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PREFACE

The work described in this report was conducted in the Image Processing Laboratory of the Earth and Space Sciences Division of the Jet Propulsion Laboratory.

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ABSTRACT

This paper discusses the relationships between the coordinates of a point on the surface of an oblate spheroid and the coordinates of the projection of that point in several common map projections. Since several of the projections are conformal, some background material is presented which summarizes the theory of conformally mapping an oblate spheroid to the plane. Then, for each projection considered, the equations which map the spheroid to the plane and their inverses are given

I. INTRODUCTION AND BASIC DEFINITIONS

The map projection program MAP2 requires the equations relating line and sample in some standard map projection to latitude and longitude on the surface of planet. The program must go in both directions; that is, given latitude and longitude it must be able to determine line and sample and also given line and sample it must be able to determine latitude and longitude. Equations for line and sample for many different projections from a sphere to a plane abound in the literature. The inverse equations are hard to find and in some cases are not easy to obtain via algebraic manipulation of the direct (line-sample from latitude-longitude) set. Equations in either direction for projections from the oblate spheroid to the plane are also rare in the literature.

Reference 1 contains the equations in the inverse direction (obtain latitude-longitude from line-sample) as they were used by MAP2 up until recently. However, those equations are valid only for projections from a sphere, except for the orthographic projection which properly handles the spheroidal case. Also Reference 1 contains no direct equations for any projection. The purpose of this document is to bring together in one place the twelve sets of equations currently used by MAP2. These are the equations for the six projections currently implemented: Polar Orthographic, Oblique Orthographic, Polar Stereographic, Oblique Stereographic, Two-Standard Lambert Conformal Conic, and Mercator projections. Equations for both directions are included. The equations are exactly correct for the spheroidal case. Only the inverse equations (latitude-longitude from line-sample) for the orthographic are identical to those in Reference 1. All the other sets are either not in the reference (all direct equations), or have been extended to correctly account for oblateness. In addition, for the oblique orthographic, the technique given in Reference 1 for the determination of which of two solutions to a quadratic equation is the desired one has been modified. The old technique failed if the pole appeared in the output picture.

Before the equations are presented, some basic definitions and standard notation will be given:

Equation of a sphere of radius R:

$$\frac{X^2}{R^2} + \frac{Y^2}{R^2} + \frac{Z^2}{R^2} = 1 \quad (1)$$

A standard three-dimensional Cartesian coordinate system is frequently used throughout this document. Its origin is at the center of the planet. Its +z-axis points to the north pole and its +x-axis points to the prime meridian. In such a system the equation of an oblate spheroid of equatorial radius R_{eq} and polar radius R_P is:

$$\frac{X^2}{R_{eq}^2} + \frac{Y^2}{R_{eq}^2} + \frac{Z^2}{R_P^2} = 1 \quad (2)$$

Throughout this discussion west longitudes will be used. East longitudes are more natural but for historical reasons MAP2 deals in west longitudes.

Latitude can be defined in several ways and on a sphere these definitions are all equivalent. On the spheroid, however, they are not. The two most commonly used latitudes are geocentric and geodetic. Geocentric is defined to be the angle at the center of the planet between the plane of the equator and the line to the point of interest. Geodetic latitude is the angle between the plane of the equator and the local normal to the surface of the spheroid at the point of interest. For clarity, west longitude will always be denoted λ ; geocentric latitude θ ; and geodetic latitude ϕ . MAP2 assumes that geocentric latitude is always desired. That is, in the direct mode west longitude and geocentric latitude are supplied and line and sample are desired. In the inverse mode line and sample are given and we are asked to compute west longitude and geocentric latitude. However, geodetic latitude and another type of latitude, called conformal, will frequently be used during intermediate calculations. Figure 1 shows the relationship between geocentric and geodetic latitude. Conformal latitude is discussed in Section II. It has no obvious geometrical significance on the spheroid.

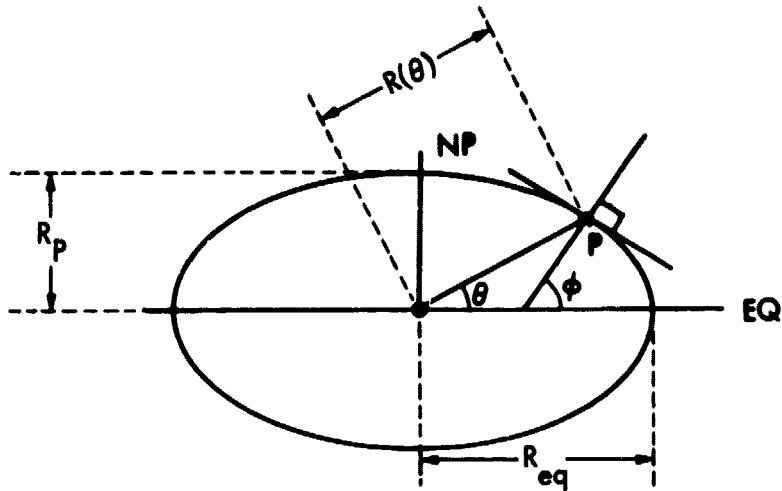


Figure 1. Geodetic vs Geocentric Latitude

It is easy to compute θ from ϕ or vice versa:

$$|\phi| = \frac{\pi}{2} - \left| \arctan \left[-\frac{R_P^2}{R_{eq}^2} \cot \theta \right] \right| \quad (3)$$

$$|\theta| = \arctan \left[-\frac{R_P^2}{R_{eq}^2} \cot \left(|\phi| - \frac{\pi}{2} \right) \right]$$

sign of ϕ = sign of θ .

