General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.
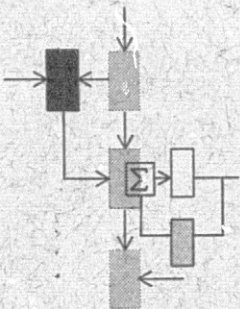
ON THE OPTIMAL MINIMUM ORDER OBSERVER-BASED COMPENSATOR AND THE LIMITED STATE VARIABLE FEEDBACK CONTROLLER

Baldomero Lloréns-Ortiz

Electronic Systems Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Department of Electrical Engineering and Computer Science

ON THE OPTIMAL MINIMUM ORDER OBSERVER-BASED
COMPENSATOR AND THE LIMITED STATE VARIABLE FEEDBACK CONTROLLER

by

Baldomero Lloréns-Ortiz

Electronic Systems Laboratory
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

ON THE OPTIMAL MINIMUM ORDER OBSERVER-BASED

COMPENSATOR AND THE LIMITED STATE VARIABLE FEEDBACK CONTROLLER

by

Baldomero Lloréns-Ortiz

S.B.E.E., Universidad de Puerto Rico
Recinto Universitario de Mayagüez
(1974)

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREES OF

MASTER OF SCIENCE

AND

ELECTRICAL ENGINEER

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September, 1976

Signature of Author....... Department of Electrical Engineering and Computer
Science, September 9, 1976

Certified by....... Thesis Supervisor

Accepted by.......
Chairman, Departmental Committee on Graduate Students

ON THE OPTIMAL MINIMUM ORDER OBSERVER-BASED

COMPENSATOR AND THE LIMITED STATE VARIABLE FEEDBACK CONTROLLER

by

Baldomero Lloréns-Ortiz

ABSTRACT

In this thesis four design problems are considered: two on the
optimal minimum order observer-based compensator design and two on the
optimal limited state variable feedback controller.

Chapter II considers the problem of designing an optimal discrete
time linear time-invariant observer-based compensator for the regulation
of an n dimensional linear discrete time time-invariant plant with m
independent outputs. This is a stochastic design problem to the extent
that the initial plant state is assumed to be a random vector with
known first and second order statistics. The compensator parameters
are obtained by minimizing the expectation, with respect to the initial
conditions, of the standard cost, quadratic in the state and control
vectors with the inclusion of cross terms. The time interval con-
sidered is $[0, +\infty)$.

The emphasis is on the necessary conditions for optimality. It
is shown that for a given plant in state-output canonical form the
optimal compensator parameters are non unique but are related by a
similarity transformation on the observer. Questions on the nature of
the Riccati equations obtained as well as the invariants of the design
under a state-output similarity transformation are answered.

Chapter III deals with the discrete-time version of the optimal
output feedback controller problem. This problem is treated on the
basis of solving two discrete-time versions of the optimal limited
state variable feedback controller problem [J1;S2]. Two different
design methods are presented. The emphasis is on the necessary condi-
tions for optimality.

In Chapter IV the optimal limited dimension control problem for
linear systems is tackled in the context of aggregation theory [A1].

The analysis is done in continuous time for the sake of notational
simplicity.

Chapter V considers the problem of designing an optimal time-
varying observer-based compensator for the regulation of a linear time
varying plant with a known input disturbance. The initial plant state
is assumed to be a random vector with known first and second order
statistics. The compensator parameters are to be obtained by mini-
mizing the expected value, with respect to the initial plant state, of
the standard cost, quadratic in the state and control vectors. The
time interval of interest is $[0, T]$. The analysis is in continuous
time and stresses the necessary conditions for optimality. The emphasis
of the chapter is in showing the separation property of the design
method proposed. That is, the control and state reconstruction problems
are decoupled and can be solved independently.

Thesis supervisor: Timothy L. Johnson

Title: Associate Professor of Electrical Engineering

ACKNOWLEDGEMENTS

The author is glad to have the opportunity to express his deep appreciation to those people who have made of the last two years an invaluable educational experience:

To Professor Timothy L. Johnson, who supervised this thesis. To Professor Michael Athans, who supervised the research for four trying months; and Professor Nils R. Sandell, Jr. for his help. Their unceasing encouragement, constructive criticism, guidance and counsel have greatly influenced the development of this work.

To Professor Sanjoy K. Mitter for his continuous moral support and advice.

To "mi hermano" Wolf Kohn for many helpful discussions related to the thesis and other topics.

And to the M.I.T. Electrical Engineering and Computer Science group and especially to the E.S.L.-family for their cooperation, efficiency and dedication to create a Great Study Environment.

I am also grateful to Mr. Horace M. Smith, Jr., administrative assistant, for helping to make a dream come true. Thanks are also due to Miss Marilyn Pierce, administrative assistant, for her patience and understanding.

I would like also to express my gratitude to Susanna Natti for her terrific job and velocity when typing this thesis; and to Miss Josefa Monserrate for previous typings.

-4a-

Thanks to Mr. Norman Darling for his excellent work on the figures.

To all these people and organizations, thank you very much, you are deep in my heart.

for

Jesus Christ

TABLE OF CONTENTS

CHAPTER I

I.I Brief Historical Review

Among the many contributions of Professor R.E. Kalman to the modern control theory is the rigorous formulation of the linear state regulator problem and its solution in the context of the calculus of variations [K1]. The philosophy behind this design criterion is to obtain an optimal closed loop system with the property that deviations from the nominal plant state are kept near zero without excessive expenditure of control energy. The problem was formulated as a completely deterministic one; namely, the initial condition vector was assumed to be known completely. Basically the solution to this problem leads to an optimal feedback control law which requires knowledge of the whole state vector. When this formulation and solution are analyzed from the practical point of view we find that they constitute an idealization of a real problem in the sense that quite often neither the initial condition vector nor the complete plant state vector are available as measurements. For this reason the problem was reformulated as a stochastic one to the extent that the initial plant state was assumed to be a random vector with known first and second order statistics. The cost was also modified to be the ensemble average cost due to the random initial state. Surprisingly enough, it turned out that when the problem was formulated in this

way, the optimal (in an "average sense") control law obtained was the same as in the completely deterministic case. The technicality of the absence of complete state measurements now had to be solved. Guided by the previous results (that the optimal control law is a linear function of the state vector) a reasonable approach for solving this problem would be to reconstruct or approximate in some sense the plant state vector. In fact, Luenberger [L3] proposed the use of an "observer" to reconstruct asymptotically the state vector based on all the available information (plant model and measurements). Of course, the state reconstructor is not the only solution to the problem of regulating the plant state. For instance, in 1969 Levine [L1] proposed the use of an optimal output feedback controller. In these design methods the optimal control law is a linear function of the measurement vector. This latter design technique may have important practical draw-backs that force the designer to use a dynamic compensator. Namely, sometimes the plant cannot be stabilized by means of output feedback, or the performance is not satisfactory for the application at hand.

The deterministic Luenberger observer design has a slight draw-back: the observer parameters may be selected rather arbitrarily. This motivated some researchers (Newmann [N1], Yüksel and Bongiorno [Y1], Miller [M1], Rom and Sarachick [R3], Blanvillain [B2] and many others) to determine the compensator parameters (control law plus observer) by minimizing a performance index. When posing this optimization problem the following two facts are strongly considered: the initial plant

state is seldom known perfectly and second, if the initial plant state
is known perfectly then a Luenberger observer, with arbitrary dynamics,
can be designed so as to reconstruct perfectly the complete state.
These are reasons for treating this optimization problem in a stochastic
context. Another important point, when this problem is being formu-
lated, is the dimension of the observer. (n-m)* was selected because
it turns out to be the minimum dimension the observer should have in
order to reconstruct asymptotically with the aid of the output vector,
the entire state vector (see Luenberger [L3]). All these elements gave
rise to what is called the optimal minimum order observer-based com-
pensator problem. An approach to this problem in continuous time was
proposed independently by Miller [M1], Blanvillain [B2] and possibly
by others. The second chapter of this thesis considers the discrete
time version of this problem. We note that very recently approaches
to this problem had been proposed independently by two authors.
Roberts [R1] mimicking Blanvillain's approach only presented the
necessary conditions for optimality. O'Reilly and Newmann [O1] tackled
the problem quite similarly to the present treatment but took one of
the design parameters to be the identity matrix.

Several approaches to the reduction of compensator dimension
have been reported. The work of Johnson and Athans [J1] is one such
contribution. In this paper, the compensator structure is assumed to

* n - dimension of the plant state vector
  m - dimension of the measurement or output vector

be of a fixed form and a quadratic cost is introduced in order that the problem be well posed. Later in 1975, Sirisena and Choi presented a slight modification in the performance index. One of the major advantages of these two formulations is that the problem can be reformulated (via augmented matrices) as in Levine's optimal output feedback controller problem. For this reason Chapter III presents the discrete time version of the optimal output feedback controller problem. Two completely different approaches are presented, so as to give more insight into the problem issues and as ways of presenting useful discrete time optimization techniques. We also present an approach to the optimal reduced order compensator problem via Aoki's aggregation theory. This is the subject of Chapter IV.

I.II Outline of contents

In Chapter II the problem of designing an optimal discrete time linear time-invariant observer-based compensator for the regulation of an n-dimensional linear discrete time time-invariant plant with m independent outputs is considered. It is assumed that the initial plant state is a random vector with known first and second order statistics. The performance index is taken to be the expectation, with respect to the initial conditions, of the standard cost, quadratic in the state and control vectors with the inclusion of cross terms. The time interval of interest is $[0, +\infty)$. The emphasis is on the necessary conditions for optimality. It is shown that for a given

plant in state output canonical form the optimal compensator parameters are non-unique but are related by a similarity transformation on the observer. Questions on the nature of the Riccati equations obtained as well as the invariants of the design under a state-output similarity transformation are answered.

Chapter III deals with the discrete-time version of the optimal output feedback controller problem. This problem is treated on the basis of solving two discrete-time versions of the optimal limited state variable feedback controller problem [J1;S2]. Two different design methods are presented. The emphasis is on the necessary conditions for optimality.

Chapter IV considers the optimal limited dimension compensator design problem in the context of aggregation theory [A1]. The analysis is done in continuous time for the sake of simplicity.

Chapter V considers the problem of designing an optimal time-varying observer-based compensator for the regulation of a linear time-varying plant with a known input disturbance. The initial plant state is assumed to be a random vector with known first and second order statistics. The performance index is taken to be the expectation, with respect to the initial condition vector, of the standard cost, quadratic in the state and control vectors. The time interval considered is [0, T]. The analysis is in continuous time and stresses the necessary conditions for optimality. The emphasis of the chapter is in showing the separation property of the design method proposed.

That is, the control and state reconstruction problems are decoupled and can be solved independently.

The conclusions and discussion of results appear in Chapter VI. Also, possible research extensions are discussed.

I.III Notation and Terminology

Small boldface Roman letters will denote vectors and capital letters the matrices unless otherwise stated. A' denotes the transpose of A, I the identity matrix and 0 the zero matrix with appropriate dimensions. Discrete time vectors function of time t are denoted with the subscript t for notational convenience. A(n x n) denotes the matrix A which is dimension n x n. It is stressed that a given matrix may have different meanings and dimensions according to each chapter. This is because the chapters are independent of each other in contents and required a large quantity of different matrices.

If R(n x n) is positive (semi-) definite, we write $R > 0$ $(R \geq 0)$; $R > Q$ means $(R - Q) > 0$.

The expected value (ensemble average) is denoted E (usually not followed by brackets). The covariance matrix of a vector-valued random variable:

$$E\{x(t)x'(t)\} - E\{x(t)\}E\{x'(t)\}$$

is denoted by:

$$cov\{x(t)\}$$

A reference for the concepts of controllability, observability and stabilizability used is W.M. Wonham [W1].

CHAPTER II

DISCRETE-TIME OPTIMAL MINIMUM

ORDER OBSERVER-BASED COMPENSATOR

II.I Introduction

In this chapter, the problem of designing a compensator with

dynamics constrained to be those of a discrete time minimum order

observer is considered. The necessary conditions will be developed

for a discrete-time linear time invariant plant with random initial

state. The cost to be minimized is the expectation, with respect to

the initial state, of the standard quadratic cost for the discrete

time linear regulator problem. The emphasis of this chapter is on

the necessary conditions for optimality. A historical account of

the problem, including salient features of observer design, is pre-

sented.

This chapter is structured as follows. In the second section

the discrete-time linear regulator problem is presented. The third

section presents the structure of the minimum order observer-based

compensator. In section II.IV the specific optimal control problem

is formulated and necessary conditions for optimality are found.

Section five discusses briefly how the sufficiency part of the proof

of optimality is worked out (based on Miller's paper [M1]). The con-

ditions for existence of positive definite and positive semidefinite

solutions of the Riccati equations obtained from the necessary
conditions are given in section II.VI. The last section deals with
peripheral issues pertinent to this problem, and discusses the rele-
vant literature.

II.II The discrete time linear regulator problem

This section considers the problem of designing a full state
feedback control for a discrete-time linear time invariant plant.
The problem is formulated as a stochastic problem because the initial
condition is assumed to be a random vector with known first and second
order statistics. The optimal control sequence is defined as that
control sequence which minimizes the expectation with respect to the
initial conditions of the discrete time standard cost, quadratic in
the states and control with the inclusion of cross terms. Only the
problem formulation and results are presented since this problem is
solved in the literature.

Optimization Problem Statement

Given:

a) the following minimal discrete-time linear time invariant
    plant:

    $$x(t+1) = A\ x(t) + B\ u(t)\ ;\ t \in [0, +\infty) \tag{2.1}$$

    where

    $x(0)$ is an R^n valued random vector with known first
    and second order moments

$x(\cdot)$ is an R^n valued random process with statistics derived from (2.1) and the initial condition statistics

$u(\cdot)$ is an R^r valued random process that is a linear function of $x(\cdot)$.

The known matrices $A(n \times n)$ and $B(n \times r)$ have real entries and are dimensioned accordingly to $x(\cdot)$ and $u(\cdot)$. The pair (A,B) is controllable.

b) the real weighting matrices $Q(n \times n)$, $S(n \times r)$ and $R(r \times r)$ such that

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0 \quad \text{and } R > 0 \tag{2.2}$$

Find: the optimal full-state feedback control*, defined as that control sequence which minimizes the following cost:

$$J(u) = E \left\{ \sum_{t=0}^{\infty} x_t' Q x_t + x_t' S u_t + u_t' S' x_t + u_t' R u_t \right\} \tag{2.3}$$

The solution to the above optimization problem is well known [D1] to be

$$u(t) = P\, x(t) = -(R + B'\Gamma B)^{-1}(B'\Gamma A + S')x(t) \tag{2.4}$$

where Γ satisfies the discrete time algebraic Riccati equation:

---

* More precisely, admissible controls are those which depend on the past history of $x(\cdot)$, $(\cdot) \in [0, t]$.

$$\Gamma = A'\Gamma A + Q - (B'\Gamma A + S')'(R + B'\Gamma B)^{-1}(B'\Gamma A + S') \qquad (2.5)$$

II.III The structure of the discrete time minimum order observer

based compensator

In the previous section the problem of designing a control

law (under the assumption of perfect full state measurements) for a

discrete time linear time invariant plant was posed and the solution

was given. In this section the assumption of perfect measurements

is no longer valid and a minimum order observer based compensator is

required. This section only presents a review of the discrete time

minimum order observer. The optimization problem for determining

the compensator parameters is treated in the next section.

The assumption of perfect full state measurements in the linear

regulator problem (see previous section) is an idealization very

seldom encountered in applications. Most of the time a linear combina-

tion of the states is available instead:

$$y(t) = C\,x(t) \quad ; \quad t \in [0, +\infty) \qquad (2.6)$$

where:

y(·) is an R^m random process with statistics derived from

those of x(·) as given by (2.1). m < n

C is a real m x n full rank matrix. The pair (A,C) is

observable.

In this case, guided by the results in section II.II (that the optimal

control law is a linear function of the states), a reasonable approach

for solving this problem is to reconstruct or approximate in some

sense the state vector. Then the "suboptimal control" would have the

same form as the optimal control, but with $\hat{x}(t)$ (the "approximation"

to $x(t)$) instead of $x(t)$. Of course this is not the only solution to

this problem; one might also use an optimal output feedback controller

(Levine [L1]). However, this control might not stabilize the plant,

or the performance obtained might not be satisfactory for the appli-

cation at hand. For these or related reasons, the designer may be

forced to use a state reconstructor as part of the compensator.

In the early sixties, D.G. Luenberger (see tutorial paper [L3])

proposed a solution to the above problem: to use what is known as an

observer. In an observer, all the available information (plant model,

measurements and external control) is used to reconstruct asymptoti-

cally a linear function of the state vector. In the present problem

observers can be used in different forms. For instance, using what

is known as an identity observer, the whole state vector is recon-

structed via a linear time invariant dynamic system of order n.

Another approach would be to put the plant model in state-output

canonical form (Luenberger [L3], Rom and Sarachick [R3]) and use an

observer of order (n-m) to reconstruct only the states missing in the

output vector. The third approach would be to use the observer to

reconstruct asymptotically the optimal control law $u = Px$ which is

another linear function of the state (see Fortmann and Williamson

[F1]). In this last approach there is the possibility of reducing the order of the observer to a number less than (n-m). The former approach has the disadvantage that the order of the observer may be too large for implementation. On the other hand, the third approach presents a series of mathematical complications (see Motazedi [M2]) which do not permit the simplicity of the first approaches. Another point is that the multi-output, multi-input case has not been worked out. In the following pages the discussion is centered around the second approach: the minimum order observer.

Given that the whole state vector is not available for implementation of the optimal control law a realization of the control signals given by the following compensator is considered:

$$z(t+1) = F\ z(t) + G\ y(t) + D\ u(t) \tag{2.7}$$

$$\hat{x}(t) = N\ y(t) + M\ z(t) \tag{2.8}$$

$$u(t) = P\ \hat{x}(t) \tag{2.9}$$

where:

$z(\cdot)$ is an R^{n-m} random process with statistics derived from $y(\cdot)$

$y(\cdot)$ is the random process described by (2.6)

$\hat{x}(\cdot)$ is an R^{n} random process with statistics derived from (2.8)

The matrices $F((n-m\ x\ (n-m))$, $G((n-m)\ x\ m)$, $D((n-m)\ x\ r)$, $N(n\ x\ m)$, $M(n\ x\ (n-m))$ and $P(r\ x\ n)$ are real valued matrices

to be designed.

For the above dynamic system to be an observer (Luenberger [L3]) the following relation must hold:

$$z(t) = T\,x(t) + e(t) \qquad\qquad (2.10)$$

where $z(t)$ approximates $T\,x(t)$ if and only if (for the continuous time version see Fortmann and Williamson [F1]):

    a)   F is an asymptotically stable matrix

    b) $TA - FT = GC$ (2.11)

    c) $D = TB$ (2.12)

$\hat{x}(t)$ in (2.8) estimates $x(t)$ if and only if T, in addition to a), b), c), satisfies the following relation:

$$NC + MT = I \qquad\qquad (2.13)$$

As a result the error dynamics described by

$$e(t+1) = F\,e(t) \qquad\qquad (2.14)$$

are guaranteed to decrease asymptotically to zero as time progresses (see condition a) above).

One useful equation that comes up as a result of using (2.6), (2.10) and (2.13) in (2.8) is:

$$\hat{x}(t) = x(t) + M\,e(t) \qquad\qquad (2.15)$$

The dynamics of the observer as well as the error is determined by the eigenvalues of F. This suggests the conclusion that the closer the eigenvalues of F are to the origin the better the observer (since

the error approaches zero much faster). It can be shown that very small eigenvalues of F do not necessarily minimize the standard quadratic cost (see Blanvillan [B2] for a historical account in continuous time). This precaution caused engineers to design observers with dynamics a little bit faster than the plant dynamics. This also excludes the possibility of having observer and plant with common eigenvalues which may preclude the existence of solutions to condition b). The reason for this technicality is the nature of equation (2.11) (for a good analysis of this equation see Gantmacher [G1] and Luenberger [L3]).

In the late 60's and early 70's (Newmann [N1], Yüksel and Bongiorno [Y1], Miller [M1], Rom and Sarachick [R3] and others) much interest in obtaining the parameters F,G,D,N,M and T by minimizing a cost was expressed. An approach in the continuous time case was proposed independently by Miller [M1] and Blanvillain [B2] (and possibly by others). Miller's approach minimizes the standard quadratic cost, constraining the control law to be an affine function of $\hat{x}(t)$. Blanvillain's derivation transforms the system to the state output canonical form, assumes the optimal control to have the same form as the optimal control for the linear regulator problem, and minimizes the increment in cost due to the use of the observer. Miller by contrast, proved that the optimal gain on the reconstructed x(t) has the same form as in the linear regulator problem. The only slight drawback of Miller's analysis is that he does not use the state-output

canonical transformation to simplify the problem. As a result his

equations are complicated and sometimes confusing. For the discrete

time problem an approach similar to Miller's will be followed, but

the simplicity obtained when the plant is put in state output canon-

ical form will be exploited.

In order to simplify (2.11), (2.12) and (2.13) the plant (2.1),

(2.6) will be transformed to state-output canonical form*. Then (2.6)

has the following form:

$$y(t) = \begin{bmatrix} I & \vdots & 0 \end{bmatrix} x(t) \tag{2.16}$$

where I is the $m \times m$ identity matrix and 0 is an $m \times (n-m)$ zero matrix.

The partitioning (2.13) in the same way (2.16) suggests:

$$N\begin{bmatrix} I & \vdots & 0 \end{bmatrix} + M\begin{bmatrix} T_1 & \vdots & T_2 \end{bmatrix} = \begin{bmatrix} I & \vdots & 0 \\ 0 & \vdots & I \end{bmatrix} \tag{2.17}$$

where:

$$T = \begin{bmatrix} \underbrace{T_1}_{(n-m) \times m} & \vdots & \underbrace{T_2}_{(n-m) \times (n-m)} \end{bmatrix}$$

$$I = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$
$$\underbrace{\quad}_{n \times m} \underbrace{\quad}_{n \times (n-m)}$$

the I's are identity matrices, in accordance with the partition. Con-

*This is no loss of generality since, given that C in (2.6) is full
rank, we can always find a similarity transformation J such that:

$$\begin{bmatrix} I & \vdots & 0 \end{bmatrix} = C J^{-1}$$

sidering the partitions in (2.17) separately, equations for M and H are obtained in terms of T_1 and T_2:

$$M = \begin{bmatrix} 0 \\ I \end{bmatrix} T_2^{-1} = LT_2^{-1} \qquad (2.18)$$

$$N = -MT_1 + \begin{bmatrix} I \\ 0 \end{bmatrix} \qquad (2.19)$$

T_2^{-1} must exist for (2.17) to make sense. Using the above partition for T and C, equation (2.11) becomes:

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} A_{1_1} & A_{1_2} \\ A_{21} & A_{2_2} \end{bmatrix} - F \begin{bmatrix} T_1 & T_2 \end{bmatrix} = G \begin{bmatrix} I & \vdots & 0 \end{bmatrix} \qquad (2.20)$$

Then, the matrices F and G can be expressed in terms of T_1 and T_2:

$$F = (T_1 A_{1_2} + T_2 A_{2_2}) T_2^{-1} = T A L T_2^{-1} \qquad (2.21)$$

$$G = T_1 A_{1_1} + T_2 A_{2_1} - F T_1 \qquad (2.22)$$

II.IV The specific control problem

As was shown in the previous section, all the parameters of the compensator (see (2.12), (2.18), (2.19), (2.21) and (2.22)) can be determined once the matrix T is known. In this section an optimal control problem for determining the matrix T and the feedback gain P of (2.9) is posed and solved. The performance index used is the expectation with respect to the initial conditions of the standard discrete-time quadratic cost. It is shown that the necessary condi-

tions to have a stationary point are satisfied by the feedback gain P

for the case of perfect measurements. The solution of the optimiza-

tion problem reduces to finding the solution of two independent dis-

crete-time algebraic Riccati equations. These are necessary for com-

puting the feedback gain P and the matrix T independently. The former

depends only on the weighting matrices and the plant model while the

latter depends on the initial condition statistics and the plant model.

