X \le V$$

or equivalently

$$\mu_o I \le V^{1/2} X^{-1} V^{1/2} \tag{G-17}$$

Now (G-15) can be rewritten as

$$(L^{1/2}\ \delta S^* L^{-1/2})\ (L^{-1/2}\ \delta S L^{1/2}) \le 1/4\ (L^{1/2}\ Z^{-1}\ L^{1/2})^2$$

and if we let μ_1 be the largest number for which

$$2\mu_1 I \le L^{1/2}\ Z^{-1}\ L^{1/2}$$

then $S+\delta S$ remains a stability matrix if

$$(L^{1/2}\delta S^* L^{-1/2})\ (L^{-1/2}\delta S L^{1/2}) < \mu_1^{\ 2}$$

or

$$\delta S_L^*\ \delta S_L < \mu_1^{\ 2}$$

where

$$\delta S = L^{1/2}\delta S_L L^{-1/2} \tag{G-18}$$

It would seem natural to refer to μ_1 as the index of insensitivity with respect to parameter variations of the form (G-18). Since X of the present discussion reduces to Z of Equation (G-11) when V of the present discussion becomes L of Equation (G-11), $2\mu_1$ should be compared with μ_o in (G-17). Hence we have the result expressed in proposition 2.

Proposition 2: For the disturbed system (G-6), the index μ_o of insensitivity with respect to external white noise disturbance v with covariance cov (v(t), v(t)) = $\delta(t-s)$ satisfies

$$\mu_o = 2\mu_1$$

where μ_1 is the index of insensitivity with respect to parameter variations

$$\delta S = L^{1/2} \delta S_L L^{-1/2}$$

TOWARD A THEORY OF MANEUVERABILITY

Let us now consider the system (G-7).

Here C is a fixed n x p matrix. For a fixed time T > 0, the condition that w should steer the system from the initial state

$$x(0) = 0 \tag{G-19}$$

to a desired final state

$$x(T) = x_1 \tag{G-20}$$

is, from the variation of parameters formula,

$$x_1 = \int_o^T e^{S(T-t)} C\, w(t)\, dt \tag{G-21}$$

Let us assume now that the plant operator uses a (possibly time varying) control mode linearly dependent on the desired final state:

$$w(t) = w(t, x_1) = A(T, t)x_1$$

Then Equation (G-19) is satisfied for all $x_1 \epsilon R^n$ just in case

$$\int_o^T e^{S(T-t)} CA(T, t)\, dt = I \tag{G-22}$$

It seems fairly plausible that human operators may, in fact, operate plants in such a manner, at least after they have adjusted to the plant's characteristics. Just what the matrix function $A(T, t)$ would be in any particular case would have to be determined by lengthy experiment.

In discussing maneuverability, however, it seems reasonable to consider that control mode which corresponds to least control effort in some sense. That control mode is obtained by letting

$$w(t) = C^* e^{S^*(T-t)}\, \xi, \quad \xi \varepsilon R^n \tag{G-23}$$

and substituting (G-23) into (G-21) with the result

$$\left(\int_o^T e^{S(T-t)}\, CC^* e^{S^*(T-t)} dt\right) \xi \equiv W(T)\, \xi = x_1 \tag{G-24}$$

Assuming that the pair (S, C) is controllable, $W(T)$ is positive define and we have

$$\xi = W(T)^{-1}\, x_1$$

$$w(t) = C^* e^{S^*(T-t)}\, W(T)^{-1}\, x_1 \equiv A_o(T, t)\, x_1 \tag{G-25}$$

The control "effort" is

$$\left[\int_o^T ||w(t)||^2 dt\right]^{1/2} = \left[x_1{}^* W(T)^{-1} \int_o^T e^{S(T-t)}\, CC^* e^{S^*(T-t)} dt W(T)^{-1} x_1\right]^{1/2}$$

$$= (\text{cf. (G-24)})\, [x_1{}^* W(T)^{-1}\, x_1]^{1/2} \tag{G-26}$$

and thus becomes small as $W(T)^{-1}$ becomes small, i.e., as $W(T)$ becomes large.

Now

$$SW(T) + W(T)S^* = \int_o^T - \frac{d}{dt}\, [e^{S(T-t)} CC^* e^{S^*(T-t)}]\, dt$$

$$= -CC^* + e^{ST}\, CC^* e^{S^*T}$$

so that W(T) satisfies

$$SW(T) + W(T)S^* + CC^* - e^{ST} CC^* e^{S^*T} = 0 \qquad \text{(G-27)}$$

For all $T > 0$ we have $0 < W(T) < W$, where

$$SW + WS^* + CC^* = 0 \qquad \text{(G-28)}$$

and, since S is a stability matrix,

$$\lim_{T \to \infty} e^{ST} CC^* e^{S^*T} = 0, \qquad \lim_{T \to \infty} W(T) = W$$

Let L be the matrix introduced in Equation (G-11).

Since L was assumed positive definite, there is a least $\mu > 0$ such that

$$\mu L \geq CC^*$$

and then

$$S(\mu Z - W) + (\mu Z - W) S^* = \mu L - CC^* \geq 0$$

and Lyapunov's theorem gives

$$\mu Z \geq W (> W(T) \text{ for any } T > 0)$$

and then

$$W(T)^{-1} > X^{-1} \geq 1/\mu \, Z^{-1}$$

We have seen that insensitivity with respect to parameter variations corresponds to keeping L large and Z^{-1} large. If L is large, μ is small and $1/\mu$ is large. So L large and Z^{-1} large makes $1/\mu \, Z^{-1}$ large. We see that $W(T)^{-1}$, which by (G-26) measures the least control effort to execute the maneuver (G-19) and (G-20), becomes large. Thus, we have less maneuverability in that a large control force is required to execute the maneuver.

We have shown, therefore, that the requirements for insensitivity with respect to parameter variations are inimical to the interests of maneuverability (a result which will surprise no one who has worked in the field, of course). Thus, control design (by which we mean the selection of K as described in (G-2)) must balance off these two factors. One seeks K such that the solution W of

$$SW + WS^* + CC^* = 0 \tag{G-28}$$

is not too small while at the same time the solution Z of

$$SZ + ZS^* + L = 0 \tag{G-11}$$

is not too large.

INSENSITIVITY WITH RESPECT TO COMMAND REQUIREMENTS

Consider the system (G-7) and also the perturbed system (G-8). We shall suppose that the plant operator has "learned" a control mode $A(T, t)$ for (G-7) for some $T > 0$; i.e., he knows how to form the command control force

$$w(t) = w(t, x_1) = A(T, t)\, x_1$$

for some $A(T, t)$ which satisfied (G-22) and hence is able to perform a maneuver $x(0) = 0$, $x(T) = x_1$.

We suppose now that the plant changes from (G-7) to (G-8). One then has

$$x(T) = \int_o^T e^{(S+\delta S)(T-t)}\, Cw(t)\, dt$$

If the operator continues to use the control mode $A(T, t)$, he will no longer steer from 0 to the desired target state x_1. Instead, with target x_1 he will reach

$$\hat{x}_1 = \int_o^T e^{(S+\delta S)(T-t)}\, CA(T, t)\, dt\, x_1$$

$$= \int_o^T e^{S(T-t)}\, CA(T, t)\, dt\, x_1$$

$$+ \int_o^T \left(e^{(S+\delta S)(T-t)} - e^{S(T-t)}\right) CA(T, t)\, dt\, x_1$$

$$= (I+E)\, x_1$$

where the operator

$$E = \int_0^T \left(e^{(S+\delta S)\,(T-t)} - e^{S(T-t)}\right) CA(T,t)\, dt \qquad \text{(G-29)}$$

represents his error at time T.

The expression (G-19) is rather difficult to deal with unless we consider limiting behavior as δS tends to zero. We define

$$\frac{\partial E}{\partial(\delta S)} = \lim_{\varepsilon \to 0} \int_0^T \left(\frac{e^{(S+\varepsilon\delta S)\,(T-t)} - e^{S(T-t)}}{\varepsilon}\right) CA(T,t)\, dt \qquad \text{(G-30)}$$

Since $E = 0$ when ε equals zero, we have

$$E(\varepsilon\delta S) = \varepsilon \frac{\partial E}{\partial(\delta S)} + o(\varepsilon), \quad \varepsilon \to 0 \qquad \text{(G-31)}$$

and $\frac{\partial E}{\partial(\delta S)}$ indicates the rate at which the error operator E grows as the plant dynamics are perturbed in the direction δS.