If $\phi = 0$, then $\theta = 0$. If $\phi = \pi/2$, $\theta = \pi/2$. For the remainder of this document angles will be specified strictly in radians.

Two radii are of importance at an arbitrary point on a spheroid. One, which will be called R , is the distance from the center of the planet to the point.

$$R(\theta) = \frac{R_P R_{eq}}{\sqrt{R_P^2 \cos^2 \theta + R_{eq}^2 \sin^2 \theta}} \quad (4)$$

The other, which shall be called N , is the local radius of curvature of the surface in the direction perpendicular to the local longitude meridian.

$$N = \frac{R_{eq}^2}{\sqrt{R_{eq}^2 \cos^2 \phi + R_P^2 \sin^2 \phi}} \quad (5)$$

Note that geocentric latitude enters into the equation for R , whereas geodetic latitude is used in the equation for N . The vector from the center of the planet to a point at (λ, θ) in our standard 3-D system is:

$$\begin{aligned} X &= R(\theta) \cos \lambda \cos \theta \\ Y &= -R(\theta) \sin \lambda \cos \theta \\ Z &= R(\theta) \sin \theta \end{aligned} \quad (6)$$

The minus sign in the equation for Y is due to our use of west longitudes.

Whenever a radius, be it R_P , R_{eq} , $R(\theta)$, or $N(\phi)$, appears in an equation involving the output (projected) image, it will be assumed to be in units of pixels, not kilometers. If F is the scale at the equator (Mercator), standard parallels (Lambert), or center of projection (Orthographic and Stereographic), in kilometers per pixel, then

$$R \text{ (pixels)} = R \text{ (km)} / F. \quad (7)$$

The 2-dimensional system used to describe the projection plane follows MAP2's and IPL's conventions. Its origin is at line 0, sample 0. Its +x-axis points to increasing sample (right); its +z-axis points toward increasing line (down).

In all projections some longitude will have special significance. In Oblique Orthographic and Stereographic it is the longitude of the center of projection. In the polar projections it is the longitude which points up (decreasing line) as seen at the projection of the pole in the projection plane. In the Lambert it is the longitude of the central meridian (which projects to a vertical line and is the only such longitude). In the Mercator it is simply

an arbitrary longitude whose sample in the projection is known in advance. In all cases the special longitude will be denoted λ_0 . In the oblique projections the latitude of the center of projection is θ_0 (or ϕ_0 for geodetic). In addition, the oblique projections have associated with them an angle, denoted ψ , which is the angle in the projection plane of north, measured at the center of projection clockwise from up (See Figure 2).



Figure 2. The North Angle ψ

In each projection there is some special point whose projected coordinates are XC , ZC (sample, line respectively). For the oblique projections the special point is the center of projection, so (λ_0, θ_0) projects to (XC, ZC) . For the Lambert and the polar projections the special point is the one (the only one) pole which is visible on the projection. These projections have associated with them a variable denoted by CAS .

$$\begin{aligned} CAS &= +1 \text{ if visible pole is north pole} \\ CAS &= -1 \text{ if visible pole is south pole} \end{aligned} \tag{8}$$

For the Mercator projection the special point is the point on the equator at longitude λ_0 . Thus the longitude λ_0 will project to sample XC , and the equator will project to line ZC . The various projections are characterized by:

- a) Polar Orthographic and Polar Stereographic: $F, \lambda_0, CAS, XC, ZC$.
- b) Oblique Orthographic and Stereographic: $F, \lambda_0, \theta_0, \psi, XC, ZC$.

- c) Lambert: θ_1, θ_2 (latitudes of standard parallels), F, XC, ZC, λ_0 (CAS is determined by θ_1 and $\theta_2 \dots$ the visible pole has the same sign latitude as θ_1 and θ_2).
- d) Mercator : λ_0, F, ZC, XC

For each projection the characterizing parameters are assumed to be known in advance. In MAP2 they are either specified as parameters or, if defaulted, are pre-computed by special subroutines which set the parameters in order to center the input picture in the output picture and fill as much of the output as possible without loss of data beyond the boundaries of the output frame. Where applicable, ψ is determined by minimizing the rotation between input and output frames. In fact, it is only for determination of default values of the characterizing parameters that MAP2 needs the direct equations at all.

