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CLOSED-LOOP STRUCTURAL STABILITY FOR LINEAR-QUADRATIC OPTIMAL SYSTEMS*

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Abstract

This paper contains an explicit parametrization of a subclass of linear constant gain feedback maps that will not destabilize an originally open-loop stable system. These results can then be used to obtain several new structural stability results for multi-input linear-quadratic feedback optimal designs.

1. Introduction and Motivation

This paper presents preliminary results which, in our opinion, represent a first necessary step in the systematic computer aided design of reliable control systems for multivariable control systems. A specific motivating example arises in the context of future high performance aircraft. It is widely recognized that advances in active control aircraft and control configured vehicles will require the automatic control of several actuators so as to be able to fly future aircraft characterized by reduced stability margins and additional flexure modes.

As a starting point for our motivation we must postulate that the design of future stability augmentation systems will have to be a multi-variable design problem. As such, traditional single-input-single-output system design tools based on classical control theory cannot be effectively used, especially in a computer aided design context. Since modern control theory provides a conceptual theoretical and algorithmic tool for design, especially in the Linear-Quadratic-Gaussian (LQG) context (see Athans [1] for example), it deserves a special look as a starting point in the investigation.

In spite of the tremendous explosion of reported results in LQG multivariable design, the robustness properties have been neglected. Experi-

ence has shown that LQG designs "work" very well if the mathematical models upon which the design is based are somewhat accurate. There are several sensitivity studies involving "small parameter perturbations" associated with the LQG problem. We submit, however, that the general problem of sensitivity and even stability of multivariable LQG designs under large parametric and structural changes is an open research area.

It is useful to reflect upon the basic methodology in classical servomechanism theory which dealt with such large parameter changes. The overall sensitivity and stability considerations were captured in the definition of gain and phase margins. If a closed-loop system was characterized by reasonable gain and phase margins, then

- (a) reasonable changes in the parameters of the open loop transfer functions
- (b) changes in the loop gains due, for example, to saturation and other non-linearities

could be accommodated with guaranteed stability and at the price of somewhat degraded performance.

Although LQG designs are time-domain oriented nonetheless their frequency-domain interpretations are important, although not universally appreciated. For example, for the case of single input single output linear-quadratic (LQ) optimal designs Anderson and Moore [2] have shown that LQ-optimal designs are characterized by

- (i) an infinite gain margin property
- (ii) a phase margin of at least 60 degrees.

Such results are valuable because it can be readily appreciated that at least in the single-input-single-output case, modern control theory designs tend to have a good degree of robustness, as measured by the classical criteria of gain and phase margin.

Advances in the multi-input-multi-output case however have been scattered and certainly have not arrived at the cookbook design stage. Multivariable system design is extremely complex*. To a

*This research was conducted in the M.I.T. Electronic Systems Laboratory with partial support extended by NASA Ames Research Center under grant NGL-22-000-114 and by the Air Force Office of Scientific Research under grant AF-AFOSR-72-2273.

*Even the notion of what constitutes a "zero" of a multivariable transfer matrix was not fully appreciated until recently.

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certain extent the numerical solution of LQ-optimal problems is very easy. However, fundamental understanding of the structural interdependencies and its interactions with the weighting matrices is not a trivial matter. We believe that such fundamental understanding is crucial for robust designs as well as for reliable designs that involve a certain degree of redundancy in control and sensors.

The recent S.M. thesis by Wong [3] represents a preliminary yet positive contribution in this area. In fact, the technical portion of the paper represents a slight modification of some of the results reported in [3]. In particular we focus our attention on the stability properties of closed loop systems designed on the basis of LQ-optimal techniques when the system matrices and loop gains undergo large variations.

The main contributions reported in this paper are the eventual results of generalizing the concepts of gain margin and of performing large-perturbation sensitivity analysis for multivariable linear systems designed via the LQ approach. For a corresponding generalization of the phase gain margin results to the multivariable problem see the paper by Safonov and Athans [7].

We warn the reader that much additional theoretical and applied research is needed before the implications of these theoretical results can (a) be fully understood and (b) translated into systematic "cookbook" procedures that have the same value as the conventional results in classical servomechanism design.

This paper is organized as follows: in Section 2 we present an explicit parametrization of a subclass of linear constant feedback maps that never destabilize an originally open-loop stable system, and establish some of its properties. In Section 3, we apply this construct to obtain several new closed-loop structural stability characterizations of multi-input LQ-optimal feedback maps. We conclude in Section 4 with a brief discussion of the relevance of the results of this paper for computer-aided iterative feedback design.

Notation

- 1) The linear time-invariant system

$$\begin{aligned}\dot{\underline{x}}(t) &= \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t) \\ \underline{z}(t) &= \underline{H}^T \underline{x}(t)\end{aligned}$$

where $\underline{x}(t) \in \mathbb{R}^n$ $\underline{x}(\cdot)$ = state vector
 $\underline{u}(t) \in \mathbb{R}^m$ $\underline{u}(\cdot)$ = control vector
 $\underline{z}(t) \in \mathbb{R}^r$ $\underline{z}(\cdot)$ = output vector

and $\underline{A} \in \mathbb{R}^{n \times n}$
 $\underline{B} \in \mathbb{R}^{n \times m}$
 $\underline{H}^T \in \mathbb{R}^{r \times n}$

will be denoted by $\Sigma(\underline{A}, \underline{B}, \underline{H}^T)$. Where $\underline{H}^T$ is irrelevant to the discussion, we will shorten the notation to $\Sigma(\underline{A}, \underline{B})$, and where the choice $\underline{A}, \underline{B}$ is

clear from the context, we will just use Σ .

