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MODELING ERROR ANALYSIS OF STATIONARY LINEAR

DISCRETE-TIME FILTERS

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16. Abstract

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LINEAR DISCRETE-TIME FILTERS

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Abstract. The performance of Kalman-type, linear, discrete-time
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sion is limited to stationary performance, and bounds are obtained for
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of a suboptimal filter with that of the optimal filter, when only the
range of errors in the elements of the model matrices is available.

1. *Introduction.* One of the problems arising in the application
of the minimum variance optimal filter of Kalman and Bucy [8] is that a
design based on imperfect knowledge of the system configuration and noise
statistics often results in poor performance. Thus, there has been con-
siderable research on the effect of modeling errors on filter performance
[3,5,6,10,11]. In particular, errors in prior information on state
statistics and noise covariances [6,10,11] and in system models [3,5]
have been considered.

This paper is concerned with providing bounds on the performance for
suboptimal as well as optimal discrete-time filters based on information

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about the range of modeling errors. These results are useful from the practical point of view, as a designer often has information on the range of modeling errors rather than a precise knowledge of the modeling errors. In this paper we limit the discussion to stationary conditions and obtain performance bounds for discrete-time filters for two types of errors: noise covariance errors and system configuration errors. Such bounds were obtained for continuous-time filters in an earlier paper [12]. The results reported here and in [12] are quite different from those of earlier work in that we do not limit the discussion to small-scale (differential) analysis as in [5], nor do we assume sign definiteness of the covariance errors as in [11].

This paper is organized as follows: the problem is formulated in Sec. 2; in Sec. 3, two general bounds are obtained. These bounds are analyzed in Sec. 4 to obtain practical expressions when the system configuration is assumed to be known and, in Sec. 5 when the noise covariances are assumed to be known. The results in Sec. 4 and 5 are illustrated by an example in Sec. 6. Section 7 concludes with some remarks on the results and comments about future research.

2. *Problem Statement.* Consider a time-invariant process described by

$$x(k + 1) = Ax(k) + Gw(k) \qquad (1)$$

where the n-dimensional state vector $x(k + 1)$ is measured by an m-dimensional vector $y(k + 1)$:

$$y(k + 1) = Hx(k + 1) + v(k + 1) \qquad (2)$$

The ℓ-dimensional process noise $w(k)$ and the m-dimensional measurement noise $v(k + 1)$ are assumed to be mutually independent Gaussian noises with zero mean and

$$E[w(k)w(j)^T] = Q\delta_{k,j} , \qquad Q \geq 0^\dagger \qquad (3)$$

$$E[v(k)v(j)^T] = R\delta_{k,j} , \qquad R > 0^\dagger \qquad (4)$$

where $\delta_{k,j}$ denotes the Kronecker delta. The system matrices A, G, and H are assumed to have appropriate dimensions, and A is assumed to be a convergent matrix.

The optimal estimates $\hat{x}(k + 1|k + 1)$ that minimizes

$$J(k + 1) \equiv E\left[\| \hat{x}(k + 1|k + 1) - x(k + 1) \|^2\right]^\ddagger$$

with observations $[y(0), y(1), \ldots, y(k + 1)]$, are given by [8]:

$$\hat{x}(k + 1|k + 1) = A\hat{x}(k|k) + K(k + 1)[y(k + 1) - HA\hat{x}(k|k)] \qquad (5)$$

where $K(k + 1)$ is the Kalman gain matrix specified by the recursive relations:

$$K(k + 1) = P_0(k + 1|k)H^T[HP_0(k + 1|k)H^T + R]^{-1} \qquad (6)$$

$$P_0(k + 1|k) = AP_0(k|k)A^T + GQG^T \qquad (7)$$

and

$$P_0(k + 1|k + 1) = [I_n - K(k + 1)H]P_0(k + 1|k) \qquad (8)$$

with $\hat{x}(0)$ and $P_0(0)$ provided by prior information on $x(0)$ and I_n, the $n \times n$ identity matrix. The minimized index $J(k + 1)$ is given by

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†A symmetric matrix W is denoted as $W > 0$ ($W \geq 0$) when W is positive definite (semidefinite). Also, $W > Z$ ($W \geq Z$) denotes $W - Z > 0$ ($W - Z \geq 0$).

$^\ddagger \| w \| \left[\| W \| \equiv \left(\sum_{i,j} |W_{ij}|^2\right)^{1/2}\right]$ denotes the Euclidean norm of a vector w (matrix W).

$$\text{Min } J(k + 1) = \text{tr}[P_0(k + 1|k + 1)] \tag{9}$$

where $\text{tr}(W)$ denotes the trace of a square matrix W.