All of the currently implemented projections except Orthographic are conformal projections. In order to preserve conformality when the object is flattened, a general theory of conformal mapping must be used. Section II summarizes the important points of this theory. The results of Section II are used repeatedly throughout subsequent sections. Throughout these sections, $\Delta\lambda = \lambda - \lambda_0$, $\Delta X = X - XC$, and $\Delta Z = Z - ZC$.

II. CONFORMAL MAPPING FROM THE SPHEROID TO THE PLANE

While it is theoretically possible to conformally project a spheroid directly to a plane, the equations are much simpler (especially in the inverse direction) if the spheroid is first conformally projected onto a sphere. Then the intermediate sphere is projected onto the plane in the usual way. The "double" projections so produced differ from the direct projections only to terms of fourth order or higher of ϵ , where

$$\epsilon = \sqrt{1 - \frac{R_P^2}{R_{eq}^2}} \quad (9)$$

For Mars this means "accuracy" to one part in 10^4 if you insist that the direct method is better. Since both methods preserve conformality exactly, there seems to be no reason to prefer the direct method, and the double projection via an intermediate sphere is used in MAP2.

In the first projection, from the spheroid to the sphere, Reference 2 shows that the transformation is given by:

$$\lambda' = \lambda \tag{10}$$

$$\phi' = \theta' = \chi,$$

where the primed coordinates refer to the intermediate sphere (thus the equality between geocentric and geodetic latitudes).

Thus the first projection preserves longitude, and latitude on the sphere equals χ , the so-called conformal latitude on the spheroid (this is the third type of latitude mentioned in the previous section). Reference 2 gives the relationship between conformal and geodetic latitudes as:

$$\tan\left(\frac{\pi}{4} + \frac{\chi}{2}\right) = \left(\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right) \left(\frac{1 - \varepsilon \sin \phi}{1 + \varepsilon \sin \phi}\right)^{\varepsilon/2} \tag{11}$$

where ε is given by Eq. 9. Throughout the rest of this document the Greek letter CHI (χ) is consistently used for conformal latitude. Obviously, if ϕ is known, χ is easy to compute. If the right side of Eq. 11 is denoted by Q , then:

$$\chi = 2 \left[\arctan Q - \frac{\pi}{4} \right] \tag{12}$$

$$\sin \chi = \frac{Q^2 - 1}{Q^2 + 1} \tag{13}$$

$$\cos \chi = \frac{2Q}{Q^2 + 1} \tag{14}$$

Unfortunately the inverse problem, obtaining ϕ given χ , is not so simple. The equation is transcendental and cannot be solved explicitly for ϕ . MAP2 solves the problem numerically via a successive approximation technique as follows:

First rearrange Eq. 11:

$$\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \left(\tan\left(\frac{\pi}{4} + \frac{\chi}{2}\right)\right) \left(\frac{1 + \epsilon \sin \phi}{1 - \epsilon \sin \phi}\right)^{\epsilon/2} \quad (15)$$

As a first guess, replace all appearances of ϕ on the right with χ . Thus

$$Q_1 = \tan\left(\frac{\pi}{4} + \frac{\chi}{2}\right) \left(\frac{1 + \epsilon \sin \chi}{1 - \epsilon \sin \chi}\right)^{\epsilon/2} \quad (16)$$

$$\phi_1 = 2\left(\arctan Q_1 - \frac{\pi}{4}\right) \quad (17)$$

Now use ϕ_1 as a guess for ϕ and compute

$$Q_2 = \tan\left(\frac{\pi}{4} + \frac{\chi}{2}\right) \left(\frac{1 + \epsilon \sin \phi_1}{1 - \epsilon \sin \phi_1}\right)^{\epsilon/2} \quad (18)$$

$$\phi_2 = 2\left(\arctan Q_2 - \frac{\pi}{4}\right). \quad (19)$$

Repeat this process until $|\phi_{i+1} - \phi_i| \leq$ some tolerance value, which MAP2 has currently set at 10^{-7} radians. The process converges extremely rapidly - on Mars, where $\epsilon \approx 0.1$, 4 iterations are almost always sufficient.

An alternative which will converge even faster in some cases is to use a Newton's method technique. This seems especially attractive since the derivative of the right side of Equation 11 has a surprisingly simple form, namely

$$\frac{d}{d\phi} \left[\tan\left(\frac{\pi}{4} + \frac{\chi}{2}\right) \right] = \frac{(1 - \epsilon^2) \sec \phi}{1 - \epsilon^2 \sin^2 \phi}. \quad (20)$$

However, sad practical experience shows that this method fails near the pole due to the fact that the second derivative of $\tan (\pi/4 + \chi/2)$ is approximately $\sec \phi$ times the first derivative. Near $\pi/2$, $\sec \phi$ approaches infinity so the basic assumption of Newton's method, that the function is approximately linear, fails. The successive approximation technique of Eqs. 15-19 seems stable at all latitudes.

The generally inferior Newton's method technique is mentioned here only because MAP2 actually uses it in the Lambert conformal conic projection. The code for this projection was the first to be written, before the method's drawbacks were discovered. Since this projection is rarely applied near the pole, it was not felt desirable to modify an already working code.