If the matrix $\underline{A}$ is stable (i.e. all eigenvalues of $\underline{A}$ have strictly negative real parts), we will refer to $\Sigma(\underline{A}, \underline{B}, \underline{H}^T)$ as a stable system.

- 2) $R(\underline{K})$ = range space of $\underline{K}$

$N(\underline{K})$ = nullspace (kernel) of $\underline{K}$

$Rk(\underline{K})$ = rank of $\underline{K}$

- 3) Given the system $\Sigma(\underline{A}, \underline{B}, \underline{H}^T)$,

$R(\underline{A}, \underline{B}) \stackrel{\Delta}{=} \underline{A}$ controllable subspace of the pair $(\underline{A}, \underline{B})$

$$\stackrel{\Delta}{=} \underline{A} R(\underline{B}) + \underline{A}^2 R(\underline{B}) + \dots + \underline{A}^{n-1} R(\underline{B})$$

$N(\underline{H}^T, \underline{A}) \stackrel{\Delta}{=} \underline{A}$ unobservable subspace of the pair $(\underline{H}^T, \underline{A})$

$$\stackrel{\Delta}{=} \bigcap_{i=1}^n N(\underline{H}^T \underline{A}^{i-1})$$

- 4) If $\underline{Q} \in \mathbb{R}^{n \times n}$ is positive semidefinite, we will write

$$\underline{Q} \geq 0$$

If $\underline{Q}$ is positive definite, we will write

$$\underline{Q} > 0.$$

- 5) $[m_{ij}]$ denotes the matrix $\underline{M}$ whose (i,j) -element is m_{ij} .

2. Parametrization of non-destabilizing feedback maps

We begin our discussion with

Definition 1

Given the stable system $\Sigma(\underline{A}, \underline{B})$, let

$$S(\Sigma) \stackrel{\Delta}{=} \{ \underline{G}^T \in \mathbb{R}^{m \times n} \mid (\underline{A} + \underline{B} \underline{G}^T) \text{ is stable} \}$$

i.e. $S(\Sigma)$ is the set of all feedback maps that will not destabilize an originally open-loop stable system, where

$$\underline{u}(t) \stackrel{\Delta}{=} \underline{G}^T \underline{x}(t)$$

Ideally, one would like to be able to explicitly parametrize $S(\Sigma)$, but as this is a well-known intractable problem, our strategy here is to look for a simple parametrization of a (hopefully) sufficiently general subset of $S(\Sigma)$.

We begin by first recalling some standard Lyapunov-type results:

Lemma 1 (Wonham)

- (i) If $\underline{A}$ is stable, then the Lyapunov equation

$$\underline{P} \underline{A} + \underline{A}^T \underline{P} + \underline{Q} = 0$$

with $Q \geq 0$ has a unique solution $P \geq 0$.

If in addition $(Q^{1/2}, A)$ is observable, then $P > 0$.

(iii) If

(1) $P \geq 0$, $Q \geq 0$ satisfy $PA + A^T P + Q = 0$

(2) $(Q^{1/2}, A)$ is detectable

Then A is stable.

(iii) If $Q \geq 0$ and $(Q^{1/2}, A)$ is observable (detectable), then for all

$P \geq 0$, $R \geq 0$ and for all B , F^T , the pair

$$(\sqrt{Q + P + F R F^T}, A + B F^T)$$

is observable (detectable).

Proof:

for (i), see [4], pp. 298

for (ii), see [4], pp. 299

for (iii), see [4], pp. 82.

To proceed, the following definition will be useful:

Definition 2

For any stable A , let

$$LP(A) \triangleq \{K \geq 0 \mid K A + A^T K \leq 0\}$$

$$LP^+(A) \triangleq \{K > 0 \mid K A + A^T K < 0\}$$

Remark: $LP(A)$ is in general a proper subset of the set of all positive-semidefinite matrices of dimension n .

Example

Suppose that

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_1 < 0, \lambda_2 < 0$$

Then

$$LP(A) = \left\{ \begin{bmatrix} K_1 & K_{12} \\ K_{12} & K_2 \end{bmatrix} \mid \begin{array}{l} K_1 \geq 0 \\ K_2 \geq 0 \end{array} \right\}$$

$$K_1 K_2 \geq \left[\frac{(\lambda_1 + \lambda_2)^2}{4\lambda_1 \lambda_2} \right] K_{12}^2$$

Note that $\begin{bmatrix} K_1 & K_{12} \\ K_{12} & K_2 \end{bmatrix} \geq 0$ iff $K_1 \geq 0$, $K_2 \geq 0$,

$$K_1 K_2 \geq K_{12}^2$$

and the $\frac{(\lambda_1 + \lambda_2)^2}{4\lambda_1 \lambda_2} \geq 1$, with equality iff $\lambda_1 = \lambda_2$.

Lemma 2

i) $LP(A)$ is a convex cone: i.e.