As a result we conclude that the control and estimation problems are

decoupled and a kind of separation theorem holds. The sufficiency

part of the proof is the subject of the next section.

Optimization Problem Statement

Given:

a) $E\{x(0)\} = m_0$ and $E\{x(0)x'(0)\} = \Sigma$ for the process

$$x(t+1) = A\ x(t) + B\ u(t) \tag{2.23}$$

$$y(t) = \begin{bmatrix} I & \vdots & 0 \end{bmatrix} x(t) \tag{2.24}$$

b) the matrices A and B for the model (2.23)-(2.24)

c) the weighting matrices Q(n x n), R(n x n) and S(n x r)

such that

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0 \text{ and } R > 0$$

Find: the matrices P(r x n), T_1 ((n-m) x m) and T_2 ((n-m) x (n-m))

and the vector z_0 (n-m) such that the functional:

$$J(P, T_1, T_2, z_0) = E\left\{\sum_{t=0}^{\infty} x_t'Qx_t + u_t'S'x_t + x_t'Su_t + u_t'Ru_t\right\}$$

(2.25)

subject to:

$$x(t+1) = A\,x(t) + B\,u(t)$$

(2.26)

$$u(t) = P\,x(t) + LT_2^{-1}e(t)$$

(2.27)

$$e(t+1) = T\,A\,L\,T_2^{-1}e(t)$$

(2.28)

$$e(0) = z_0 - T\,x(0)$$

(2.29)

is minimized (Recall that $T = \left[T_1 \;\vdots\; T_2\right]$).

The above optimization problem can be simplified as follows:

Using (2.28) in (2.25) the cost becomes:

$$J(P, T_1, T_2, z_0) = E\left\{\sum_{t=0}^{\infty} [x_t'\,e_t'] \cdot \right.$$

$$\left. \cdot \begin{bmatrix} Q+SP+P'S'+P'RP & SPLT_2^{-1}+P'RPLT_2^{-1} \\ (T_2^{-1})'L'P'S' + (T_2^{-1})'L'P'RP & (T_2^{-1})'L'P'RPLT_2^{-1} \end{bmatrix} \begin{bmatrix} x_t \\ e_t \end{bmatrix}\right\}$$

(2.30)

Constraints (2.26)-(2.28) can be written in closed loop system form:

$$\begin{bmatrix} x(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} A+BP & BPLT_2^{-1} \\ 0 & TALT_2^{-1} \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}.$$

(2.31)

Using (2.29) the initial state second moment matrix can be obtained:

$$E\left\{\begin{bmatrix} x(0) \\ e(0) \end{bmatrix} \begin{bmatrix} x'(0) & e(0) \end{bmatrix}\right\} = \begin{bmatrix} \Sigma & mz_0'-\Sigma T' \\ z_0m'-T\Sigma & z_0z_0'-z_0m'T'-Tmz_0'+T\,\Sigma\,T' \end{bmatrix}$$

(2.32)

It is important to note that using the observer approach the closed loop system dynamics (see (2.31)) is composed of two decoupled sets of eigenvalues: closed loop system eigenvalues for the perfect measurement case plus the error or compensator eigenvalues. Also, we should keep in mind that the optimization problem considered makes sense only when these sets of eigenvalues can be placed in the left half complex plane. This restriction imposes certain properties on the system matrices A, B, C that will be discussed later.

The above dynamic optimization problem can be transformed to a static optimization problem of the following form (the steps taken are similar to the ones in Appendix D for the output feedback controller):

$$
\bar{J}(P,T_1,T_2,z_0,\Gamma,K) = \text{tr}\left\{ \begin{bmatrix} \Gamma_{1_1} & \Gamma_{1_2} \\ \Gamma'_{1_2} & \Gamma_{2_2} \end{bmatrix} \cdot \right.
$$

$$
\cdot \begin{bmatrix} \Sigma & mz_0' - \Sigma T' \\ z_0 m' - T\Sigma & z_0 z_0' - z_0 m' T' - Tmz_0' + T\Sigma T' \end{bmatrix} + \left\{ - \begin{bmatrix} \Gamma_{1_1} & \Gamma_{1_2} \\ \Gamma'_{1_2} & \Gamma_{2_2} \end{bmatrix} + \right.
$$

$$
+ \begin{bmatrix} (A+BP)' & 0 \\ (BPLT_2^{-1})' & (TALT_2^{-1})' \end{bmatrix} \begin{bmatrix} \Gamma_{1_1} & \Gamma_{1_2} \\ \Gamma'_{1_2} & \Gamma_{2_2} \end{bmatrix} \cdot
$$

$$
\cdot \begin{bmatrix} (A+BP) & (BPLT_2^{-1}) \\ 0 & (TALT_2^{-1}) \end{bmatrix} +
$$

$$
+ \begin{bmatrix} Q+SP+P'S'+P'RP & SPLT_2^{-1}+P'RPLT_2^{-1} \\ (T_2^{-1})'L'P'S'+(T_2^{-1})'L'P'RP & (T_2^{-1})'L'P'RPLT_2^{-1} \end{bmatrix} \left. \begin{bmatrix} K_{1_1} & K_{1_2} \\ K'_{1_2} & K_{2_2} \end{bmatrix} \right\} \right\}
$$

$$(2.33)$$

where $K = \begin{bmatrix} K_{1_1} & K_{1_2} \\ K'_{1_2} & K_{2_2} \end{bmatrix}$, $m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$,

$\Gamma = \begin{bmatrix} \Gamma_{1_1} & \Gamma_{1_2} \\ \Gamma'_{1_2} & \Gamma_{2_2} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \Sigma_{1_1} & \Sigma_{1_2} \\ \Sigma'_{1_2} & \Sigma_{2_2} \end{bmatrix}$

with dimensions accordingly.

The necessary conditions for (2.33) to have a stationary point at P^*, T_1^*, T_2^*, z_0^*, K^*, $\ ^*$ are the following:

$$\left. \frac{\partial \bar{J}}{\partial P} \right|_* = 0 \tag{2.34}$$

$$\left. \frac{\partial \bar{J}}{\partial T_1} \right|_* = 0 \tag{2.35}$$

$$\left. \frac{\partial \bar{J}}{\partial T_2} \right|_* = 0 \tag{2.36}$$

$$\left. \frac{\partial \bar{J}}{\partial z_0} \right|_* = 0 \tag{2.37}$$

$$\left. \frac{\partial \bar{J}}{\partial K} \right|_* = 0 \tag{2.38}$$

$$\left. \frac{\partial \bar{J}}{\partial \Gamma} \right|_* = 0 \tag{2.39}$$

where $\big|_*$ means "evaluated at P^*, T_1^*, T_2^*, z_0^*, K^*, Γ^*".

Using Athans [A3] the above conditions become:

From (2.34):

$$[S' + RP* + B'\Gamma^*_{1_1} (A + BP*)]K^*_{1_1} + [S' + RP* + B'\Gamma^*_{1_1} (A + BP*)] \cdot$$

$$\cdot K^*_{1_2} (T^{*-1}_2)'L' + [RP* + B'\Gamma^*_{1_1} BP*]LT^{*-1}_2 K^{*'}_{1_2} +$$

$$+ [RP* + B'\Gamma^*_{1_1} BP*]LT^{*-1}_2 K^*_{2_2} (T^{*-1}_2)'L' + B'\Gamma^*_{1_2} T*ALT^{*-1}_2 \cdot$$

$$\cdot [K^{*'}_{1_2} + K^*_{2_2} (T^{*-1}_2)'L'] = 0 \tag{2.40}$$

From (2.35):

$$\Gamma^{*'}_{1_2} \left[- \begin{bmatrix} \Sigma_{1_1} \\ \Sigma'_{1_2} \end{bmatrix} + (A + BP*)K^*_{1_2} (T^{*-1}_2)'A'_{12} + BP*LT^{*-1}_2 K^*_{2_2} (T^{*-1}_2)'A_{1_2} \right]$$

$$+ \Gamma^*_{2_2} \left[-z^*_0 m'_1 + T^*_1 \Sigma_{1_1} + T^*_2 \Sigma'_{12} + (T^*_1 A_{1_2} + T^*_2 A_{2_2})T^{*-1}_2 K^*_{2_2} (T^{*-1}_2)' \cdot \right.$$

$$\left. \cdot A'_{1_2} \right] = 0 \tag{2.41}$$

From (2.36):

$$\Gamma^{*'}_{1_2} \left[- \begin{bmatrix} \Sigma_{1_2} \\ \Sigma_{2_2} \end{bmatrix} + (A + BP*)K^*_{2_2} (T^{*-1}_2)'A'_{2_2} + BP*LT^{*-1}_2 K^*_{2_2} (T^{*-1}_2)'A'_{2_2} \right]$$

$$+ \Gamma_{2_2} \left[-z^*_0 m'_2 + T^*_1 \Sigma_{1_2} + T^*_2 \Sigma_{2_2} + (T^*_1 A_{1_2} + T^*_2 A_{2_2})T^{*-1}_2 K^*_{2_2} (T^{*-1}_2)'A'_{2_2} \right]$$

$$- \left[T^{*-1}_2 K^{*'}_{1_2} (SP*L + P^{*'}RP*L + (A' + P^{*'}B')\Gamma^*_{1_2} (T^*_1 A_{1_2} + T^*_2 A_{2_2}) \right.$$

$$\left. + (A' + P^{*'}B')\Gamma^*_{1_1} BP*L)T^{*-1}_2 \right]' -$$

$$-\left[T_{2_2}^{*-1} K_{2_2}^* (T_{2_2}^{*-1})' \left(L'P^{*'}RP^*L + L'P^{*'}B'\Gamma_{1_1}^* BP^*L + \right.\right.$$

$$+ L'P^{*'}B'\Gamma_{1_2}^* (T_1^* A_{1_2} + T_2^* A_{2_2}) + (A_{1_2}' T_1^{*'} + A_{2_2}' T_2^{*'})\Gamma_{1_2}^{*'} BP^*L +$$

$$\left.\left. + (A_{1_2}' T_1^{*'} + A_{2_2}' T_2^{*'})\Gamma_{2_2}^* (T_1^* A_{1_2} + T_2^* A_{2_2}) \right) T_{2_2}^{*-1} \right]' = 0 \qquad (2.42)$$

From (2.37)

$$\Gamma_{1_2}^{*'}m + \Gamma_{2_2}^* (z_0^* - T^*m) = 0 \qquad (2.43)$$

From (2.38):

$$
\begin{bmatrix} \Gamma_{1_1}^* & \Gamma_{1_2}^* \\ \Gamma_{1_2}^{*'} & \Gamma_{2_2}^* \end{bmatrix}
= \begin{bmatrix} (A + BP^*)' & 0 \\ (BP^*LT_2^{*-1})' & (T^*ALT_2^{*-1})' \end{bmatrix}
\begin{bmatrix} \Gamma_{1_1}^* & \Gamma_{1_2}^* \\ \Gamma_{1_2}^{*'} & \Gamma_{2_2}^* \end{bmatrix} \cdot
$$

$$
\cdot \begin{bmatrix} (A + BP^*) & (BP^*LT_2^{*-1}) \\ 0 & (T^*ALT_2^{*-1}) \end{bmatrix} +
$$

$$
+ \begin{bmatrix} Q + SP^* + P^{*'}S' + P^{*'}RP^* & SP^*LT_2^{*-1} + P^{*'}RP^*LT_2^{*-1} \\ (T_2^{*-1})'L'P^{*'}S' + (T_2^{*-1})'L'P^{*'}RP^* & (T_2^{*-1})'L'P^{*'}RP^*LT_2^{*-1} \end{bmatrix}
$$

$$\qquad (2.44)$$

From (2.39):

$$
\begin{bmatrix} K_{1_1}^* & K_{1_2}^* \\ K_{1_2}^{*'} & K_{2_2}^* \end{bmatrix}
= \begin{bmatrix} (A + BP^*) & BP^*LT_2^{*-1} \\ 0 & T^*ALT_2^{*-1} \end{bmatrix}
\begin{bmatrix} K_{1_1}^* & K_{1_2}^* \\ K_{1_2}^{*'} & K_{2_2}^* \end{bmatrix} \cdot
$$

$$
\cdot
\begin{bmatrix}
(A + BP*)' & 0 \\
(BP*LT_2^{*-1})' & (T*ALT_2^{*-1})'
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
\Sigma & mz_0^{*'} - \Sigma T*' \\
z_0^* m' - T*\Sigma & z_0^* z_0^{*'} - z_0^* m' T*' - T* m z_0^{*'} + T*\Sigma T*
\end{bmatrix}
\tag{2.45}
$$

The above conditions can be simplied as follows:

Expanding (2.44) the following matrix equation is obtained:

$$
\Gamma_{1_1}^* = (A + BP*)'\Gamma_{1_1}^*(A + BP*) + Q + SP* + P*'S' + P*'RP*
\tag{2.46}
$$

comparing this equation with (2.5) we can see that _if_ P* is given by:

$$
P* = -(R + B'\Gamma_{1_1}^* B)^{-1}(B'\Gamma_{1_1}^* A + S')
\tag{2.47}
$$

Equation (2.5) is obtained with $\Gamma_{1_1}^* = \Gamma$. We will suggest such a solution to the necessary conditions. Whether the results will be sufficient for optimality is a question to be answered later.

Another partition of (2.44) is as follows:

$$
\Gamma_{1_2}^* = (A + BP*)'\Gamma_{1_1}^*(BP*LT_2^{*-1}) + (A + BP*)'\Gamma_{1_2}^*(T*ALT_2^{*-1}) +
$$
$$
+ SP*LT_2^{*-1} + P*'RP*LT_2^{*-1} = 0
\tag{2.48}
$$

Using (2.47) in (2.48):

$$
\Gamma_{1_2}^* = (A + BP*)'\Gamma_{1_2}^*(T*ALT_2^{*-1})
\tag{2.49}
$$

If the analysis of this problem is done for the finite time interval

[0, T] and then the limit is taken we will find that the boundary

condition for (2.49) is Γ_{1_2} (T) = 0, which implies that

$$\Gamma_{1_2} (t) = 0 \tag{2.50}$$

for all t.

With Γ_{1_2} = 0 and P given by (2.47) the third matrix equation

from (2.44) becomes:

$$\Gamma^*_{2_2} = (T*ALT^{*-1}_2)'\Gamma^*_{2_2} (T*ALT^{*-1}_2) + (T^{*-1}_2)'[L'P^{*'}B'\Gamma^*_{1_1}BP*L +$$

$$+ L'P^{*'}RP*L]T^{*-1}_2 \tag{2.51}$$

Studying this equation (see Appendix B) we will find that $\Gamma^*_{2_2}$ has to

be positive definite in order that a minimal order observer be

obtained.

Using the above argument and $\Gamma^*_{1_2}$ = 0, from (2.43) is obtained

that:

$$z_0 = Tm \tag{2.52}$$

since $\Gamma^{*-1}_{2_2}$ exists. Even if one cannot conclude that $\Gamma^{*-1}_{2_2}$ exists,

(2.52) is sufficient for (2.43) to hold. As will be shown later this

condition is sufficient for optimality.

Using (2.47) and $\Gamma^*_{1_2}$ = 0, (2.40) reduces to:

$$[RP* + B'\Gamma^*_{1_1}BP*]LT^{*-1}_2K^{*'}_{1_2} + [RP* + B'\Gamma^*_{1_1}BP*]LT^{*-1}_2K^*_{2_2}(T^{*-1}_2)'L = 0$$

$$\tag{2.53}$$

And (2.41) to:

$$T_1^*(\Sigma_{1_1} - m_1 m_1') + T_2^*(\Sigma_{1_2}' - m_2 m_1') + (T_1^* A_{1_2} + T_2^* A_{2_2})T_2^{*-1} K_{2_2}^* \cdot$$

$$\cdot \; (T_2^{*-1})' A_{1_2}' = 0 \tag{2.54}$$

Equation (2.42) reduces to:

$$\Gamma_{2_2}^* \, [T_1^*(\Sigma_{1_2} - m_1 m_2') + T_2^*(\Sigma_{2_2} - m_2 m_2') +$$

$$+ \; (T_1^* A_{1_2} + T_2^* A_{2_2})T_2^{*-1} K_{2_2}^* (T_2^{*-1})' A_{2_2}']$$

$$- \; [T_2^{*-1} K_{2_2}^* (T_2^{*-1})' \Big(L'P^{*'}RP^{*'}L + L'P^{*'}B'\Gamma_{1_1}^* BP^*L +$$

$$+ \; (A_{1_2}' T_1^{*'} + A_{2_2}' T_2^{*'})\Gamma_{2_2}^* (T_1^* A_{1_2} + T_2^* A_{2_2})\Big) T_2^{*-1}]' = 0 \tag{2.55}$$

Using equation (2.51), (2.55) becomes:

$$\Gamma_{2_2}^* \, [T_1^*(\Sigma_{1_2} - m_1 m_2') + T_2^*(\Sigma_{2_2} - m_2 m_2') + (T_1^* A_{1_2} + T_2^* A_{2_2})T_2^{*-1} K_{2_2}^* \cdot$$

$$\cdot \; (T_2^{*-1})' A_{2_2}'] - (T_2^{*-1} K_{2_2}^* \Gamma_{2_2}^*)' = 0 \tag{2.56}$$

Therefore:

$$T_1^*(\Sigma_{1_2} - m_1 m_2') + T_2^*(\Sigma_{2_2} - m_2 m_2') + (T_1^* A_{1_2} + T_2^* A_{2_2})T_2^{*-1} K_{2_2}^* \cdot$$

$$\cdot \; (T_2^{*-1})' A_{2_2}' - K_{2_2}^* (T_2^{*-1})' = 0 \tag{2.57}$$

or:

$$K_{2_2}^* = (T_1^* A_{1_2} + T_2^* A_{2_2})T_2^{*-1} K_{2_2}^* (T_2^{*-1})' A_{2_2}' T_2^{*'} + T_2^*(\Sigma_{2_2} - m_2 m_2')T_2^{*'} +$$

$$+ T_1^*(\Sigma_{1_2} - m_1 m_2') T_2^*' \tag{2.58}$$

If one expands (2.45) in three matrix equations, from one of them it can be shown that (2.58) has been satisfied already if (2.54) holds. That is, from (2.45)

$$z_0^* z_0^*' - z_0^* m' T^*' - T^* m z_0^*' + T^* \Sigma T^*' + (T^* A L T_2^{*-1}) K_{2_2}^* (T^* A L T_2^{*-1})'$$

$$= K_{2_2} \tag{2.59}$$

Using (2.52), (2.59) becomes:

$$[T_1^*(\Sigma_{1_1} - m_1 m_2) + T_2^*(\Sigma_{1_2}' - m_2 m_1') + (T_1^* A_{1_2} + T_2^* A_{2_2}) \cdot$$

$$\cdot T_2^{*-1} K_{2_2}^* (T_2^{*-1})' A_{1_2}'] T_1^* + T_1^*(\Sigma_{1_2} - m_1 m_2') T_2^*' +$$

$$+ T_2^*(\Sigma_{2_2} - m_2 m_2') T_2^*' - K_{2_2}^* + (T_1^* A_{1_2} + T_2^* A_{2_2}) T_2^{*-1} K_{2_2}^* \cdot$$

$$\cdot (T_2^{*-1})' A_{2_2}' T_2^*' = 0 \tag{2.60}$$

It is clear that given that (2.54) and (2.60) are satisfied, (2.58) is satisfied already. This suggests that the optimal cost as well as the optimal compensator structure are invariant with respect to the matrix T_2.[†] A formal proof of this statement would require a similarity transformation on the compensator to show that the optimal cost as well as the optimal compensator structure are invariant under such transformation. For instance, a similarity transformation T_2^{-1} would make T to

---

[†] Remark: The solution T^*, z_0^* is <u>nonunique</u> to the extent that any choice of T_2 (full rank) gives a corresponding T_1^*.

be of the form:

$$T = \left[\overline{T}_1 \;\vdots\; I\right] \tag{2.61}$$

In the continuous time version of the above problem an identical situation to the one explained above occurs. For instance, in the thesis of Blanvillain, T_2 was assumed identity while in Miller's it is arbitrary.

Assuming T_2 to be the identity matrix the equations to be solved in order to obtain the optimal discrete time observer based compensator are:

Feedback gain:

$$P^* = -(R + B'\Gamma^*_{1_1} B)^{-1}(B'\Gamma^*_{1_1} A + S') \tag{2.62}$$

where:

$$\Gamma^*_{1_1} = A'\Gamma^*_{1_1} A + Q - (B'\Gamma^*_{1_1} A + S')'(R + B'\Gamma^*_{1_1} B)^{-1}(B'\Gamma^*_{1_1} A + S') \tag{2.63}$$

For parameters of the compensator:

$$T^*_1 = -(\Sigma'_{1_2} - m_2 m'_1 + A_{2_2} K^*_{2_2} A'_{1_2})(\Sigma_{1_1} - m_1 m'_1 + A_{1_2} K^*_{2_2} A'_{1_2})^{-1} \tag{2.64}$$

where:

$$
\begin{aligned}
K^*_{2_2} = A_{2_2} K^*_{2_2} A'_{2_2} + \Sigma_{2_2} - m_2 m'_2 - \\
- (\Sigma'_{1_2} - m_2 m'_1 + A_{2_2} K^*_{2_2} A'_{1_2})(\Sigma_{1_1} - m_1 m'_1 + A_{1_2} K^*_{2_2} A'_{1_2})^{-1} \cdot \\
\cdot (\Sigma_{1_2} - m_1 m'_2 + A_{1_2} K^*_{2_2} A'_{2_2})
\end{aligned}
\tag{2.65}
$$

Before proceeding with the next section some discussion on the existence of the inverses $(R + B'\Gamma^*_{1_1} B)^{-1}$ and $(\Sigma_{1_1} - m_1 m'_1 + A_{1_2} K^*_{2_2} A'_{1_2})^{-1}$ is necessary. To guarantee their existence, the matrices R and $(\Sigma_{1_1} - m_1 m'_1)$ should be constrained to be positive definite. This is because, given that the added terms are quadratic and the matrices $K^*_{2_2}$, $\Gamma^*_{1_1}$ are at least positive semidefinite, then the resulting matrix sums are invertible.