We assume that complete information on the system matrices A and H and the noise covariances Q and R is not available, and that the optimal estimator is replaced by a suboptimal estimator based on a model:

$$\hat{x}_M(k + 1|k + 1) = A_M\hat{x}_M(k|k) + K_M(k + 1)[y(k + 1) - H_M A_M\hat{x}_M(k|k)] \tag{10}$$

where $A_M \equiv A + \Delta A$ and $H_M \equiv H + \Delta H$ are model representations of A and H, respectively, and ΔA and ΔH denote modeling errors. The gain $K_M(k + 1)$ can be computed from the model matrices using the following relations:

$$K_M(k + 1) = P_M(k + 1|k)H_M^T[H_M P_M(k + 1|k)H_M^T + R_M]^{-1} \tag{11}$$

$$P_M(k + 1|k) = A_M P_M(k|k)A_M^T + GQ_M G^T \tag{12}$$

and

$$P_M(k + 1|k + 1) = [I_n - K_M(k + 1)H_M]P_M(k + 1|k) \tag{13}$$

where $Q_M \equiv Q + \Delta Q$ and $R_M \equiv R + \Delta R$ are the model representations of Q and R, respectively. The mean-squared error of this estimate is expressed by

$$E\left[\| \hat{x}_M(k + 1) - x(k + 1) \|^2\right] = \text{tr}[P(k + 1)] \tag{14}$$

In (14) and in the remainder of this section, the index $(j|j)$ is denoted by (j) wherever appropriate. The covariance matrix $P(k + 1)$ is described by [11]

$$P(k + 1) = L(k)A_M P(k)A_M^T L(k)^T + \Delta C(k)V(k)A_M^T L(k)^T + L(k)A_M V(k)^T \Delta C(k)^T$$

$$+ \Delta C(k)U(k)\Delta C(k)^T + [L(k) + K_M(k)\Delta H]GQG^T[L(k) + K_M(k)\Delta H]^T$$

$$+ K_M(k)RK_M(k)^T \tag{15}$$

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$$V(k + 1) = AV(k)A_M^T L(k)^T + AU(k)\Delta C(k)^T - GQG^T[L(k) + K_M(k)\Delta H]^T \quad (16)$$

$$U(k + 1) = AU(k)A^T + GQG^T \quad (17)$$

where

$$L(k) \equiv I_n - K_M(k)H_M \quad (18)$$

$$\Delta C(k) \equiv L(k)\Delta A - K_M(k)\Delta H A_M + K_M(k)\Delta H\Delta A \quad (19)$$

$$P(k) \equiv E[\tilde{x}(k)\tilde{x}(k)^T]$$

$$V(k) \equiv E[x(k)\tilde{x}(k)^T]$$

$$U(k) \equiv E[x(k)x(k)^T]$$

$$\tilde{x}(k) \equiv \hat{x}_M(k) - x(k)$$

$$P(0) = -V(0) = U(0) = E\left\{[x(0) - \bar{x}(0)][x(0) - \bar{x}(0)]^T\right\}$$

$$\bar{x}(0) \equiv E[x(0)]$$

Since the modeling error matrices ΔA, ΔH, ΔQ, and ΔR are generally not known exactly, (15) to (17) cannot be solved to obtain $tr[P(k + 1)]$, the performance of the suboptimal filter. However, a designer usually has estimates of the magnitude of errors in the model matrices, e.g., $|\Delta A_{ij}|$ in the (i,j)th element of A_M. Therefore, it is reasonable to obtain upper bounds for $tr[P(k + 1)]$ based on such estimates. It is also helpful to obtain lower bounds for $tr[P_o(k + 1)]$ as functions of the estimates so that the designer can evaluate the performance degradation, $tr[P(k + 1)] - tr[P_o(k + 1)]$, which he should expect with the possible modeling errors. Such bounds are obtained in later sections for stationary conditions.

3. *Performance Analysis for Stationary Conditions.* The covariances of estimation errors are constant matrices for stationary conditions, i.e., in (6) to (8), $P_o(k + 1|k) = P_o(k|k - 1)$ and $P_o(k + 1|k + 1) = P_o(k|k)$

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and, in (15) to (17), $P(k + 1) = P(k)$, $V(k + 1) = V(k)$, and $U(k + 1) = U(k)$ for stationary conditions. Hence the stationary filtering error covariance matrix for the suboptimal filter is denoted by P and the stationary filtering and (one-step) prediction error covariances for the optimal filter are denoted by P_F and P_P, respectively. In this section, general expressions for an upper bound for $tr(P)$ and a lower bound for $tr(P_F)$ are obtained. These are specialized in later sections to obtain more practical expressions.