Now $e^{(S+\varepsilon\delta S)\tau}$ satisfies the differential equation

$$\frac{d}{d\tau}\left(e^{(S+\varepsilon\delta S)\tau}\right) = Se^{(S+\varepsilon\delta)\tau} + \varepsilon\delta Se^{(S+\varepsilon\delta S)\tau}$$

and reduces to the identity for $\tau = 0$. Regarding $\varepsilon\delta Se^{(S+\varepsilon\delta S)\tau}$ as an "inhomogeneous" term and invoking the variation of parameters formula, we have

$$e^{(S+\varepsilon\delta S)\,(T-t)} = e^{S(T-t)} + \int_0^{T-t} e^{S(T-t-\tau)}\, \varepsilon\delta Se^{(S+\varepsilon\delta S)\tau}\, d\tau$$

and then

$$\lim_{\varepsilon \to 0} \frac{e^{(S+\varepsilon\delta S)\,(T-t)} - e^{S(T-t)}}{\varepsilon} \qquad \text{(G-32)}$$

$$= \int_0^{T-t} e^{S(T-t-\tau)}\, \delta S\, e^{S\tau}\, d\tau$$

Consequently,

$$\frac{\partial E}{\partial(\delta S)} = \int_o^T \int_o^{T-t} e^{S(T-t-\tau)}\, \delta S\, e^{S\tau}\, d\tau\, CA(T,t)\, dt \tag{G-33}$$

$$= \int_o^T e^{S(T-t)} \int_p^{T-t} e^{-S\tau}\, \delta S\, e^{S\tau}\, d\tau\, CA(T,t)\, dt$$

One possible measure of the system's insensitivity with respect to command requirements is

$$\sup_{\|\delta S\| \neq 0} \frac{\left\| \frac{\partial E}{\partial(\delta S)} \right\|}{\|\delta S\|}$$

i.e., the maximum growth rate of the error operator $E(\varepsilon\delta S)$ as compared with that of the perturbation $\varepsilon\delta S$. Eventually we shall obtain an estimate for this quantity. But our real thrust here is to proceed a step further and let

$$A(\varepsilon\delta S,\ T,\ t) = A(T,t) + \varepsilon \frac{\partial A}{\partial(\delta S)}(T,t) + o(\varepsilon) \tag{G-34}$$

denote a corrected control mode, i.e., one for which

$$\int_o^T e^{(S+\varepsilon\delta S)(T-t)}\, CA(\varepsilon\delta S,\ T,\ t)\, dt = I \tag{G-35}$$

Subtracting (G-22) from (G-35) we have

$$0 = \int_o^T e^{(S+\varepsilon\delta S)(T-t)}\, CA(\varepsilon\delta S,\ T,\ t)\, dt - \int_o^T e^{S(T-t)}\, CA(t,\ t)\, dt$$

$$= \int_o^T e^{(S+\varepsilon\delta S)(T-t)} C\, [A(\varepsilon\delta S, T, t) - A(T,t)]\, dt$$

$$+ \int_o^T [e^{(S+\varepsilon\delta S)(T-t)} - e^{S(T-t)}]\, CA(T,t)\, dt$$

(cf. (G-31), (G-32), (G-34)

$$= \varepsilon \int_0^T e^{(S+\varepsilon\delta S)(T-t)} C \frac{\partial A}{\partial(\delta S)} (T,t)\, dt$$

$$+ \varepsilon \frac{\partial E}{\partial(\delta S)} + o(\varepsilon)$$

and dividing by ε and taking the limit as $\varepsilon \to 0$, we have

$$\int_0^T e^{S(T-t)} C \frac{\partial A}{\partial(\delta S)} (T,t)\, dt + \frac{\partial E}{\partial(\delta S)} = 0$$

which must be satisfied by the first order correction term in (G-34).

The least norm solution of the above equation is provided by setting

$$\frac{\partial F}{\partial(\delta S)} (T,t) = C^* e^{S^*(T-t)} \Gamma$$

yielding

$$\int_0^T e^{S(T-t)} CC^* e^{S^*(T-t)}\, dt\, \Gamma = -\frac{\partial E}{\partial(\delta S)}$$

or (cf. (G-24))

$$W(T)\Gamma = -\frac{\partial E}{\partial(\delta S)}\,, \quad \Gamma = -W(T)^{-1} \frac{\partial E}{\partial(\delta S)}$$

and thus

$$\frac{\partial F}{\partial(\delta S)} (T,t) = -C^* e^{S^*(T-t)} W(T)^{-1} \frac{\partial E}{\partial(\delta S)}$$

An appropriate norm for $\frac{\partial A}{\partial(\delta S)}$ is

$$\left\| \frac{\partial A}{\partial(\delta S)} \right\|^2_{L^2[0,T]} = \mathrm{Tr} \int_0^T \frac{\partial A}{\partial(\delta S)} (T,t)^* \frac{\partial A}{\partial(\delta S)} (T,t)\, dt$$

$$= \mathrm{Tr}\left[\left(\frac{\partial E}{\partial(\delta S)}\right)^* W(T)^{-1} \int_0^T e^{S(T-t)} CC^* e^{S^*(T-t)}\, dt\, W(T)^{-1} \left(\frac{\partial E}{\partial(\delta S)}\right)\right] \qquad \text{(G-36)}$$

$$= \mathrm{Tr}\left[\left(\frac{\partial E}{\partial(\delta S)}\right)^* W(T)^{-1} \left(\frac{\partial E}{\partial(\delta S)}\right)\right] = \mathrm{Tr}\left[W(T)^{-1} \left(\frac{\partial E}{\partial(\delta S)}\right)\left(\frac{\partial E}{\partial(\delta S)}\right)^*\right]$$

Thus, on the one hand, $\| \frac{\partial A}{\partial(\delta S)} \|^2$, which measures the minimal rate at which the control mode must be changed in order to adapt to the change from $\dot{x} = Sx+Cw$ to $\dot{x} = (S+\epsilon\,\delta S)x + Cw$, varies directly with $\|W(T)^{-1}\|$ (and becomes small as maneuverability is improved, large as maneuverability suffers) and, on the other hand, varies directly with $\|\frac{\partial E}{\partial(\delta S)}\|^2$.

Thus, whether we choose to define insensitivity with respect to command requirements in terms of $\frac{\partial A}{\partial(\delta S)}$ or $\frac{\partial E}{\partial(\delta S)}$ we shall have to estimate the latter in any case.

We take the formula (G-33) for $\frac{\partial E}{\partial(\delta S)}$ and effect a change of variable $\tau = s+t$ to get

$$\frac{\partial E}{\partial(\delta S)} = \int_0^T \int_t^T e^{S(T-\tau)}\delta S e^{S(\tau-t)} CA(T,t)\, d\tau\, dt$$

$$= \int_0^T e^{S(T-\tau)}\delta S \int_0^\tau e^{S(\tau-t)}\, CA(T,t)\, dt\, d\tau \tag{G-37}$$

Put

$$B(\tau) = \int_0^\tau e^{S(\tau-t)}\, CA(T,t)\, dt$$

and (G-37) becomes

$$\frac{\partial E}{\partial(\delta S)} = \int_0^T e^{S(T-\tau)}\, \delta S\, B(\tau)\, d\tau$$

and then, using the Schwartz inequality, we have

$$\| \frac{\partial E}{\partial(\delta S)} \|^2 = \mathrm{Tr}\, \left(\int_0^T \int_0^T e^{S(T-\tau)}\, \delta S\, B(\tau)\, B(\sigma)^*\, \delta S^* e^{S^*(T-\sigma)}\, d\sigma\, d\tau\right)$$

$$= \mathrm{Tr}\, \left(\int_0^T \int_0^T B(\tau)\, B(\sigma)^* \delta S^* e^{S^*(T-\sigma)} e^{S(T-\tau)}\, \delta S\, d\sigma\, d\tau\right)$$

$$\leq \mathrm{Tr}\, \left[\left(\int_0^T \int_0^T B(\sigma)\, B(\tau)^*\, B(\tau)\, B(\sigma)\, d\sigma\, d\tau\right)\right]^{1/2}$$

$$\times\left[\mathrm{TR}\left(\int_0^T\int_0^T \delta S^* e^{S^*(T-\tau)} e^{S(T-\sigma)}\,\delta S\delta S^* e^{S^*(T-\sigma)} e^{S(T-\tau)}\delta S\, d\sigma\, d\tau\right)\right]^{1/2} \tag{G-38}$$

$$= \left[\mathrm{Tr}\left(\int_0^T B(\tau)^* B(\tau)\, d\tau\right)^2\right]^{1/2} \left[\mathrm{Tr}\left(\int_0^T e^{S(T-\tau)}\,\delta S\,\delta S^* e^{S^*(T-\tau)}\, d\tau\right)^2\right]^{1/2}$$