It is also important to point out the separation property implicit in the above equations. Notice that the optimal control law gain P* given by (2.62)-(2.63) depends only on the plant model and weighting matrices while the observer parameters (see (2.64)-(2.65)) depend only on the plant model and initial condition statistics. As a consequence the control and estimation parts of the problem are completely decoupled and can be solved independently.

II.V Sufficiency part of the proof of optimality

As pointed out in section II.IV, the equation (2.47) was assumed of special form and then the consequences were examined. This value of P* was proposed because of the known similarity of equation (2.46) with equation (2.5) for the case of perfect measurements. In other words, the solution (2.62-(2.65) satisfies the necessary conditions, but these may not be sufficient for the minimization of (2.33). The original approach proposed to answer this question is due to R.A. Miller for the continuous time version [M1]. The discrete time

version mimics such results, so they are not repeated here. Instead
the following paragraph outlines the most important issues of the
proof.

In order to prove the sufficiency part, Miller [M2] expressed
the cost functional (2.3) as a linear combination of the optimal
cost for the linear regulator problem with perfect measurements plus
an increment in cost due to the use of a control different from it.
This is done using a least squares approach analogous to Brockett's
[B4]. Then by means of a similarity transformation the deterministic
and stochastic aspects of the control problem may be separately exam-
ined. In this way it is shown that the increment in cost consists
of two parts: one dependent on u(t) and another completely independent
of u(t). Also a set of constraints on u(t) are obtained. Since the
second term of the increment in cost is independent of the optimal
control law, only the first term is considered in the minimization.
Thus, an expression for u(t) which minimizes this term to zero is
obtained. Recall that this u(t) is subject to a set of constraints.
Then, what is left to prove is that the observer-based control law
proposed in section II.IV realizes this optimal control law and
satisfies the constraints.

II.VI Conditions for the existence of positive definite and positive
       semidefinite solutions of the Riccati equations (2.63),(2.65)

       Traditionally, the problem in modern control theory that has

been solved and studied first has been the linear regulator problem with no cross term in the cost. In part this has been so because the linear regulator problem with a cross term in the cost can be reformulated as a linear regulator problem with no cross term and because in this format the problem is simpler to solve. For instance the necessary conditions for existence of positive definite and positive semidefinite solutions of algebraic discrete time Riccati equations in Appendix A are for linear regulator problems with no cross term in the cost.

In order to show that the equations (2.64)-(2.65) can be understood as an optimal feedback gain for a discrete time linear regulator problem with a cross term in the cost they are written as follows:

$$T_1^{*\prime} = -(\Sigma_{1_1} - m_1 m_1^\prime + A_{1_2} K_{2_2}^* A_{1_2}^\prime)^{-1}(\Sigma_{1_2} - m_1^\prime m_2 + A_{1_2} K_{2_2}^* A_{2_2}^\prime)$$

$$(2.66)$$

where:

$$K_{2_2}^* = A_{2_2} K_{2_2}^* A_{2_2}^\prime + \Sigma_{2_2} - m_2 m_2^\prime - (\Sigma_{1_2}^\prime - m_2 m_1^\prime + A_{2_2} K_{2_2}^* A_{1_2}^\prime) \cdot$$

$$\cdot (\Sigma_{1_1} - m_1 m_1^\prime + A_{1_2} K_{2_2}^* A_{1_2}^\prime)^{-1}(\Sigma_{1_2} - m_1 m_2^\prime + A_{1_2} K_{2_2}^* A_{2_2})$$

$$(2.67)$$

Comparing these with (2.62) and (2.63) the following equivalence is obtained:

$$R \sim \Sigma_{1_1} - m_1 m_1^\prime \qquad (2.68)$$

$$B \sim A'_{1_2} \tag{2.69}$$

$$\Gamma \sim K^*_{2_2} \tag{2.70}$$

$$\overline{A} \sim A'_{2_2} \tag{2.71}$$

$$S \sim \Sigma'_{1_2} - m_2 m'_1 \tag{2.72}$$

$$Q \sim \Sigma_{2_2} - m_2 m'_2 \tag{2.73}$$

$$P^* \sim T^{*'}_{1} \tag{2.74}$$

where $\sim$ means "equivalent to". Then using the following matrix identity reported by Sain [S1] the problem with a cross term in the cost can be formulated as one with no cross term in the cost:

$$(I_n + AB)^{-1} = I_n - A(I_r + BA)^{-1}B \tag{2.75}$$

where:

I_n : n x n identity matrix

A : n x r matrix

B : r x n matrix

the following results are obtained:

$$\overline{Q} = Q - SR^{-1}S' \tag{2.76}$$

$$\overline{u}(t) = u(t) + R^{-1}S'x(t) \tag{2.77}$$

$$\overline{A} = A - BR^{-1}S' \tag{2.78}$$

$$\overline{B} = B \tag{2.79}$$

$$\overline{R} = R \tag{2.80}$$

where the matrices $\bar{Q}$, $\bar{A}$ and so forth are the appropriate matrices for the discrete time linear regulator problem with <u>no</u> cross term (the formulation of this problem as well as the optimal control law can be obtained making $S = 0$ in the material presented in section II.II).

Now the results in Appendix A can be applied. These conditions require among <u>other things</u> that for positive definiteness of $\Gamma_{1_1}^*$ and $K_{1_1}^*$:

a) $(A - BR^{-1}S', B)$ be a controllable pair,

b) $(A - BR^{-1}S', (Q - SR^{-1}S')^{\frac{1}{2}})$ be an observable pair

and

c) $(A_{2_2}' - A_{1_2}'(\Sigma_{1_1} - m_1 m_1')^{-1}(\Sigma_{1_2} - m_1 m_2'), A_{1_2}')$ be a controllable pair,

d) $\Big(A_{2_2}' - A_{1_2}'(\Sigma_{1_1} - m_1 m_1')^{-1}(\Sigma_{1_2} - m_1 m_2'),$

$(\Sigma_{2_2} - m_2 m_2' - (\Sigma_{1_2}' - m_2 m_1')(\Sigma_{1_1} - m_1 m_1')^{-1}(\Sigma_{1_2} - m_1 m_2'))^{\frac{1}{2}}\Big)$

be an observable pair

respectively. The controllable conditions a) and c) are satisfied if the following conditions (see Gopinath [G2]):

e) (A, B) be a controllable pair

and

f) (A_{2_2}, A_{1_2}) be an observable pair

are satisfied respectively. The f) condition is satisfied if the

pair (A, C)* is observable (see Gopinath [G2]).

II.VI <u>Other issues related to this problem and comments on the</u>
<u>literature</u>

As pointed out in section II.III, considerable simplicity is
gained in section II.IV if the plant is initially transformed to a
state-output canonical form. It happens that the transformation used
to obtain such a canonical form is not unique although the plant
dynamics are conserved. It can be shown that any two such realiza-
tions of a plant yield the same compensator dynamics (eigenvalue
structure) and even more, any two realizations of the optimal compen-
sator yield the same performance. For continuous systems this is the
subject of a paper by P. Blanvillain and T.L. Johnson [B3]. In the
discrete time domain the analysis is quite similar (see Appendix C).

Recently it was discovered that the necessary conditions
obtained in section II.IV were reported independently by G. Roberts
[R1] and J. O'Reilly-M.M. Newmann [O1]. The approach followed by
Roberts mimics the approach taken by Blanvillain [B2]. He only
reports the necessary conditions. O'Reilly and Newmann, on the other
hand, take the same approach of section II.IV but assuming $T_2 = I$.
They also prove in detail the sufficiency of the above solution and
present sufficient conditions for the existence of positive definite

---

* Recall that the triple A,B,C was transformed to state-output canoni-
cal form and therefore C has the form [I : 0].

and positive semidefinite solutions of the resulting Riccati

equations.

Chapter III

THE DISCRETE OPTIMAL OUTPUT FEEDBACK CONTROLLER

III.I Introduction

One way of posing (via augmented matrices) the problem of reg-
ulating a linear time invariant plant with a linear compensator of
limited dimension is to use the format established by Levine [L1] for
the output feedback controller problem. This approach was taken by
Johnson and Athans [J1] in 1970. Later in 1975, Sirisena and Choi [S2]
presented a "new way" of defining the performance cost.* As will be
discussed below, the only difference between these two approaches is
the way the performance criterion is structured.

This chapter is structured as follows. In section III.II the
discrete time versions of the optimal constrained dynamic compensator
problem by Johnson-Athans and Sirisena-Choi are presented and briefly
compared. Later in that section the differences between these
approaches and the one suggested by Luenberger observer theory [L3]
are pointed out. Section III.III deals with the formulation of the
optimal constrained dynamic compensator problem as an optimal output
feedback controller problem. In the fourth section the optimal output

---

*The performance cost of the system composed of the plant and com-
 pensator.

feedback controller problem is solved by posing the problem as a static optimization problem. In section III.V a "Direct Approach" is presented.

III.II The Optimal Constrained Dynamic Compensator (Discrete Time)

The problem studied by Johnson and Athans in 1970 came up as a result of studying the regulation of a high order deterministic system with a small number of outputs. This problem can be attacked using an optimal compensator which incorporates a minimum order observer, but the order $(n-m)*$ of this compensator may still be too high for implementation. One possible approach to this problem would be to mimic Miller's [M1] results for an observer of dimension less than $(n-m)$, but this is not possible due to the fact that a crucial rank condition[†] is not satisfied. Johnson and Athans proposed a formulation in which the dynamic part of the compensator is not constrained to be an observer of the plant state vector. This formulation in discrete time yields the following results:

Consider the linear time invariant plant:

$$x(t+1) = A\, x(t) + B\, u(t); \quad E\{x(0)\} = m_0, \quad t \in [0, +\infty) \qquad (3.1)$$

with output equation:

$$y(t) = C\, x(t) \qquad (3.2)$$

---

*n = dimension of the plant state vector space
  m = dimension of the output vector space
[†]See equation (2.12) in Miller's [M1].

where:

$x(\cdot)$ is an R^n-valued random process with known first and second order statistics.*

$u(\cdot)$ is a linear function of $x(\cdot)$ generated by (3.3) and (3.4).

$y(\cdot)$ is an R^m-valued random process with statistics derived from $x(\cdot)$.

The matrices A, B, C have real valued entries and dimensions accordingly.

The compensator dynamics are represented by the following linear time-invariant system:

$$z(t+1) = F\,z(t) + G\,y(t) \tag{3.3}$$

with output equation (control law for the plant):

$$u(t) = M\,z(t) + N\,y(t) \tag{3.4}$$

where:

$$z(t) \in R^S \qquad \forall\ t \in [0,+\infty)$$

and the matrices F, G, M, N have real valued entries and dimensions accordingly to $z(\cdot)$, $y(\cdot)$ and $u(\cdot)$.

Given the form of the compensator, the choice of performance index for this design problem is crucial for obtaining a well posed

---

*We will assume that $x(0)$ is vector valued random variable with known mean and covariance and therefore (3.1) is a representation of a Markov process and it's first and second order statistics can be computed recursively.

problem as will be indicated later.

Optimization Problem Statement I (see Appendix B):

> Given: a) $E\{x(0)\}$ and $cov(x(0))$ for the process in (3.1)
>
> b) the matrices A, B, C for the model (3.1)-(3.2)
>
> c) the weighting matrices $\begin{bmatrix} Q & S \\ S' & R_1 \end{bmatrix} \geq 0$, $R_i > 0$
> for $i = 1,2$

Find: The matrices with real entries: $F(sxs)$, $G(sxm)$,

M(rxs) and N(rxm).

such that the functional:

$$J(G,N,F,M) = E\left\{\sum_{t=0}^{\infty} x_t'Qx_t + u_t'S'x_t + x_t'Su_t + \\ + y_t'[G'R_2G + N'R_1N]y_t + z_t'[F'R_2F + M'R_1M]z_t\right\} \quad (3.5)$$

with constraints:

$$x(t+1) = A \, x(t) + B \, u(t) \quad\quad\quad\quad (3.6)$$

$$y(t) = C \, x(t) \quad\quad\quad\quad (3.7)$$

$$z(t+1) = F \, z(t) + G \, y(t) \quad\quad\quad\quad (3.8)$$

$$u(t) = M \, z(t) + N \, y(t) \quad\quad\quad\quad (3.9)$$

is minimized.

Notice the presence of the terms $y_t'G'R_2Gy_t$, $z_t'F'R_2Fz_t$ and $z_t'M'R_1Mz_t$ in the cost functional (3.5). In the usual formulation of the time-invariant Linear Plant Quadratic Cost regulator problem [A4] these terms are not included. However for the problem formulated

above, with the absence of these terms, the corresponding solution becomes a trivial one. That is, since no constraints are imposed on F, G and N then these matrices in the closed loop plant-compensator system will tend to zero in order to minimize the cost functional. Another reason for selecting the cost functional structure as in (3.5) is that this problem, when reformulated, is equivalent to the output feedback controller problem [L2].

The choice of the weighting matrices in (3.5) is not a trivial task. As usual the matrices Q and S can be chosen based on the physical limitations on the plant state and control vectors. On the other hand, the matrices R_1 and R_2 can be chosen based on the limitations of the physical elements responsible for producing the gains N-M and G-F respectively. R_1 is the same for Ny_t and Mz_t because we are assuming that the same kind of physical elements are used for the gains N and M.

One slight drawback of the above formulation is that the control law obtained is more difficult to compute than in Sirisena and Choi's approach [S2]. The only difference in formulation between these two approaches is the structure of the cost functional. The discrete-time version of Sirisena and Choi's approach is presented next.

Optimization Problem Statement II:

      Given: (a) $E\{x(0)\}$ and $cov(x(0))$ for the process in (3.1)

                 (b)   the matrices A, B, C for the model (3.1)-(3.2)
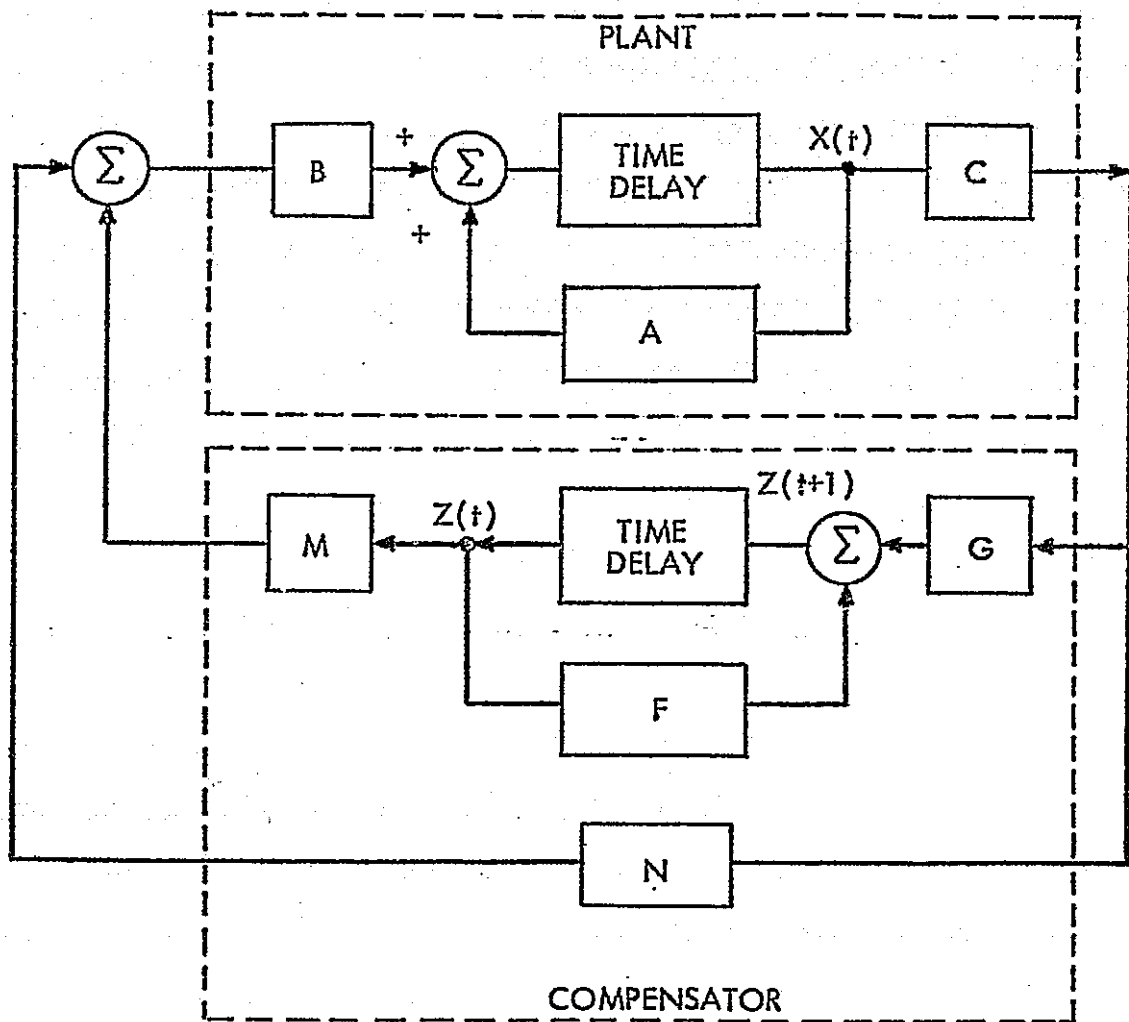
Figure 3.1

(c) the weighting matrices $\begin{bmatrix} Q_1 & S \\ S' & R_1 \end{bmatrix} \geq 0,$

$Q_2 \geq 0$, $R_i > 0$ for $i = 1,2$

Find: the real valued matrices $F(sxs)$, $G(sxm)$, $M(rxs)$ and

   $N(rxm)$

such that: the cost functional

$$J(G,N,F,M) = E\left\{\sum_{t=0}^{\infty} x_t'Q_1x_t + u_t'S'x_t + x_t'Su_t + z_t'Q_2z_t + \nu_t'R_2\nu_t \right.$$
$$\left. + u_t'R_1u_t\right\} \qquad (3.10)$$

subject to:

$$\nu(t) = F\, z(t) + G\, y(t) \qquad (3.11)$$

and (3.6)-(3.9) is minimized.

In this formulation it is easier to determine the appropriate weighting matrices. As usual the matrices Q_1, S and R_1 can be chosen based on standard considerations (see Bryson and Ho [B5]). R_2 is chosen chosen as to limit the input to the time delay element (see Fig. 3.1) in the compensator dynamics. The only problem with the above cost functional is that the state of the compensator dynamics is also being penalized. There is no physical reason for weighting this term since we are not interested in minimizing the deviations of $z(t)$ from zero.

In the two formulations described above, no specific constraining relation among the plant and compensator states is established.

This fact constitutes the crucial difference between these design criteria and the one based on the theory of minimal Luenberger observers [B2; M1]. However, it should be noted that any system (3.6) may be interpreted as an observer [R2]. When an observer is used as the dynamic part of the compensator a linear relation between the observer and plant state is implicitly assumed. Using observer theory one of the design objectives is to reconstruct asymptotically a linear combination of the plant states.

III.III Formulation of the optimal constrained dynamic compensator controller problem as an optimal output feedback controller problem

In this section the central problem analyzed in this chapter is posed and then a discussion of how to formulate this problem as an optimal output feedback controller problem is given.

Consider the optimization problem statement II. This problem can be suitably modified as to correspond to an output feedback controller design problem as follows:

(a) Note that the dynamics of the closed loop system (using (3.6)-(3.9)) is given by:

$$\begin{bmatrix} x(t+1) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} A + BNC & BM \\ GC & F \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{3.12}$$

(b) Using (3.6)-(3.9) and (3.12) in (3.10) and after some algebraic manipulation the cost functional becomes:

$$J(G,N,F,M) = E\left\{ \sum_{t=0}^{\infty} [x'_t z_t] \begin{bmatrix} Q_1 + C'N'R_1NC + C'N'S' + SNC + C'G'R_2GC & \vdots & C'N'R_1M + SM + C'G'R_2F \\ M'R_1NC + M'S' + F'R_2GC & \vdots & Q_2 + M'R_1M + F'R_2F \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} \right\}$$

(3.13)

(c) Let:

$$P \triangleq \begin{bmatrix} N & M \\ G & F \end{bmatrix}, \quad \bar{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} \triangleq \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}$$

$$\bar{C} \triangleq \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad \bar{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}, \quad \bar{Q} \triangleq \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

$$\bar{S} \triangleq \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

(3.14)

(d) Then (3.12) becomes:

$$\begin{bmatrix} x(t+1) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} \bar{A} + \bar{B} P \bar{C} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

(3.15)

and (3.13),

$$J(P) = E\left\{ \sum_{t=0}^{\infty} [x'_t z'_t] [\bar{Q} + \bar{S} P \bar{C} + \bar{C}'P'\bar{S}' + \bar{C}'P'\bar{R} P \bar{C}] \begin{bmatrix} x_t \\ z_t \end{bmatrix} \right\}$$

(3.16)

Remark: The optimization problem:

$$\min_{P} \quad J(P)$$

(3.17)

such that

$$\begin{bmatrix} x(t+1) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} \overline{A} + \overline{B} \, P \, \overline{C} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \qquad (3.18)$$

is in the format of the optimal output feedback control problem for which a solution is given in Section III.IV.

A refinement to the optimization problem II is obtained by replacing the constraint (3.11) by the following:

$$v(t) = F \, z(t) + G \, y(t) - z(t) \qquad (3.19)$$

This cost component (see (3.10)) constrains the "size" of the jumps allowed to the compensator state in successive time intervals which is usually a physically desired condition. However, a heavily penalized compensator state difference $v(t)$ decreases the speed of response (to perturbations in the nominal plant state) of the compensator. Thus the choice of R_2 should be made taking in consideration this tradeoff.

With the modification suggested above the corresponding formulation of the optimal output feedback controller problem develops as follows:

(a) From (3.19):

$$v(t) = (F-I)z(t) + G \, y(t) \qquad (3.20)$$

(b) Let:

$$\overline{F} = F - I \qquad (3.21)$$

(c) Then the closed loop system equation becomes:

$$
\begin{bmatrix} x(t+1) \\ z(t+1) \end{bmatrix} = \begin{bmatrix} A + BNC & BM \\ GC & \overline{F} + I \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{3.22}
$$

(d)   After some matrix manipulations (3.10) becomes,

$$
J(G,N,F,M) = E \left\{ \sum_{t=0}^{\infty} [x_t' \; z_t'] \cdot \right.
$$

$$
\cdot \begin{bmatrix} Q_1 + C'N'R_1NC + C'N'S' + SNC + C'G'R_2GC & C'N'R_1M + SM + C'G'R_2\overline{F} \\ M'R_1NC + M'S' + \overline{F}'R_2GC & Q_2 + M'R_1M + \overline{F}'R_2\overline{F} \end{bmatrix} \cdot
$$

$$
\left. \cdot \begin{bmatrix} x_t \\ z_t \end{bmatrix} \right\} \tag{3.23}
$$

Note that if matrices P and $\overline{A}$ defined as

$$
P \triangleq \begin{bmatrix} N & M \\ G & \overline{P} \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \tag{3.24}
$$

and $\overline{B}$, $\overline{C}$, $\overline{R}$, $\overline{Q}$, $\overline{S}$ are as in (3.14) then the formulation of the optimal output feedback controller design corresponding to this case is identical to the one given before.