$K_1, K_2 \in LP(A)$ implies

$\alpha_1 K_1 + \alpha_2 K_2 \in LP(A)$ for all $\alpha_1 \geq 0$, $\alpha_2 \geq 0$

ii) $K \in LP(A_1) \cap LP(A_2)$ implies $K \in LP(A_1 + A_2)$

iii) $K \in LP(A)$ implies $K \in LP(A + B(S - L)B^T K)$

for all S, L such that $S = -S^T$, $L \geq 0$,

provided $(A + B(S - L)B^T K)$ is stable.

Proof

Straightforward.

We are now ready to introduce our first crucial result:

Lemma 3

Let A be stable.

Then $(A + (N - M)K)$ is stable for all $K \in LP(A)$ and for all $N \geq 0$, $N = -N^T$ such that $R(N) \subset R(M)$.

If $K \in LP^+(A)$, the condition $R(N) \subset R(M)$ can be omitted.

Proof

Let

$$Q \triangleq -(K A + A^T K)$$

Since $K \in LP(A)$, we have $Q \geq 0$, and A stable implies $(Q^{1/2}, A)$ is always detectable.

Now

$$K A + A^T K + Q = 0,$$

so $K(A + (N - M)K) + (A + (N - M)K)^T K + 2K N K + Q - (K N K + K N^T K) = 0$ but $K N K + K N^T K = 0$ since $N = -N^T$.

If $K \in LP^+(A)$, then $Q > 0$

so $(\sqrt{Q + 2K N K}, (A + (N - M)K))$ is observable, which implies $(A + (N - M)K)$ is stable by Lemma 1 (ii).

Otherwise, assume $R(N) \subset R(M)$ which implies that there exists V such that $N = V M$ or that $(N - M)K = (V - I)M K$.

By defining $B \triangleq (V - I)M^{1/2}$

$$F^T \triangleq N^{1/2} K$$

$$P \triangleq 0$$

$$R \triangleq I$$

in Lemma 1 (iii), we have that

$(Q + 2K N K, A + (N - M)K)$ is detectable.

By Lemma 1 (ii), we therefore have $(A + (N - M)K)$ is stable.

Q.E.D.

Remark

A special case of Lemma 3 was established by Barnett and Storey in [5].

By specializing Lemma 3, we immediately obtain an explicit parametrization of a subclass of stabilizing feedback. First we introduce:

Definition 3

Given the stable system $\bar{L}(\underline{A}, \underline{B})$, let

$$S_1(\bar{L}) \triangleq \{ \underline{G}^T \in \mathbb{R}^{n \times n} \mid \underline{G}^T = (\underline{S} - \underline{L})\underline{B}^T \underline{K}, \underline{S} = -\underline{S}^T, \\ \underline{L} \geq \underline{0}, \text{ and either } \underline{K} \in LP^+(\underline{A}) \text{ or else } \underline{K} \in LP(\underline{A}) \text{ with } R(\underline{S}) \subset R(\underline{L}) \}$$

We can now state the result as:

Theorem 1

Given the stable system $\bar{L}(\underline{A}, \underline{B})$, then

(i) $\underline{G}^T \in S_1(\bar{L})$ implies $(\underline{A} + \underline{B} \underline{G}^T)$ is stable

$$(ii) \int_0^\infty e^{\underline{A}^T t} \underline{Q} e^{\underline{A} t} dt \\ \geq \int_0^\infty e^{(\underline{A} + \underline{B} \underline{G}^T)^T t} \underline{Q} e^{(\underline{A} + \underline{B} \underline{G}^T) t} dt$$

where $\underline{Q} \geq \underline{0}$ is such that $\underline{K} \underline{A} + \underline{A}^T \underline{K} + \underline{Q} = \underline{0}$ and $\underline{G}^T \triangleq (\underline{S} - \underline{L})\underline{B}^T \underline{K} \in S_1(\bar{L})$.

Proof

(i) Let $\underline{M} = \underline{B} \underline{L} \underline{B}^T$, $\underline{N} = \underline{B} \underline{S} \underline{B}^T$ in Lemma 3, and the result follows directly.

(ii) Let $\underline{Q} \geq \underline{0}$ be such that

$$\underline{K} \underline{A} + \underline{A}^T \underline{K} + \underline{Q} = \underline{0} \quad (*)$$

Then we have

$$\underline{K} = \int_0^\infty e^{\underline{A}^T t} \underline{Q} e^{\underline{A} t} dt$$

Next rewrite (*) as

$$\underline{K}(\underline{A} + \underline{B} \underline{G}^T) + (\underline{A} + \underline{B} \underline{G}^T)^T \underline{K}$$

$$+ (2\underline{K} \underline{B} \underline{L} \underline{B}^T \underline{K} + \underline{Q}) = \underline{0}$$

where $\underline{G}^T \triangleq (\underline{S} - \underline{L})\underline{B}^T \underline{K} \in S_1(\bar{L})$

which implies $\underline{K} = \int_0^\infty e^{(\underline{A} + \underline{B} \underline{G}^T)^T t} \cdot$

$$\cdot (2\underline{K} \underline{B} \underline{L} \underline{B}^T \underline{K} + \underline{Q}) e^{(\underline{A} + \underline{B} \underline{G}^T) t} dt$$