Now for any positive definite symmetric matrix M with (positive) eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$, we have

$$(\mathrm{Tr}\, M^2)^{1/2} = \left(\sum_{k=1}^{n} (\lambda_k)^2\right)^{1/2} \leq \left(\left(\sum_{k=1}^{n} \lambda_k\right)^2\right)^{1/2} = \sum_{k=1}^{n} \lambda_k = \mathrm{Tr} M$$

so (G-38) implies

$$\left\| \frac{\delta E}{\delta(\delta S)} \right\|^2 \leq \left(\mathrm{Tr}\int_0^T B(\tau)^* B(\tau)\, d\tau\right)\left(\mathrm{Tr}\int_0^T e^{S(T-\tau)}\delta S\,\delta S^* e^{S^*(T-\tau)}\, d\tau\right) \tag{G-39}$$

Now

$$\mathrm{Tr}\int_0^T B(\tau)^* B(\tau)\, d\tau = \int_0^T \mathrm{Tr}\, B(\tau)^* B(\tau)\, d\tau$$

and, using the same reasoning as in the steps above,

$$\mathrm{Tr}\, B(\tau)^* B(\tau)\, d\tau = \mathrm{Tr}\left[\int_0^\tau\int_0^\tau e^{S(\tau-t)}\, CA(T,t)\, A(T,s)^*\, C^* e^{S^*(\tau-s)}\, ds\, dt\right]$$

$$\leq \left[\mathrm{Tr}\left(\int_0^\tau A(T,t)^* A(T,t)\, dt\right)^2\right]^{1/2} \left[\mathrm{Tr}\left(\int_0^\tau e^{S(\tau-t)}\, CC^* e^{S^*(\tau-t)}\, dt\right)^2\right]^{1/2}$$

$$\leq \mathrm{TR}\left(\int_0^\tau A(T,t)^* A(T,t)\, dt\right)\ \mathrm{TR}\left(\int_0^\tau e^{S(\tau-t)}\, CC^* e^{S^*(\tau-t)}\, dt\right)$$

$$\leq \mathrm{TR}\left(\int_0^T A(T,t)^* A(T,t)\, dt\right)\ \mathrm{Tr}\left(\int_0^\tau e^{S(\tau-t)}\, CC^* e^{S^*(\tau-t)}\, dt\right)$$

and returning to (G-39) we have

$$\left\| \frac{\delta E}{\delta(\delta S)} \right\|^2 \leq \mathrm{Tr}\left(\int_0^T A(T,t)^* A(T,t)\, dt\right)\left(\mathrm{Tr}\int_0^T e^{S(T-\tau)}\,\delta S\,\delta S^* e^{S^*(T-\tau)}\, d\tau\right)$$

$$\times \int_0^T \mathrm{Tr}\left(\int_0^\tau e^{S(\tau-t)} CC^* e^{S^*(\tau-t)}\, dt\right) d\tau$$

Now

$$\int_0^T \mathrm{Tr}\left(\int_0^\tau e^{S(\tau-t)} CC^* e^{S^*(\tau-t)}\, dt\right) d\tau$$

$$\le \int_0^T \mathrm{Tr}\left(\int_0^T e^{S(T-t)} CC^* e^{S^*(T-t)}\, dt\right) d\tau$$

$$= (\text{cf. (G-24)}) = T\,[\mathrm{Tr}\, W(T)]$$

and we finally have

$$\left\| \frac{\delta E}{\delta(\delta S)} \right\|^2 \le (T\ \mathrm{Tr}\,(W(T)))\ (\mathrm{Tr} \int_0^T A(T,t)^* A(T,t)\, dt)$$

$$\times (\mathrm{Tr} \int_0^T e^{S(T-\tau)}\, \delta S\ \delta S^* e^{S^*(T-\tau)}\, d\tau)$$

$$= T\,[\mathrm{Tr}\,(W(T))]\ \|A\|^2\ [\mathrm{Tr}\, Q(T)] \qquad \text{(G-40)}$$

where $Q(T)$ satisfies

$$SQ(T) + Q(T)S^* + \delta S\ \delta S^* - e^{ST}\, \delta S\ \delta S^* e^{S^*T} = 0$$

and therefore obeys the inequality

$$Q(T) \le Q, \quad T \ge 0$$

where

$$SQ + QS^* + \delta S\ \delta S^* = 0 \qquad \text{(G-41)}$$

Similarly we have $W(T) \le W$, where W satisfies (G-28) and we have the weaker but more convenient estimate

$$\left\| \frac{\delta E}{\delta(\delta S)} \right\|^2 \le T\ \mathrm{Tr}W\ \mathrm{Tr}Q\ \|A\|^2 \qquad \text{(G-42)}$$

Thus, if we measure insensitivity with respect to command requirements by means of $|| \frac{\partial E}{\partial(\delta S)} ||$, we want the product TrW TrQ $||A||^2$ to be small. Now, letting A_o be the least norm control defined in (G-25), we have

$$\text{TrW TrQ } ||A||^2 \geq \text{TrW TrQ } ||A_o||^2 \geq \text{TrW TrQ TrW}^{-1}$$

with equality holding in the first instance when $A = A_o$ and in the second instance when we let $T \rightarrow \infty$.

For many purposes we can assume that TrW and TrW^{-1} will balance each other out leaving us with something like

$$|| \frac{\partial E}{\partial(\delta S)} ||^2 \leq \gamma T \text{ Tr } (Q) \qquad \text{(G-43)}$$

in the case $A = A_o$, T large.

Hence insensitivity with respect to command requirements (ICR) amounts, in this instance, to the condition that Q, which is the unique positive definite solution of (G-41), should be small. We see that if we measure ICR in this way it will correspond to IPV with $\delta S\ \delta S^*$ taking the place of L in (G-11). At this point, therefore, we have established an essential unity of IPV, IED, and ICR if ICR is measured via $|| \frac{\partial E}{\partial(\delta S)} ||$. In this connection we should note how the control mode A enters the picture here. The estimate on $|| \frac{\partial E}{\partial(\delta S)} ||^2$ is directly proportional to $||A||^2$ showing that the manner in which the human operator carries out his maneuvers has some bearing on the sensitivity question. The more efficient his control mode, the more the system fulfills the ICR requirement as defined above.

As we have indicated earlier, however, our main thrust is to measure ICR in terms of $\frac{\partial A}{\partial(\delta S)}$. Returning to (G-36) we have

$$|| \frac{\partial A}{\partial(\delta S)} ||^2_{L^2[0,T]} \equiv \text{Tr} \int_o^T \frac{\partial A}{\partial(\delta S)} (T,t) \frac{\partial A}{\partial(\delta S)} (T,t)^* \, dt$$

$$\leq \text{Tr } (W(T)^{-1} \frac{\partial E}{\partial(\delta S)} (\frac{\partial E}{\partial(\delta S)})^*)$$

$$\le \mathrm{Tr}\,(W(T)^{-1})\ \mathrm{Tr}\,\left(\frac{\partial E}{\partial(\delta S)}\ \frac{\partial E}{\partial(\delta S)}^{*}\right)$$

$$= \mathrm{Tr}\,(W(T)^{-1})\ \left\|\frac{\partial E}{\partial(\delta S)}\right\|^{2}$$

If we accept (G-43) and assume T large enough so that $W(T)^{-1}$ is close to W^{-1}, we obtain an estimate of the form

$$\left\|\frac{\partial A}{\partial(\delta S)}\right\|^{2} \le \hat{\gamma}\, T\ \mathrm{Tr}\,(W^{-1})\ \mathrm{Tr}\,(Q)$$

for some $\hat{\gamma} > 0$. To fulfill the ICR requirement when given in terms of $\left\|\frac{\partial A}{\partial(\delta S)}\right\|^{2}$, we see that we want to

Maximize W: $SW + WS^{*} + CC^{*} = 0$

Minimize Q: $SQ + QS^{*} + \delta S\ \delta S^{*} = 0$

effecting in the end some compromise between these contradictory requirements. We see that ICR, as measured by $\left\|\frac{\partial A}{\partial(\delta S)}\right\|^{2}$, involves the same tradeoff which was introduced in connection with the conflicting requirements of maneuverability and insensitivity.

SUMMARY REMARKS ON THE THEORETICAL DEVELOPMENT

In the foregoing analysis, we have seen that there are a number of approaches to insensitivity controllers which make use of solutions of Lyapunov type equations. Three of these, IPV, IED, and ICR (measured by $\left\|\frac{\partial E}{\partial(\delta S)}\right\|$), have been seen to be essentially equivalent, and they are in some sense inimical to the interests of maneuverability. Because of this latter problem, it is the author's feeling that ICR (measured by $\frac{\partial A}{\partial(\delta S)}$) is the most realistic measure of insensitivity. This is reinforced by the fact that $\frac{\partial A}{\partial(\delta S)}$ represents, in however crude a sense, the degree to which the plant operator must be sensitive to parameter changes, measuring, as it does, the minimal necessary change in open-loop control strategy which the operator must effect to preserve the same maneuvering capability in the face of parameter fluctuations in the plant.