III.IV   <u>A Solution of the Discrete Optimal Output feedback controller</u>

<u>problem via Lagrange Multipliers</u>

In this section a procedure for solving the optimization problem (3.17)-(3.18) is discussed.

Recall that the problem to be solved has the following form:

Given: (a) $E\{x(0)\}$ and cov(x(0))

(b)   the weighting matrices:

$$\begin{bmatrix} \overline{Q} & \overline{S} \\ \overline{S} & \overline{R} \end{bmatrix} \geq 0, \quad \overline{R} > 0$$

(c)   the system matrices:

$$\overline{A}((n+s)\times(n+s)) \quad , \quad \overline{B}((n+s)\times(s+r))$$

$$\overline{C}((s+m)\times(n+s))$$

Minimize with respect to $P((r+s)\times(s+m))$ the following cost functional:

$$J(P) = E\left\{ \sum_{t=0}^{\infty} [x_t' \; z_t'][\overline{Q} + \overline{S}\,P\,\overline{C} + \overline{C}'P'\overline{S}' + \overline{C}'P'\overline{R}\,P\,C]\begin{bmatrix} x_t \\ z_t \end{bmatrix}\right\}$$

(3.25)

subject to:

$$\begin{bmatrix} x(t+1) \\ z(t+1) \end{bmatrix} = [\overline{A} + \overline{B}\,P\,\overline{C}]\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

(3.26)

It is easy to show that this problem can be solved by solving the following equivalent problem:

Given (a), (b), (c) as above, minimize the following cost functional:

$$J(P) = E\left\{ \sum_{t=0}^{\infty} \tilde{x}_t'\overline{Q}\tilde{x}_t + \tilde{x}_t'\overline{S}\tilde{u}_t + \tilde{u}_t'\overline{S}'\tilde{x}_t + \tilde{u}_t'\overline{R}\tilde{u}_t \right\}$$

(3.27)

subject to:

$$\tilde{x}(t+1) = \overline{A}\,\tilde{x}(t) + \overline{B}\,\tilde{u}(t)$$

(3.28)

$$\tilde{y}(t) = \overline{C}\ \tilde{x}(t) \tag{3.29}$$

$$\tilde{u}(t) = P\ \tilde{y}(t) \tag{3.30}$$

where:

$\tilde{x}(t)$ corresponds to $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$ and the matrices $\overline{Q}$, $\overline{S}$, $\overline{R}$, $\overline{A}$, $\overline{B}$, $\overline{C}$, P

have the same properties and dimensions as before. Notice

$$E\{\tilde{x}(0)\} = \begin{bmatrix} E\{x(0)\} \\ z(t) \end{bmatrix} \text{ and } cov\{\tilde{x}(0)\} = \begin{bmatrix} cov(x(0)) & 0 \\ 0 & 0 \end{bmatrix}.$$

The above dynamic optimization problem can be posed as the

following static optimization problem (for details see Appendix D):

Given: (a) $E\{\tilde{x}(0)\tilde{x}'(0)\} = \tilde{x}_0$

(b) the matrices with real entries:

$$\overline{A}((n+s)x(n+s)), \ \overline{B}((n+s)x(s+r)), \ \overline{C}((s+m)x(n+s))$$

$$\overline{Q}((n+s)x(n+s)), \ \overline{S}((n+s)x(s+r)), \ \overline{R}((s+r)x(s+r))$$

with the properties as before.

Minimize:

$$J(\Gamma, P) = tr\ \{\Gamma\ X_0\} \tag{3.31}$$

subject to:

$$\Gamma = \overline{Q} + \overline{S}\ P\ \overline{C} + \overline{C}'P'\overline{S}' + \overline{C}'P'\overline{R}\ P\ \overline{C} +$$

$$(\overline{A} + \overline{B}\ P\ \overline{C})'\Gamma(\overline{A} + \overline{B}\ P\ \overline{C}) \tag{3.32}$$

The optimization problem (3.31)-(3.32) can be solved by a standard

Lagrange Multiplier technique:

Let:

$$\vec{J}(\Gamma,\ P,\ \Sigma) = \text{tr}\ \{\Gamma\tilde{x}_0 + \Sigma'[-\Gamma + \overline{Q} + \overline{S}\ P\ \overline{C} + \overline{C}'P'\overline{S}' +$$

$$+ \overline{C}'P'\overline{R}\ P\ \overline{C} + (\overline{A} + \overline{B}\ P\ \overline{C})'\Gamma(\overline{A} + \overline{B}\ P\ \overline{C}) \qquad (3.33)$$

The necessary conditions for (3.33) to have a stationary point at $\Gamma*$, $P*$, $\Sigma*$ are:

$$\left.\frac{\partial \overline{J}}{\partial \Gamma}\right|_* = 0 \qquad\qquad (3.34)$$

$$\left.\frac{\partial \overline{J}}{\partial P}\right|_* = 0 \qquad\qquad (3.35)$$

$$\left.\frac{\partial \overline{J}}{\partial \Sigma}\right|_* = 0 \qquad\qquad (3.36)$$

$\Big|_*$ means "evaluated at $\Gamma*$, $P*$, $\Sigma*$". From (3.34) it follows that

$$\Sigma* = (A + B\ P*C)\Sigma*(A + B\ P*\overline{C})' + \tilde{x}_0 \qquad (3.37)$$

From (3.35):

$$\overline{S}'\Sigma*\overline{C}' + \overline{R}\ P*\overline{C}\ \Sigma*\overline{C}' + \overline{B}'\Gamma*\overline{A}\ \Sigma*\overline{C}' + \overline{B}'\Gamma*\overline{B}\ P*\overline{C}\ \Sigma*\overline{C}' = 0 \qquad (3.38)$$

(3.36) gives (3.32) with $\Gamma = \Gamma*$ and $P = P*$.

Solving simultaneously algebraic equations (3.37), (3.38) and (3.32) with $\Gamma = \Gamma*$ and $P = P*$ we obtain $\Gamma*$, $P*$, $\Sigma*$.

If $\left.\dfrac{\partial^2 \overline{J}}{\partial \alpha}\right|_{\alpha*} > 0$ for $\alpha = \Gamma,\ P,\ \Sigma$ then the above solution is a minimum. This condition is not easy to test due to the fact that it is a tensor.

III.V A solution to the discrete optimal output feedback controller problem by a calculus of variation method

In section III.IV a solution was given for the discrete optimal output feedback controller problem equivalent to the reduced compensator scheme described in III.III. In this section an alternate procedure for solving this problem is given. The basic difference between this technique and the one given in III.IV is that in the latter a two step parameter optimization procedure was followed in order to determine the unknown parameters of the compensator (the matrix P) while in this section a direct optimization procedure is followed.

The technique to be presented is very similar to the one used by Levine [L2] when he solved the output feedback controller problem. It is presented for the sake of giving more insight to the problem and as a way of presenting several tools which are useful when solving discrete time optimization problems. It uses the discrete time version of Bellman's result on perturbation theory [B1, p. 174] and Kleinman's results [K4] for finding gradient matrices. These are presented in Appendix E for the sake of completeness.

The approach used here is termed a direct method because the closed loop system equation (obtained combining (3.28)-(3.30)) is used directly in the cost. Then, given that the cost is only a function of the unknown matrix P, we proceed to find the necessary condition for the existence of a stationary point obtaining:

-57-

$$\left.\frac{\partial J}{\partial P}\right|_{P*} = 0 \tag{3.39}$$

In order to obtain this condition Kleinman's lemma [K4] is used. It reads as follows:

$$tr\left(\left.\frac{\partial J'}{\partial P}\right|_{P*} \Delta P\right) = \frac{1}{\varepsilon}\left(J(P* + \varepsilon\Delta P) - J(P*)\right) \tag{3.40}$$

where ε is a perturbation parameter and ΔP a perturbation matrix. This is where the result by Bellman on perturbation theory for

$$(\overline{A} + \overline{B} P \overline{C} + \varepsilon \overline{B} \Delta P \overline{C})^t$$

is used.

The necessary condition obtained is the following (for details see Appendix E):

$$\left.\frac{\partial J}{\partial P}\right|_{P*} = 2 \sum_{t=1}^{\infty} \sum_{\tau=0}^{t-1} \left\{\overline{C}(\overline{A} + \overline{B} P*\overline{C})^{\tau}\tilde{X}_0(\overline{A}' + \overline{C}'P*'\overline{B}')^t \cdot \right.$$

$$\cdot \; [\overline{Q} + \overline{C}'P*'\overline{R} P*\overline{C} + \overline{S} P*\overline{C} + \overline{C}'P*'\overline{S}'](\overline{A} + \overline{B} P*\overline{C})^{t-(\tau+1)}\overline{B}\Big\}$$

$$+ \sum_{t=0}^{\infty} \left\{\overline{C}(\overline{A} + \overline{B} P*\overline{C})^t\tilde{X}_0(\overline{A}'+ \overline{C}'P*'\overline{B}')^t(\overline{C}'P*\overline{R} + \overline{S})\right\}$$

$$= 0 \tag{3.41}$$

In order to compare (3.41) with the necessary conditions obtained for the equivalent problem in section III.IV, equations (3.32), (3.37) and (3.38) are written in compact form:

For (3.32):

$$\Gamma^* = \sum_{t=0}^{\infty} (\overline{A}{}' + \overline{C}{}'P^*{}'\overline{B}{}')^t [\overline{Q} + \overline{S} P^*\overline{C} + \overline{C}{}'P^*{}'\overline{S}{}' + \overline{C}{}'P^*{}'R P^* \overline{C}] \cdot$$

$$\circ (\overline{A} + \overline{B} P^* \overline{C})^t \tag{3.42}$$

For (3.37):

$$\Sigma^* = \sum_{t=0}^{\infty} (\overline{A} + \overline{B} P^*\overline{C})^t \tilde{X}_0 (\overline{A}{}' + \overline{C}{}'P^*{}'\overline{B}{}')^t \tag{3.43}$$

It can be seen that (3.43) is identical to the second term in the L.H.S. of (3.41). The first term in the L.H.S. of (3.41) is, if we expand all the involved terms and use (3.42)-(3.43), identical to:

$$\overline{C} \Sigma^* (\overline{C}{}'P^* \overline{B}{}' + \overline{A}{}')\Gamma^*{}'\overline{B}$$

After adding the above term to (3.43) we obtain (3.38).

Therefore it has been proven that the set of equations (3.32), (3.37) and (3.38) are equivalent to the compact equation (3.41).

CHAPTER IV

REDUCED ORDER TIME INVARIANT COMPENSATOR DESIGN

IN THE CONTEXT OF AGGREGATION THEORY

IV.I Introduction

In the last chapter a method for designing suboptimal reduced

order linear time invariant compensators was given. The design method

consisted of assuming linear reduced order dynamics for the compensator

and determining its parameters so as to minimize a quadratic design

criterion. In this chapter the optimal reduced order compensator

problem is tackled in the context of aggregation theory. The chapter

is inspired by M. Aoki's paper "Control of Large Scale Dynamic Systems

by Aggregation" [A1].

For notational convenience, all the analysis will be done in

continuous time. Although there are important technical differences

between discrete and continuous time systems, the techniques developed

in this chapter can be easily translated to corresponding techniques

for discrete time systems.

This chapter is divided into four sections. In the second sec-

tion the heart of the design philosophy named "aggregation theory" is

briefly reviewed. Section IV.III presents a solution to the optimal

output feedback control problem via aggregation theory. The subject

of the fourth section is limited dimension control (using suboptimal

observers) via the aggregated model.

IV.II Review of Aggregation Theory for Linear Systems

It has become an almost standard step in systems oriented engineering analysis and/or design to develop a mathematical representation (called "a model") of the dynamics of the system. Models may assume many different forms, e.g., time domain or frequency domain. For instance, in applying modern control methods, it is often advantageous to use a set of first order linear differential equations (see Kalman [K2]) as a model. In obtaining the parameters of such differential equations a compromise between simplicity and accuracy must be made. Sometimes certain inherent physical properties of the system must be ignored and a reduced model is sought. The topic of interest in this section is one class of linear reduced order models called "aggregated models".
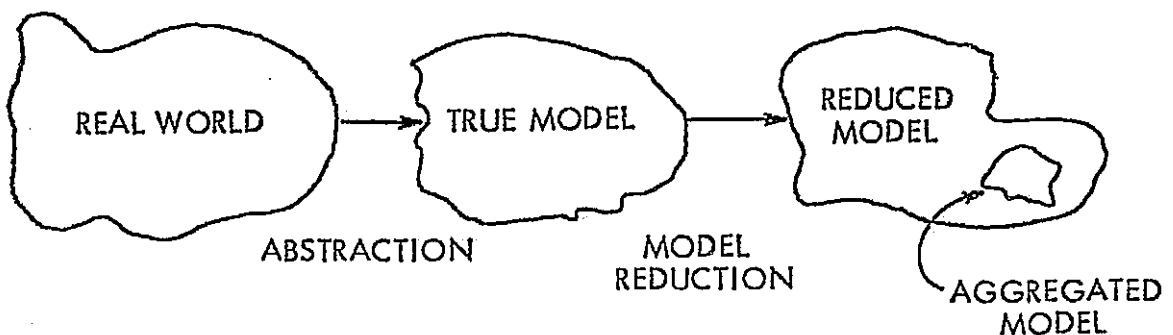

Figure 4.1

-61-

The main idea of aggregation theory, in the context of linear dynamical systems, is to obtain a model of state space dimension less than the dimension of the "true model" (see Fig. 4.1). The state of this "aggregated" model is to be linearly related to the state of the true model. The aggregate model must reflect the most significant portion of the dynamics of the true system.

Consider the plant described by the true model given by:

$$\dot{x}(t) = A\,x(t) + B\,u(t); \quad t \in [0, +\infty) \tag{4.1}$$

where:

$x(\cdot)$ is an R^n-valued random process with statistics computed

using (4.1) given $E\{x(0)\}$ and $E\{x(0)x'(0)\}$

$u(\cdot)$ is a linear function of $x(\cdot)$ contained in R^r

and the reduced order model given by:

$$\dot{z}(t) = F\,z(t) + D\,u(t); \quad t \in [0, +\infty) \tag{4.2}$$

where

$z(\cdot)$ is an R^p-valued random process with statistics computed

using (4.2) and the knowns $E\{z(0)\}$ and $E\{z(0)z'(0)\}$; $p < n$

$u(\cdot)$ is a linear function of $z(\cdot)$ contained in $R^{r\dagger}$

The matrices A, B, F, and D have real entries and are dimensioned accordingly with $z(\cdot)$, $x(\cdot)$ and $u(\cdot)$. Assume (A,B) and (F,D) be

---

$\dagger$Recall that the control vectors in the reduced and true models are
in the same vector space.

controllable pairs.

Following Aoki [A1], a linear reduced order model (4.2) for a given linear true model (4.1) is called an aggregated model if there exists a full rank matrix H(p x n) such that the true state x(·) and the aggregated state z(·) are related by:

$$z(t) = H x(t) \qquad (4.3)$$

Thus, the following conditions hold:

$$FH = HA \qquad (4.4)$$

$$D = HB \qquad (4.5)$$

where A(n x n) and F(p x p) are the system matrices for the true model and aggregated model respectively. B(n x r) is the control matrix for the true model and D(p x r) is the resulting aggregated control matrix. See (4.1) and (4.2). The matrix H(p x n) is the so-called aggregation matrix because, given the relative dimensions of x(·) and z(·), the vector z(·) aggregates the properties of an n-dimensional system in a p-dimensional space.

Given the matrix F, a nontrivial solution H for (4.4) exists only if the matrices F and A have eigenvalues in common (Gantmacher [G1, p. 215-220]). Intuitively this result suggests that if F and A have the same dominant eigenvalues then the true and aggregated models represent similar dynamics. For a method for obtaining H see Appendix G.

Given that H is known and H and A satisfy the matrix equation:

$$H \ A = H \ A \ H'(H \ H')^{-1}H \tag{4.6}$$

then F is given by:

$$F = H \ A \ H'(H \ H')^{-1} \tag{4.7}$$

Remark: In both cases, whether H or F is known, equations (4.4) and (4.5) have to be satisfied exactly in order for (4.2) to be called an aggregated model.

The initial condition statistics for the aggregated model can be computed from the initial condition statistics of the true model as follows:

$$E\{z(0)\} = H \cdot E\{x(0)\} \tag{4.8}$$

$$E\{z(0)z'(0)\} = H \ E\{x(0)x'(0)\}H' \tag{4.9}$$

The notion of "the most significant portion of the dynamics of the true model" (see introduction to this section) is left to the engineer's judgement. However, three properties which must be considered are:

i)   all eigenvalues with positive real parts must be included in the aggregated model

ii)   the dominant eigenvalues (very close to the imaginary axis) must be strongly considered for the aggregated model

iii) a "good" representation of the weak eigenvalues* may

---

*The initial condition distribution may be such that the behaviour of the plant is strongly influenced by the weak eigenvalues.

also be included in the aggregated model.

A possible approach for considering these three factors is to study the relative position of the true eigenvalues* and to make an educated judgement (i.e., based on simulations and/or experience) to allocate the position of the eigenvalues of the aggregated model accordingly.

IV.III Suboptimal output feedback control via the aggregation method

In this section, the problem of controlling a plant with a linear combination of the available measurements is studied from the aggregation theory point of view. First, the problem is posed and solved as suggested by aggregation theory (see previous section). Then, the method is compared with the one developed by Levine [L1] when a suboptimal control, similar to the one proposed by aggregation theory, is used. The main advantages of the aggregation method over the one proposed by Levine are its computational simplicity and independence of initial condition statistics.

One way of posing the limited dimension compensator problem is augmenting the state and compensator dynamics in such a way that the results of the problem considered in this section can be used, e.g. as in Johnson and Athans [J1] or Sirisena and Choi [S2, p. 662-663].

---

*There exists software that computes the eigenvalues according to magnitude starting with the largest one. Such programs may prove to be useful in determining the most significant part of the true model dynamics without having to compute all the eigenvalues of the system.

Suppose that the linear regulator problem (Athans and Falb [A4]) has been posed for the true model. That is, for the standard quadratic cost, the true state and control weighting matrices are known (the time interval of interest is $[0, +\infty)$). Assume that a control law constrained to be a linear function of the output vector is to be used. Then, the approach suggested by aggregation theory would be to find an aggregated model with state vector equal to the output vector of the true model. Suppose the original quadratic cost is modified to weight the aggregated state and the linear regulator problem for the aggregated model is solved. The suboptimal output feedback control is then taken to be the suboptimal aggregated state feedback control.

In mathematical terms the method suggested above requires:

i)   the output vector and the aggregated state vector are the same:

$$z(t) = H(t)x(t) = y(t) = C(t)x(t) \tag{4.10}$$

where $y(t)$ is a p-dimensional output vector of the plant described by (4.1).

ii)   After obtaining the aggregated model (see section IV.II)

$$\dot{z}(t) = F\, z(t) + D\, u(t) \tag{4.11}$$

we have to modify the quadratic cost given for the true model as follows (using the Penrose inverse, e.g. see Aoki [A2]):

$$\overline{J}(u) = E \int_0^\infty \{z'\overline{Q}\, z + u'R\, u\}dt \qquad (4.12)$$

where:

$$\overline{Q} = (H\ H')^{-1}H\ Q\ H'(H\ H')^{-1} \qquad (4.13)$$

and R, Q are known from the linear regulator problem for the true model.

iii)   It is well known [A4,K6] that the optimal steady state feedback control for (4.12), (4.11) is given by:

$$u = -R^{-1}D'\Gamma\, z \qquad (4.14)$$

where $\Gamma(p \times p)$ is the solution of the following algebraic Riccati equation:

$$F'\Gamma + \Gamma\ F - \Gamma\ D\ R^{-1}D'\Gamma + \overline{Q} = 0 \qquad (4.15)$$

Sufficient conditions for the existence of positive definite and semidefinite solutions to (4.15) are given in Kucera [K5].

In order to compare these results with the ones proposed by Levine [L2, p. 787-788], premultiplication by H' and postmultiplication by H of (4.15) is very convenient:

$$H'F'\ \Gamma\ H + H'\ \Gamma\ F\ H - H'\ \Gamma\ D\ R^{-1}D'\Gamma\ H + H'\overline{Q}\ H = 0 \qquad (4.16)$$

Notice that if (4.15) is satisfied then (4.16) is satisfied, but the converse is not necessarily true. Using (4.4) and (4.5), (4.16) is simplified:

$$A'(H'\bar{\Gamma} H) + (H'\Gamma H)A - (H'\Gamma H)BR^{-1}B'(H'\Gamma H) + H'\bar{Q} H = 0$$

(4.17)

and the optimal feedback control:

$$u = -R^{-1}B'(H'\Gamma H)x$$

(4.18)

In order to compare the above results with the equations of the optimal output feedback controller obtained by Levine, his results are repeated here*:

Optimal output feedback control:

$$u(t) = P H x(t)$$

(4.19)

where:

$$P = -R^{-1}B'K \Sigma H'(H \Sigma H')^{-1}$$

(4.20)

and K, Σ satisfy the following two equations:

Riccati equation: $K(A + B P H) + (A' + H'P'B')K + Q + H'P'R P H = 0$

(4.21)

Liapunov equation: $\Sigma(A' + H'P'B') + (A + B P H)\Sigma + X_0 = 0$

where $X_0 = E\{x(0)x'(0)\}$.

(4.22)

Let:

$$K = H'\Gamma H$$

(4.23)

be a solution[†] of (4.21), then (4.19-4.21)** become:

---

*The notation has been changed to facilitate the comparison.

[†]This is a suboptimal control in the sense that the optimal K may be of rank greater than the number of outputs while this K is of rank equal to the number of outputs.

**Note: Given that the coupling between (4.22) and (4.20)-(4.21)

$$u(t) = -R^{-1}B'(H'\Gamma H)x(t) \tag{4.24}$$

$$(H'\Gamma H)A + A'(H'\Gamma H) - (H'\Gamma H)B R^{-1}B'(H'\Gamma H) + Q = 0 \tag{4.25}$$

Equations (4.24) and (4.25) are identical to equations (4.18) and (4.17) respectively if Q is chosen equal to $H'\bar{Q} H$. Of course this equality is not possible unless Q is chosen with rank equal to the number of outputs. For instance, if the cost of the true model penalizes only the output vector then there are no problems.