To summarize, all conformal projections from a spheroid are easily handled by using the standard spherical equations with two easily implemented modifications:

- 1) Replace all latitudes in the spherical equations with χ , the conformal latitude on the spheroid.
- 2) For the radius of the sphere in the equations use the radius of the intermediate sphere. This radius, R_{int} , is given in Reference 2 as

$$R_{int} = \frac{N_0 \cos \phi_0}{\cos \chi_0}, \quad (21)$$

where the zero subscript refers to the center of projection.

Equation 21 breaks down at $\phi = \theta = \chi = 0$ and at $\phi = \theta = \chi = \pi/2$. So

$$\text{if } \phi = 0, \quad R_{int} = R_{eq} \quad (22)$$

$$\text{if } \phi = \frac{\pi}{2}, \quad R_{int} = \frac{R_{eq}^2}{R_P} \left(\frac{1 + \epsilon}{1 - \epsilon} \right)^{\epsilon/2}$$

The remainder of this document is a description of the equations on a projection by projection basis. For all except the orthographic, the

techniques of this section make the problem conceptually simple, though algebraically complex. For the orthographic projection, which is not conformal, Section II does not apply. But the simple geometrical definition of the orthographic projection saves us and allows the problem to be solved.

III. POLAR ORTHOGRAPHIC

A. Direct Equations

On the spheroid we define the orthographic projection to be one of true perspective. The plane of the projection is tangent to the spheroid at the center of projection. The perspective point is at infinity, so the perspective lines are all parallel to each other, perpendicular to the plane of projection, and thus parallel to the local surface normal at the center of projection.

The oblateness of the spheroid hardly raises any complications in the polar case, although it causes considerable trouble in the oblique case. In the polar case longitude meridians are straight lines radiating from the pole at their true orientation. Latitude lines are circles having their true radius. Thus we have

$$\begin{aligned} X &= R(\theta) \cos \theta \sin \Delta\lambda * CAS + XC \\ Z &= - R(\theta) \cos \theta \cos \Delta\lambda + ZC \end{aligned} \tag{23}$$

B. Inverse Equations

The above equations are easily inverted algebraically to produce

$$RAD = \sqrt{\Delta X^2 + \Delta Z^2} \tag{24}$$

$$\Delta\lambda = \arccos \left(- \frac{\Delta Z}{RAD} \right) \tag{25}$$

$$\lambda = \text{CAS} * \Delta\lambda - \lambda_0$$

$$\theta = \text{CAS} * \arctan\left(\sqrt{\frac{R_P^2}{\text{RAD}^2} - \frac{R_P^2}{R_{\text{eq}}^2}}\right) \quad (26)$$

If the quantity under the radical is negative, the point specified is not on the planet. Remember that the orthographic projection does not fill the projection plane.

IV. OBLIQUE ORTHOGRAPHIC

A. Direct Equations

Consider Figure 3.

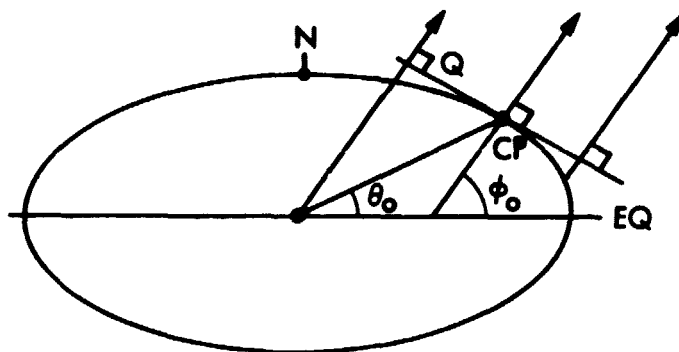


Figure 3. Oblique Orthographic Projection on the Spheroid

First compute the vector from the planet center to the point of interest in our standard 3-D coordinate system.

$$\begin{aligned} X &= R(\theta) \cos \lambda \cos \theta \\ Y &= -R(\theta) \sin \lambda \cos \theta \\ Z &= R(\theta) \sin \theta \end{aligned} \quad (27)$$

Next rotate these coordinates to a system whose z-axis is parallel to the perspective lines, and whose +x-axis points south (in the projection plane, which is now parallel to the x-y plane). It should be clear that in the new coordinate system the x and y coordinates are directly related to line and sample in the projection plane. The required rotation is a double one. First rotate by $-\lambda_0$ about the z-axis; then by $\pi/2 - \phi_0$ about the new y-axis. Note that the geodetic latitude of the center of projection is involved here. Thus we obtain

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} \sin \phi_0 \cos \lambda_0 & -\sin \phi_0 \sin \lambda_0 & -\cos \phi_0 \\ \sin \lambda_0 & \cos \lambda_0 & 0 \\ \cos \phi_0 \cos \lambda_0 & -\cos \phi_0 \sin \lambda_0 & \sin \phi_0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad (28)$$