An obvious weak spot in our analysis is the dependence of our criteria upon a fixed time interval T--a dependence which we have tried to avoid by saying that if T is reasonably large and the eigenvalues of S are significantly displaced to the left of the imaginary axis then little error is introduced by taking $T = +\infty$. Clearly this device will be inadmissible when we are dealing with short time period maneuvers in a plant which is only slightly damped. In such situations, for example, we should feel obliged to use the solution W(t) of (G-27) rather than the solution W of (G-28). We are then immediately confronted with the question of which value of T to use. One possibility here would be to look for some approach which treats the time T in which a maneuver is to be performed as a random variable with an appropriate probability distribution assigned on the interval $(0, \infty)$.

RELATIONS BETWEEN THE LINEAR QUADRATIC AND LYAPUNOV EQUATION APPROACHES

The Lyapunov equation approach developed up to this point may be briefly summarized as follows. Two matrices, Z and W, were introduced as solutions of the Lyapunov equations, (G-11) and (G-28). To achieve insensitivity it is desired to choose the feedback gain matrix, K, so that Z is small and W is large.

In the linear quadratic approach, the feedback gain matrix, K, of the linear controller is chosen to minimize the quadratic performance index

$$J = \int_0^\infty (x^*Qx + u^*Ru)\, dt \qquad \text{(G-44)}$$

The resulting feedback gain matrix is given by

$$K = -R^{-1} G^*P \qquad \text{(G-45)}$$

where P is the unique positive definite solution of the Riccati matrix equation

$$0 = F^*P + PF + Q - PGR^{-1} G^*P \qquad \text{(G-46)}$$

The value of the performance index obtained with the optimal control is $x_o^*Px_o$. We may rewrite Equation (G-46) is the form

$$0 = (F - GR^{-1} G^*P)^*P + P(F-GR^{-1}G^*P) + Q + PGR^{-1} G^*P \qquad \text{(G-47)}$$

If we multiply (G-47) by P^{-1} on both sides, we obtain

$$\begin{aligned} 0 &= (F - GR^{-1} G^*P)P^{-1} + P^{-1} (F-GR^{-1}G^*P)^* + P^{-1}QP^{-1} + GR^{-1}G^* \\ &= (F + GK)P^{-1} + P^{-1} (F+GK)^* + P^{-1}QP^{-1} + GR^{-1}G^* \end{aligned} \tag{G-48}$$

Let Z be the solution of the Lyapunov equation

$$(F+GK)\, Z+Z\, (F+GK)^* + P^{-1}\, QP^{-1} = 0$$

with K given by (G-45). This Lyapunov equation is the same as (G-11) if we put $L = P^{-1}\, QP^{-1}$. Thus, $W = P^{-1} - Z$ satisfies

$$(F+GK)\, W+W\, (F+GK)^* + GR^{-1}\, G^* = 0 \tag{G-28'}$$

which is the same as (G-28) if $GR^{-1}G^* = CC^*$. Thus, the solution of (G-46) leads rather naturally to a pair of solutions of (G-11) and (G-28') and we have the relationship

$$P = (W+Z)^{-1}$$

In general, this process is not reversible. If we have (G-11) and (G-28) for stable F+GK, we can set.

$$L = (W+Z)\, Q\, (W+Z),\ P = (W+Z)^{-1}$$

and (G-11) and (G-28') add to give

$$(F+GK)P^{-1} + P^{-1}\, (F+GK)^* + P^{-1}\, QP^{-1} + GR^{-1}G^* = 0$$

Multiplying on the right and left by P gives

$$(F+GK)^*P+P\, (F+GK) + Q + PGR^{-1}\, G^*P = 0$$

which is not of the form (G-46) unless

$$K = -R^{-1}\, G^*P$$

Here $\|K+R^{-1}G^*\, (W+Z)^{-1}\|$ can be regarded as a measure of the departure of (G-11) and (G-28') from an optimal linear quadratic system.

Because the null space of $GR^{-1/2}$ is the orthogonal complement of the range of $R^{-1/2}G^*$, one may assume without loss of generality for any feedback matrix K that

$$K = -R^{-1} G^*\hat{P}$$

for some n x n matrix $\hat{P}$. If one now computes

$$- (F+GK)^*\hat{P} - \hat{P}^*(F+GK) + K^*RK =$$

$$- (F-GH^{-1} G^*\hat{P})^* \hat{P} - \hat{P}^* (F-GR^{-1}G^*\hat{P}) + \hat{P}^*GR^{-1} G^*\hat{P} = Q(\hat{P}) \qquad (G-49)$$

we have a Riccati equation of the form (G-46) if $Q(\hat{P})$ turns out to be positive definite and $\hat{P}$ is symmetric (and, *a fortiori*, positive definite if F+GK is stable). Modifying $\hat{P}$, and hence K, so as to increase $Q(\hat{P})$ tends to move the system toward the set of Riccati systems. Equation (G-49) is important for general K since the properties of the sensitivity index $\$(\lambda) = (I-R(\lambda)GK)^{-1}$ depend[39, 40] to a very large degree on $Q(\hat{P})$ where $R(\lambda) = (\lambda I-F)^{-1}$.

Our double Lyapunov equation approach has a number of relationships with the sensitivity analysis based on $\$(\lambda)$. The matrix $\$(\lambda)$ measures the ratio of output disturbances in the open-loop and closed-loop systems

$$\dot{x} = F(p)x + GK\, x_o(t), \quad x(0) = x_o \qquad (G-50)$$

$$\dot{\tilde{x}} = (F(p) + GK)\tilde{x}, \; \tilde{x}(0) = x_o \qquad (G-51)$$

where $x_o(t)$ is the solution of

$$\dot{x}_o(t) = (F(p_o) + G(p_o)K)\, x_o(t), \quad x_o(0) = x_o \qquad (G-52)$$

Equation (G-11) can be interpreted as providing the limiting (as $t \to \infty$) covariance matrix corresponding to a white noise disturbance with covariance $\delta(t-s)L$.

In this sense (G-11) defines a "Lyapunov transform" relating input, v, and output, x, in a particular statistical manner. It compares with the relationship

$$\tilde{x}(s) = (sI - (F+GK))^{-1}\, \tilde{v}(s)$$

based on the Laplace transform approach but, whereas the vector functions $\tilde{x}(s)$, $\tilde{v}(s)$ lie in infinite dimensional spaces, Z and L lie in finite $(\frac{n(n+1)}{2})$ dimensional spaces. In both cases transient effects are neglected. With this comparison in mind we can use (G-11) to study insensitivity in the "Lyapunov transform" setting much as one uses $\$(\lambda)$ to study it in the Laplace transform setting. Suppose we start with a stable nominal open-loop system $\dot{x} = F(p_o)x$ and solve the corresponding version of (G-11),

$$FZ + ZF^* + L = 0 \qquad \text{(G-53)}$$

Introducing a small parameter variation $\delta F = F(p) - F(p_o)$, we produce a corresponding variation δZ in Z which satisfies

$$F\delta Z + \delta Z F^* + \delta F Z + Z\delta F^* = 0 \qquad \text{(G-54)}$$

With feedback $u = Kx$, one can solve

$$(F+GK)\, Y + Y\, (F+GK)^* + L = 0 \qquad \text{(G-55)}$$

but, to mimic the line of reasoning which is followed with respect to the sensitivity matrix $\$(\lambda) = (I-R(\lambda)GK)^{-1}$, we adjust the "input" L so that the "responses" Z and Y agree when $\delta F = 0$. This means we replace L by

$$\hat{L} = L - GKZ - ZK^*G^* \qquad \text{(G-56)}$$

Then the solution $\hat{Y}$ of

$$(F+GK)\, \hat{Y} + \hat{Y}\, (F+GK)^* + \hat{L} = 0$$

is given by $\hat{Y} = Z$. (We remark that $\hat{L}$ will remain positive definite if K is not too large.) Bringing in the parameter variation δF again we produce a change $\delta\hat{Y}$ in $\hat{Y}$,

$$(F+GK)\, \delta\hat{Y} + \delta\hat{Y}(F+GK)^* + \delta F Y + Y\delta F^* = 0$$

which, since $\hat{Y} = Z$, is just

$$(F+GK)\, \delta\hat{Y} + \delta\hat{Y}\, (F+GK)^* + \delta F Z + Z\delta F^* = 0 \qquad \text{(G-57)}$$

The "adjustment" (G-56) in $\hat{L}$ allows us to have the same inhomogeneous term $\delta F Z + Z\delta F^*$ in both (G-54) and (G-57). The criterion for insensitivity improvement should now be that $\delta\hat{Y}$ is smaller, in some sense, than δZ. A really satisfactory criterion for smallness here remains to be developed, but we can give a fairly persuasive argument. Suppose K is selected so that the solution Y of (G-55) satisfies

$$Y \leq \rho Z, \quad 0 < \rho < 1 \tag{G-58}$$

where Z is the solution of the Equation (G-53).