In the sequel, the notation $W \otimes Z$ is used for the Kronecker product of matrices W and Z. The column string of an $n \times n$ matrix W, denoted by $cs(W)$, is defined by the following n^2-dimensional column vector:

$$cs(W) \equiv [w_{11} \cdots w_{n1}, w_{12} \cdots w_{n2}, \ldots, w_{1n} \cdots w_{nn}]^T$$

where w_{jk} is the (j,k)th element of matrix W. Note that

$$tr(W) = [cs(I_n)]^T cs(W) \tag{20}$$

and

$$\| cs(W) \| = \| W \| \tag{21}$$

Theorem 1: An upper bound for $tr(P)$ is given by

$$tr(P) \leq \pi_1 \equiv i^T M^{-1} b + \| (M^{-1})^T i \| \ \| D \| \tag{22}$$

where

$$M \equiv I_{n^2} - LA_M \otimes LA_M \tag{23}$$

$$L \equiv I_n - K_M H_M \tag{24}$$

$$i \equiv cs(I_n)$$

$$b \equiv cs(B), \quad B \equiv LGQ_M G^T L^T + K_M R_M K_M^T \tag{25}$$

$$D \equiv LGQ_M G^T \Delta H^T K_M^T + K_M \Delta H GQ_M G^T (L + K_M \Delta H)^T + \Delta C V A_M^T L^T + LA_M V^T \Delta C^T + \Delta C U \Delta C^T$$

$$- (L + K_M \Delta H) G \Delta Q G^T (L + K_M \Delta H)^T - K_M \Delta R K_M^T \tag{26}$$

$$\Delta C \equiv L \Delta A - K_M \Delta H A_M + K_M \Delta H \Delta A \tag{27}$$

Remark: Note that in the expression for π_1, only D contains the error matrices ΔA, ΔH, ΔQ, and ΔR. In Secs. 4 and 5, the term $\| D \|$ is analyzed further to obtain more explicit expressions.

Proof of Theorem 1: Equation (15) can be written as

$$P = LA_M P (LA_M)^T + B + D$$

and, in Kronecker form, as

$$cs(P) = [(LA_M) \otimes (LA_M)] cs(P) + cs(B) + cs(D)$$

Recalling (20), we obtain

$$tr(P) = i^T cs(P) = i^T M^{-1} b + i^T M^{-1} cs(D)$$

$$\leq i^T M^{-1} b + \| (M^{-1})^T i \| \, \| cs(D) \|$$

where the Schwartz inequality was used to obtain the inequality. Noting (21), we obtain (22), thereby completing the proof.

Next, we obtain a lower bound for the filtering error covariance matrix of the optimal filter.

Theorem 2: A lower bound for $tr(P_F)$ is given by

$$tr(P_F) \geq \pi_2 \equiv \frac{-a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_1} \tag{28}$$

where

$$a_1 \equiv \| A^T H^T R^{-1} HA \|_s \,[\S] \tag{29}$$

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[§] $\| W \|_s \equiv \max[\lambda(WW^T)]^{1/2}$ denotes the spectral norm (maximum singular value) of a matrix W; $\max \lambda(W) [\min \lambda(W)]$ denotes the maximum [minimum] eigenvalue of a symmetric matrix W.

$$a_2 \equiv n + tr(H^T R^{-1} H G Q G^T) - \sum_{i=1}^{n} |\lambda_i(A)|^2 \qquad (30)$$

and

$$a_3 \equiv tr(G Q G^T) \qquad (31)$$

Proof of Theorem 2: From (6) to (8), we obtain

$$I_n + H^T R^{-1} H (A P_F A^T + G Q G^T) = P_F^{-1} (A P_F A^T + G Q G^T)$$

Taking the trace of both sides yields

$$n + tr[H^T R^{-1} H (A P_F A^T + G Q G^T)] = tr(P_F^{-1} A P_F A^T) + tr(P_F^{-1} G Q G^T)$$

Applying Lemmas 1 and 2 (see Appendix), we obtain the following inequality:

$$n + \| A^T H^T R^{-1} H A \|_s tr(P_F) + tr(H^T R^{-1} H G Q G^T) \geq \sum_{i=1}^{n} |\lambda_i(A)|^2 + \frac{tr(G Q G^T)}{tr(P_F)} \qquad (32)$$

which can be rearranged in the form

$$a_1 [tr(P_F)]^2 + a_2 tr(P_F) - a_3 \geq 0$$

Solving the above inequality for $tr(P_F)$, we obtain (28), thereby completing the proof.

4. *Performance Bounds with Incorrect Noise Covariances.* We now consider the case for which A and H are known exactly, i.e., $\Delta A = 0$ and $\Delta H = 0$, and we obtain bounds for $tr(P)$ and $tr(P_F)$ in terms of ΔQ and ΔR explicitly.

Theorem 3: If $\Delta A = 0$ and $\Delta H = 0$, then an upper bound for $tr(P)$ is given by

$$tr(P) \leq \pi_3 \equiv i^T M^{-1} b + \| (M^{-1})^T i \| \left(\| L G \|_s^2 \| \Delta Q \| + \| K_M \|_s^2 \| \Delta R \| \right) \qquad (33)$$

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<u>Proof of Theorem 3</u>: Since $\Delta A = 0$ and $\Delta H = 0$ in (26),

$$\| D \| = \| LG\Delta QG^TL^T + K_M\Delta RK_M^T \|$$

$$\leq \| LG \|_s^2 \| \Delta Q \| + \| K_M \|_s^2 \| \Delta R \|$$

where the last inequality follows from the inequality $\| WZ \| \leq \| W \|_s \| Z \|$ [2, p. 37]. The result in (33) then follows from (22).