Multiplying out Eq. 28, substituting Eq. 27 into the result, ignoring Z' , algebraically simplifying, and renaming variables to conform more closely to the MAP2 2-D standard X - Z system, results in

$$\begin{aligned} X'' &= -R(\theta) \cos \theta \sin \Delta \lambda \\ Z'' &= R(\theta) [\sin \phi_0 \cos \theta \cos \Delta \lambda - \cos \phi_0 \sin \theta] \end{aligned} \quad (29)$$

The double primes in (29) are there as reminders that we have not yet reached our standard system in which X points to increasing sample, Z to increasing line, with the center at (0, 0). Due to the rotation in (28), Z'' points south, not to increasing line. Also the origin is not (0, 0) or even the center of projection, but instead is the intersection of the projection plane and the perspective line passing through the center of the planet (point Q in Fig. 3). First, translate so that the center of projection is at the origin:

$$\begin{aligned} X' &= X'' \\ Z' &= Z'' - R(\theta_0) \sin(\phi_0 - \theta_0) \end{aligned} \quad (30)$$

Finally, rotate about an axis perpendicular to the projection plane by $-\psi$ and translate to $(0, 0)$ as origin:

$$\begin{aligned} X &= X' \cos \psi - Z' \sin \psi + XC \\ Z &= X' \sin \psi + Z' \cos \psi + ZC \end{aligned} \quad (31)$$

Equations 29, 30, and 31 suffice to compute X and Z for any specified (λ, θ) in the oblique orthographic projection.

B. Inverse Equations

These equations are presented here in cookbook style for the sake of completeness. They were originally derived by Arnie Schwarz and are described in detail in Reference 1. The basic plan is to determine, using analytic geometry, the point of intersection between the spheroid and the perspective line passing through the given point on the plane of projection.

$$\Omega \equiv \theta_0 + \arccos \left[\frac{\frac{\cos^2 \theta_0}{R_{eq}^2} + \frac{\sin^2 \theta_0}{R_P^2}}{\sqrt{\frac{\cos^2 \theta_0}{R_{eq}^4} + \frac{\sin^2 \theta_0}{R_P^4}}} \right] \quad (32)$$

$$L \equiv \theta_0 - \Omega \quad (33)$$

$$\delta = \psi + \arctan \frac{-\Delta X}{-\Delta Z} \quad (\text{Remember, } \psi = \text{north angle}) \quad (34)$$

$$DD = \sqrt{(\Delta X)^2 + (\Delta Z)^2} \quad (35)$$

$$A1 = -\sin \delta \sin \lambda_0 - \cos \delta \cos \lambda_0 \sin \Omega$$

$$B1 = -\sin \delta \cos \lambda_0 + \cos \delta \sin \lambda_0 \sin \Omega$$

$$C1 = \cos \delta \cos \Omega$$

$$\begin{aligned} D1 = & -DD \sin^2 \delta + R(\theta_0) \cos \delta \sin \Omega \cos \theta_0 - R(\theta_0) \sin \theta_0 \cos \Omega \cos \delta \\ & -DD \cos^2 \delta \cos^2 \lambda_0 \sin \theta_0 \sin \Omega - DD \cos^2 \delta \cos^2 \Omega \\ & -DD \sin^2 \delta \cos^2 \delta \sin^2 \lambda_0 \end{aligned} \quad (36)$$

$$A2 = -\cos \delta \sin \lambda_0 \cos L + \sin \delta \sin \theta_0 \cos \lambda_0$$

$$B2 = -\cos \delta \cos \lambda_0 \cos L - \sin \delta \sin \theta_0 \sin \lambda_0$$

$$C2 = -\cos \theta_0 \sin \delta$$

$$\alpha = A2 \cdot C1 - A1 \cdot C2$$

$$\beta = A2 \cdot B1 - A1 \cdot B2$$

$$\gamma = B1 \cdot C2 - B2 \cdot C1$$

$$\epsilon = -\gamma$$

$$G2 = R_{eq}^2 \gamma^2$$

$$DRPSQ = R_P^2 \gamma^2$$

$$Z1 = DRPSQ (\alpha^2 + \gamma^2) + G2 \cdot \beta^2$$

$$K1 = \frac{\alpha \cdot C2 \cdot D1 \cdot DRPSQ + \beta \cdot B2 \cdot D1 \cdot G2}{Z1}$$

$$K2 = \frac{\gamma \epsilon}{Z1}$$

$$\begin{aligned}
K3 = & \left[2\alpha \cdot C2 \cdot \beta \cdot B2 \cdot R_{eq}^2 - (C2)^2 DRPSQ - (B2)^2 G2 - \alpha^2 (B2)^2 R_{eq}^2 \right. \\
& \left. - \beta^2 (C2)^2 R_{eq}^2 \right] (D1)^2 + G2 \cdot DRPSQ + R_{eq}^2 (DRPSQ) \alpha^2 \\
& + R_{eq}^2 \beta^2 \cdot G2
\end{aligned}$$

At this point check to see if $K3 < 0$. If it is, the point originally specified is off the planet - there is no intersection between the spheroid and the perspective line.