Let $\mu \geq 0$ be selected so that

$$\delta F Z + Z \delta F^* \leq \mu L \tag{G-59}$$

for all δF in a certain class, say $\delta F \delta F^* \leq \mathcal{E} I$. Then, comparing (G-54) and (G-53) and using (G-59), we have, for all such δF,

$$\delta Z \leq \mu Z \tag{G-60}$$

On the other hand, comparing (G-57) and (G-55) and using (G-58) and (G-59, we have

$$\delta\hat{Y} \leq \mu Y \leq \rho\mu Z = \rho\mu\hat{Y} \tag{G-61}$$

for all such δF, which is an improved bound as compared with (G-60). Thus we see that selecting K so as to reduce Y as compared with Z has as a consequence a type of insensitivity improvement comparable in a sense to the more familiar one based on the Laplace transform approach and the sensitivity matrix $\$(\lambda)$.

NON-SPECIFIC APPROACHES TO INSENSITIVITY IMPROVEMENT

By "non-specific" approaches to insensitivity improvement we mean methods which do not take into account the specific form of the perturbations encountered. An example coming to mind very quickly is that of linear quadratic control. It is very well known that heavier weighting on Q, as compared with R, leads to a less sensitive system. Convincing arguments to this effect are given in References 39 and 40.

We present here an alternate, but related, approach to non-specific insensitivity improvement. We have noted that if we let L be an n x n symmetric positive definite matrix and solve (G-11) for Z, which will also be symmetric and positive definite if S is stable, then $S+\delta S$ will remain stable, provided only that

$$\delta S^* L^{-1} \delta S \leq (1/4) Z^{-1} L Z^{-1}$$

If we do not have any specific ideas about the form of δS, it is clear that our objective should be to make Z small relative to L or, equivalently, Z^{-1} large relative to L^{-1}. The method which we will now describe does precisely this by making Z smaller while keeping L fixed. The method is also notable for the light which it sheds on the relationship between the Equations (G-11) and (G-28').

For ease of discussion let us assume that $G(p) \equiv G(p_o) \triangleq G$, that is, G is known with certainty. We keep L fixed and allow the feedback matrix $K = K(s)$ and the solution matrix $Z = Z(s)$ of

$$[F(p_o) + GK(s)]\, Z(s) + Z(s)\, [F(p) + GK(s)]^* + L = 0 \qquad \text{(G-62)}$$

to depend on a scalar parameter s. We shall require K(s) to be differentiable with respect to s. It is then easy to show that Z(s) is also differentiable in any interval in which $F(p_o) + GK(s)$ remains stable. Differentiating (G-62) with respect to s, we have

$$[F(p_o) + GK(s)]\, \frac{dZ}{ds} + \frac{dZ}{ds}\, [F(p_o) + GK(s)]^* + G\, \frac{dK}{ds}\, Z(s) + Z(s)\, \frac{dK^*}{ds}\, G^* = 0 \qquad \text{(G-63)}$$

To make Z(s) move toward smaller values as s increases, we need to have $\frac{dZ}{ds}$ negative definite. There are a number of ways in which this can be done but certainly the most obvious is to put

$$\frac{dK}{ds} = -R^{-1} G^* Z(s)^{-1} \qquad \text{(G-64)}$$

so that (G-63) becomes

$$[F(p_o) + GK(s)]\, \frac{dZ}{ds} + \frac{dZ}{ds}\, [F(p_o) + GK(s)]^* - GR^{-1}G^* = 0 \qquad \text{(G-65)}$$

Comparing (G-65) with (G-28') we conclude that the choice (G-64) for $\frac{dK}{ds}$ leads to

$$\frac{dZ}{ds} = -2\, W(s) \qquad \text{(G-66)}$$

where (cf. (G-28'))

$$[F(p_o) + GK(s)]\, W(s) + W(s)\, [F(p_o) + GK(s)]^* + GR^{-1}G^* = 0 \qquad \text{(G-67)}$$

Since W(s) is positive definite, $\frac{dZ}{ds}$ is negative definite and Z(s) is made smaller while no change is made in L. Thus the two differential Equations (G-64) and (G-66) together with the Equations (G-62) and (G-67) constitute a non-specific method for insensitivity improvement.

There are very definite limitations on how much improvement can be effected by this method. With K(s), Z(s) being the gain matrix and Lyapunov solution matrix, we know that $F(p_o) + \delta F + GK(s)$ is stable where $\delta F = F(p) - F(p_o)$ satisfies

$$\delta F^* L^{-1} \delta F \leq (1/4) Z(s)^{-1} L Z(s)^{-1} \tag{G-68}$$

If we select δF in such a way that (F(p), G) becomes an uncontrollable pair, then, for some $\lambda_o \geq 0$, $(F(p_o) + \delta F + \lambda_o I, G)$ is not stabilizable and we cannot have

$$(\delta F + \lambda_o I) L^{-1} (\delta L^* + \lambda_o I) < (1/4)\, Z(s)^{-1}\, L Z(s)^{-1}$$

for any value of s. For L = I we see that this limits the degree to which the largest eigenvalue of Z(s) can be reduced.

It is always true that one does not get something for nothing. The above method of defining K(s) makes Z(s) monotone decreasing. But clearly $Z(s) \geq 0$ for all s. We conclude therefore that

$$\lim_{s \to \infty} \frac{dZ}{ds} = -2 \lim_{s \to \infty} W(s) = 0$$

But W(s) measures the maneuverability or degree of controllability of the system. So it is clear that we would never want to push s to $+\infty$ in this method. Also, since

$$\frac{dK}{ds} = -G^* Z(s)^{-1}$$

it is clear that the gain matrix will tend to become large as $s \to +\infty$. In implementing the method, it seems reasonable that one should decide upon a priori bounds ρ_K, ρ_W and stop the process as soon as

$$\|K(s)\| \geq \rho_K$$

or

$$W(s)^{-1} \geq \rho_W I \qquad (I = n \times n \text{ identity matrix})$$

Since the rate at which $Z(s)$ can be decreased is equal to $-2W(s)$, we see that, at the outset, it is not only important that $Z(0)$ should be small but also that $W(0)$ should be large. We are therefore right back to the original problem with respect to selection of $K(0)$: we want $K(0)$ to be such that $Z(0)$ is not too large while $W(0)$ is not too small.

We remark that (G-62), (G-64), (G-66), and (G-67) can be solved numerically by familiar methods. The Bartels-Stewart method[41] is particularly appropriate for (G-62) and (G-67) since the same matrix, $F(p_o) + GK(s)$, is involved in both cases.

A further question which occurs rather naturally is this. If we follow the above procedure, do we decrease sensitivity as measured by the so-called "sensitivity matrix"?

As one examines the theory of this matrix as it related to linear quadratic regulator theory (see, for example, Reference 39 and in particular Reference 40), we observe the following. First of all we may, without any loss of generality, assume

$$K(0) = -R^{-1}G^*\hat{P}(0)$$

for some $\hat{P}(0)$. Then letting

$$\hat{P}(s) = \hat{P}(0) + \int_o^s Z(s)^{-1}\, ds \tag{G-69}$$

we have

$$K(s) = -R^{-1}G^*\hat{P}(s)$$

We now set

$$Q(s) = -F(p_o)^*P(s) - \hat{P}(s)^*F(p_o) + \hat{P}(s)^*GR^{-1}G^*\hat{P}(s) \tag{G-70}$$

Following the approach used in Reference 39, we see that

$$(S(i\omega)^*)^{-1} K(s)^*K(s)S^*(i\omega)^{-1} \geq K(s)^* [I+G^*R(i\omega)^* (G(s) + i\omega(\hat{P}(0) - \hat{P}(0)^*)) R(i\omega)G] K(s)$$

The criterion for improvement of insensitivity in this setting is that Q(s) should be positive, and the larger Q(s) is, the better. Now, differentiating (G-69), we have

$$\frac{dQ}{ds} = -F^*Z(s)^{-1} -Z(s)^{-1} F+Z(s)^{-1}GR^{-1}G^*\hat{P}(s) + \hat{P}(s)^*GG^*Z(s)^{-1}$$

$$= -(F-GK(s))^*Z(s)^{-1} -Z(s)^{-1} (F-GK(s))$$

But, multiplying (G-67) on the left and right by $Z(s)^{-1}$ then shows that

$$\frac{dQ}{ds} = Z(s)^{-1} L Z(s)^{-1} \qquad \text{(G-71)}$$

and therefore Q(s) increases, the rate being greater as $Z(s)^{-1}$ becomes greater, i.e., as Z(s) becomes smaller. Thus our method also improves sensitivity as measured by the sensitivity matrix $\$(\lambda)$.