Theorem 4: If $\Delta A = 0$ and $\Delta H = 0$, then a lower bound for $tr(P_F)$ is given by

$$tr(P_F) \geq \pi_4 \equiv \frac{-\alpha_2 + \sqrt{\alpha_2^2 + 4\alpha_1\alpha_3}}{2\alpha_1} \tag{34}$$

where

$$\alpha_1 \equiv \frac{\| HA \|_s^2}{\min \lambda(R_M) - \| \Delta R \|} \tag{35}$$

$$\alpha_2 \equiv n + \frac{\| HG \|^2 \left(\| Q_M \|_s + \| \Delta Q \| \right)}{\min \lambda(R_M) - ||\Delta R||} - \sum_{i=1}^{n} |\lambda_i(A)|^2 \tag{36}$$

and

$$\alpha_3 \equiv tr(GQ_MG^T) - \ell \| G \|_s^2 \| \Delta Q \| \tag{37}$$

<u>Proof of Theorem 4</u>: In (28), π_2 decreases monotonically as a_1 and a_2 increase and as a_3 decreases. Hence, to obtain the bound π_4, we bound a_1 and a_2 from above and a_3 from below in terms of $\| \Delta Q \|$ and $\| \Delta R \|$. This is done by bounding $\| A^TH^TR^{-1}HA \|_s$ and $tr(H^TR^{-1}HGQG^T)$ from above and $tr(GQG^T)$ from below.

An upper bound for $\| A^TH^TR^{-1}HA \|_s$ is given by

$$\| A^TH^TR^{-1}HA \|_s \leq \| HA \|_s^2 \| R^{-1} \|_s = \frac{\| HA \|_s^2}{\min \lambda(R)} \leq \frac{\| HA \|_s^2}{\min \lambda(R_M) - \| \Delta R \|} \tag{38}$$

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where Weyl's inequality [4, p. 157] is used to obtain the last inequality. An upper bound for $tr(H^T R^{-1} HGQG^T)$ is given by

$$tr(H^T R^{-1} HGQG^T) \leq tr(G^T H^T R^{-1} HG) \| Q \|_s$$

$$\leq tr(HGG^T H^T) \| R^{-1} \|_s \| Q \|_s$$

$$\leq \frac{\| HG \|^2 \left(\| Q_M \|_s + \| \Delta Q \| \right)}{\min \lambda(R_M) - \| \Delta R \|} \tag{39}$$

where Lemma 1(i) in the Appendix is used to obtain the first two inequalities. A lower bound for $tr(GQG^T)$ is obtained as follows:

$$tr(GQG^T) = tr[G(Q_M - \Delta Q)G^T]$$

$$= tr(GQ_M G^T) - tr(G^T G \Delta Q)$$

$$\geq tr(GQ_M G^T) - \ell \| G^T G \Delta Q \|_s$$

$$\geq tr(GQ_M G^T) - \ell \| G \|_s^2 \| \Delta Q \| \tag{40}$$

where the first inequality follows since $G^T G \Delta Q$ is an $\ell \times \ell$ matrix. Using (38) to (40) in (29) to (31) results in (34), thereby completing the proof.

    __Remark:__ For $\Delta Q \geq 0$ and $\Delta R \geq 0$ (respectively, $\Delta Q \leq 0$ and $\Delta R \leq 0$), Nishimura [11] has given bounds on the matrix P of the form $\bar{P}_M \geq P$ (respectively, $P \geq \underline{P}_M$). These results also hold for the nonstationary case. Note that the results of Theorems 3 and 4 do not require any sign definiteness for ΔQ and ΔR, although they are limited to the stationary case. However, unlike the analysis in [11], the results of the theorems do not require that the filter gain K_M be obtained by a Kalman filter design ((11) to (13)).

    5. *Performance Bounds with Incorrect Process Configuration.* We now assume that $\Delta Q = 0$ and $\Delta R = 0$, i.e., Q and R, are known exactly. Theorems 5 and 6 then give an upper bound for $tr(P)$ and a lower bound

for $tr(P_F)$, respectively. The analysis used to obtain the upper bound for $tr(P)$ is more involved than that in Sec. 4 since (15) is coupled with (16) and (17). To derive an upper bound, we first analyze (17) for the stationary case to obtain the following.