$$X1 = K1 + K2 \sqrt{K3}$$

$$X2 = K1 - K2 \sqrt{K3}$$

$$Y1 = \frac{-D1 \cdot C2 + \alpha \cdot X1}{\gamma}$$

(40)

$$Y2 = \frac{-D1 \cdot C2 + \alpha \cdot X2}{\gamma}$$

$$Z1 = \frac{-B2 \cdot D1 + \beta \cdot X1}{\epsilon}$$

$$Z2 = \frac{-B2 \cdot D1 + \beta \cdot X2}{\epsilon}$$

You now have two vectors, $(X1, Y1, Z1)$ and $(X2, Y2, Z2)$, which are the lines from the planet center to the two points of intersection between the spheroid and the perspective line, expressed in our standard 3-D coordinate system. Call the vectors \vec{V}_1 and \vec{V}_2 respectively.

Next compute the vector \vec{P} from the planet center to the center of projection

$$P_X = R(\theta_0) \cos \theta_0 \cos \lambda_0$$

$$P_Y = -R(\theta_0) \cos \theta_0 \sin \lambda_0 \quad (41)$$

$$P_Z = R(\theta_0) \sin \theta_0$$

Choose the vector from \vec{V}_1 and \vec{V}_2 which minimizes the quantity $|\vec{V}_1 - \vec{P}|$ and $|\vec{V}_2 - \vec{P}|$. Call the winning vector's components X, Y, and Z.

$$\lambda = -\arctan\left(\frac{Y}{X}\right) \quad (42)$$

$$\theta = \arctan\left(\frac{Z}{\sqrt{X^2 + Y^2}}\right)$$

V. POLAR STEREOGRAPHIC

A. Direct Equations

The equations in Subsection A are taken directly from Reference 3, which used the principles of Section II. First compute ρ , the length in pixels on the projection of the line from the pole to the point of interest.

$$\rho = 2 R_{eq} (1 + \epsilon)^{-(1-\epsilon)/2} (1 - \epsilon)^{-(1+\epsilon)/2} \tan\left(\frac{\pi}{4} - \frac{CAS*\phi}{2}\right) \left(\frac{1 + \epsilon \sin(CAS*\phi)}{1 - \epsilon \sin(CAS*\phi)}\right)^{\epsilon/2} \quad (43)$$

$$X = CAS \cdot \rho \sin \Delta\lambda + XC$$

$$Z = -\rho \cdot \cos \Delta\lambda + ZC \quad (44)$$

B. Inverse Equations

These equations follow from simple algebraic manipulation of the equations of Subsection A, above.

First, compute

$$C = 2 R_{eq} (1 + \epsilon)^{-(1-\epsilon)/2} (1 - \epsilon)^{-(1+\epsilon)/2} \quad (45)$$

and

$$\rho = \sqrt{(\Delta X)^2 + (\Delta Z)^2} \quad (46)$$

Then

$$\lambda = \arctan \left(\frac{-CAS \cdot \Delta X}{-\Delta Z} \right) + \lambda_0 \quad (47)$$

$$\chi = \frac{2}{CAS} \left[\frac{\pi}{4} - \arctan \left(\frac{\rho}{C} \right) \right] \quad (48)$$

Use the equations of Sections I and II to obtain ϕ and θ from χ .

VI. OBLIQUE STEREOGRAPHIC

A. Direct Equations

The spherical equations were taken from Reference 3 and modified according to the principles of Section II. First compute line and sample in a coordinate system centered at the center of projection and which has north at the top.

$$X' = \frac{-2R_{int} \cos \chi \sin \Delta \lambda}{1 + \sin \chi_0 \sin \chi + \cos \chi_0 \cos \chi \cos \Delta \lambda} \quad (49)$$

$$Z' = \frac{-2 R_{\text{int}} [\cos \chi_0 \sin \chi - \sin \chi_0 \cos \chi \cos \Delta \lambda]}{1 + \sin \chi_0 \sin \chi + \cos \chi_0 \cos \chi \cos \Delta \lambda}$$

Now rotate so north is at the desired angle and translate to the usual origin:

$$X = X' \cos \psi - Z' \sin \psi + XC \quad (50)$$

$$Z = X' \sin \psi + Z' \cos \psi + ZC$$

B. Inverse Equations

Direct algebraic inversion of the equations of Subsection A is very difficult so explicit advantage of the existence of the intermediate sphere is taken. Analytic geometry is used to find the intersection of the given perspective line of interest and the intermediate sphere. The latitude of the point of intersection on the sphere is the conformal latitude of the point of interest on the spheroid. The method works because the stereographic projection from a sphere is a true perspective projection with plane of projection tangent to the sphere and perspective point on the opposite side of the sphere from the center of projection, at coordinates $(\lambda_0 + \pi, -\chi_0)$. See Figure 4.