$$\text{hence } \int_0^\infty e^{\underline{A}^T t} \underline{Q} e^{\underline{A} t} dt = \int_0^\infty e^{(\underline{A} + \underline{B} \underline{G}^T)^T t} \cdot$$

$$\cdot \underline{Q} e^{(\underline{A} + \underline{B} \underline{G}^T) t} dt + 2 \int_0^\infty e^{(\underline{A} + \underline{B} \underline{G}^T)^T t} \cdot$$

$$\cdot \underline{K} \underline{B} \underline{L} \underline{B}^T \underline{K} e^{(\underline{A} + \underline{B} \underline{G}^T) t} dt \quad \text{or}$$

$$\int_0^\infty e^{\underline{A}^T t} \underline{Q} e^{\underline{A} t} dt \geq \int_0^\infty e^{(\underline{A} + \underline{B} \underline{G}^T)^T t} \cdot$$

$$\cdot \underline{Q} e^{(\underline{A} + \underline{B} \underline{G}^T) t} dt$$

Q.E.D.

Remark

It can be easily shown that all the eigenvalues of the feedback term $\underline{B}(\underline{S} - \underline{L})\underline{B}^T \underline{K}$ have non-positive real parts. This observation, and the content of Theorem 1(ii), makes it convenient to interpret $S_1(\bar{L})$ as a natural generalization of the concept of 'negative' feedback to the multi-variable and multi-input case.

The next two corollaries are easy consequences of Theorem 1.

Corollary 1.1

Let $\bar{L}(\underline{A}, \underline{B})$ be a system with a single input, i.e. let $\underline{B}$ be a column $(n \times 1)$ vector $\underline{b}$. If $\underline{g}_1^T, \dots, \underline{g}_j^T \in S_1(\bar{L}(\underline{A}, \underline{b}))$, then

$$\sum_{i=1}^j \alpha_i \underline{g}_i^T \in S_1(\bar{L}) \quad \text{for all } \alpha_i \geq 0, i=1, \dots, j$$

Proof

Each $\underline{g}_i^T$ is of the form $-\underline{r}_i \underline{b}^T \underline{K}_i$ for some admissible $\underline{r}_i > 0$, $\underline{K}_i \geq \underline{0}$ so

$$\sum_{i=1}^j \alpha_i \underline{g}_i^T = \sum_{i=1}^j -\alpha_i \underline{r}_i \underline{b}^T \underline{K}_i = -\underline{b}^T \left(\sum_{i=1}^j \alpha_i \underline{r}_i \underline{K}_i \right).$$

But from Lemma 2(i), $\underline{K}_i \in LP(\underline{A})$ implies

$$\sum_{i=1}^j \alpha_i \underline{r}_i \underline{K}_i \in LP(\underline{A}) \quad \text{for all } \alpha_i \underline{r}_i \geq 0 \text{ hence}$$

$$\sum_{i=1}^j \alpha_i \underline{g}_i^T \in S(\bar{L}) \quad \text{for all } \alpha_i \geq 0. \quad \text{Q.E.D.}$$

Corollary 1.2

Suppose there exists $\underline{L} \geq \underline{0}$ such that $\underline{B} \underline{L} \underline{B}^T \in LP(\underline{A}^T)$.

Then $(\underline{A} - \underline{B} \underline{L} \underline{B}^T (\underline{K} + \underline{N}))$ is stable for all $\underline{K} \geq \underline{0}$ and $\underline{N} = -\underline{N}^T$ such that $R(\underline{K}) \supset R(\underline{N})$.

If $\underline{B} \underline{L} \underline{B}^T \in LP^+(\underline{A}^T)$ actually, then the condition $R(\underline{K}) \supset R(\underline{N})$ can be omitted.

Proof

Immediate from 'taking the transpose' in Lemma 3.

Q.E.D.

Theorem 1 has illustrated the importance of $LP(\underline{A})$. It is therefore useful to have an alternative characterization of $LP(\underline{A})$:

Proposition 1

$LP(A)$ is A^T -invariant, i.e. for all $K \in LP(A)$

$$A^T K(K) \subset R(K)$$

Proof

$K \in LP(A)$ iff $K A + A^T K + H H^T = 0$
for some H

We claim that

$N(K) = N(H^T, A) = \text{unobservable subspace of } (H^T, A)$

For $K = \int_0^\infty e^{A^T t} H H^T e^{A t} dt$

so $x \in N(H^T, A)$ implies $H^T e^{A^T t} x = 0$ for all $t \in R$
which implies $x \in N(K)$. Conversely, $x \in N(K)$

implies $x^T K x = 0$ which implies $\int_0^\infty |H^T e^{A^T t} x|^2 dt = 0$
or $H^T e^{A^T t} x = 0$ for all $t \in R$, i.e. $x \in N(H^T, A)$.

To complete the proof, note that

$$\begin{aligned} R(K) &= R(K^T) = N(K)^{\perp} \\ &= N(H^T, A)^{\perp} \\ &= R(A^T, H) \\ &= \text{controllable subspace of } (A^T, H). \end{aligned}$$

But any controllable subspace of A^T is necessarily an A^T -invariant subspace.

Q.E.D.