Lemma: If A_M is diagonalizable, then, for stationary U,

$$\|U\|_s \leq \beta \equiv \frac{1}{2} \|U_M\|_s \exp\left[-\frac{\kappa^2}{2\sigma} \|I_n - F_M\|_s^2 \left(\|\Delta A\|^2 + 2\|\Delta A\|\ \|A_M\|_s\right)\right]$$

(41)

where

$$F_M \equiv (A_M^T + I_n)^{-1}(A_M^T - I_n)$$

(42)

$\sigma \equiv \max[\text{Re } \lambda(F_M)] < 0$ (real part of dominant eigenvalue of F_M)

$\kappa \equiv \|T\|_s \|T^{-1}\|_s$ (spectral condition number of matrix T)

T similarity transformation matrix to diagonalize A_M

and U_M is the solution of the Lyapunov equation:

$$F_M^T U_M + U_M F_M = -(I_n - F_M^T)GQG^T(I_n - F_M)$$

Remarks: (a) It is easy to show that a similarity transformation matrix T that diagonalizes A_M also diagonalizes F_M.

(b) The diagonalizability condition in the above lemma can be removed by the following procedure: when a model matrix A_M' is not diagonalizable, we can always find a diagonalizable matrix A_M such that $\|A_M - A_M'\|_s$ is as small as we wish [4, p.111]. The lemma can then be used for such A_M.

(c) The (Cayley) transformation in (42) maps the eigenvalues of A_M inside the unit circle to the left-half complex plane, $\text{Re}(\lambda) < 0$.

Proof of Lemma: For stationary U, (17) can be written as

$$U = A_M U A_M^T + \Delta A U \Delta A^T - \Delta A U A_M^T - A_M^T U \Delta A + G Q G^T \tag{43}$$

From (42),

$$A_M = (I_n - F_M^T)^{-1}(I_n + F_M^T) \tag{44}$$

Pre- and postmultiplying (43) by $(I_n - F_M^T)$ and $(I_n - F_M)$ respectively, and substituting for A_M from (44) yields:

$$F_M^T U + U F_M = -\frac{1}{2}(I_n - F_M^T)(GQG^T + \Delta A U \Delta A^T - \Delta A U A_M^T - A_M U \Delta A^T)(I_n - F_M)$$

Therefore, U can be expressed by the following integral [1, p. 239]:

$$U = \frac{1}{2} U_M - \frac{1}{2} \int_0^\infty [\exp(F_M^T t)](I_n - F_M^T)(\Delta A U A_M^T + A_M U \Delta A^T$$

$$- \Delta A U \Delta A^T)(I_n - F_M)[\exp(F_M t)] dt$$

which yields the norm inequality

$$\|U\|_s \leq \frac{1}{2}\left[\|U_M\|_s + \int_0^\infty \|I_n - F_M\|_s^2\left(\|\Delta A\|^2 \right.\right.$$

$$\left.\left. + 2\|\Delta A\| \|A_M\|_s\right)\|\exp(F_M t)\|_s^2 \|U\|_s \, dt\right]$$

Applying the Bellman-Gronwall lemma [7, p. 420] yields

$$\|U\|_s \leq \frac{1}{2}\|U_M\|_s \exp\left[\|I_n - F_M\|_s^2\left(\|\Delta A\|^2\right.\right.$$

$$\left.\left. + 2\|\Delta A\| \|A_M\|_s\right)\int_0^\infty \|\exp(F_M t)\|_s^2 \, dt\right] \tag{45}$$

Since T diagonalizes F_M, it follows that

$$\exp(F_M t) = T \exp(\Lambda t) T^{-1}$$

where Λ is a diagonal matrix with the eigenvalues of F_M on its diagonal. Therefore,

$$\|\exp(F_M t)\|_s \leq \|T\|_s \|T^{-1}\|_s \|\exp(\Lambda t)\|_s = \kappa \exp(\sigma t)$$

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Hence

$$\int_0^\infty \| \exp(F_M t) \|_s^2 \, dt \le \int_0^\infty \kappa^2 \exp(2\sigma t)dt = -\frac{\kappa^2}{2\sigma}$$

Substituting the above result into (45) yields (41).

We can now derive the following upper bound for $tr(P)$ from Theorem 1.

<u>Theorem 5</u>: If $\Delta Q = 0$ and $\Delta R = 0$, then an upper bound for $tr(P)$ is given by

$$tr(P) \le \pi_5 \equiv \left(\mu_1 + \sqrt{\mu_1^2 + \mu_2} \right)^2 \tag{46}$$

where

$$\mu_1 \equiv \sqrt{\beta} \, \| (M^{-1})^T i \| \, \| LA_M \|_s \left(\| L \|_s \| \Delta A \| \right.$$

$$+ \| K_M \|_s \| A_M \|_s \| \Delta H \| + \| K_M \|_s \| \Delta H \| \, \| \Delta A \| \big)$$

and

$$\mu_2 \equiv i^T M^{-1} b + \| (M^{-1})^T i \| \left[\beta \left(\| L \|_s \| \Delta A \| + \| K_M \|_s \| A_M \|_s \| \Delta H \| \right. \right.$$

$$+ \| K_M \|_s \| \Delta H \| \, \| \Delta A \| \big)^2 + \| K_M \|_s^2 \| GQG^T \|_s \| \Delta H \|^2$$