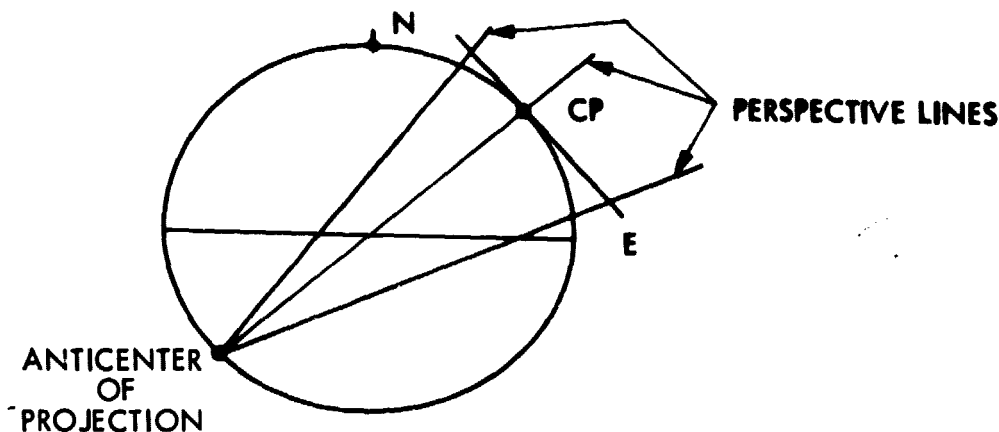


Figure 4. Oblique Stereographic Projection of a Sphere

Phase 1 – Translate the given projection plane coordinates to a new origin at (XC, ZC) and rotate so north is at the top (X' points east, Z' south) from the origin. For ease in conversion to our standard 3-D system, the Z' axis will be renamed Y' . This rotation is by $+\psi$.

$$\begin{aligned} X' &= \Delta X \cos \psi + \Delta Z \sin \psi \\ Y' &= -\Delta X \sin \psi + \Delta Z \cos \psi \end{aligned} \quad (51)$$

Phase 2 – Now consider X' , Y' to be two of the three coordinates in a non-standard 3-dimensional system whose origin is the center of the intermediate sphere. From the definition of X' and Y' above, and in order to obtain a right-handed system, the Z' axis of this system points to the anticenter of projection (see Figure 4).

Thus, the coordinates of the projection of the point of interest in this new system are:

$$\begin{aligned} X_1 &= X' \\ Y_1 &= Y' \\ Z_1 &= -R_{int} \end{aligned} \quad (52)$$

Similarly the coordinates of the anti-C. P. are:

$$\begin{aligned} X_2 &= 0 \\ Y_2 &= 0 \\ Z_2 &= +R_{int} \end{aligned} \quad (53)$$

We wish to find the intersection of the perspective line of interest, which passes through (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) , and the intermediate sphere. Equation of the sphere is

$$\frac{X^2}{R_{int}^2} + \frac{Y^2}{R_{int}^2} + \frac{Z^2}{R_{int}^2} = 1 \quad (54)$$

The equation of a line in 3-space is

$$\frac{X - X_1}{X_2 - X_1} = \frac{Y - Y_1}{Y_2 - Y_1} = \frac{Z - Z_1}{Z_2 - Z_1} \quad (55)$$

or, in our case,

$$\frac{X - X'}{-X'} = \frac{Y - Y'}{-Y'} = \frac{Z + R_{\text{int}}}{2 R_{\text{int}}} \quad (56)$$

There are two intersections of the perspective line and the sphere. One is just the anti-C. P. and is not of interest. Simple but tedious algebra shows that the coordinates of the interesting intersection are:

$$Z = R_{\text{int}} \frac{\left[X'^2 + Y'^2 - 4 R_{\text{int}}^2 \right]}{\left[X'^2 + Y'^2 + 4 R_{\text{int}}^2 \right]}$$

$$X = \frac{-X'}{2 R_{\text{int}}} (Z + R_{\text{int}}) + X' \quad (57)$$

$$Y = \frac{-Y'}{2 R_{\text{int}}} (Z + R_{\text{int}}) + Y'$$

Phase 3 — Rotate the coordinates just found to our standard 3-D system with +z-axis pointing to the north pole. This involves first a rotation about the x-axis by $\pi/2 + \chi_0$, then about the new z-axis by $\lambda_0 - \pi/2$. Thus

$$X_F = X \sin \lambda_0 + Y \sin \chi_0 \cos \lambda_0 - Z \cos \chi_0 \cos \lambda_0$$

$$Y_F = X \cos \lambda_0 - Y \sin \chi_0 \sin \lambda_0 + Z \cos \chi_0 \sin \lambda_0 \quad (58)$$

$$Z_F = -Y \cos \chi_0 - Z \sin \chi_0$$

(X_F, Y_F, Z_F) are our desired coordinates, and the latitude and longitude follow immediately.

$$\lambda = \arctan \left(\frac{-Y_F}{X_F} \right) \quad (59)$$

$$\chi = \arcsin \left(\frac{Z_F}{R_{int}} \right)$$

As usual, ϕ and θ must then be computed from χ . To summarize, the given X and Z plus the characterizing parameters of the projection allow Eq. 51 to be used to determine X' , Y' . Then Eq. 57 determines X , Y , Z . Equation 58 determines X_F , Y_F , Z_F and finally Eq. 59 gives χ and λ .