Aggregation theory simplifies the problem since now only a Riccati equation has to be solved to determine the optimal control, compared with three coupled equations ((4.19)-(4.22)) in the solution proposed by Levine. Recall that even if we find a solution to Levine's set of equations, there would always be the doubt of whether the solution obtained is optimal or not. Another advantage of the design method derived from aggregation theory is that the suboptimal control does not depend on the statistics of the initial conditions. One may argue, however, that if the aggregated model is fixed by the measurement equation and if the cost used to obtain the optimal control only weights those modes selected by the aggregated model, design deficiencies may result. Notice also that H has to be such that (4.4) is satisfied exactly and this will not always be possible. Even if (4.4) is satisfied exactly, H fixes the dynamics of the aggregated model

---

disappears, equation (4.22) is not used anymore in obtaining the feedback control gain. Equation (4.22) is used only for obtaining the steady state second moment: Σ.

and F will not necessarily have "the most significant portion of the
dynamics of the true model". It may also happen that some unstable
eigenvalues in the true model are not included in the aggregated
model. In this case, since no control is applied for these unstable
modes, the plant will be unstable under the control law (4.24).

IV.IV Limited Dimension control using suboptimal observers via the
       aggregated model

In this section an approach to the optimal linear reduced
order compensator design problem by means of aggregation theory is
presented. In this methodology the nature of the system matrix A
and observations matrix C (see (4.10)) is not so restricted as in the
approach presented in the previous section (see (4.4)). Assuming the
linear regulator problem has been posed for the true model, the main
idea of the method consists of the following:

i)   Obtain an aggregated model with the most important
     part of the dynamics of the true model. Leave out only
     asymptotically stable modes.

ii) Design an optimal minimum order observer-based compen-
     sator for the aggregated model (the required matrices are
     obtained using the Penrose inverse in the appropriate
     true model linear regulator parameters).

It is shown that when this control law designed for the aggregated
model is used on the true model the resulting closed loop system is

asymptotically stable.

Consider the problem of designing an optimal observer based compensator for a large-scale dynamic system. In this problem there are two major difficulties:

i)   computation of the Kalman gain (usual Riccati equation)

ii) computation of the parameters of the optimal observer

     [B2;M1]; this requires solution of another Riccati

     equation.

One may consider the possibility of reducing either the number of states of the true model or the number of states to be reconstructed so that the dimension of one or both Riccati equations mentioned above is reduced. Of course, the resulting control law would not be optimal but its effects might be acceptable for the application at hand. To prove whether the particular results are acceptable or not would require simulations and comparison with the optimal control law.

The difficulties pointed out above may be avoided by use of aggregation theory. When analyzing the approach taken, we have to keep in mind one of the most important results (in the author's opinion) of Aoki's paper [A1, p. 250]. The true model will be stable under a control law designed for an aggregated model if (see Appendix F for proof):

i)   the control law is stable for the aggregated model

ii) the system matrix A for the true model is in Jordan

     block diagonal form with all the unstable modes included

in the aggregated model

iii)   H is of the form $[\bar{H} \;\vdots\; 0]$, where $\bar{H}$ is full rank.

Even if the above three conditions are not satisfied the plant may yet be stable under a control law designed for the aggregated model. The importance of the above three conditions is that they are suffi-cient to guarantee the stability of the plant under any stable control law designed for the aggregated model. This is extremely important when designing any control law. The three basic conditions mentioned above are satisfied in the following design:

Consider a plant described by a true model given by:

$$\dot{\bar{x}}(t) = \bar{A}\,\bar{x}(t) + \bar{B}\,\bar{u}(t); \quad t \in [0, +\infty) \tag{4.26}$$

$$y(t) = \bar{C}\,\bar{x}(t) \tag{4.27}$$

where:

$\bar{x}(t) \in R^n$ is a random process with known $E\{\bar{x}(0)\} = \bar{m}_0$
and $E\{\bar{x}(0)\,\bar{x}'(0)\} = \bar{X}_0$

$\bar{u}(t) \in R^r$ is a linear function of $y(\cdot)$ and the state of a minimum order observer.*

$y(t) \in R^m$ is a random process with statistics derived from (4.27)

and $\bar{A}$, $\bar{B}$, $\bar{C}$ are known real valued matrices dimensioned accordingly.

---

*Here we are talking about a minimum order observer design as in Miller's paper [M1]. The structure of such compensator is not dis-cussed here because the design to be presented avoids its use.

The optimal feedback control law is to be obtained by minimizing the following performance index:

$$\bar{J}(u) = E \int_0^\infty \{ \bar{x}'(t)\bar{Q}\ \bar{x}(t) + u'(t)R\ u(t) \} dt \qquad (4.28)$$

where $\bar{Q} \geq 0$ and $R > 0$ are known real valued matrices dimensioned accordingly.

Using the Jordan transformation obtain the Jordan equivalent true model given by:

$$\dot{x}(t) = A\ x(t) + B\ u(t) \qquad (4.29)$$

$$y(t) = C\ x \qquad (4.30)$$

where A is in Jordan diagonal form. The Jordan blocks in A are organized so as to have the unstable and other important eigenvalues in the first Jordan blocks along the diagonal. Also, the cost (4.28) has been transformed:

$$J = E \int_0^\infty \{ x'(t)Q\ x + u'R\ u \} dt \qquad (4.31)$$

The matrices A, B, C, Q are related to $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{Q}$ as follows (using $x(t) = J_0^{-1}\bar{x}(t)$, where J_0 is the Jordan transformation discussed above):

$$A = J_0^{-1}\bar{A}\ J_0$$

$$B = J_0^{-1}\bar{B}$$

$$C = \bar{C}\ J_0$$

$$Q = J_0'\bar{Q}\ J_0$$

Because of the order imposed on A when diagonalizing use $H = [I \; \vdots \; 0]$ (see section IV.II) to extract the most significant portion of the dynamics of the true model and in this way obtain the aggregated model.

In order to completely transform the problem (4.29)-(4.31) so that the aggregated model can be used the following must be done:

i)   transform the output vector using the Penrose Inverse:

$$y(t) = C(H'H)^{-1}H'z(t) = \tilde{C} \, z(t) \tag{4.32}$$

ii)   and performance index:

$$J = E \int_0^\infty \{z'(t)[H(H'H)^{-1}Q(H'H)^{-1}H']z(t) + u'(t)R \, u(t)\}dt$$
$$\tag{4.33}$$

Now a new optimization problem is defined:

Given:

a)   the real valued matrices F(p x p), D(p x r) for the aggregated model:

$$\dot{z}(t) = F \, z(t) + D \, u(t); \quad t \, \varepsilon \, [0, +\infty) \tag{4.34}$$

where $E\{z(0)\}$ and $E\{z(0)z'(0)\}$ are obtained as in (4.8)-(4.9).

b)   the real valued matrix $\tilde{C}$(m x p) for the measurement vector:

$$y(t) = \tilde{C} \, z(t) \tag{4.35}$$

c)   the weighting matrices:

$$\tilde{Q} = H(H'H)^{-1}Q(H'H)^{-1}H' \geq 0$$

$$R > 0$$

Minimize:

$$J(u) = E \int_0^\infty \{z'(t)\tilde{Q}\, z(t) + u'(t)R\, u(t)\}dt \qquad (4.36)$$

subject to: (4.34) and (4.35) and the following minimum order observer-based compensator structure (Miller [M1] and Blanvillain [B2]):

$$\dot{w}(t) = \tilde{F}\, w(t) + \tilde{D}\, u(t) + \tilde{G}\, y(t) \qquad (4.37)$$

with control law:

$$\hat{z}(t) = \tilde{M}\, w(t) + \tilde{N}\, y(t) \qquad (4.38)$$

$$u(t) = \tilde{P}\, \hat{z}(t) \qquad (4.39)$$

where:

$w(t) \in R^{(p-m)}$ is an observer of $z(t)$

$\hat{z}(t) \in R^p$

The real value matrices $\tilde{F}((p-m) \times (p-m))$, $\tilde{D}((p-m) \times r)$, $\tilde{G}((p-m) \times m)$, $\tilde{M}(p \times p)$, $\tilde{N}(p \times m)$ and $\tilde{P}(r \times p)$ are to be determined by minimizing (4.36).

Using this approach the compensator will have dimension (p-m) which is less that the dimension (n-m) of the Luenberger observer proposed by Miller and others.

If the above control is applied to the true model the resulting closed loop system is asymptotically stable. This is proved in Appendix F following the method used by Aoki.

CHAPTER V

OPTIMAL MINIMUM ORDER OBSERVER-BASED COMPENSATOR FOR

A TIME-VARYING PLANT WITH A KNOWN INPUT DISTURBANCE

V.I Introduction

This chapter considers the problem of designing an optimal mini-

mum order time varying observer-based compensator for a linear time

varying plant with a known input disturbance. The compensator para-

meters are to be designed by minimizing the standard performance index,

quadratic in the state and control vectors for the time interval

[0, T]. The problem is treated in the context of stochastic processes

due to the nature of the initial condition vector. The initial condi-

tion is assumed to be a random vector with known first and second

order statistics. The problem is considered in continuous time for

the sake of simplicity. Although there exist significant differences

between discrete time and continuous time problems, the results

obtained can be easily translated to discrete time using a similar

approach.

It will be shown that a kind of separation theorem holds for

this case. That is, the control and state reconstruction parts of

the design problem can be solved independently. The approach is based

on minimizing the performance index with respect to the control para-

meters and using the necessary conditions of optimality to simplify

the original problem. The necessary conditions are simplified based
on our knowledge of the solution for the complete state measurement
case. In this way the control and estimation parts may be separately
examined. The result obtained is that the observer parameters are
determined by solving an optimization problem which is independent of
of the control law parameters; the control law is the same as in the
presence of complete state measurements.

This chapter is structured as follows. In section V.II the state
regulator problem for a plant with a known input disturbance vector is
posed and and solved under the assumption of complete state measure-
ments. In section V.III the problem in section V.II is solved with
two additional constraints: only an incomplete state measurement is
available and the missing states are to be reconstructed asymptotically
by means of a minimum order time-varying observer-based compensator.
The solutions proposed only satisfy the necessary conditions for
optimality.

V.II Optimal control for a plant with a known input disturbance

In this section necessary conditions for the existence of an
optimal control for a linear time-varying plant with a deterministic
disturbance and perfect observations are derived. The problem is
posed as a stochastic problem due to the nature of the initial condi-
tions. The initial condition is assumed to be a random vector with
known first and second order statistics. The performance index mini-
mized is the standard cost quadratic in the state and control vectors

on the time interval $[0, T]$. The optimal control is assumed to be a linear combination of a feedback term and a bias control term. The parameters of the optimal control are determined using the Minimum Principle.

Optimization problem statement

Given: a) the linear time-varying plant described by:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + H(t)w(t); \quad t \in [0, T] \quad (5.1)$$

where

$x(\cdot)$ is an R^n-valued random process with known first and and second order statistics.*

$u(\cdot)$ is an R^r-valued linear function of $x(\cdot)$ generated by (5.2)

$w(\cdot)$ is an R^s-valued known disturbance vector.

The matrices $A(t)$, $B(t)$, $H(t)$ are known matrices with real continuously time varying entries dimensioned accordingly to $x(\cdot)$, $u(\cdot)$, and $w(\cdot)$.

b) the weighting matrices $Q_T \geq 0$, $Q(t) \geq 0$ and $R(t) > 0$ are known,

$Q_T \in R^{n \times n}$

$Q(t) \in R^{n \times n}$

$R(t) \in R^{r \times r}$

---

*We will assume that $x(0)$ is a random vector with known $E\{x(0)\}$ and $E\{x(0)x'(0)\}$. Therefore (5.1) generates a Markov process whose first and second order statistics can be computed recursively.

Assume: the optimal control law to have the following form:

$$u(t) = P(t)x(t) + g(t) \qquad\qquad (5.2)$$

where the first term on the R.H.S. is the linear state feedback
term. The second term is the bias control term and has the
purpose of counteracting the effects of the disturbance term
H(t)w(t) in (5.1). P(t) and g(t) are real valued, deterministic
and are dimensioned accordingly to $x(\cdot)$ and $u(\cdot)$.

Find: u(t) of the form (5.2) such that the following performance
index is minimized:

$$J(u) = E\{x'(T)Q_T x(T) + \int_0^T x'(t)Q(t)x(t) + u'(t)R(t)u(t)\,dt\}$$

$$(5.3)$$

subject to (5.1).

The expectation is taken with respect to the initial conditions.

Note: The matrices A, B, P and H will be written without explicit
dependence on t for the sake of notational simplicity.

Solution:

Using (5.2) in (5.1) the state correlation $E\{x(t)x'(t)\}$ and
mean $E\{x(t)\}$ can be obtained:

$$\dot{x}(t) = (A + B P)x(t) + B g(t) + H w(t) \qquad\qquad (5.4)$$

therefore:

$$x(t) = \phi(t, 0)x(0) + \int_0^t \phi(t, \tau)(Bg(\tau) + H(\tau))\,d\tau \qquad (5.5)$$

where:

$\Phi(t, \tau)$ is the transition matrix of $(A + BP)$. It allows calculation of the state vector at some time t, given complete knowledge of the state vector at time τ in the absence of the term $Bg + Hw$.

From (5.5) the mean of $x(t)$ can be obtained:

$$E\{x(t)\} = m(t) = \Phi(t, 0)m_0 + \int_0^t \Phi(t, \tau)(Bg(\tau) + Hw(\tau))d\tau$$

$$(5.6)$$

where $m_0 = E\{x(0)\}$.

Taking the derivative of (5.6) with respect to time*:

$$\dot{m}(t) = (A + B P)m(t) + (Bg(t) + Hw(t)) \qquad (5.7)$$

The correlation is obtained using (5.5):

$$E\{x(t)x'(t)\} = \Phi(t, 0)E\{x(0)x'(0)\}\Phi'(t, 0) +$$

$$+ \int_0^t \Phi(t, \tau)(Bg(\tau) + Hw(\tau))d\tau E\{x'(0)\}\Phi'(t, 0) +$$

$$+ \Phi(t, 0)E\{x(0)\}\int_0^t (Bg(\tau) + Hw(\tau))'\Phi'(t, \tau)d\tau +$$

$$+ \int_0^t \Phi(t, \tau)(Bg(\tau) + Hw(\tau))dt \int_0^t (Bg(\tau) +$$

$$+ Hw(\tau))'\Phi'(t, \tau)d\tau \qquad (5.8)$$

---

*Recall two important properties of the transition matrix:

i) $\dfrac{d}{dt} \Phi(t, \tau) = (A + B P)\Phi(t, \tau)$

ii) $\Phi(t, t) = I$
    where I is the identity matrix.

Taking the derivative of (5.8) with respect to time*:

$$\dot{\Sigma}(t) = (A + B\,P)\Sigma(t) + \Sigma(t)(A' + P'B) +$$

$$+ (Bg(t) + Hw(t))m'(t) + m(t)(Bg(t) + Hw(t))' \qquad (5.9)$$

where

$$\Sigma(t) = E\{x(t)x'(t)\} \qquad (5.10)$$

And m(t) is given by (5.6).

Using (5.2) in (5.3) the cost functional can be expressed in terms of the correlation and the mean of x(t):

$$J(P,\, g(t)) = \int_0^T [tr\{(Q + P'R\,P)\Sigma(t)\} + g'(t)R\,P\,m(t) +$$

$$+ m'(t)P'R\,g(t) + g'(t)Rg(t)]dt + tr\{Q_T\Sigma(T)\} \qquad (5.11)$$

Now the optimization problem (5.1)-(5.2) can be expressed as follows:

Obtain the (r x n) matrix P, the (n x n) matrix $\Sigma(t)$, the r-dimensional vector g(t) and the n-dimensional vector m(t) such that the following performance index is minimized:

$$J(P,\, g(t)) = \int_0^T tr\{(Q + P'R\,P)\Sigma(t)\} + g'(t)R\,Pm(t) +$$

$$+ m'(t)P'Rg(t) + g'(t)Rg(t)dt + tr\{Q_T\Sigma(T)\} \qquad (5.12)$$

subject to:

*Usually what is found is the autocovariance so that a differential equation with no driving term is obtained. In our case we only find $E\{x(t)x'(t)\}$ because this is the expression we will need later.

$$\dot{m}(t) = (A + B \; P)m(t) + Bg(t) + Hw(t) \qquad (5.13)$$

$$\dot{\Sigma}(t) = (A + B \; P)\Sigma(t) + \Sigma(t)(A + B \; P)' +$$

$$+ (Bg(t) + Hw(t))m'(t) + m(t)(Bg(t) + Hw(t))' \qquad (5.14)$$

where: $m(0)$ and $\Sigma(0)$ are known.

Necessary conditions for (5.12) to have a stationary point at P^* and $g^*(t)$ can be obtained using the Minimum principle (Athans and Falb [A4]). In order to obtain such conditions a Hamiltonian is defined:

$$H(P, \; \Sigma(t), \; m(t), \; g(t), \Gamma(t), \; f(t)) = \text{tr}\{(Q + P'R \; P)\Sigma(t)\} +$$

$$+ g'(t)RPm(t) + m'(t)P'Rg(t) + g'(t)Rg(t) +$$

$$+ \Gamma'(t)[(A + BP)\Sigma(t) + \Sigma(t)(A + Bp)' +$$

$$+ (Bg(t) + Hw(t))m'(t) + m(t)(Bg(t) + Hw(t))'] +$$

$$+ 2f'(t)[(A + BP)m(t) + Bg(t) + Hw(t)] \qquad (5.15)$$

where:

$\Gamma(t)$ is an n x n costate matrix

$2f(t)$ is an n-dimensional costate vector

The necessary conditions for (5.12) to have a stationary point at P^*, $\Sigma^*(t)$, $m^*(t)$, $g^*(t)$, $\Gamma^*(t)$, $f(t)$ are:

$$\left.\frac{\partial H}{\partial P}\right|_* = 0 \qquad (5.16)$$

$$\frac{\partial H}{\partial \Sigma(t)}\bigg|_{*} = -\dot{\Gamma}*(t) \tag{5.17}$$

$$\frac{\partial H}{\partial m(t)}\bigg|_{*} = -\dot{f}*(t) \tag{5.18}$$

$$\frac{\partial H}{\partial g(t)}\bigg|_{*} = 0 \tag{5.19}$$

$$\frac{\partial H}{\partial \Gamma(t)}\bigg|_{*} = \dot{\Sigma}*(t) \tag{5.20}$$

$$\frac{\partial H}{\partial f(t)}\bigg|_{*} = \dot{m}*(t) \tag{5.21}$$

where the notation $\bigg|_{*}$ means "evaluated at: P*, Σ*(t), m*(t), g*(t), Γ*(t), f*(t)."

From (5.16) it follows that (using Athans [A3]):

$$R \; P*\Sigma*(t) + Rg*(t)m'*(t) + B'f*(t)m'*(t) + B'\Gamma*(t)\Sigma*(t) = 0 \tag{5.22}$$

(5.17) becomes:

$$-\dot{\Gamma}*(t) = (A + B \; P*)'\Gamma*(t) + \Gamma*(t)(A + BP*) + Q + P*'R \; P* \tag{5.23}$$

From (5.18) it follows that:

$$\dot{f}*(t) = -(A + B \; P*)'f*(t) - \Gamma*(t)(Bg*(t) + Hw(t)) - P*'Rg*(t) \tag{5.24}$$

(5.19) is:

$$R P^*m^*(t) + R g^*(t) + B'\Gamma^*(t)m^*(t) + B'f^*(t) = 0 \qquad (5.25)$$

From (5.20):

$$\dot{\Sigma}^*(t) = (A + B P^*)\Sigma^*(t) + \Sigma^*(t)(A + B P^*)' +$$

$$+ (Bg^*(t) + Hw(t))m^{*'}(t) + m^*(t)(Bg^*(t) + Hw(t))'$$

$$(5.26)$$

And (5.21):

$$\dot{m}^*(t) = (A + B P^*)m^*(t) + Bg^*(t) + Hw(t) \qquad (5.27)$$

Equations (5.22)-(5.27) can be further simplified in the following way:

Using the value of $Rg^*(t)$ from (5.25) in (5.22):

$$[RP^* + B'\Gamma^*(t)][\Sigma^*(t) - m^*(t)m^{*'}(t)] = 0 \qquad (5.28)$$

Since this equation has to be valid for any $\Sigma(0)$ and $m(0)$, and therefore for any $\Sigma(t)$ and $m(t)$, as long as an optimal solution exists we conclude that:

$$P^* = -R^{-1}B'\Gamma^*(t) \qquad (5.29)$$

Using (5.29) in (5.25):

$$g^*(t) = -R^{-1}B'f^*(t) \qquad (5.30)$$

Using (5.29) and (5.30), (5.24) is reduced to:

$$\dot{f}^*(t) = -(A - B R^{-1}B'\Gamma^*(t))'f^*(t) - \Gamma^*(t)Hw(t) \qquad (5.31)$$

Therefore the solution to (5.1)-(5.3) can be summarized as follows:

$$u(t) = P^*(t)x(t) + g^*(t) \qquad (5.32)$$

where P*(t) satisfies:

$$P*(t) = -R^{-1}(t)B'(t)\Gamma*(t) \qquad (5.33)$$

and:

$$\dot{\Gamma}*(t) = -A'(t)\Gamma*(t) - \Gamma*(t)A(t) + \Gamma*(t)B(t)R(t)B'(t)\Gamma*(t) - Q(t)$$

$$(5.34)$$

obtained using (5.29) in (5.23).

g*(t) is given by:

$$g*(t) = -R^{-1}(t)B'(t)\bar{f}*(t) \qquad (5.35)$$

where:

$$\dot{f}*(t) = -[A(t) - B(t)R^{-1}(t)B'(t)\Gamma(t)]'f*(t)$$

$$- \Gamma*(t)H(t)w(t) \qquad (5.36)$$

where $\Gamma*(t)$ is computed using (5.34).