Remark: The significance of Proposition 1 is that it provides a systematic means for generating members of $LP(A)$. For example, if A has distinct, real eigenvalues, then every $K \in LP(A)$ is of the form

$$K = P^T M P$$

where the rows of P are left eigenvectors of A , i.e.,

$$P A = \Lambda P, \Lambda = \text{diagonal } (\lambda_1, \dots, \lambda_n)$$

and $M \geq 0$ is such that $[-(\lambda_i + \lambda_j)m_{ij}] \geq 0$, where $M = [m_{ij}]$.

Thus, all members of $LP(A)$ can be generated once P is known.

While membership in $S_1(E)$ is sufficient to guarantee closed-loop stability, it is of course not necessary, i.e. $S_1(E)$ is a strictly proper subset of $S(E)$. Intuitively, if the open-loop system is stable 'enough' to begin with, it can tolerate a certain amount of 'positive' feedback without leading to closed-loop instability. In other words, the poles of the open-loop system can be shifted to the right by feedback without destroying stability

so long as none of them get shifted into the closed right-half plane. By allowing such additional nondestabilizing feedback, therefore, we ought to be able to 'enlarge' $S_1(E)$. More precisely, we have:

Definition 4

Given the stable system $\Sigma(A, B)$ and any $L \geq 0$, $L \in R^{n \times n}$, let

$$LP(E, L) \triangleq \{K \geq 0 | K A + A^T K + 2K B L B^T K \leq 0\}$$

$$LP^+(E, L) \triangleq \{K > 0 | K A + A^T K + 2K B L B^T K < 0\}$$

Definition 5

Given the stable system $\Sigma(A, B)$, let

$$S_2(E) \triangleq \{G^T \in R^{m \times n} | G^T = (\hat{L} + S) B^T K, \hat{L} = \hat{L}^T,$$

$$S = -S^T, L \geq 0, L \geq \hat{L}, \text{ and either}$$

$$K \in LP^+(E, L) \text{ or else } K \in LP(E, L)$$

$$\text{with } R(\hat{L} + S) \subset R(L - \hat{L})\}$$

Theorem 2

Given the stable system $\Sigma(A, B)$, then

$$G^T \in S_2(E) \text{ implies } (A + B G^T) \text{ is stable.}$$

Proof

The proof follows by a straightforward extension of the proof of Lemma 3 and Theorem 1, and hence is omitted.

Q.E.D.

Remark

It can be easily seen that Theorem 1 is just a special case of Theorem 2 (with $L \equiv 0$ and $L \leq 0$, $S_2(E)$). Note that in the general case covered by Theorem 2, no definiteness assumption is made of L , and thus various 'mixtures' of 'positive' and 'negative' feedbacks are allowed.

The next proposition provides further clarification on our parametrization scheme. First define:

$$F_1(B) \triangleq \{G^T \in R^{m \times n} | G^T = D B^T K, D \in R^{m \times m} \text{ arbitrary,}$$

$$K \in R^{n \times n} \text{ and } K \geq 0\}$$

$$F_2(B) \triangleq \{G^T \in R^{m \times n} | RK(G^T B) < RK(G^T)\}.$$

Proposition 2

$$F_1(B) \cap F_2(B) = \emptyset$$

$$F_1(B) \cup F_2(B) = R^{m \times n}$$

i.e. any feedback map $G^T \in R^{m \times n}$ is either in the set $F_1(B)$ or else $F_2(B)$.

Proof

We need only to show that

$$F_2(B) = \{G^T \in R^{n \times n} | RK(G^T B) = RK(G^T)\}$$

Necessity:

Suppose $G^T \in F_1(B)$, i.e. there exists $D \in R^{n \times n}$ and $K \in R^{n \times n}$, $K \geq 0$ such that $G^T = D B^T K$. Then

$$G^T B D^T = D B^T K B D^T \geq 0$$

so $RK(G^T B) \supseteq RK(G^T B D^T) = RK(D B^T K B D^T) = RK(D B^T K) = RK(G^T)$.

Sufficiency:

Take $D = G^T B$ and observe that the equation

$$G^T = G^T B B^T K$$

has a solution $K \geq 0$ if $RK(G^T B) = RK(G^T)$.

Q.E.D.

We now relate the content of Proposition 2 to Theorem 2. Observe first that $S_2(\underline{E}) \subset F_1(\underline{B})$, and hence our parametrization scheme fails to capture any non-destabilizing feedback map $\in F_2(\underline{B})$. That $S(\underline{E}) \cap F_2(\underline{B}) \neq \emptyset$ is demonstrated by the following trivial example:

Example

$$\underline{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \lambda_1, \lambda_2 < 0,$$

$$\underline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \underline{g}^T = [0 \quad 1] \in F_2(\underline{B})$$

$$\text{and } (\underline{A} + \underline{B} \underline{g}^T) = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \text{ is stable.}$$

Note, however, that if $\underline{B}$ is of full rank, then the set $F_2(\underline{B})$ is NOT generic in $R^{n \times n}$.

The more interesting question, 'is $S_2(\underline{E})$ generic (i.e. dense) in $S(\underline{E}) \cap F_1(\underline{B})$?' is at present unsolved.