VII. TWO-STANDARD LAMBERT CONFORMAL CONIC PROJECTION

A. Direct Equations

These equations come directly from Ref. 3. First compute K , the constant of the cone, as follows:

$$K = \frac{\ln [N_1 \cos \phi_1] - \ln [N_2 \cos \phi_2]}{\ln \left[\left(\frac{1 + \epsilon \sin (\text{CAS}^* \phi_1)}{1 - \epsilon \sin (\text{CAS}^* \phi_1)} \right)^{\epsilon/2} \tan \left(\frac{\pi}{4} - \frac{\text{CAS}^* \phi_1}{2} \right) \right] - \ln \left[\left(\frac{1 + \epsilon \sin (\text{CAS}^* \phi_2)}{1 - \epsilon \sin (\text{CAS}^* \phi_2)} \right)^{\epsilon/2} \tan \left(\frac{\pi}{4} - \frac{\text{CAS}^* \phi_2}{2} \right) \right]} \quad (60)$$

where subscripts 1 and 2 refer to the latitudes of the standard parallels.

Next, compute C , the distance in pixels on the projection from the visible pole to any point on the equator.

$$C = \frac{N_1 \cos \phi_1}{K \left[\left(\frac{1 + \epsilon \sin (\text{CAS}^* \phi_1)}{1 - \epsilon \sin (\text{CAS}^* \phi_1)} \right)^{\epsilon/2} \tan \left(\frac{\pi}{4} - \frac{\text{CAS}^* \phi_1}{2} \right) \right]} \quad (61)$$

Next, get ρ = distance on projection from visible pole to point of interest:

$$\rho = C \left[\left(\frac{1 + \epsilon \sin(CAS*\phi)}{1 - \epsilon \sin(CAS*\phi)} \right)^{\epsilon/2} \tan \left(\frac{\pi}{4} - \frac{CAS*\phi}{2} \right) \right]^K \quad (62)$$

Finally:

$$\begin{aligned} X &= -\rho \sin(K \cdot \Delta\lambda) + XC \\ Z &= CAS* \rho \cos(K \cdot \Delta\lambda) + ZC \end{aligned} \quad (63)$$

B. Inverse Equations

These follow from direct algebraic manipulation of the equations of Subsection A.

$$\rho = \sqrt{(\Delta X)^2 + (\Delta Z)^2} \quad (64)$$

If $\rho = 0$, you are at the visible pole.

Check to see if

$$Z = \frac{\arctan \left(\frac{\Delta X}{CAS*\Delta Z} \right)}{K} \quad (65)$$

has an absolute value greater than π .

If it does, the point of interest is not on the planet (the Lambert projection does not fill the plane). If it does not, then

$$\begin{aligned} \lambda &= \lambda_0 - Z \\ \cot \left(\frac{\pi}{4} + \frac{CAS*\lambda}{2} \right) &= \left[\frac{\rho}{C} \right]^{1/K} \end{aligned} \quad (66)$$

Compute θ from ν in the usual way.

VIII. THE MERCATOR PROJECTION

A. Direct Equations

These were obtained from Reference 3.

$$X = -R_{eq} \cdot \Delta\lambda + XC \tag{67}$$

$$Z = -R_{eq} \ln \left[\left(\frac{1 - \epsilon \sin \phi}{1 + \epsilon \sin \phi} \right)^{\epsilon/2} \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right) \right] + ZC$$

B. Inverse Equations

These are obtained by inversion of Eq. 67:

$$\lambda = -\frac{\Delta X}{R_{eq}} + \lambda_0 \tag{68}$$

$$\tan \left(\frac{\pi}{4} + \frac{\chi}{2} \right) = e^{-\Delta Z/R_{eq}}$$

Compute θ from χ in the usual way.

Care must be taken in the Mercator projection to insure that the projection, which does not fill the plane, is not accidentally repeated indefinitely by ignoring this fact. The absolute value of $\Delta\lambda$ must never be allowed to exceed π . Thus in Eq. 67, if $|\lambda - \lambda_0| = |\Delta\lambda|$ is greater than π , add or subtract 2π to it to ensure that $|\Delta\lambda| \leq \pi$. In Eq. 68, if the quantity $|\Delta X|/R_{eq} > \pi$, the point specified is off the planet. If these precautions are taken, the longitude λ_0 becomes the central meridian, with exactly half the planet on its left and half on its right. Alternatively, you could restrict $\Delta\lambda$ to the range $-2\pi \leq \Delta\lambda \leq 0$ in Eq. 67, and have the off-planet criterion for Eq. 68 be: $XC = 1$ at all times, $\Delta X < 0$ or $\Delta X > 2\pi R_{eq}$. If this route is

taken, λ_0 becomes the longitude at the extreme left edge of the picture (sample 1). This latter technique is used by MAP2.

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