The above reasoning also shows that if Q(0) is positive definite, corresponding to

$$K(0) = -R^{-1}G^* \hat{P} (0)$$

which is the optimal control for the cost

$$\int_0^\infty (x^*Q(0)x + u^*Ru) \, dt$$

then

$$K(s) = -R^{-1}G^* P(s)$$

remains the optimal control relative to

$$\int_0^\infty (x^*Q(s)x + u^*Ru) \, dt$$

Thus the class of optimal linear quadratic controls is invariant under our method and, in the context of such systems, corresponds to increasing Q(s) in a particular way, namely (G-71). To express this in a different way, suppose we have a symmetric positive solution P(s) of the Riccati equation

$$F^*P(s) + P(s)F + Q(s) - P(s)GR^{-1}G^*P(s) = 0 \tag{G-72}$$

corresponding to the feedback control law

$$u = -R^{-1} G^*P(s)x$$

We now let Z(s) be the solution of

$$(F-GR^{-1} G^*P(s))\, Z(s) + Z(s)\, (F-GR^{-1}G^*P(s))^* + L = 0 \tag{G-73}$$

or, equivalently,

$$(F-GR^{-1}G^*P(s))^*\, Z(s)^{-1} + Z(s)^{-1}\, (F-GR^{-1} G^*P(s)) + Z(s)^{-1}\, LP(s)^{-1} = 0$$

In (G-73) we let

$$\frac{dP}{ds} = Z(s)^{-1}, \quad \frac{dQ}{ds} = Z(s)^{-1}\, L\, Z(s)^{-1} \tag{G-74}$$

Then (G-73) remains invariant; i.e.,

$$\frac{d}{ds}\, [F^*P(s) + P(s)F + Q(s) - P(s)\, GR^{-1}G^*P(s)] \equiv 0$$

This means that the differential Equations (G-74) combined with (G-73) provide a methodical way of moving through the class of Riccati Equations (G-72) in a manner which

(1) Decreases sensitivity as measured by the sensitivity matrix

(2) Decreases sensitivity as measured by our criterion in the sense that in

$$(F-GR^{-1}G^*P)\,(s))\, Z(s) + Z(s)\, (F-GR^{-1}G^*P(s))^* + L = 0$$

the matrix L remains fixed but Z(s) becomes progressively smaller.

Now property (1) can be made true in Riccati Equations (G-73) just be increasing Q in any manner so that (1) is not all that startling. But, as noted previously, just increasing Q in an arbitrary manner does not, in general, allow one to conclude anything similar to (2). So what we have appears to be a particularly advantageous way to vary the weighting matrix Q in Riccati equations. Our method is also more general since the system of Equations (G-62), (G-64), (G-66), and (G-67) (which leaves (G-72) invariant and accomplishes (2) above) makes sense for any starting gain matrix K(0), whether derived from a Riccati equation or not.

"SPECIFIC" APPROACHES TO INSENSITIVITY IMPROVEMENT

By a "specific" approach to insensitivity improvement, we mean a method which is geared to the perturbations which are known to affect the system at hand. Thus, we consider the nominal system

$$\dot{x} = F(p_o)x + Gu, \ x(0) = x_o, \ x \epsilon R^n, \ u \epsilon R^n \tag{G-75}$$

and the perturbed system

$$\dot{x} = F(p)x + Gu = (F(p_o) + \delta F)x + Gu, \ x(0) = x_o \tag{G-76}$$

and we suppose that there are finitely many matrices $F_1, F_2, \ldots, F_\ell$ such that δF can be written

$$\delta F = \sum_{i=1}^{\ell} \alpha_i F_i, \ \alpha_i \geq 0, \quad i = 1, 2, \ldots, \ell \tag{G-77}$$

and we will suppress the dependence of $F(p_o)$ on p_o. We suppose now that K is an m x n matrix such that F+GK is stable, and we solve

$$(F+GK)\,Z+Z\,(F+GK)^* + L = 0 \tag{G-11'}$$

for some positive definite symmetric matrix L. Then we also have

$$(F + \sum_{i=1}^{\ell} \alpha_i F_i + GK)\,Z+Z\,(F + \sum_{i=1}^{\ell} \alpha_i F_i + GK)^* + L - \sum_{i=1}^{\ell} \alpha_i (F_i Z + Z F_i^*) = 0$$

A sufficient condition for $F + \sum_{i=1}^{\ell} \alpha_i F_i + GK$ to be stable is that

$$L - \sum_{i=1}^{\ell} \alpha_i(F_iZ + ZF_i^*) > 0 \qquad \text{(G-78)}$$

We assume the matrix F_i to be scaled in such a way that the range of likely system matrix perturbations is the convex set

$$\left\{ \sum_{i=1}^{\ell} \alpha_i F_i \,\middle|\, \alpha_i \geq 0,\ i = 1, 2, \ldots, \ell,\ \sum_{i=1}^{\ell} \alpha_i \leq 1 \right\}$$

Now (G-78) is true for $\alpha_i \geq 0$, $i = 1, 2, \ldots, \ell$, $\sum_{i=1}^{\ell} \alpha_i \leq \alpha$ if and only if the matrices $L - \alpha(F_iZ + ZF_i^*)$ are positive definite for $i = 1, 2, \ldots, \ell$. Our immediate objective can be taken to be (for fixed L and K) to find:

$$\alpha_o = \text{maximum } \alpha \qquad \text{(G-79)}$$

such that

$$L - \alpha(F_iZ + ZF_i^*) \geq 0, \quad i = 1, 2, \ldots, \ell \qquad \text{(G-80)}$$

subject to (G-11).

Let $L = MM^*$. Then (G-80) becomes

$$M(I - \alpha[M^{-1} F_iZ(M^{-1})^* + M^{-1} ZF_i^* (M^{-1})^*]) M^* \equiv M(I - \hat{F}_i) M^* \geq 0,\ i = 1, 2, \ldots, \ell \qquad \text{(G-81)}$$

and we see that

$$\alpha_o = [\max_{i \in \lambda_i > 0} \{\lambda_i\}]^{-1} \qquad \text{(G-82)}$$

where λ_i is the largest eigenvalue of the matrix $\hat{F}_i$ which has been defined in formula (G-80). Having determined α_o, which is a standard problem of numerical matrix theory, we let

$$I = \{ i \mid \frac{1}{\lambda_i} = \alpha_o \}$$

and we let ϕ_{ij}, $i \in I$, $j = 1, 2, \ldots, m_i$ be the eigenvectors of $\hat{F}_i$ corresponding to the eigenvalue λ_i. Here m_i is the multiplicity of λ_i. Since $\hat{F}_i$ is symmetric, the number of eigenvectors corresponding to λ_i equals m_i and, in addition, we may assume orthonormality:

$$\phi_{ik}^* \phi_{ij} = \delta_j^k = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$$

Now, suppose a change $K \rightarrow K+\delta K$ is effected in the feedback matrix with a resultant change $Z \rightarrow Z+\delta Z$ occurring in Z. Then $\hat{F}_i$(cf. (G-80)) is changed to

$$\hat{F}_i + \delta\hat{F}_i = \hat{F}_i + M^{-1} [F_i \delta Z + \delta Z F_i^*] (M_o^{-1})^* \qquad \text{(G-83)}$$

The question of prime interest to us is this: how do the λ_i, $i\epsilon I$ (and hence α_o) change? We must divide our answer into two parts, according to the multiplicity of λ_i.