Our results so far have been on systems $\underline{E}(\underline{A}, \underline{B})$ which are open-loop stable; the question next arises as to what the situation would be for systems which are NOT open-loop stable (i.e. $\underline{A}$ has unstable poles). For $\underline{A}$ unstable it is of course not possible to write Lyapunov-type equations. One is reminded, however, of the algebraic Riccati equations; indeed, we have the following interpretation of the traditional LQ-optimization problem:

Definition 6

Given $(\underline{A}, \underline{B})$ a stabilizable pair, let

$LQ(\underline{A}, \underline{B}) \triangleq \{K \geq 0 | K = K(\underline{A}, \underline{B}, \underline{R}, \underline{H}^T) \text{ for some } \underline{R} > 0 \text{ and some } \underline{H}^T \text{ such that } (\underline{H}^T, \underline{A}) \text{ is a detectable pair}\}$

where $K(\underline{A}, \underline{B}, \underline{R}, \underline{H}^T)$ denotes the unique positive semidefinite solution to the algebraic Riccati equation:

$$\underline{K} \underline{A} + \underline{A}^T \underline{K} - \underline{K} \underline{B} \underline{B}^T \underline{K} + \underline{H} \underline{H}^T = 0$$

For $\underline{R}$ fixed, we will denote the corresponding set as $LQ(\underline{A}, \underline{B}, \underline{R})$.

Definition 7

$$S_3(\underline{E}) \triangleq \{G^T \in R^{n \times n} | G^T = -\underline{R}^{-1} \underline{B}^T K, \underline{R} > 0,$$

$$K \in LQ(\underline{A}, \underline{B}, \underline{R})\}$$

Proposition 3

Given any stabilizable system $\underline{E}(\underline{A}, \underline{B})$, $G^T \in S_3(\underline{E})$ implies $(\underline{A} + \underline{B} G^T)$ is stable.

Remark

The above proposition merely summarizes the well-known 'standard' results of LQ-optimal feedback theory (see [1], [4]). However, the interpretation here of the LQ-optimal feedback class $(S_3(\underline{E}))$ as a parametrization of a subclass of stabilizing feedback is interesting.

3. Structural Stability Characterization of Linear Quadratic (LQ) Optimal Feedback Maps

In this section we show how the parametrization scheme developed in the previous section can be applied to obtain characterization of the closed-loop structural stability properties of systems under LQ-optimal feedback. More precisely, we establish an explicit parametrization of a general class of structural perturbations in the control feedback gains as well as in the control actuation matrix ($\underline{B}$) that leave the closed-loop system stabilized. These new results, we believe, are the natural generalizations and nontrivial extensions of some earlier results of Anderson and Moore [2].

We begin by first recalling from Lemma 2(iii) that, for $\underline{A}$ stable, $K \in LP(\underline{A})$ always implies $K \in LP(\underline{A} - \underline{B} \underline{L} \underline{B}^T K)$; however, for $\underline{A}$ unstable and $K \geq 0$ such that $(\underline{A} - \underline{B} \underline{L} \underline{B}^T K)$ is stable, it need NOT be true that $K \in LP(\underline{A} - \underline{B} \underline{L} \underline{B}^T K)$. The following example underscores this unfortunate state of affairs:

Example

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $(A - B L B^T K) = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$ is stable, but

$$K(A - B L B^T K) + (A - B L B^T K)^T K = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix} \not\leq 0$$

However, we have the following interesting observation:

Lemma 4

If $K \in LQ(A, B; R)$ then $K \in LP(A - B \hat{R}^{-1} \hat{B}^T K)$

Proof

Immediate from the Riccati equation.

Q.E.D.

In other words, the above unfortunate state of affairs cannot occur if K is an LQ-solution.

We are now ready to state our first main result of the section:

Theorem 3 (Infinite Gain Margin Property)

Let $K \in LQ(A, B; R)$

Then

$(A - [B(S + L)B^T + \hat{B}(N + M)\hat{B}^T]K)$ is stable

for all

$$L \geq R^{-1}, M \geq 0, S = -S^T, R(S) \subset R(L - R^{-1})$$

$$N = -N^T, R(N) \subset R(M)$$

$\hat{B}$ arbitrary

Proof

We have $K \in LP(A - B \hat{R}^{-1} \hat{B}^T K)$, so by Lemma 3,

$(A - B \hat{R}^{-1} \hat{B}^T K + (V - W)K)$ is stable for all $W \geq 0$, $V = -V^T$ such that $R(V) \subset R(W)$. Take $W = B(L - R^{-1})B^T + \hat{B}M\hat{B}^T$ and $V = B(S + L)B^T + \hat{B}N\hat{B}^T$ and we are done.

Q.E.D.

Remark

For $\hat{B} \equiv 0$, Theorem 3 is a generalization of the 'infinite gain margin' property of LQ-optimal feedback for single-input systems first noted by Anderson and Moore [1], who showed that the feedback gain vector $g^T = -\frac{1}{r} b^T K$ can be multiplied by any scalar $\alpha \geq 1$ without destroying stability; the proof they used involves classical Nyquist techniques. Theorem 3 not only generalizes this property to multi-input systems, but allows more complicated alterations of the feedback gain vectors; moreover,

it makes the proof of this property much more transparent.

Remark

For $\hat{B} \neq 0$, Theorem 3 allows for changes in the $\hat{B}$ matrix itself without destroying stability. One useful interpretation is the following.