Boundary conditions of the above solution:

i)   For the equations (5.26)-(5.27) the knowns m(0) and

$\Sigma(0)$ constitute the initial conditions.

ii) For equation (5.34) the transversality conditions of

the Minimum principle give the boundary conditions

(Athans and Falb [A4])*:

$$\frac{\partial\{J(P, g(t))\}}{\partial\Sigma(T)} = \Gamma*(T)$$

therefore:

* Here we use the part of the cost (5.12) that depends only on the terminal state second moment: $tr\{Q_T\Sigma(T)\}$

$$\Gamma^*(T) = Q_T \tag{5.37}$$

iii)   For (5.36) the boundary conditions are obtained in a

similar way:

$$\frac{\partial\{J(P, g(t))\}}{\partial m(T)} = f^*(T)$$

which gives:

$$f^*(T) = 0 \tag{5.38}$$

Calculation of the optimal cost

Using (5.29) and (5.30), the performance index (5.12) can be

expressed as:

$$J(P^*, g^*(t)) = \int_0^T tr\{(Q(t) + P^{*\prime}(t)R(t)P^*(t))\Sigma^*(t)\}$$

$$+ f^{*\prime}(t)B(t)R^{-1}(t)B'(t)\Gamma^*(t)m^*(t)$$

$$+ m^{*\prime}(t)\Gamma^*(t)B(t)R^{-1}(t)B'(t)f^*(t)$$

$$+ f^{*\prime}(t)B(t)R^{-1}(t)B'(t)f^*(t)dt + tr\{Q_T\Sigma(T)\}$$

$$\tag{5.39}$$

Using the trace identity:

$$tr\{A\ B\} = tr\{B\ A\} \tag{5.40}$$

and the value of

$$B(t)g^*(t)m^{*\prime}(t) = -B(t)R^{-1}(t)B'(t)f^*(t)m^{*\prime}(t) \tag{5.41}$$

from (5.26), (3.39) becomes:

$$J(P^*, g^*(t)) = \int_0^T [tr\{(Q(t) + P^{*\prime}(t)R(t)P^*(t))\Sigma^*(t)\}$$

$$+ tr\{-\Gamma^*(t)\dot{\Sigma}^*(t) + \Gamma^*(t)(A(t) + B(t)P^*(t))\Sigma^*(t) +$$

$$+ \quad \Gamma*(t)\Sigma*(t)(A(t) + B(t)P*(t))'$$

$$+ \Gamma*(t)H(t)w(t)m*'(t) + \Gamma*(t)m*w'(t)H'(t)\}$$

$$+ f*'(t)B(t)R^{-1}(t)B'(t)f*(t)]dt$$

$$+ tr\{Q_T\Sigma(T)\} \tag{5.42}$$

which is equal to:

$$= \int_0^T tr\Big\{[Q(t) + P*'(t)R(t)P*(t) + \Gamma*(t)[A(t) + B(t)P*(t)]$$

$$+ [A(t) + B(t)P*(t)]'\Gamma*(t)]\Sigma*(t)\Big\} - tr\{\Gamma*(t)\dot{\Sigma}*(t)\} \cdot$$

$$\cdot tr\{\Gamma*(t)H(t)w(t)m*'(t) + \Gamma*(t)m*(t)w'(t)H'(t)$$

$$+ f*'(t)B(t)R^{-1}(t)B'(t)f*(t)\}dt + tr\{Q_T\Sigma(T)\} \tag{5.43}$$

Using (5.23) and:

$$\frac{d}{dt}\{\Gamma*(t)\Sigma*(t)\} = \dot{\Gamma}*(t)\Sigma*(t) + \Gamma*(t)\dot{\Sigma}*(t) \tag{5.44}$$

(5.43) becomes:

$$= \int_0^T -tr\{\frac{d}{dt}(\Gamma*(t)\Sigma*(t))\} + tr\{\Gamma*(t)H(t)w(t)m*'(t) +$$

$$+ \Gamma*(t)m*(t)w'(t)H'(t) + f*'(t)B(t)R^{-1}(t)B'(t)f*(t)dt$$

$$+ tr\{Q_T\Sigma(T)\} \tag{5.45}$$

Therefore, using (5.37):

$$J(P*, g*(t)) = tr\{\Gamma*(0)\Sigma*(0)\} +$$

$$+ \int_0^T tr\{\Gamma*(t)H(t)w(t)m*'(t) +$$

$$+ \Gamma*(t)m*(t)w'(t)H'(t)\} +$$

$$+ f^{*\prime}(t)B(t)R^{-1}(t)B^{\prime}(t)f^{*}(t)dt \qquad (5.46)$$

Equation (5.46) can be further reduced using (5.36) and (5.38), but it will be left the way it is for the sake of simplicity of the derivations that follow in the next section.

V.III Optimal minimum order observer based compensator for a linear time-varying plant with a known input disturbance

In the previous section the optimal control for a linear time varying plant with quadratic cost was designed. It was shown that the optimal control law requires knowledge of the whole state vector. This is an assumption that is seldom satisfied in practice. Usually what is known are linear combinations of the states*. In this sec- tion the problem of the previous section is considered under the assumption of incomplete state measurements. It is assumed that the control law has the same form as in the previous section but $x(t)$ is replaced by $\hat{x}(t)$, where $\hat{x}(t)$ is the state vector reconstructed asymptotically by means of a minimum order observer-based compensator. The parameters of the compensator are to be determined by minimizing the standard cost quadratic in the state and control vectors over the time interval $[0, T]$. The optimization problem is tackled by minimizing the cost with respect to those parameters "independent" of the observer parameters and then using the equations obtained to

---

*We are assuming that these linear combinations are independent and less in number than the dimension of the state vector.

simplify the cost functional. The result obtained is a cost functional

dependent only on the parameters of the compensator. It has the same

form as the increment in cost due to the use of the observer obtained

by Yuksel and Bongiorno [Y1, p. 611] for a plant with no input distur-

bance. The section is organized as follows. First the compensator

structure is briefly reviewed. Then the optimization problem is

defined. A solution follows.

The plant considered in this section is the same as in (5.1)

with measurements described by:

$$y(t) = C(t)x(t); \quad t \in [0, T] \tag{5.47}$$

where:

y(·) is an R^m-valued random process with statistics derived

from x(·)

C(·) is an (m x n) time varying matrix.

The structure of the Luenberger observer [L3] is the following:

$$\dot{z}(t) = F(t)z(t) + G(t)y(t) + D(t)u(t) + E(t)w(t) \tag{5.48}$$

$$\hat{x}(t) = N(t)y(t) + M(t)z(t) \tag{5.49}$$

where:

z(t) in an R^{n-m}-valued random process with statistics derived

from:

$$z(t) = T(t)x(t) - e(t) \tag{5.50}$$

and the matrices F(t), G(t), D(t), E(t), N(t), M(t), T(t) and

the vector e(t) are dimensioned accordingly. These matrices

are parameters that will be determined by minimizing the quadratic cost functional (5.63).

The control law is assumed to have the same form as in (5.32):

$$u(t) = P(t)\hat{x}(t) + g(t) \tag{5.51}$$

The following equalities [Y1, p. 140-141] are obtained using (5.50) and (5.47) in (5.1) and (5.48):

$$F(t)T(t) + G(t)C(t) = \dot{T}(t) + T(t)A(t) \tag{5.52}$$

$$D(t) = T(t)B(t) \tag{5.53}$$

$$E(t) = T(t)H(t) \tag{5.54}$$

and the error equation:

$$\dot{e}(t) = F(t)e(t) \tag{5.55}$$

Equations (5.52)-(5.55) considerably simplify the optimization problem. The problem is simplified even more if all the matrices involved in equations (5.48)-(5.49) are expressed as a function of T(t). The analysis of these derivations is presented in Yuksel and Bongiorno [Y1].

Notice that if the term E(t)w(t) were not in (5.48) then the error (5.55) would become:

$$\dot{e}(t) = F(t)e(t) + T(t)H(t)w(t) \tag{5.56}$$

which means that the nature of the term H(t)w(t) must be restricted if we want the error to decrease asymptotically to zero.

Equation (5.49) can be simplified using (5.50) and the condition that $\hat{x}(t)$ is equal to x(t) plus an error-dependent term. That is:

$$N(t)C(t) + M(t)T(t) = I \qquad (5.57)$$

where I is the identity matrix. And:

$$\hat{x}(t) = x(t) - M(t)e(t) \qquad (5.58)$$

Then the control law (5.51) becomes:

$$u(t) = P(t)[x(t) - M(t)e(t)] + g(t) \qquad (5.59)$$

The closed loop system equation can be obtained using (5.59) in (5.1) and (5.48):

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A(t)+B(t)P(t) & -B(t)P(t)M(t) \\ 0 & F(t) \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B(t)g(t) \\ 0 \end{bmatrix} + $$
$$+ \begin{bmatrix} H(t)w(t) \\ 0 \end{bmatrix} \qquad (5.60)$$

In deriving (5.60) (see Miller [M1] for a similar approach) the error equations were used instead of equations involving $z(t)$, for the sake of simplification. Another reason is that (5.60) gives insight to the structure of the closed loop system. That is, the poles of the closed loop system are the poles of the plant for the case of complete state measurement, plus the poles of the observer.

Optimization Problem Statement

Given: a) $E\{x(0)\} = m_0$ and $E\{x(0)x'(0)\} = X$ for the process*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + H(t)w(t) \qquad (5.61)$$

---

*For details as dimensions, etc. see section V.II and the introduction to this section.

$$y(t) = C(t)x(t) \tag{5.62}$$

b) the matrices $A(\cdot)$, $B(\cdot)$, $H(\cdot)$, $C(\cdot)$ for the model (5.61)-(5.62)

c) the time varying weighting matrices $Q(\cdot) \geq 0$, $Q_T \geq 0$ and $R(\cdot) > 0$ with dimensions $(n \times n)$, $(n \times n)$ and $(r \times r)$ respectively.

Find: the matrices[†] $F^*(t)$, $T^*(t)$, $N^*(t)$, $G^*(t)$, $M^*(t)$, $P^*(t)$ and the vector $g^*(t)$

such that: the cost functional

$$J(P(t), T(t), M(t), F(t), N(t), G(t), g(t)) =$$

$$= E\{x'(T)Q_T x(T) + \int_0^T x'(t)Q(t)x(t) + u'(t)R(t)u(t)\,dt\} \tag{5.63}$$

subject to:

$$u(t) = P(t)x(t) - P(t)M(t)e(t) + g(t) \tag{5.64}$$

$$\dot{e}(t) = F(t)e(t) \tag{5.65}$$

$$F(t)T(t) + G(t)C(t) = \dot{T}(t) + T(t)A(t) \tag{5.66}$$

$$N(t)C(t) + M(t)T(t) = I \tag{5.67}$$

$$e(0) = T(0)m_0 - z_0 \tag{5.68}$$

and (5.61)-(5.62)

is minimized.

[†]The matrices $D(t)$ and $E(t)$ are not included here because they are determined from (5.5) and (5.6) once $T(t)$ is obtained.

Note: The matrices A, B, P, M, F, H, Q, R, etc. are time functions, but their dependence on time will not be explicitly written.

Solution:

For computing the closed loop system mean and correlation:

From (5.60):

$$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \Phi(t, 0) \begin{bmatrix} x(0) \\ e(0) \end{bmatrix} + \int_0^t \Phi(t, \tau) \begin{pmatrix} Bg(\tau) + Hw(\tau) \\ 0 \end{pmatrix} d\tau \tag{5.69}$$

Thus,

$$\dot{m}_{x,e} = \begin{pmatrix} A + BP & -BPM \\ 0 & F \end{pmatrix} m_{x,e} + \begin{pmatrix} Bg + Hw(t) \\ 0 \end{pmatrix} \tag{5.70}$$

and

$$\dot{\Sigma}_{x,e} = \begin{pmatrix} A + BP & -BPM \\ 0 & F \end{pmatrix} \Sigma_{x,e} + \Sigma_{x,e} \begin{pmatrix} A + BP & -BPM \\ 0 & F \end{pmatrix}'$$

$$+ \begin{pmatrix} Bg + Hw(t) \\ 0 \end{pmatrix} m'_{x,e} + m_{x,e} \begin{pmatrix} Bg + Hw(t) \\ 0 \end{pmatrix}' \tag{5.71}$$

where:

$\Phi(t, \tau)$ is the transition matrix of the system matrix:

$$\begin{pmatrix} A + BP & -BPM \\ 0 & F \end{pmatrix}$$

$$m_{x,e} = E \begin{Bmatrix} x(t) \\ e(t) \end{Bmatrix} = \begin{bmatrix} m_x \\ e_x \end{bmatrix} \tag{5.72}$$

$$\Sigma_{x,e} = E \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \begin{bmatrix} x'(t) & e'(t) \end{bmatrix} = \begin{bmatrix} \Sigma_{1_1} & \Sigma_{1_2} \\ \Sigma'_{1_2} & \Sigma_{2_2} \end{bmatrix} \qquad (5.73)$$

The cost (5.63) can be expressed in terms of $m_{x,e}$ and $\Sigma_{x,e}$ as follows:

$$J(P, T, M, F, N, G, g) = E\{x'(T)Q_T x(T) +$$

$$+ \int_0^T [x'(t) \quad e'(t)] \begin{bmatrix} Q + P'RP & -P'RPM \\ -M'P'RP & M'P'RPM \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} +$$

$$+ \begin{bmatrix} x'(t) & e'(t) \end{bmatrix} \begin{bmatrix} P'Rg \\ -M'P'Rg \end{bmatrix} + \begin{bmatrix} g'RP & -g'RPM \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} +$$

$$+ g'Rg \ dt \qquad (5.74)$$

which equals, using the trace operator and interchanging it with the expectation operator:

$$= tr\{Q_T \Sigma_{1_1}(T)\} + \int_0^T tr\left\{ \begin{bmatrix} Q + P'RP & -P'RPM \\ 'M'P'RP & M'P'RPM \end{bmatrix} \Sigma_{x,e} \right\} +$$

$$+ m'_{x,e} \begin{bmatrix} P'Rg \\ -M'P'Rg \end{bmatrix} + \begin{bmatrix} g'RP & -g'RPM \end{bmatrix} m_{x,e} + g'Rg \ dt$$

$$\qquad (5.75)$$

Now the optimization problem (5.61)-(5.68) can be tackled using

equations (5.70), (5.71) and (5.75) instead of (5.61)-(5.65).

Following Athans and Falb [A4] the Hamiltonian becomes:

$$H(P, T, M, F, N, G, g, \Sigma_{x,e}, m_{x,e}, \Gamma, f, K_1, K_2) =$$

$$= tr\left\{ \begin{bmatrix} Q + P'RP & -P'RPM \\ -M'P'RP & M'P'RPM \end{bmatrix} \Sigma_{x,e} \right\} + m'_{x,e} \begin{bmatrix} P'Rg \\ -M'P'Rg \end{bmatrix} +$$

$$+ \ [g'RP - g'RPM]m_{x,e} + g'Rg + tr\left\{\Gamma'\left[\begin{pmatrix} A + BP & -BPM \\ 0 & F \end{pmatrix}\Sigma_{x,e} +\right.\right.$$

$$+ \ \Sigma_{x,e}\begin{pmatrix} A + BP & -BPM \\ 0 & F \end{pmatrix}' + \begin{pmatrix} Bg + Hw(t) \\ 0 \end{pmatrix}m'_{x,e} +$$

$$\left.+ \ m_{x,e}\begin{pmatrix} Bg + Hw(t) \\ 0 \end{pmatrix}'\right]\right\} + 2f'\left[\begin{pmatrix} A + BP & -BPM \\ 0 & F \end{pmatrix}m_{x,e} +\right.$$

$$\left.+ \ \begin{pmatrix} Bg + Hw(t) \\ 0 \end{pmatrix}\right] + tr\left\{K_1'[FT + GC - TA] + K_2'[NC + MT - I]\right\} \qquad (5.76)$$

where the multipliers $\Gamma((2n-m) \times (2n-m))$, $f((2n-m) \times 1)$, $K_1((n-m) \times n)$, $K_2(n \times n)$ are time varying functions. Note that (5.68) is not in the Hamiltonian since this constraint is a boundary condition. It has to be considered when the transversality conditions are found.

In the analysis that follows the necessary conditions for the existence of a stationary point P^*, m_x^*, g^*, $\Sigma_{i_1}^*$ and $\Sigma_{i_2}^*$ will be obtained. The resulting equations will be compared with (5.32)-(5.38) for the complete state measurements case as to simplify them. The resulting equations will be used to express the cost (5.75) as a function of the compensator parameters only. Following this procedure the problem is considerably simplified. The necessary conditions for a stationary point M^*, F^*, T^*, N^*, G^*, $\Sigma_{2_2}^*$, m_e^*, etc. are to be obtained by minimizing this simplified cost.

Necessary conditions for the existence of a stationary point P^*, m_x^*, g^*, $\Sigma_{i_1}^*$ and $\Sigma_{i_2}^*$:

$$\left.\frac{\partial H}{\partial P}\right|_* = 0 \qquad (5.77)$$

$$\left.\frac{\partial H}{\partial m_x}\right|_* = -2\dot{f}_1^* \qquad (5.78)$$

where the partition:

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad (5.79)$$

has been used.

$$\left.\frac{\partial H}{\partial g}\right|_* = 0 \qquad (5.80)$$

And using the partition:

$$\Gamma = \begin{bmatrix} \Gamma_{1_1} & \Gamma_{1_2} \\ \Gamma'_{1_2} & \Gamma_{2_2} \end{bmatrix} \qquad (5.81)$$

we obtain:

$$\left.\frac{\partial H}{\partial \Sigma_{1_1}}\right|_* = -\dot{\Gamma}_{1_1}^* \qquad (5.82)$$

$$\left.\frac{\partial H}{\partial \Sigma_{1_2}}\right|_* = -\dot{\Gamma}_{1_2}^* \qquad (5.83)$$

$$\left.\frac{\partial H}{\partial \Gamma_{1_1}}\right|_* = \dot{\Sigma}_{1_1}^* \qquad (5.84)$$

$$\left.\frac{\partial H}{\partial \Gamma_{1_2}}\right|_* = \dot{\Sigma}^*_{1_2} \tag{5.85}$$

where $\Big|_*$ means "evaluated at P^*, m^*_x, g^*, $\Sigma^*_{1_1}$, $\Sigma^*_{1_2}$, f^*_1, $\Gamma^*_{1_1}$, $\Gamma^*_{1_2}$".

With transversality conditions (where J is given by the cost (5.75)):

$$\frac{\partial J}{\partial \Sigma_{1_1}(T)} = \Gamma^*_{1_1}(T) \tag{5.86}$$

$$\frac{\partial J}{\partial \Sigma_{1_2}(T)} = \Gamma^*_{1_2}(T) \tag{5.87}$$

From (5.77) is obtained:

$$RP^*\bar{\Sigma}^*_{1_1} - RP^*M\Sigma^{*'}_{1_2} - RP^*\Sigma^*_{1_2}M' + RP^*M\Sigma_2 M' + Rg^*m^{*'}_x$$

$$- Rg^*m'_e M' + B'f^*_1 m^{*'}_x - B'f^*_1 m'_e M' + B'\Gamma^*_{1_1}\Sigma^*_{1_1} - B'\Gamma^*_{1_1}\Sigma^*_{1_2}M' +$$

$$+ B'\Gamma^*_{1_2}\Sigma^{*'}_{1_2} - B'\Gamma^*_{1_2}\Sigma_2 M' = 0 \tag{5.88}$$

From (5.78):

$$-\dot{f}^*_1 = (A + BP^*)'f^*_1 + \Gamma^*_{1_1}(Bg^* + Hw) + P^{*'}Rg^* \tag{5.89}$$

From (5.80):

$$RP^*m^*_x - RP^*Mm_e + Rg^* + B'\Gamma^*_{1_1}m^*_x + B'\Gamma^*_{1_2}m_e + B'f^*_1 = 0 \tag{5.90}$$

From (5.82):

$$-\dot{\Gamma}^*_{1_1} = (A + BP^*)'\Gamma^*_{1_1} + \Gamma^*_{1_1}(A + BP^*) + Q + P^{*'}RP^* \tag{5.91}$$

From (5.83):

$$-\dot{\Gamma}^*_{1_2} = (A + BP^*)'\Gamma^*_{1_2} + \Gamma^*_{1_2} F - \Gamma_{1_1} BP^*M - P^{*'}RP^*M \qquad (5.92)$$

From (5.84):

$$\dot{\Sigma}^*_{1_1} = (A + BP^*)\Sigma^*_{1_1} - BP^*M\Sigma^{*'}_{1_2} + \Sigma^*_{1_2}(A + BP^*)'$$

$$- \Sigma^*_{1_2} M'P^{*'}B' + (Bg + Hw)m^{*'}_{x} + m^*_{x}(Bg + Hw)' \qquad (5.93)$$

From (5.85):

$$\dot{\Sigma}^*_{1_2} = (A + BP^*)\Sigma^*_{1_2} - BP^*M\Sigma_{2_2} + \Sigma^*_{1_2} F' + (Bg + Hw)m'_e \qquad (5.94)$$

And boundary conditions:

$$\Gamma^*_{1_1}(T) = Q_T \qquad (5.95)$$

$$\Gamma^*_{1_2}(T) = 0 \qquad (5.96)$$

The above equations can be simplified as follows:

Comparing (5.91) with (5.34) and (5.95) with (5.37) is clearly

seen that if

$$P^* = -R^{-1}B'\Gamma^*_{1_1} \qquad (5.97)$$

then the solutions $\Gamma^*_{1_1}$, P^* given by (5.97), (5.91) and Γ^*, P^* for

the complete state measurements case (see (5.33)-(5.34)) are identical

respectively.

If (5.97) is used in (5.92), then

$$\dot{\Gamma}^*_{1_2} = -(A + BP*)'\Gamma^*_{1_2} - \Gamma^*_{1_2} F \qquad (5.98)$$

with boundary condition: (5.96).

From (5.98):

$$\Gamma_{1_2}(T) = \Phi'_1(T, t)\Gamma^*_{1_2}(t)\Phi_2(T, t) \qquad (5.99)$$

where: Φ'_1 is the transition matrix for $(A + BP*)'$ and

Φ_2 is the transition matrix for F.

Using the boundary condition (5.96) it is concluded that:

$$\Gamma^*_{1_2}(t) = 0 \quad , t \in [0, T] \qquad (5.100)$$

since (5.99) has to be valid for all t.

Using (5.97) in (5.89):

$$-\dot{f}^*_1 = (A + BP*)'f^*_1 + \Gamma^*_{1_1} Hw \qquad (5.101)$$

which is identical to (5.36) for the complete measurements case. It
is clear that if we use:

$$g* = -R^{-1}B'f^*_1 \qquad (5.102)$$

then the parameters $P(t)$ and $g(t)$ of (5.51) are identical to the ones
obtained for the perfect measurements case (see (5.32)-(5.36)).

Using (5.97), (5.100) and (5.102) in (5.88):

$$P*M\Sigma^{*'}_{1_2} = P*M\Sigma_{2_2} M' \qquad (5.103)$$

Using (5.97), (5.100) and (5.102), (5.90) becomes:

$$P^*Mm_e = 0 \tag{5.104}$$

Using (5.97), (5.102), (5.93) and (5.104), (5.75) is simplified as follows:

$$J(t,M,F,N,G) = tr\{Q_T \Sigma_{1_1}(T)\} + \int_0^T tr\{[Q + \Gamma^*_{1_1} BR^{-1} N' \Gamma^*_{1_1} +$$

$$+ \Gamma^*_{1_1}(A + BP^*) + (A + BP^*)'\Gamma^*_{1_1}]\Sigma^*_{1_1} - \Gamma^*_{1_1} \dot{\Sigma}^*_{1_1}\}$$

$$+ tr\{\Gamma^*_{1_1} Hw\, m^{*'}_x + \Gamma^*_{1_1} m^*_x w'H'\} +$$

$$+ tr\{M'P^{*'}RP^*M\Sigma_{2_2}\} + f^{*'}_1 BR^{-1} B' f^*_1 dt \tag{5.105}$$

Using (5.97), (5.91) and (5.44), (5.105) becomes:

$$J(T,M,F,N,G) = tr\{\Gamma^*_{1_1}(0)\Sigma^*(0)\} + \int_0^T tr\{\Gamma^*_{1_1} Hw\, m^{*'}_x +$$

$$+ \Gamma^*_{1_1} m^*_x w'H'\} + f^{*'}_1 BR^{-1} B' f^*_1 dt +$$

$$+ \int_0^T tr\{M'P^{*'}RP^*M\Sigma_{2_2}\}dt \tag{5.106}$$

where a trace identity has been used to put it in a format similar to (5.46).