$$+ 2 \| K_M \|_s \| LGQG^T \|_s \| \Delta H \| \Big]$$

<u>Proof of Theorem 5</u>: Since $\Delta Q = 0$ and $\Delta R = 0$ in (26),

$$\| D \| \le 2 \| LGQG^T \Delta H^T K_M^T \| + \| K_M \Delta HGQG^T \Delta H^T K_M^T \| + 2 \| \Delta CVA_M^T L^T \|$$

$$+ \| \Delta CU\Delta C^T \|$$

$$\tag{47}$$

$$\le 2 \| LGQG^T \|_s \| K_M \|_s \| \Delta H \| + \| K_M \|_s^2 \| GQG^T \|_s \| \Delta H \|^2$$

$$+ 2 \| \Delta C \| \, \| V \|_s \| LA_M \|_s + \| \Delta C \|^2 \| U \|_s$$

From (27),

$$\| \Delta C \| \leq \| L \|_s \| \Delta A \| + \| K_M \|_s \| A_M \|_s \| \Delta A \| + \| K_M \|_s \| \Delta A \| \ \| \Delta H \|$$

$$(48)$$

Since

$$\begin{bmatrix} P & V^T \\ V & U \end{bmatrix} = E \begin{bmatrix} \tilde{\tilde{x}}\tilde{\tilde{x}}^T & \tilde{\tilde{x}}\tilde{x}^T \\ \tilde{x}\tilde{\tilde{x}}^T & \tilde{x}\tilde{x}^T \end{bmatrix}$$

is a covariance matrix, it is positive semidefinite, and with $P > 0$,

Lemma 3 (see Appendix) yields

$$\| V \|_s \leq \| P \|_s^{1/2} \| U \|_s^{1/2} \leq [tr(P)]^{1/2} \| U \|_s^{1/2} \qquad (49)$$

Using inequalities (47) to (49) in (22), we get a quadratic inequality
in $[tr(P)]^{1/2}$:

$$tr(P) - 2\mu_1 [tr(P)]^{1/2} - \mu_2 \leq 0$$

which yields (46).

Theorem 6: If $\Delta Q = 0$ and $\Delta R = 0$, then a lower bound for $tr(P_F)$
is given by

$$tr(P_F) \geq \pi_6 \equiv \frac{-\gamma_2 + \sqrt{\gamma_2^2 + 4\gamma_1\gamma_3}}{2\gamma_1} \qquad (50)$$

where

$$\gamma_1 \equiv \left(\| H_M \|_s + \| \Delta H \| \right)^2 \| R^{-1} \|_s \left(\| A_M \|_s + \| \Delta A \| \right)^2$$

$$\gamma_2 \equiv n + \left(\| H_M \|_s + \| \Delta H \| \right)^2 \| R^{-1} \|_s \ tr(GQG^T)$$

$$- \frac{1}{n} \left[|tr(A_M)| - \sum_{i=1}^{n} |\Delta A_{ii}| \right]^2$$

and

$$\gamma_3 \equiv tr(GQG^T)$$

where ΔA_{ii} denotes the (i,i)th term of ΔA.

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<u>Proof of Theorem 6</u>: As in the proof of Theorem 4, we bound a_1 and a_2 from above and a_3 from below in (28) to obtain π_6. Therefore,

$$a_1 = \| A^T H^T R^{-1} H A \|_s \leq \| H \|_s^2 \| R^{-1} \|_s \| A \|_s^2$$

$$\leq \left(\| H_M \|_s + \| \Delta H \| \right)^2 \| R^{-1} \|_s \left(\| A_M \|_s + \| \Delta A \| \right)^2 = \gamma_1$$

$$a_2 = n + \text{tr}(H^T R^{-1} H G Q G^T) - \sum_{i=1}^{n} |\lambda_i(A)|^2$$

$$\leq n + \| H^T R^{-1} H \|_s \text{tr}(G Q G^T) - \frac{1}{n} [\text{tr}(A)]^2$$

$$\leq n + \left(\| H_M \|_s + \| \Delta H \| \right)^2 \| R^{-1} \|_s \text{tr}(G Q G^T)$$

$$- \frac{1}{n} [\text{tr}(A_M) - \text{tr}(\Delta A)]^2$$

where Lemma 1 (i) and Lemma 2 are used to obtain the first inequality for a_2. Since

$$[\text{tr}(A_M) - \text{tr}(\Delta A)]^2 \geq \Big[|\text{tr}(A_M)| - |\text{tr}(\Delta A)| \Big]^2$$

and

$$|\text{tr}(\Delta A)| \leq \sum_{i=1}^{n} |\Delta A_{ii}|$$

it follows that $a_2 \leq \gamma_2$. Finally, we note that $a_3 = \gamma_3$ and that in (28) π_2 decreases monotonically as a_1 and a_2 increase. Consequently, the above bounds suffice to obtain the bound π_6.