Suppose that the optimal feedback gain matrix has been computed for a nominal B_0 , but that the actual value of B during system operation is changed to $B = B_0 + B_1$. Then the feedback term becomes $(B_0 K^{-1} B_0^T K + B_1 K^{-1} B_0^T K)$. As long as $B_1 = B_0(N + M)R$ for some $N = -N^T$, $M \geq 0$, Theorem 3 will guarantee us that the system will remain stable. (For example, $B_1 = \alpha B_0$, $\alpha > 0$). More complicated cases are allowed.

Remark

Alternatively, the case $\hat{B} \neq 0$ can be interpreted as allowing for the possibility of adding extra controllers, and using these extra feedbacks to 'fine-tune' the closed-loop behavior of the original system. (A more systematic exploitation of this idea will be dealt with in a future publication; see also [3]).

Theorem 3 has dealt with the case when the 'negative' feedback gains, etc. are allowed to increase in magnitude; the converse situation, when the 'negative' feedback gains are reduced in magnitude (or when there is system structural parameter changes) is examined in the next proposition:

Theorem 4 (Gain Reduction and Robustness Property)

Let $K \geq 0$ be the Riccati solution to the LQ-problem (A, B, R, Q) where $R > 0$ and $(Q^{1/2}, A)$ detectable. Then

(i) $(A - B(M + N)B^T K)$ is stable

for all $M \geq 0$ such that $M \geq \frac{1}{2} R^{-1}$

$$N = -N^T$$

(ii) If $(Q^{1/2}, A)$ is actually observable, then

$(A - B(M + N)B^T K + K^{-1}(\hat{Q} + \hat{N}))$ is stable

where M, N are as above, and

$\hat{Q} = \hat{Q}^T$ is such that $\hat{Q} \leq \frac{1}{2} Q$, $R(\frac{1}{2} Q - \hat{Q}) \supset R(Q)$

and $\hat{N} = -\hat{N}^T$ such that $R(\frac{1}{2} Q - \hat{Q}) \supset R(\hat{N})$.

Proof

(i) Let $\Sigma_c = \Sigma(A - B \hat{R}^{-1} \hat{B}^T K)$, $B = \Sigma(A_c, B)$

Then we have $K \in LP(\Sigma_c; \frac{1}{2} R^{-1})$ from the

Riccati equation, and so by Theorem 2,

$(\underline{A}_c + \underline{B}(\underline{M} - \underline{N})\underline{K})$ is stable for all $\underline{M} < \frac{1}{2}\underline{R}^{-1}$
 $\underline{N} = -\underline{N}^T$

or $(\underline{A} - \underline{B}(\underline{R}^{-1} - \underline{M} + \underline{N})\underline{K})$ is stable

let $\underline{M} \hat{=} \underline{R}^{-1} - \underline{N} > \frac{1}{2}\underline{R}^{-1}$, and the proof is complete.

(ii) Let $\underline{A}_c \hat{=} (\underline{A} - \underline{B}(\underline{M} + \underline{N})\underline{K})$. From the Riccati equation we have

$$\underline{K}\underline{A}_c + \underline{A}_c^T\underline{K} + \underline{K}\underline{B}(2\underline{M} - \underline{R}^{-1})\underline{B}^T\underline{K} + \underline{Q} = 0$$

Since $(\underline{Q}^{1/2}, \underline{A})$ -observable implies $\underline{K} > 0$,

$\underline{K}^{-1}$ exists, so we have $\underline{K}(\underline{A}_c + \underline{K}^{-1}(\underline{Q} + \underline{N}))$

$$+ (\underline{A}_c + \underline{K}^{-1}(\underline{Q} + \underline{N}))^T\underline{K} + \underline{K}\underline{B}(2\underline{M} - \underline{R}^{-1})\underline{B}^T\underline{K}$$

$$+ (\underline{Q} - 2\underline{Q}) = 0$$

Hence, subject to the condition $\frac{1}{2}\underline{Q} \geq \hat{\underline{Q}}$,

$$\underline{R}(\frac{1}{2}\underline{Q} - \hat{\underline{Q}}) - \underline{R}(\underline{Q} + \underline{N}) \text{ it can be shown that}$$

$$((\underline{Q} - 2\underline{Q}) + \underline{K}\underline{B}(2\underline{M} - \underline{R}^{-1})\underline{B}^T\underline{K}, \underline{A}_c + \underline{K}^{-1}(\underline{Q} + \underline{N}))$$

is observable. Thus by Lemma 1(iii),

$$(\underline{A}_c + \underline{K}^{-1}(\underline{Q} + \underline{N})) \text{ is stable.}$$

Q.E.D.

Remark

Theorem 4(i) is a generalization of the known 'gain reduction tolerance' property of LQ-optimal feedback. This interpretation is most transparent in the special case when $\underline{R}^{-1} = \text{diag. } (a_1, \dots, a_m)$ and $\underline{M} = \text{diag. } (\hat{a}_1, \dots, \hat{a}_1)$, $\underline{N} \hat{=} 0$. Then the original individual feedback loops are of the form

$$\underline{u}_i = -\hat{a}_i \underline{b}_i^T \underline{K}, i = 1, \dots, m$$

The theorem states that, in this special case, the system remains stable if the feedback gains are reduced to

$$\underline{u}_i = -\hat{a}_i \underline{b}_i^T \underline{K}$$

so long as $\hat{a}_i > \frac{1}{2} a_i$.