In order to compare (5.106) with (5.46) the following analysis is done:

From (5.70):

$$\dot{m}^*_x = (A + BP^*)m^*_x - BP^*Mm_e + Bg^* + Hw \tag{5.107}$$

Using (5.104), (5.107) becomes:

$$\dot{m}_x^* = (A + BP^*)m_x^* + Bg^* + Hw \tag{5.108}$$

Comparing (5.108) with (5.13) for the complete state measurements case it is clear that m_x^* and m^* are identical since A, B, P^*, g^*, H, w are the same for both cases.

Now it is clear that the first four terms on the R.H.S of (5.106) are identical to the optimal cost (5.46) for the complete state measurements case. Therefore these are due to the control law (5.51) and are independent of the state reconstructor structure. Even without comparing (5.106) and (5.46) it can be noticed that the necessary conditions for obtaining the observer parameters are invariant with respect to these four terms. Therefore the optimization problem for obtaining the compensator parameters can be posed as follows:

Minimize:

$$\tilde{J}(T,M,F,N,G,z_0) = \int_0^T \text{tr}\{M'P^{*'}RP^*M\Sigma_{2_2}\}dt \tag{5.109}$$

subject to:

$$\dot{\Sigma}_{2_2} = F\Sigma_{2_2} + \Sigma_{2_2}F' \tag{5.110}$$

$$FT + GC = \dot{T} + TA \tag{5.111}$$

$$NC + MT = I \tag{5.112}$$

$$\dot{m}_e = F\,m_e \tag{5.113}$$

$$e(0) = T(0)m_0 - z_0 \tag{5.114}$$

where the constraints (5.108)-(5.112) are obtained from (5.65)-(5.68) and constraints in (5.76) not considered in the previous necessary

conditions.

The optimization problem (5.109)-(5.114) is equivalent to the

following:

minimize:

$$\tilde{J}(T,M,F,N,G,z_0) = E \int_0^T e'(t) M'P*'RP*Me(t) dt \qquad (5.115)$$

subject to:

$$\dot{e}(t) = Fe(t) \qquad (5.116)$$

$$FT + GC = \dot{T} + TA \qquad (5.117)$$

$$NC + MT = I \qquad (5.118)$$

$$e(0) = T(0)m_0 - z_0 \qquad (5.119)$$

The importance of the solution proposed above is the separation

property implicit in it. That is, the control parameters $P*(t)$ and

$g*(t)$ in the control law

$$u(t) = P*(t)\hat{x}(t) + g*(t) \qquad (5.120)$$

are independent of the state reconstructor optimization problem. This

is concluded because the equations for obtaining these parameters

are identical to the corresponding ones obtained in section V.II for

the complete state measurements case. On the other hand, the state

reconstructor parameters are obtained defining an optimization problem

which is completely decoupled of $P*(t)$ and $g*(t)$. In fact, the opti-

mization problem (5.115)-(5.119) is identical to the one formulated

by Yüksel and Bongiorno [Y1] for the case of no input disturbance.

Therefore, from (5.97), (5.91), (5.102), (5.101) and the optimization
problem (5.115)-(5.119) we conclude that the control and state recon-
structor optimization problems can be analyzed and solved independently
since no coupling between these two problems exists.

We have to emphasize that the solution proposed satisfies the
necessary conditions but sufficiency has not been shown yet. Given
that this part of the proof of optimality is lengthy and similar to
Miller's [M1, p. 150-152] it will not appear here.

CHAPTER VI

VI.I Conclusions and Discussion of Results

Chapter II

The optimal discrete time minimum order observer-based compensator problem has been solved. The necessary conditions for optimality were obtained and it was shown that a control law identical to the one for the complete state measurements case satisfies these conditions. The sufficiency part of the proof has been briefly discussed on the basis of Miller's results [M1]. The Riccati equations obtained have been analyzed, based on a comparison with the Riccati equation for the discrete time linear regulator problem with no cross term in the cost. In fact it has been shown that the stabilizability of the pair [A, B] and observability of [A, C] play an important role in the obtainment of an asymptotically stable closed loop system.

It has been shown that, as in the continuous time case, there is a separation property implicit in the design. That is, the optimal compensator structure turns out to be obtained by solving two decoupled and independent design problems: finding the optimal feedback control gains and finding of the state reconstructor parameters. The optimal control gains have the same form as for the case of complete state measurements and as such only depend on the plant model and the cost weighting matrices. On the other hand the compensator parameters

(obtained once two matrices are determined, i.e. T_1 and T_2) depend only
on the plant model and the initial condition statistics. It was shown
that the optimal cost is invariant with respect to one of these matrices.
From this result it was shown that the optimal compensator parameters
are nonunique since this matrix is arbitrary. Nevertheless the com-
pensator dynamics are unique since any two optimal observers are related
by a similarity transformation. Given this equivalency between optimal
compensators the compensator structure was fixed (taking this arbitrary
matrix as the identity). Then it was shown that a change in state-out-
put canonical form changes the optimal observer structure by a simi-
larity transformation. Even more, the compensator dynamics as well as
the total optimal cost are invariant under this kind of transformation.

Chapter III

The necessary conditions for optimality of the discrete time
version of the optimal output feedback controller have been obtained.
The results were achieved by two different methods which illustrated
important optimization techniques in discrete time systems design.
Also the discrete time versions of the optimal limited dimension com-
pensator design by Johnson-Athans and Sirisena-Choi have been presented
and briefly discussed. These kind of optimization problems are solved
by reformulating them, via augmented matrices, as output feedback
control problems.

Chapter IV

The output feedback control problem has been analyzed in the context of aggregation theory. It has been shown that is is possible to simplify Levine's results as to decouple the Riccati and Liapunov equations and achieve independence of initial conditions. There exist various difficulties with this approach. For instance, the resulting aggregated model may not have the most significant portion of the plant dynamics and second, the measurement and system matrices are constrained by an equality that may not always be satisfied.

A way of reducing the compensator dimension by means of aggregation theory has been proposed. It was shown that to provide control only for the unstable and other significant modes of the plant dynamics reduces the order of the compensator. It has been shown that, fixing the design method using Aoki's results, the design for the aggregated model can be applied to the true plant yielding an asymptotically stable closed loop system.

Chapter V

The regulation of a linear time varying plant with a known input disturbance by means of an optimal control law with an optimal minimum order observer has been studied. The necessary conditions for the optimality of the compensator structure have been obtained. It has been shown that the control gains for the case of complete state measurement satisfy these conditions. Moreover the result obtained is a

decoupling and independence of the control and state reconstructor

parts of the problem. The sufficiency of these results can be proved

by an approach similar to Miller's [M1]. It also has been shown that

the optimization problem for obtaining the observer parameters is

identical to the one obtained by Yüksel and Bongiorno [Y1] for the

case of no input disturbance. An important remark is that these

results are only valid when the observer has knowledge of the plant

input disturbance for all t.

VI.II Possible Extensions and the Related Research

A possible extension of the results obtained in Chapters II

and III would be a comparison of the methods presented, based on

performance and sensitivity analysis. Also a step response comparison

may give insight into the problem. The implementation of these

designs in a discrete time linear time-invariant plant, or appropriate

simulation might enhance the analysis.

By means of a matrix partition similar to the one used in

Chapter II the optimal reduced order observer-based compensator can

be studied. The invariance of the compensator dynamics under a simi-

larity transformation on the observer may be analyzed. For instance,

working out the example of a reduced order compensator by Fortmann and

Williamson [F1] using a matrix partition, some insight in the problem

is gained. For this particular case it turns out that the compensator

parameters which reconstruct asymptotically the optimal control law

are completely determined from a set of constraints obtained from the partitions. Changing the partition, it can be shown that the compensator parameters are non unique. The only difference between them is a similarity transformation on the observer. Surprisingly enough, the compensator dynamics are always the same. It is clear that in cases like this to define an optimization problem so as to find the parameters of the compensator has no meaning unless the class of admissible gains is properly stated.

Concerning the output feedback control problem in the context of aggregation theory, the results obtained in Chapter IV indicate that considerable simplification is obtained if an equality constraint on the system and measurement matrices is satisfied. A possible extension of the results that appear on this chapter would be to define an approximation so as to validate the results obtained for a broader class of systems. This is a problem that can be studied in further detail up to the point that significant simplification of Levine's results is obtained.

Also, further research effort can be dedicated to implement the results proposed on the reduced order observer-based compensator via aggregation theory. A comparison of this methodology with the ones presented in Chapters II and III is desired in order to demonstrate the validity of this approach.

APPENDIX A

Conditions for the existence and uniqueness of positive
definite and positive semidefinite solutions of the standard steady
state Riccati equation (see Kwakernaak and Sivan [K6, p. 495]).

For the following discrete steady state Riccati equation
(obtained when there is no cross term in the cost (2.3): S = 0)

$$\bar{\Gamma} = \bar{A}'\bar{\Gamma}\bar{A} + \bar{Q} - (\bar{B}'\bar{\Gamma}\bar{A})'(\bar{R} + \bar{B}'\bar{\Gamma}\bar{B})^{-1}(\bar{B}'\bar{\Gamma}\bar{A}) \qquad (A.1)$$

or equivalently (see Appendix D for case C = I):

$$\bar{\Gamma} = \sum_{t=0}^{\infty} (\bar{A}' + \bar{P}'\bar{B}')^t (\bar{Q} + \bar{P}'\bar{R}\bar{P})(\bar{A} + \bar{B}\bar{P})^t \qquad (A.2)$$

$$= (\bar{A}' + \bar{P}'\bar{B}')\bar{\Gamma}(\bar{A} + \bar{B}\bar{P}) + \bar{Q} + \bar{P}'\bar{R}\bar{P} \qquad (A.3)$$

where:

$$\bar{P} = (\bar{R} + \bar{B}'\bar{\Gamma}\bar{B})^{-1}(\bar{B}'\bar{\Gamma}\bar{A}) \qquad (A.4)$$

the following results hold. These are derived as a limiting case of
the following equation:

$$\bar{\Gamma}(t) = (\bar{A}' + \bar{P}'(t+1)\bar{B}')\bar{\Gamma}(t+1)(\bar{A} + \bar{B}\bar{P}(t+1)) + \bar{Q} +$$

$$+ \bar{P}'(t+1)\bar{R}\bar{P}(t+1) \qquad (A.5)$$

where:

$$\bar{P}(t) = -(\bar{R} + \bar{B}'\bar{\Gamma}(t)\bar{B})^{-1}(\bar{B}'\bar{\Gamma}(t)\bar{A}) \qquad (A.6)$$

with boundary condition $\bar{\Gamma}(T)$.

Theorem A.1

If $(\overline{A}, \overline{B})$ is a controllable pair, $\overline{\Gamma}(T) \geq 0$ is boundary condition, then there exists a convergent sequence $\{\overline{\Gamma}(t)\}$ which is elementwise positive semidefinite. The limit, as T goes to infinity, $\overline{\Gamma}$ satisfies A.1 and is positive semidefinite.

Theorem A.2

If theorem A.1 is satisfied and $(\overline{A}, \overline{Q}^{1/2})$ is an observable pair, then there exists a unique positive definite solution to (A.1)-(A.3) and it yields an asymptotically stable closed loop system.

The above two sufficient conditions, originally proposed by Kalman [K1;K3] for continuous time version, were weakened by Wonham [W1] for continuous time in 1968. The discrete time results were presented in 1970 by Caines and Maine [C1].

Theorem A.3

If the iteration (A.5) is started at $\Gamma(T) \geq 0$ and the pair $(\overline{A}, \overline{B})$ is stabilizable then there exists $0 \leq \overline{\Gamma} < +\infty$ such that the sequence $\{\overline{\Gamma}(t)\}$ converges to a constant steady state solution $\overline{\Gamma}$. This $\overline{\Gamma}$ satisfies (A.1)-(A.3).

Theorem A.4

If theorem A.3 is satisfied and the pair $(\overline{A}, \overline{Q}^{1/2})$ is observable then $\overline{\Gamma} > 0$ is unique and yields an asymptotically stable closed loop system.

APPENDIX B

Analysis of equation (2.51)

Consider the recursion:

$$\Gamma^*_{2_2}(t) = (T^*ALT^{*-1}_2)'\Gamma^*_{2_2}(t+1)(T^*ALT^{*-1}_2) +$$

$$(T^{*-1}_2)'L'P^{*'}B'\Gamma^*_{1_1} B P^*LT^{*-1}_2 + (T^{*-1}_2)'L'P^{*'}RP^*LT^{*-1}_2$$

(B.1)

notice that if $m+r \geq n$, where r is the number of inputs, m is the number of outputs and n is the number of states, then $\{\Gamma^*_{2_2}(t)\}$ is automatically positive definite for any $\Gamma_{2_2}(T) \geq 0$ since the sum of the last two terms is positive definite. The series $\{\Gamma^*_{2_2}(t)\}$ is a convergent series if $(T^*ALT^{*-1}_2)$ has eigenvalues of moduli less than 1. The latter condition is an implicit assumption in the well posedness of the problem.

If the above conditions are not satisfied, then an analysis in terms of observability and controllability has to be done to determine conditions sufficient for the positive definiteness of the steady state solution Γ^*_2. Putting (B.1) in the following form (using equations (2.18) and (2.21)):

$$\Gamma^*_{2_2} = \sum_{t=0}^{\infty} (F^{*'})^t (M^{*'}P^{*'}B'\Gamma^*_{1_1} BP^*M^* + M^{*'}P^{*'}RP^*M^*)(F^*)^t$$

(B.2)

we can clearly see (using Appendix A) that if the pair (F^*, P^*M^*) is

observable then $\{\Gamma_{2_2}(t)\}$ is a convergent positive definite sequence

provided that F is a stable matrix and $\Gamma_{2_2}(T)$ is positive semidefinite.

All this condition says is that the observer has to be minimal in

order to have $\Gamma_{2_2} > 0$. This is an implicit assumption in section

II.III when the dimension of the observer was fixed to be n-m and

the plant assumed minimal.

APPENDIX C

Invariants of discrete time optimal minimum

order observer-based compensator

In this appendix it is proved that any two similar state-output discrete time canonical plant models yield the same optimal compensator dynamics and optimal cost. The analysis follows quite closely the analysis for the continuous time case followed by Blanvillain and Johnson [B3].

The appendix is structured as follows. First a review of a method to obtain the state-output canonical plant model from any other realization is presented. Then, optimal minimal order observer-based compensators are designed for two equivalent state output canonical plant models. As in Blanvillain and Johnson a matrix decomposition and a matrix equivalence are used in order to compare these two designs. In this way it is shown that different state output plant canonical forms yield observer design parameters which are related by a similarity transformation. Also it is shown that these two designs yield the same optimal cost.

For a plant described by the following linear time invariant model (see section II.II for details on dimensions, etc.):

$$x(t+1) = A x(t) + B u(t) \qquad (C.1)$$

$$y(t) = C x(t) \qquad (C.2)$$

with initial condition statistics:

$$E\{x(0)x'(0)\} = \Sigma \tag{C.3}$$

$$E\{x(0)\} = m_0 \tag{C.4}$$

with optimal control given by (see Section[†] II.IV):

$$u(t) = P \hat{x}(t) \tag{C.5}$$

$$\hat{x}(t) = N y(t) + M z(t) \tag{C.6}$$

$$z(t+1) = F z(t) + G y(t) + D u(t); \quad z(0) = [T_1 \vdots I]m_0 \tag{C.7}$$

designed by minimizing the following performance index:

$$J = E\left\{\sum_{t=0}^{\infty} x_t'Q x_t + u_t'S'x_t + x_t'Su_t + u_t'Ru_t\right\} \tag{C.8}$$

Consider the following similarity transformation which puts (C.1)-

(C.4) in state-output canonical form:

$$\overline{x}(t) = H x(t) \tag{C.9}$$

where H(n x n) is obtained as follows:

$$H = \begin{bmatrix} C \\ \Lambda \end{bmatrix} \tag{C.10}$$

where Λ((n-m) x n) is an arbitrary matrix such that H is full rank.
The freedom in selecting this Λ is what makes this transformation non-
unique. The transformation is selected as above because then the

[†]The asterisks indicating the optimality of the parameters have been
dropped for notational convenience.

measurement equation (C.2) becomes:

$$y(t) = C H^{-1} \bar{x}(t) = [I \vdots 0] \bar{x}(t) \tag{C.11}$$

since:

$$H H^{-1} = I \tag{C.12}$$

and:

$$C H^{-1} = [I \vdots 0] \tag{C.13}$$

As shown in Blanvillain and Johnson, there exists a unique

matrix decomposition for Λ:

$$\Lambda = VC + \tilde{N} \; ; \; \tilde{N} C' = 0 \tag{C.14}$$

where $V((n-m) \times m)$ and $\tilde{N}((n-m) \times n)$ are given by:

$$V = \Lambda C'(C C')^{-1} \tag{C.15}$$

$$\tilde{N} = \Lambda(I - C'(C C')^{-1}C) \tag{C.16}$$

H^{-1} can be partitioned in an analogous form to (C.10):

$$H^{-1} = [\tilde{C} \quad \tilde{\Lambda}] \tag{C.17}$$

where the matrices $\tilde{C}(n \times m)$ and $\tilde{\Lambda}(n \times (n-m))$ are given by:

$$\tilde{C} = C'(C C')^{-1} - \tilde{N}'(\tilde{N} \tilde{N}')^{-1}V \tag{C.18}$$

$$\tilde{\Lambda} = \tilde{N}'(\tilde{N} \tilde{N}')^{-1} \tag{C.19}$$

It can be shown [B3] that for any two Λ's: Λ^1 and Λ^2 with unique

decompositions as in (C.14), there exists a unique nonsingular matrix

$W((n-m) \times (n-m))$ such that

$$\tilde{N}^2 = W \tilde{N}^1 \tag{C.20}$$

where $\tilde{N}^2$, $\tilde{N}^1$ correspond to Λ^2 and Λ^1 respectively.

The relations (C.14)-(C.20) play a _very important_ role in the proof that follows.

Under the transformation (C.9), the plant model (C.1)-(C.4) becomes:

$$\overline{x}(t+1) = H A H^{-1}\overline{x}(t) + H B u(t) \tag{C.21}$$

$$y(t) = [I \; \vdots \; 0] \, \overline{x}(t) \tag{C.22}$$

with initial condition statistics given by:

$$E\{\overline{x}(0)\} = H \, m_0 \tag{C.23}$$

$$E\{\overline{x}(0)\overline{x}'(0)\} = H \, \Sigma \, H' \tag{C.24}$$

The optimal control (C.5)-(C.7) is also modified:

$$u(t) = P H^{-1} \hat{\overline{x}} \tag{C.25}$$

$$\hat{\overline{x}}(t) = H N[I \; \vdots \; 0]\overline{x} + H M z(t) \tag{C.26}$$

$$z(t+1) = F z(t) + G y(t) + D u(t); \; z(0) = [T_1 \; \vdots \; I]H \, m_0 \tag{C.27}$$

for the performance index:

$$J = E \left\{ \sum_{t=0}^{\infty} \overline{x}'_t (H')^{-1}Q \, H^{-1}\overline{x}_t + u'_t S' H^{-1}\overline{x}_t + \overline{x}'_t (H')^{-1}Su_t \right.$$

$$\left. + u'_t R \, u_t \right\} \tag{C.28}$$

Using the partitions (C.10) and (C.17) some of the above matrices can be expressed as follows:

$$H A H^{-1} = \begin{bmatrix} C A \tilde{C} & C A \tilde{\Lambda} \\ \Lambda A C & \Lambda A \tilde{\Lambda} \end{bmatrix} \qquad \text{(C.29)}$$

$$H B = \begin{bmatrix} C B \\ \Lambda B \end{bmatrix} \qquad \text{(C.30)}$$

$$H m_0 = \begin{bmatrix} C m_0 \\ \Lambda m_0 \end{bmatrix} \qquad \text{(C.31)}$$

$$H \Sigma H' = \begin{bmatrix} C \Sigma C' & C \Sigma \Lambda' \\ \Lambda \Sigma C' & \Lambda \Sigma \Lambda' \end{bmatrix} \qquad \text{(C.32)}$$

In the analysis that follows optimal minimal order observer-based compensators as in (C.25)-(C.27) are designed for two equivalent systems of the form (C.21)-(C.24). The only difference between these two systems is the Λ-matrix used in the transformation (C.10), (C.9). For notational convenience these systems and corresponding designs are differentiated by the superscripts 1 and 2.

Following the results obtained in Section II.IV the compensator design parameters corresponding to Λ^i, where $i = 1, 2$, are:

From (C.25)-(C.26) and (2.18)-(2.19) with $T_2 = I$:

$$P^i N^i = P^i \left[-\begin{bmatrix} 0 \\ I \end{bmatrix} T_1^i + \begin{bmatrix} I \\ 0 \end{bmatrix} \right] \qquad \text{(C.33)}$$

$$P^i M^i = P^i \begin{bmatrix} 0 \\ I \end{bmatrix} \qquad \text{(C.34)}$$

From (C.27) and (2.12), (2.21), (2.22) for $T_2 = I$:

$$F^i = T_1^i A_{1_2}^i + A_{2_2}^i \tag{C.35}$$

$$G^i = T_1^i A_{1_1}^i + A_{2_1}^i - F^i T_1^i \tag{C.36}$$

$$D^i = T_1^i B_1^i + B_2^i \tag{C.37}$$

$$z^i(0) = T_1^i m_1^i + m_2^i \tag{C.38}$$

where the partitions to be used are:

$$\begin{bmatrix} A_{1_1} & A_{1_2} \\ A_{2_1} & A_{2_2} \end{bmatrix} = \begin{bmatrix} C\,A\,\tilde{C} & C\,A\,\tilde{\Lambda} \\ \Lambda\,A\,\tilde{C} & \Lambda\,A\,\tilde{\Lambda} \end{bmatrix} \tag{C.39}$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} C\,B \\ \Lambda\,B \end{bmatrix} \tag{C.40}$$

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} C\,m \\ \Lambda\,m \end{bmatrix} \tag{C.41}$$

obtained from (C.29)-(C.31) respectively; with the appropriate super-scripts. According to Section II.IV, P^i and T^i are given by:

From (2.62):

$$P^i = -(R + B^{i'} \Gamma_1^i B^i)^{-1} (B^{i'} \Gamma_1^i A^i + S^{i'}) \tag{C.42}$$

where $\Gamma^i_{1_1}$ is given by (2.63):

$$\Gamma^i_{1_1} = A^{i'} \Gamma^i_{1_1} A^i + Q^i - (B^{i'} \Gamma^i_{1_1} A^i + S^{i'})' \cdot$$

$$\cdot (R + B^{i'} \Gamma^i_{1_1} B^i)^{-1} (B^{i'} \Gamma^i_{1_1} A^i + S^{i'}) \tag{C.43}$$

Finding P^1 from (C.42) and using:

$$B^1 = H^1 B \tag{C.44}$$

$$A^1 = H^1 A (H^1)^{-1} \tag{C.45}$$

$$S^1 = (H^{1'})^{-1} S \tag{C.46}$$

obtained from (C.21)-(C.32), we obtain:

$$P^1 = -(R + B'H^{1'} \Gamma^1_{1_1} H^1 B)^{-1} (B'H^{1'} \Gamma^1_{1_1} H^1 A (H^1)^{-1} + S' (H^1)^{-1}) \tag{C.47}$$

Let:

$$\Gamma_{1_1} = H^{1'} \Gamma^1_{1_1} H^1 \tag{C.48}$$

and:

$$P = -(R + B' \Gamma_{1_1} B)^{-1} (B' \Gamma_{1_1} A + S') \tag{C.49}$$

then

$$P = P^1 H^1 \tag{C.50}$$

(C.48) can also be obtained from (C.43) for $i = 1$.