6. *Example.* To illustrate the results presented in the preceding sections, we consider a process with the following state space description:

$$x(k + 1) = \begin{bmatrix} a_{M1} & 0 \\ a_{M2} & a_{M3} \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k)$$

$$y(k + 1) = [1 \quad 0] x(k + 1) + v(k + 1)$$

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The numerical values for a model are specified as $a_{M1} = -1/3$, $a_{M2} = 1/10$, and $a_{M3} = -1/4$. The modeled noise variances of the zero mean white noises are

$$q_M = E[w(k)^2]_{model} = 10 \; ; \quad r_M \equiv E[v(k)^2]_{model} = 5$$

The parameters a_{M2} and a_{M3} are assumed to be correct and the modeling error bounds for the parameters a_{M1}, q_M, and r_M are given by

$$|\Delta a_{M1}| \leq 0.1 \; , \quad |\Delta q| \leq 1 \; , \quad \text{and} \quad |\Delta r| \leq 0.5$$

For the numerical values specified, Theorems 3 to 6 yield the following bounds:

Theorem 3:

    Case 1 $\pi_3 = 3.567 \; (3.519)$

    Case 2 $\pi_3 = 3.748 \; (3.644)$

Theorem 4:

    Case 1 $\pi_4 = 2.206 \; (3.295)$

    Case 2 $\pi_4 = 2.432 \; (3.171)$

Theorem 5: $\pi_5 = 4.246 \; (3.450)$

Theorem 6: $\pi_6 = 2.506 \; (3.390)$

For comparison, the least upper bounds for $tr(P)$ and the greatest lower bounds for $tr(P_F)$ are given in parentheses. Cases 1 and 2 for Theorems 3 and 4 denote the following:

    Case 1: $|\Delta q| \leq 1 \; , \quad \Delta r = 0$

    Case 2: $|\Delta r| \leq 0.5, \quad \Delta q = 0$

7. *Conclusions.* The performance of Kalman-type, stationary, linear, discrete-time filters in the presence of modeling errors has been analyzed. The mean-square error of the estimates was used as the performance measure; modeling errors in the system configuration (ΔA and ΔH) and in the noise covariances (ΔQ and ΔR) were considered. Upper bounds for the performance measure of suboptimal filters with modeling errors are given in Theorems 1, 3, and 5 and lower bounds for the optimal filters without errors are given in Theorems 2, 4, and 6. The bounds in Theorems 3 to 6 require knowledge of only the model matrices and the range of errors of these matrices. Consequently, these bounds are useful in practice, as a designer often has information on the range of modeling errors rather than on the exact values of the error matrices.

It has been implicitly assumed in the derivation of the bounds that stability of the system and the filter is preserved in the presence of modeling errors ΔA and ΔH and that sign definiteness of Q and R is preserved in the presence of modeling errors ΔQ and ΔR. It should also be noted that the bounds obtained in this paper may be conservative for some systems, e.g., those with very small stability margins, and it may be desirable to obtain tighter bounds for specific cases.

8. *Appendix: Inequalities for Positive Semidefinite Matrices.* Some of the inequalities used in the proofs of the theorems presented in the text are proved below.

Lemma 1: (i) If $A \geq 0$ and $B \geq 0$, then

$$\text{tr}(AB) \leq \|A\|_s \, \text{tr}(B) \leq \text{tr}(A)\text{tr}(B)$$

(ii) If $A > 0$ and $B \geq 0$, then

$$\text{tr}(A^{-1}B) \geq \frac{\text{tr}(B)}{\|A\|_s} \geq \frac{\text{tr}(B)}{\text{tr}(A)}$$

Proof of Lemma 1: (i) Since $A \geq 0$ and $B \geq 0$, $A^{1/2}$ and $B^{1/2}$ exist; hence

$$\text{tr}(AB) = \text{tr}(A^{1/2}B^{1/2}B^{1/2}A^{1/2})$$

$$= \|A^{1/2}B^{1/2}\|^2 \leq \|A^{1/2}\|_s^2 \|B^{1/2}\|^2 = \|A\|_s \text{tr}(B)$$

and since $A \geq 0$, $\|A\|_s \leq \text{tr}(A)$, thereby completing the proof.

(ii) To show the second inequality, we write

$$\|A\|_s \, \text{tr}(A^{-1}B) = \|A^{1/2}\|_s^2 \, \text{tr}(A^{-1/2}B^{1/2}B^{1/2}A^{-1/2})$$

$$= \|A^{1/2}\|_s^2 \|A^{-1/2}B^{1/2}\|^2 \geq \|B^{1/2}\|^2 = \text{tr}(B)$$

and since $A > 0$, $\text{tr}(A) \geq \|A\|_s$, thereby completing the proof.