More complicated cases are of course allowed.

Remark

By interpreting the additional term $\underline{K}^{-1}(\underline{Q} + \underline{N})$ as a model perturbation term $\underline{A}$ of the open-loop matrix $\underline{A}$, we can use Theorem 4(ii) to perform finite perturbation sensitivity analysis.

The following simple example illustrates the usefulness of this approach:

Example

$$\text{Let } \underline{A} = \begin{bmatrix} .5 & 0 \\ 0 & -2 \end{bmatrix}, \underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{If we take } \underline{Q} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}, \underline{R} = \frac{1}{2}$$

Then we obtain the algebraic Riccati solution

as

$$\underline{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the optimal feedback gain $\underline{q}^{*T} = -2[1 \quad 1]$

$$\text{For any } \delta \underline{A} = \begin{bmatrix} \beta_1 & \beta_{12} + \gamma \\ \beta_{12} - \gamma & \beta_2 \end{bmatrix}$$

$$\text{where } \gamma \in \mathbb{R}, \begin{bmatrix} \beta_1 & \beta_{12} \\ \beta_{12} & \beta_2 \end{bmatrix} < \begin{bmatrix} .5 & 1 \\ 1 & 3 \end{bmatrix}$$

we are assured by Theorem 4(ii) that

$$\underbrace{\begin{bmatrix} .5 + \beta_1 & \beta_{12} + \gamma \\ \beta_{12} - \gamma & -2 + \beta_2 \end{bmatrix}}_{\underline{A} + \delta \underline{A}} + \alpha \underline{b} \underline{q}^{*T}$$

is stable for all $\alpha > \frac{1}{2}$.

Consider the following special cases:

$$(a) \gamma = \beta_{12}, \beta_1 = \beta_2 = 0$$

We have

$$\underbrace{\begin{bmatrix} .5 & 2\beta_{12} \\ 0 & -2 \end{bmatrix}}_{\underline{A} + \delta \underline{A}} + \alpha \underline{b} \underline{q}^{*T} \text{ stable for all } \alpha > \frac{1}{2}$$

and β_{12} such that

$$(1 - \beta_{12})^2 < 1.5 \text{ or } 1 - \frac{3}{2} < \beta_{12} < 1 + \frac{3}{2}$$

$$(b) \gamma = \beta_{12} = 0,$$

We have

$$\underbrace{\begin{bmatrix} .5 + \beta_1 & 0 \\ 0 & -2 + \beta_2 \end{bmatrix}}_{\underline{A} + \delta \underline{A}} + \alpha \underline{b} \underline{q}^{*T} \text{ stable for all}$$

$\alpha > \frac{1}{2}$ and β_1, β_2 such that

$$i) \beta_1 < .5, \beta_2 < 3$$

$$ii) (.5 - \beta_1)(3 - \beta_2) > 1$$

thus if $\beta_1 = 0$, the perturbed system is stable for all $\beta_2 < 1$.

Other more general cases are of course allowed.

The above example thus shows that the combined effect of feedback gain reduction and perturbation, or uncertainty of the open-loop system parameters (poles and coupling terms) can be tolerated by a linear quadratic design without leading to closed-loop instability. This robustness property of the LQ-feedback design deserves more attention.

4. Concluding Remarks

Since further applications of the parametrization results established in this paper to reliable stabilization synthesis and decentralized stabilization coordination will be made in a further publication, we will reserve a fuller discussion of the implications of our approach until then. At this point, however, we would like to point out an important implication for practical design that is immediate: the ability to perform feedback 'loop-shaping' analysis.

In any realistic synthesis problem (keeping a system stabilized, localizing particular disturbances, etc.) there is usually a large number of feasible solutions. While the use of cost-criterion optimization (e.g. LQ) in theory allows the designer to pick exactly one such solution, in practice, the difficulties of judging or fully incorporating the relevant cost considerations and their trade-offs as well as the often gross model uncertainties and physical variabilities of the system and the controllers, necessitate further sensitivity analysis or trial-and-error 'hedging' about the nominal solution. It is therefore very important in the computer-aided design context that the 'feasible solution space' structure be known in some details to facilitate and guide the conduct of iterative search. In this regard, a major merit of a 'classical' design technique like root-locus is that it provides an explicit functional dependence of the closed-loop system structures (distribution of poles and zeros) on the control structure (feedback gain). However, such classical approaches become totally intractable when there is a multiple number of controllers, while modern 'state-space' linear feedback design techniques like 'pole-placement' algorithm and 'diagonal-feedback' design suffer the serious drawback of providing little structural information about the solutions they generate, and moreover such techniques are guided more by mathematical convenience than by physical interpretation.

From this perspective, the parametrization results established earlier appear to be promising in providing the basis for a new iterative design algorithm that will overcome the last-mentioned drawbacks.

Several years ago Rosenbrock [6] suggested a frequency-domain multi-loop feedback design tech-

nique (the 'inverse Nyquist array' method) which he motivated also as an attempt to overcome some of the above-mentioned drawbacks. His approach is in contrast with ours, which is a 'time-domain' approach. It will be interesting to investigate the connection, if any, between the two approaches.

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