If λ_i has multiplicity 1, it is easily established to first order

$$\delta\lambda_i = \phi_i^* \, \delta\hat{F}_i \, \phi_i$$

If this multiplicity condition holds for all $i\epsilon I$, our goal of increasing α_o can be achieved if

$$\max_{i\epsilon I} \; \phi_i^* \, \delta\hat{F}_i \, \phi_i < 0 \qquad \text{(G-84)}$$

In (G-83) we have indicated that $\delta\hat{F}_i$ arises from a change δZ in Z which arises in turn from a change in the feedback matrix K, K passing to $K+\delta K$. Now δZ and δK are approximately related by

$$(F+GK)\,\delta Z+\delta Z\,(F+GK)^* + G\,\delta K Z + Z\delta K^* G^* = 0 \qquad \text{(G-85)}$$

Thus the requirement (G-84) becomes

$$\max_{i\epsilon I} \; \phi_i^* \, [M^{-1} F_i \, \delta Z \, (M^{-1})^* + M^{-1} \, \delta Z F_i^* \, (M^{-1})^*] \, \phi_i < 0$$

subject to δZ, δK obeying (G-85). Now let the permissible variations in K be linear combinations of matrices K_ν, e.g., $K = \sum_{\nu=1}^{r} \beta_\nu K_\nu$, and set

$$\delta K = \mathcal{E} \sum_{\nu=1}^{r} \mathcal{E}_\nu K_\nu$$

(Note that here we allow for the possibility of limited observational ability.) Define matrices Z_ν by

$$(F+GK)\,Z_\nu + Z_\nu\,(F+GK)^* + GK_\nu\,Z + ZK_\nu{}^*G^* = 0$$

Then the steepest descent method for varying K becomes

$$\min_{\substack{(\varepsilon_1,\ \dots,\ \varepsilon_r)\\ (\varepsilon_1)^2+\dots+(\varepsilon_r)^2\le 1}} \left(\max_{i\in I} \left(\sum_{\nu=1}^{r} \varepsilon_\nu\, \phi_i{}^* \,[M^{-1}F_i\, Z_\nu (M^{-1})^* + M^{-1}\, Z_\nu\, F_i{}^*\, (M^{-1})^*]\phi_i \right)\right)$$

and the requirement for being able to improve insensitivity is that this quantity be negative. With

$$\gamma_{i\nu} \triangleq \phi_i{}^*\,[M^{-1}\, F_i Z_\nu\, (M^{-1})^* + M^{-1} Z_\nu F_i{}^*\, (M^{-1})^*]\,\phi_i$$

this reads

$$\min_{\substack{(\varepsilon_1,\ \dots,\ \varepsilon_r)\\ (\varepsilon_1)^2+\dots+(\varepsilon_r)^2\le 1}} \left(\max_{i\in I} \sum_{\nu=1}^{r} \varepsilon_\nu\, \gamma_{i\nu}\right)$$

This can be reduced to a standard mathematical programming problem by introducing a non-negative parameter γ and solving

$$\max \gamma$$

subject to

$$\begin{aligned} &\gamma + \sum_{\nu=1}^{r} \varepsilon_\nu\, \gamma_{i\nu} \le 0, \quad i\in I,\\ &(\varepsilon_1)^2 + \dots + (\varepsilon_r)^2 \le 1 \end{aligned} \qquad \text{(G-86)}$$

For computational purposes it might be well to replace the last condition by

$$|\varepsilon_\nu| \le 1, \qquad \nu = 1,\ 2,\ \dots,\ r$$

which leaves us with a linear programming problem.

Once the mathematical programming problem has been solved and we have $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r$, we let

$$\delta K = \mathcal{E} \left(\sum_{\nu=1}^{r} \mathcal{E}_\nu K_\nu \right)$$

where $\mathcal{E}$ is an appropriately small positive "step size" and we have a new feedback matrix

$$K_1 = K + \delta K = K + \mathcal{E} \left(\sum_{\nu=1}^{r} \mathcal{E}_\nu K_\nu \right)$$

Our analysis guarantees that, provided $\mathcal{E} > 0$ is sufficiently small, the number α_1 corresponds to K_1 as α_o (cf. (G-79)) did to K, and satisfies

$$\alpha_1 > \alpha_o$$

The whole procedure outlined above is now repeated with K replaced by K_1 to obtain $K_2 = K_1 + \delta K_1$ with $\alpha_2 > \alpha_1 > \alpha_o$, etc. We iterate to obtain a sequence of feedback matrices K, K_1, K_2, K_3, ... and a corresponding monotonic sequence $\alpha_o < \alpha_1 < \alpha_2 < \alpha_3 < \ldots$. The process stops when $\alpha_i \geq 1$ or when it is no longer possible to increase α_i by a significant amount.

The above procedure must be modified somewhat if the eigenvalue λ_i of $\hat{F}_i$ has multiplicity $m_i > 1$. The formula

$$\delta\lambda_i = \phi_i^* \, \delta\hat{F}_i \, \phi_i$$

obviously no longer holds since every unit vector ϕ in the m_i-dimensional eigenspace corresponding to λ_i is an eigenvector. About the most which can be said is that the m_i-fold eigenvalue λ_i of $\hat{F}_i$ passes into the m_i eigenvalues of the m_i x m_i matrix

$$\begin{pmatrix} \phi_1^* \\ \vdots \\ \phi_{m_i}^* \end{pmatrix} (\hat{F}_i + \delta\hat{F}_i) (\phi_1, \ldots, \phi_{m_i}) \equiv \Phi_i^* (\hat{F}_i + \delta\hat{F}_i) \Phi_i$$

The condition that all of the eigenvalues of this matrix should be less than λ_i is that the matrix

$$\Phi_i^* \, \delta \hat{F}_i \, \Phi_i$$

should be negative definite. So to improve our value of α_o we need

$$\Phi_i^* \, \delta \hat{F}_i \, \Phi_i < 0, \quad i \epsilon I$$

Using K_ν, Z_ν as defined earlier, we now have the problem of selecting $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r$ so that each of the matrices

$$\sum_{\nu=1}^{r} \mathcal{E}_\nu \, \Phi_i^* M^{-1} [F_i Z_\nu + Z_\nu F_i^*] (M^{-1})^* \Phi_i$$

is negative definite. Defining the matrices $\Gamma_{i\nu}$ by

$$\Gamma_{i\nu} = \Phi_i^* M^{-1} [F_i Z_\nu + Z_\nu F_i^*] (M^{-1})^* \Phi_i$$

the problem is this: given the linear combinations of matrices

$$\Gamma_i(\mathcal{E}_1, \ldots, \mathcal{E}_r) = \sum_{\nu=1}^{r} \mathcal{E}_\nu \Gamma_{i\nu}, \; i \epsilon I$$

do there exist values of $\mathcal{E}_1, \ldots, \mathcal{E}_r$ such that all of the $\Gamma_i(\mathcal{E}_1, \ldots, \mathcal{E}_r)$ are negative definite?

Thus posed, we have a very difficult problem which we shall not attempt to solve. Instead, one proceeds as follows. Let a positive number γ be selected which is, roughly speaking, the minimal improvement in α_o which we consider worthwhile. For $i \epsilon I$ we select m_i dimensional unit vectors

$$\xi_1^i, \xi_2^i, \ldots, \xi_{\mu_i}^i$$

(One would likely start with $\mu_i = m_i$ and the ξ_j^i the usual basis for R^{m_i}.) We require

$$\sum_{\nu=1}^{r} \mathcal{E}_\nu \, \xi_j^{i*} \Gamma_{i\nu} \xi_j^i \geq -\gamma, \; i \epsilon I, \; j = 1, 2, \ldots, \mu_i$$

$$|\mathcal{E}_\nu| \leq 1, \quad \nu = 1, 2, \ldots, r \tag{G-87}$$

Whether or not this set is empty is a standard linear programming problem. If it is empty our procedure terminates and we accept whatever value of α_o we have. If the set is not empty, we denote it by $S(\xi_j^i,\ i\epsilon I,\ j = 1, \ldots, \mu_i) \equiv S$ and solve

$$\begin{aligned}&\min\ \mathcal{E}\\ &\mathcal{E}_\nu - \mathcal{E} \geq 0, \quad \nu = 1, \ldots, r \quad (\mathcal{E}_1, \ldots, \mathcal{E}_r)\ \epsilon S\end{aligned} \tag{G-88}$$

We then have the matrices $\Gamma_i(\mathcal{E}_1, \ldots, \mathcal{E}_r)$ satisfying (G-87). For each i we now compute the largest eigenvalue of $\Gamma_i(\mathcal{E}_1, \ldots, \mathcal{E}_r)$, call it θ_i. If $\theta_i < 0$, $\Gamma_i(\mathcal{E}_1, \ldots, \mathcal{E}_r)$ is negative definite. If $\theta_i > 0$, we adjoin to the set $\xi_1^i, \ldots, \xi_{\mu_i}^i$ the unit eigenvector corresponding to θ_i and increase μ_i by 1. Having done this for all i, either all $\Gamma_i(\mathcal{E}_1, \ldots, \mathcal{E}_r)$ are found to be positive, in which case our objective has been achieved and we proceed to change K as in the earlier single multiplicity case, or else we must go though the linear program (G-87), (G-88) again with a larger set of constraints corresponding to the new vectors we have adjoined. It can be shown that this procedure eventually terminates with (G-87) an empty set, in which case we are all through, or else with the $\Gamma_i(\mathcal{E}_1, \ldots, \mathcal{E}_r)$ all negative definite, in which case we improve our value of α_o.