Lemma 2: If $A \geq 0$ and ϕ is an $n \times n$ matrix,

$$\text{tr}(A^{-1}\phi A\phi^T) \geq \sum_{i=1}^{n} |\lambda_i(\phi)|^2 \geq \frac{1}{n} [\text{tr}(\phi)]^2$$

Proof of Lemma 2:

$$\text{tr}(A^{-1}\phi A\phi^T) = \text{tr}(A^{-1/2}\phi A^{1/2}A^{1/2}\phi^T A^{-1/2})$$

$$= \|A^{-1/2}\phi A^{1/2}\|^2$$

Since $\|W\|^2 \geq \sum_{i=1}^{n} |\lambda_i(W)|^2$ for any $n \times n$ matrix W and

$$\lambda_i(A^{-1/2}\phi A^{1/2}) = \lambda_i(\phi),$$

$$\text{tr}(A^{-1}\phi A\phi^T) \geq \sum_{i=1}^{n} |\lambda_i(\phi)|^2$$

Using Cauchy's inequality [9, p. 42], we have

$$\sum_{i=1}^{n} |\lambda_i(\phi)|^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} |\lambda_i(\phi)|\right)^2$$

The result of the lemma then follows since

$$\sum_{i=1}^{n} |\lambda_i(\phi)| \geq \sum_{i=1}^{n} \lambda_i(\phi) = \text{tr}(\phi)$$

<u>Lemma 3</u>: If $A \geq 0$ is partitioned as

$$\begin{bmatrix} A_1 & A_3^T \\ A_3 & A_2 \end{bmatrix},$$

where $A_1 > 0$, then

$$\| A_1 \|_s \| A_2 \|_s \geq \| A_3 \|_s^2$$

<u>Proof of Lemma 3</u>: Since A_1^{-1} exists:

$$A = \begin{bmatrix} I_n & 0 \\ A_3 A_1^{-1} & I_n \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 - A_3 A_1^{-1} A_3^T \end{bmatrix} \begin{bmatrix} I_n & A_1^{-1} A_3^T \\ 0 & I_n \end{bmatrix}$$

Therefore, $A \geq 0$ implies that $A_2 - A_3 A_1^{-1} A_3^T \geq 0$. Using Weyl's inequality for eigenvalues [4, p. 157], we have

$$\max \lambda(A_2) - \max \lambda(A_3 A_1^{-1} A_3^T) \geq \min \lambda(A_2 - A_3 A_1^{-1} A_3^T) \geq 0$$

i.e.,

$$\| A_2 \|_s \geq \max \lambda(A_3 A_1^{-1} A_3^T)$$

$$= \max \lambda(A_1^{-1/2} A_3^T A_3 A_1^{-1/2})$$

$$= \| A_1^{-1/2} A_3^T A_3 A_1^{-1/2} \|_s$$

Therefore,

$$\| A_1 \|_s \| A_2 \|_s \geq \| A_1^{1/2} \|_s \| A_1^{-1/2} A_3^{T} A_3 A_1^{-1/2} \|_s \| A_1^{1/2} \|_s$$

$$\geq \| A_3^{T} A_3 \|_s = \| A_3 \|_s^2$$

thereby completing the proof.

REFERENCES

[1] R. BELLMAN, Introduction to Matrix Analysis, 2nd ed., McGraw-Hill Book Co., New York, 1970.

[2] E. BODEWIG, Matrix Calculus, North-Holland, Amsterdam, 1956.

[3] S. L. FAGIN, Recursive linear regression theory, optimal filter theory and error analysis of optimal systems, 1964 IEEE Intern. Conv. Rec., pp. 216-240.

[4] J. N. FRANKLIN, Matrix Theory, Prentice-Hall, Englewood Cliffs, 1968.

[5] R. E. GRIFFIN and A. P. SAGE, Large and small scale sensitivity analysis of optimum estimation algorithms, IEEE Trans. Automatic Control, AC-13 (Aug. 1968), pp. 320-329.

[6] H. HEFFES, The effect of erroneous models on the Kalman filter response, IEEE Trans. Automatic Control, AC-11 (July 1966), pp. 541-543.

[7] J. C. HSU and A. U. MEYER, Modern Control Principles and Applications, McGraw-Hill Book Co., New York, 1968.

[8] R. E. KALMAN and R. S. BUCY, New results in linear filtering and prediction theory, Trans. ASME, J. Basic Eng., Ser. D, 83 (March 1961), pp. 95-108.

[9] D. S. MITRINOVIĆ, Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[10] T. NISHIMURA, <u>On the a priori information in sequential estimation problems</u>, IEEE Trans. Automatic Control, AC-11 (April 1966), pp. 197-204.

[11] T. NISHIMURA, <u>Modeling errors in Kalman filters</u>, Ch. 4 in Theory and Application of Kalman Filtering, Editor: C. T. Leondes, AGARDograph 139, Feb. 1970.

[12] M. TODA and R. V. PATEL, <u>Performance bounds for stationary linear continuous filters in the presence of modeling errors</u>, submitted to IEEE Trans. Automatic Control.