We remark here that, in the generic case, only one matrix $\hat{F}_i$ will have an eigenvalue λ_i with $1/\lambda_i = \alpha_o$ and it will be of single multiplicity. So our discussion of the case of higher multiplicity might seem a bit of pedantry. However, the effect of our procedure is to decrease this eigenvalue while ignoring others so that, eventually, we must expect two or more eigenvalues which are equal. If and when they occur in the same matrix $\hat{F}_i$, we shall be confronted with the problem which we have been taking pains to discuss here.

Some Variations

We have noted that in the system of Equations (G-11) and (G-28') it is desirable to keep Z small and W large. There is a certain inconsistency in this to the extent that, since L is positive definite, we can find $\mu > 0$ such that

$$\mu L \geq GR^{-1}G^*$$

and we then clearly have $\mu Z \geq W$, assuming F+GK is a stability matrix. But it is not at all inconsistent to attempt to decrease the largest eigenvalue of Z while, at the same time, we attempt to increase the smallest eigenvalue of W, or perhaps, to avoid making this smallest eigenvalue of W any smaller. Nor is it inconsistent to attempt to carry out the mathematical program involving Z while at the same time we attempt to keep the smallest eigenvalue of W as far away from zero as we can. The methods for carrying this out are essentially the same as those used to deal with Z.

If we consider only the single perturbation (cf. (G-77))

$$F_1 = \frac{1}{2} I$$

the objective (G-79) and (G-80) becomes

$$\max \alpha$$

subject to (G-11) and

$$L - \alpha Z \geq 0$$

When L = I also, this objective amounts to minimization of the largest eigenvalue of Z. If we consider the perturbation

$$F_2 = \frac{1}{2} W^{-1} Z^{-1}$$

where W solves (G-28'), then the objective (G-79) and (G-80) becomes

$$\max \beta$$

subject to (G-28') and

$$L = \beta W^{-1} \geq 0$$

Again, when L = I, solution of this problem amounts to minimization of the largest eigenvalue of W^{-1}, i.e., maximization of the smallest eigenvalue of W. The two objectives can be combined in a fairly reasonable way by attempting to solve the problem

$$\min \frac{1}{\alpha\beta}$$

subject to (G-11) and (G-28') and

$$\alpha > 0, \ \beta > 0$$

$$L - \alpha Z \geq 0$$

$$L - \beta W^{-1} \geq 0$$

In the iterative procedure, we again let $L = MM^*$ (trivial, of course, if we take $L = I$) so that

$$L - \alpha Z = M(I - \alpha M^{-1} Z(M^{-1})^*) M^*$$

$$L - \beta W^{-1} = M(I - \beta M^{-1} W^{-1} (M^{-1})^*) M^*$$

We let ϕ_j, $j = 1, \ldots, m_1$ be the orthonormal eigenvectors corresponding to the largest eigenvalue, λ, of $M^{-1}Z(M^{-1})^*$ and we let ψ_j, $j = 1, \ldots, m_2$ be the orthonormal eigenvectors corresponding to the largest eigenvalue, μ, of $M^{-1} W^{-1} (M^{-1})^*$. Allowing perturbations

$$K \rightarrow K+\delta K = K+\varepsilon(\varepsilon_1 K_1 + \ldots + \varepsilon_r K_r)$$

as before, we let

$$(F+GK) Z_\nu + Z_\nu (F+GK)^* + GK_\nu Z + ZK_\nu^* G^* = 0 \tag{G-89}$$

$$(F+GK) W_\nu + W_\nu (F+GK)^* + GK_\nu W + WK_\nu^* G^* = 0 \tag{G-90}$$

We further note that

$$\delta W^{-1} = - W^{-1} \delta W W^{-1}$$

$$\delta\left(\frac{1}{\alpha\beta}\right) = \delta(\lambda\mu) = \lambda\delta\mu + \mu\delta\lambda$$

In the single multiplicity case we have

$$(\delta\lambda) = \phi^* (M^{-1} \delta Z (M^{-1})^*) \phi$$

$$(\delta\mu) = - \psi^* (M^{-1} W^{-1} \delta W W^{-1}) \lambda$$

and our objective becomes

$$\min_{\substack{\varepsilon_1, \ldots, \varepsilon_r \\ (\varepsilon_1)^2 + \ldots + (\varepsilon_r)^2 \leq 1}} \left(\max \left\{ \lambda \sum_{\nu=1}^{r} \varepsilon_\nu \phi^* (M^{-1} Z_\nu (M^{-1})^*) \phi - \mu \sum_{\nu=1}^{r} \varepsilon_\nu \psi^* (M^{-1} W^{-1} W_\nu W^{-1} (M^{-1})^*) \psi \right\} \right)$$

improvement will be realized if the min-max is negative. We are left with a mathematical programming problem of the same type as before and we obtain a linear program again if we replace $(\varepsilon_1)^2 + \dots + (\varepsilon_r)^2 \le 1$ by

$$|\varepsilon_\nu| \le 1, \quad \nu = 1, 2, \dots, r$$

The higher multiplicity case is treated as before.

If we wish to combine the objectives on Z with a requirement that the smallest eigenvalue of W should not be decreased, we would adjoin to (G-86) the constraint

$$\sum_{\nu=1}^{r} \varepsilon_\nu \psi^* W_\nu \psi \ge 0$$

where ψ is the unit eigenvector corresponding to the smallest eigenvalue of W. The W_ν are again determined by (G-90). Again modifications are required for the higher multiplicity case.

Finally, there is the question of what matrix L should be used in (G-11). This question is by no means trivial because it is entirely possible to have a stable matrix, F_o, a positive definite symmetric matrix Z and yet have

$$L = -F_oZ - ZF_o^*$$

not positive definite. Thus positivity of L in addition to that of Z is sufficient, but not necessary, for stability of F_o. This means, carried a little further, that the criteria for persistence of stability of $F + \delta F$ are somewhat pessimistic, i.e., in

$$(F + \delta F)\, Z + Z\, (F + \delta F)^* + L - \delta F Z - Z \delta F^* = 0$$

it may very well happen that $F + \delta F$ remains stable even through $L - \delta F Z - Z \delta F^*$ is not positive definite.

Suppose F_o is stable and diagonalizable so that there is a non-singular T with

$$T^{-1} F_o T = \mathrm{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n) \equiv \Lambda$$

Look at

$$\Lambda Y + Y\Lambda^* + I = 0 \qquad \text{(G-91)}$$

Then

$$Y = \text{diag}\left(\frac{-1}{2\text{Re}(\lambda_1)}\right), \quad \left(\frac{-1}{2\ \text{Re}\ (\lambda_2)}\right), \ \ldots, \quad \left(\frac{-1}{2\ \text{Re}\ (\lambda_v)}\right)$$

and we see that the smallest eigenvalue of $\frac{1}{2}\ Y^{-1}$ is precisely the stability margin for F_o. Now (G-91) is

$$T^{-1}\ F_o T\ Y{+}Y\ T^*\ F_o{}^*\ (T^{-1})^* + I = 0$$

and, multiplying by T on the left, T* on the right, we have

$$F_o\ (TYT^*) + (TYT^*)\ F_o^* + TT^* = 0$$

The stability margin remains the same: it is the smallest eigenvalue of

$$\frac{1}{2}\ Y^{-1} = \frac{1}{2}\ T^*\ (TYT^*)^{-1}\ T$$

i.e., the largest number λ such that

$$\lambda I \leq \frac{1}{2} T^*\ (TYT^*)^{-1} T$$

or, equivalently,

$$\lambda^2\ (TT^*)^{-1} \leq \frac{1}{2}\ (TYT^*)^{-1}\ TT^*\ (TYT^*)^{-1}$$

With

$$Z = TYT^*, \quad L = TT^* \tag{G-92}$$

this is just

$$\lambda^2\ L^{-1} \leq \frac{1}{2}\ Z^{-1}\ LZ^{-1}$$

which is the stability margin derived from our theory. Thus the choice (G-92) for L and resulting Z accurately reflect the stability properties of F_o.

We can take $F_o = F{+}GK(0)$, our initial closed-loop matrix. If we then choose L as above, our conclusions, based on the preceeding material in this Appendix, should not be excessively pessimistic for variations δF in F that are not too large. Choosing L to give the best estimate for large δF remains an open question.

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*U.S. GOVERNMENT PRINTING OFFICE: 1977 - 735-004/1