The case for $i = 2$ is treated in the same way as above. The

results obtained are as follows:

For:

$$\Gamma_{1_1} = H^{2'} \; \Gamma^2_{1_1} \; H^2 \tag{C.51}$$

and P as in (C.49):

$$P = P^2 \; H^2 \tag{C.52}$$

From (2.64):

$$T^i_1 = -(\Sigma^{i'}_{1_2} - m^i_2 m^{i'}_1 + A^i_{2_2} K^i_{2_2} A^{i'}_{1_2})(\Sigma^i_{1_1} - m^i_1 m^{i'}_1 + A^i_{1_2} K^i_{2_2} A^{i'}_{1_2})^{-1} \tag{C.53}$$

where K_{2_2} is given by (2.65)

$$K^i_{2_2} = A^i_{2_2} K^i_{2_2} A^i_{2_2} + \Sigma^i_{2_2} - m^i_2 m^{i'}_2 - (\Sigma^{i'}_{1_2} - m^i_2 m^{i'}_1 + A^i_{2_2} K^i_{2_2} A^{i'}_{1_2})$$

$$\cdot (\Sigma^i_{1_1} - m^i_1 m^{i'}_1 + A^i_{1_2} K^i_{2_2} A^{i'}_{1_2})^{-1}(\Sigma^i_{1_2} - m^i_1 m^{i'}_2 + A^i_{1_2} K^i_{2_2} A^{i'}_{2_2}) \tag{C.54}$$

where the partitions to be used are (obtained from (C.31)-(C.32)):

$$\begin{bmatrix} \Sigma_{1_1} & \Sigma_{1_2} \\ \Sigma'_{1_2} & \Sigma_{2_2} \end{bmatrix} = \begin{bmatrix} C \; \Sigma \; C' & C \; \Sigma \; \Lambda' \\ \Lambda \; \Sigma \; C' & \Lambda \; \Sigma \; \Lambda' \end{bmatrix} \tag{C.55}$$

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} C & m_0 \\ \Lambda & m_0 \end{bmatrix} \tag{C.56}$$

and (C.39)-(C.40) with the appropriate superscripts.

Using (C.14), (C.19), (C.39) and (C.55) in (C.53) the following
is obtained:

$$T_1^i = -\tilde{N}^i(\Sigma C' - mm'C' + A'\tilde{N}^{i'}(\tilde{N}^i\tilde{N}^{i'})^{-1}K_{2_2}^i(\tilde{N}^i\tilde{N}^{i'})^{-1}\tilde{N}^iA'C') \cdot$$

$$\cdot (C \Sigma C' - C mm'C' + CA\tilde{N}^{i'}(\tilde{N}^i\tilde{N}^{i'})^{-1}K_{2_2}^i(\tilde{N}^i\tilde{N}^{i'})^{-1}\tilde{N}^iA'C')^{-1}-v^i \tag{C.57}$$

Using (C.14), (C.19), (C.39) and (C.55), (C.54) becomes:

$$K_{2_2}^i = \tilde{N}^iA \tilde{N}^{i'}(\tilde{N}^i\tilde{N}^{i'})^{-1}K_{2_2}^i(\tilde{N}^i\tilde{N}^{i'})^{-1}\tilde{N}^iA'\tilde{N}^{i'} + \tilde{N}^i(\Sigma - mm')\tilde{N}^{i'}$$

$$- \tilde{N}^i(\Sigma C' - mm'C' + A \tilde{N}^{i'}(\tilde{N}^i\tilde{N}^{i'})^{-1}K_{2_2}^i(\tilde{N}^i\tilde{N}^{i'})^{-1}\tilde{N}^iA'C') \cdot$$

$$\cdot(C \Sigma C' - C mm'C' + CA\tilde{N}^{i'}(\tilde{N}^i\tilde{N}^{i'})^{-1}K_{2_2}^i(\tilde{N}^i\tilde{N}^{i'})^{-1}\tilde{N}^iA'C')^{-1} \cdot$$

$$\cdot(C \Sigma - C mm' + CA\tilde{N}^{i'}(\tilde{N}^i\tilde{N}^{i'})^{-1}K_{2_2}^i(\tilde{N}^i\tilde{N}^{i'})^{-1}\tilde{N}^iA')\tilde{N}^{i'} \tag{C.58}$$

Comparing (C.57) for i = 1 and 2 based on (C.20), we conclude:

$$T_1^2 = W(T_1^1 + v^1) - v^2 \tag{C.59}$$

Comparing (C.58) for i = 1 and 2 based on (C.20), we obtain:

$$K_{2_2}^2 = W K_{2_2}^1 W' \tag{C.60}$$

The relations (C.50), (C.52) and (C.59)-(C.61) are useful in simpli-
fying (C.33)-(C.37) for i = 2 as follows:

a) From (C.33) with i = 2 and using (C.14):

$$P^2 N^2 = P\, H^{2^{-1}} \left[- \begin{bmatrix} 0 \\ I \end{bmatrix} T_1^2 + \begin{bmatrix} I \\ 0 \end{bmatrix} \right] \tag{C.61}$$

Using (C.17)-(C.19) and (C.20) we obtain:

$$P^2 N^2 = P[-\tilde{N}^1 (\tilde{N}^1 \tilde{N}^{1'})^{-1} (T_1^2 + V^2) + C'(C\ C')^{-1}] \tag{C.62}$$

Finding $P^1 N^1$ the same expression on the R.H.S. of (C.62)

is obtained; therefore:

$$P^2 N^2 = P^1 N^1 \tag{C.63}$$

b) Using (C.19), (C.34) for $i = 2$ becomes:

$$P^2 M^2 = P\, H^{2^{-1}} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{C.64}$$

Using (C.17), (C.19) and (C.20):

$$P^2 M^2 = P\, \tilde{N}^{1'} (\tilde{N}^1 \tilde{N}^{1'})^{-1}\, W^{-1} \tag{C.65}$$

If $P^1 M^1$ is found as above and compared with the

R.H.S. of (C.65), we conclude that:

$$P^2 M^2 = P^1 M^1\, W^{-1} \tag{C.66}$$

c) (C.35) for $i = 2$ equals:

$$F^2 = T_1^2 A_{1_2}^2 + A_{2_2}^2 \tag{C.67}$$

Using (C.59), (C.39) and (C.14), (C.67) becomes:

$$F^2 = W(T_1^1 + V^1)\ C\ A\ \tilde{\Lambda}^2 + \tilde{N}^2 A\ \tilde{\Lambda}^2 \tag{C.68}$$

Using (C.19), (C.20) and (C.14):

$$F^2 = W(T_1^1 \tilde{C} A + \Lambda^1 A)\tilde{N}^{1'}(\tilde{N}^1 \tilde{N}^{1'})^{-1}W^{-1} \tag{C.69}$$

Using (C.19) and (C.39) for $i = 1$ we conclude:

$$F^2 = W(T_1^1 A_{1_2}^1 + A_{2_2}^1)W^{-1} = W\ F^1 W^{-1} \tag{C.70}$$

d) In a similar way as above, (C.36) for $i = 2$ can be

simplified:

$$G^2 = T_1^2 A_{1_1}^2 + A_{2_1}^2 - F^2 T_1^2 \tag{C.71}$$

using (C.59), (C.39), (C.14), (C.19), (C.18) and (C.20)

as:

$$G^2 = W\ G^1 \tag{C.72}$$

e) (C.37) for $i = 2$ becomes:

$$D^2 = T_1^2 B_1^2 + B_2^2 \tag{C.73}$$

Using (C.59), (C.40) and (C.14), (C.73) is simplified:

$$D^2 = W\ D^1 \tag{C.74}$$

f) and (C.38) for $i = 2$:

$$Z^2(0) = T_1^2 m_1^2 + m_2^2 \tag{C.75}$$

Using (C.59), (C.41), (C.14), (C.20) and (3.38) for $i = 1$

we can show that:

$$Z^2(0) = W\ Z^1(0) \tag{C.76}$$

It is clear from equations (C.63), (C.66), (C.70), (C.72), (C.74) and (C.76) that different state-output plant canonical models (obtained by changing Λ in (C.10)) yield optimal observers related by a similarity transformation. From (C.70) it is also clear that the optimal observer dynamics are invariant to a change in state-output plant canonical form.

In the analysis that follows we intend to show that the optimal cost for this optimal compensator design problem is independent of the state-output transformation used. Before proceeding with this proof the previous results are tabulated:

TABLE 1

Equation reference	Corresponding matrices for design i = 1	Corresponding matrices for design i = 2
A^i from (C.21)	$H^1 A (H^1)^{-1}$	$H^2 A (H^2)^{-1}$
B^i from (C.21)	$H^1 B$	$H^2 B$
$[I : 0]$ from (C.22)	$[I : 0]$	$[I : 0]$
Σ^i from (C.24)	$H^1 \Sigma H^{1'}$	$H^2 \Sigma H^{2'}$
m_0^i from (C.23)	$H^1 m_0$	$H^2 m_0$
S^i from (C.28)	$(H^1)^{-1} S$	$(H^2)^{-1} S$
Q^i from (C.28)	$(H^{1'})^{-1} Q (H)^{-1}$	$(H^{2'})^{-1} Q (H^2)^{-1}$
R from (C.28)	R	R
P^i from (C.25)	$P (H^1)^{-1}$	$P (H^2)^{-1}$
$\Gamma_{1_1}^i$ from (C.43)	$(H^{1'})^{-1} \Gamma_{1_1} (H^1)^{-1}$	$(H^{2'})^{-1} \Gamma_{1_1} (H^2)^{-1}$
T_1^i from (C.53)	T_1^1	$T_1^2 = W(T_1^1 + V^1) - V^2$

$K_{2_2}^i$ from (C.54) $\qquad$ $K_{2_2}^1$ $\qquad$ $K_{2_2}^2 = WK_{2_2}^1 W'$

$P^i N^i$ from (C.33) $\qquad$ $P^1 N^1$ $\qquad$ $P^2 N^2 = P^1 N^1$

$P^i M^i$ from (C.34) $\qquad$ $P^1 M^1$ $\qquad$ $P^2 M^2 = P^1 M^1 W^{-1}$

F^i from (C.35) $\qquad$ F^1 $\qquad$ $F^2 = WF^1 W^{-1}$

G^i from (C.36) $\qquad$ G^1 $\qquad$ $G^2 = WG^1$

D^i from (C.37) $\qquad$ D^1 $\qquad$ $D^2 = WD^1$

Z_0^i from (C.38) $\qquad$ Z_0^1 $\qquad$ $Z_0^2 = WZ_0^1$

The optimal cost, when an optimal minimum order observer-based compensator (see Chapter II) is used, is given by:

$$J^* = tr\{\Gamma_{1_1} \Sigma\} + tr\{\Gamma_{2_2} (T \Sigma T')\} \tag{C.77}$$

This expression is obtained from (2.33) using (2.52) and $\Gamma_{1_2} = 0$. Recall that the T's have the form:

$$T = [T_1 \; : \; I] \tag{C.78}$$

In (C.77), Γ_{1_1} is given by (C.43) and Γ_{2_2} by:

$$\Gamma_{2_2} = F'\Gamma_{2_2} F + M'(B'\Gamma_{1_1} B + R)M \tag{C.79}$$

which is obtained using (2.18), (2.21) and $T_2 = I$ in (2.51). Analyzing (C.79) for designs $i = 1$ and $i = 2$ the following results are obtained:

For design $i = 1$:

$$\Gamma_{2_2}^1 = F^{1'}\Gamma_{2_2}^1 F^1 + M^{1'}(B^{1'}\Gamma_{1_1}^1 B^1 + R)M^1 \tag{C.80}$$

For design 2:

$$\Gamma^2_{2_2} = F^{2'}\Gamma^2_{2_2} F^2 + M^{2'}(B^{2'}\Gamma^2_{1_1} B^2 + R)M^2 \qquad\qquad (C.81)$$

Using results on Table 1, (C.81) becomes

$$\Gamma^2_{2_2} = (W')^{-1}F^{1'}W'\Gamma^2_{2_2} W\, F^1(W)^{-1} +$$

$$(W')^{-1}M^{1'}(B'H^{2'}\Gamma^2_{1_1} H^2B + R)M^1W^{-1} \qquad\qquad (C.82)$$

Using $\Gamma^1_{1_1}$, $\Gamma^2_{1_1}$ and B^1 from Table 1, premultiplying (C.82) by W' and

postmultiplying by W we can compare (C.82) with (C.80) and obtain the

following:

$$\Gamma^1_{2_2} = W'\Gamma^2_{2_2} W \qquad\qquad (C.83)$$

Calculating the optimal cost for designs $i = 1$ and $i = 2$ the following

results are obtained:

For design $i = 1$:

$$J^{1*} = \text{tr}\{(H^{1'})^{-1}\Gamma_{1_1} (H^1)^{-1}(H^1 \Sigma H^{1'})\} +$$

$$\text{tr}\{\Gamma^1_{2_2} (T^1H^1 \Sigma H^{1'}T^{1'})\} \qquad\qquad (C.84)$$

This was obtained using $\Gamma^1_{1_1}$ and Σ^1 from Table 1 in (C.79). Using a

trace identity and (C.32), (C.78), (C.84) becomes:

$$J^{1*} = \text{tr}\{\Gamma_{1_1} \Sigma\} + \text{tr}\{\Gamma^1_{2_2} (T^1_1 C \Sigma C' T^{1'}_1 + \Lambda^1 \Sigma C'T^{1'}_1 +$$

$$+ T^1_1 C \Sigma \Lambda^{1'} + \Lambda^1 \Sigma \Lambda^{1'})\} \qquad\qquad (C.85)$$

Using (C.14), (C.85) can be expressed as:

$$J^{1*} = \text{tr}\{\Gamma_{1_1} \Sigma\} + \text{tr}\{\Gamma_{2_2}^1 \,[(T_1^1 + v^1)C \,\Sigma\, C'(T_1^1 + v^1)'$$

$$+ \tilde{N}^1 \,\Sigma\, C'(T_1^1 + v^1)' + (T_1^1 + v^1)C \,\Sigma\, \tilde{N}^{1'} + \tilde{N}^1 \,\Sigma\, \tilde{N}^{1'}] \quad \text{(C.86)}$$

For design i = 2 the same expression as in (C.86) is obtained but with superscripts changed:

$$J^{2*} = \text{tr}\{\Gamma_{1_1} \Sigma\} + \text{tr}\{\Gamma_{2_2}^2 \,[(T_1^2 + v^2)C \,\Sigma\, C'(T_1^2 + v^2)'$$

$$+ \tilde{N}^2 \,\Sigma\, C'(T_1^2 + v^2)' + (T_1^2 + v^2)C \,\Sigma\, \tilde{N}^{2'} + \tilde{N}^2 \,\Sigma\, \tilde{N}^{2'}]\}$$

$$\text{(C.87)}$$

Using T_1^2 from Table 1 and (C.20), (C.87) becomes:

$$J^{2*} = \text{tr}\{\Gamma_{1_1} \Sigma\} + \text{tr}\{\Gamma_{2_2}^2 \, W[(T_1^1 + v^1)C \,\Sigma\, C'(T_1^1 + v^1)'$$

$$+ \tilde{N}^1 \,\Sigma\, C'(T_1^1 + v^1)' + (T_1^1 + v^1)C \,\Sigma\, \tilde{N}^{1'} + \tilde{N}^1 \,\Sigma\, \tilde{N}^{1'}]W'\}$$

$$\text{(C.88)}$$

Using a trace identity and (C.83) it is obtained that:

$$J^{1*} = J^{2*}$$

as desired!

APPENDIX D

Formulation of a specific dynamic optimization

problem as a static optimization problem

The following dynamic optimization problem can be posed as a static optimization problem.

$$\min J(u_t) = E\left\{ \sum_{t=0}^{\infty} x_t'\bar{Q}x_t + x_t'\bar{S}u_t + u_t'\bar{S}'x_t + u_t'\bar{R}u_t \right\} \tag{D.1}$$

subject to:

$$x(t+1) = A\ x(t) + B\ u(t) \tag{D.2}$$

$$y(t) = C\ x(t) \tag{D.3}$$

$$u(t) = P\ y(t) \tag{D.4}$$

$$E\{x(0)x'(0)\} = \overset{\circ}{x}_0 \tag{D.5}$$

$$E\{x(0)\} = m_0$$

Solution:

Using (D.2)-(D.4) the closed loop system equation is:

$$x(t+1) = (A + B\ P\ C)x(t) \tag{D.6}$$

In compact form:

$$x(t) = (A + B\ P\ C)^t x(0) \tag{D.7}$$

Therefore the cost functional (D.1) becomes:

$$J(P) = E\left\{ \sum_{t=0}^{\infty} x'(0)(A' + C'P'B')^t(Q + S\ P\ C + C'P'S' + C'P'R\ P\ C) \right.$$

$$\left. \cdot (A + B\ P\ C)^t x(0) \right\} \tag{D.8}$$

Using the trace operator and interchanging it with the expectation operator:

$$J(P) = tr \left\{ \sum_{t=0}^{\infty} (A' + C'P'B')^t (Q + S\ P\ C + C'P'S' + C'P'R\ P\ C)(A + B\ P\ C)^t X_0 \right\} \qquad (D.9)$$

Let:

$$\Gamma = \sum_{t=0}^{\infty} (A' + C'P'B')^t (Q + S\ P\ C + C'P'S' + C'P'R\ P\ C) \cdot$$

$$\cdot\ (A + B\ P\ C)^t \qquad (D.10)$$

Then (D.9) can be expressed as:

$$\min J(P,\ \Gamma) = tr\{\Gamma\ X_0\} \qquad (D.11)$$

subject to (D.10).

Analyzing (D.10) the following identities can be obtained:

$$\Gamma = Q + S\ P\ C + C'P'S' + C'\ P'R\ P\ C + \sum_{t=1}^{\infty} (A' + C'P'B')^t \cdot$$

$$\cdot\ (Q + S\ P\ C + C'P'S' + C'P'R\ P\ C)(A + B\ P\ C)^t \qquad (D.12)$$

$$= Q + S\ P\ C + C'P'S' + C'P'R\ P\ C + (A' + C'P'B') \cdot$$

$$\cdot \left\{ \sum_{t=1}^{\infty} (A' + C'P'B')^{t-1} (Q + S\ P\ C + C'P'S' + C'P'R\ P\ C) \cdot \right.$$

$$\left. \cdot\ (A + B\ P\ C)^{t-1} \right\} (A + B\ P\ C) \qquad (D.13)$$

Using the following transformation:

$$\tau = t-1$$

then using (D.10), (D.13) becomes:

$$\Gamma = Q + S \, P \, C + C'P'S' + C'P'R \, P \, C + (A' + C'P'B')\Gamma(A + B \, P \, C)$$

$$(D.14)$$

Adjoining the constraint (D.14) to the cost functional (D.11) via the Lagrange multiplier Σ the optimization problem considered becomes:

$$\min \bar{J}(\Gamma, \Sigma, P) = tr\Big\{\Gamma \, X_0 + \Sigma[-\Gamma + Q + S \, P \, C + C'P'S' +$$

$$+ C'P'R \, P \, C + (A' + C'P'B')\Gamma(A + B \, P \, C)]\Big\}$$

$$(D.15)$$

which is a static optimization problem.

APPENDIX E

Kleinman's Lemma and Bellman's Approximation to $(A + \varepsilon B)^t$

E.I   Kleinman's lemma [K4, Appendix F]

Let $f(\cdot)$ be a trace function operating on the $r \times n$ space:

$$f(x) = \text{trace } (F(x)) \tag{E.1}$$

here $F(\cdot)$ is a continuously differentiable mapping from $R^{r \times n}$, where x lies, into $R^{n \times n}$.

If we can write:

$$f(x + \varepsilon \Delta x) - f(x) = \varepsilon \text{tr}\{M(x) \cdot \Delta x\} \tag{E.2}$$

where ε is an arbitrary small scalar and $M(x)$ an $n \times r$ matrix functional then:

$$\frac{\partial f(x)}{\partial x} = M'(x) \tag{E.3}$$

E.II   Discrete time version of Bellman's result on perturbation theory for $e^{(A + \varepsilon B)}$ [B1, p. 174].

Problem statement:

Find an expression for $(A + \varepsilon B)^t$ for ε arbitrary small.

Solution:

The expression $(A + \varepsilon B)^t$ is obtained as a compact form of the following sequential equation:

$$x(t+1) = A\, x(t) + \varepsilon B\, x(t) \tag{E.4}$$

-130-

Another compact solution of (E.4) is obtained taking $\varepsilon B\, x(t)$ as an input:

$$x(t) = A^t x(0) + A^t \left(\sum_{\tau=0}^{t-1} A^{-(\tau+1)} \varepsilon B\, x(\tau) \right) \qquad (E.5)$$

Iterating (E.5):

$$x(t) = A^t x(0) + A^t \left(\sum_{\tau=0}^{t-1} A^{-(\tau+1)} \varepsilon B\, A^\tau x(0) \right) + H.O.T.(\varepsilon)$$

$$(E.6)$$

If ε is arbitrary small:

$$x(t) = \left(A^t + \varepsilon A^t \sum_{\tau=0}^{t-1} A^{-(\tau+1)} B\, A^\tau \right) x(0) \qquad (E.7)$$

Therefore:

$$(A + \varepsilon B)^t = A^t + \varepsilon A^t \sum_{\tau=0}^{t-1} \left\{ A^{-(\tau+1)} B\, A^\tau \right\} \qquad (E.8)$$

E.III Derivation of Equation (3.41):

J(P + εΔP) and J(P) can be obtained from (D.8). The increment J(P + εΔP) − J(P) can be simplified using the discrete-time version of Bellman's result given in E.II. The result obtained is as follows:

$$J(P + \varepsilon \Delta P) - J(P) = 2\,\varepsilon \sum_{t=0}^{\infty} \operatorname{tr}\Big\{ (A' + C'P'B')^t (Q + C'P'R\,P\,C +$$

$$+ S\,P\,C + C'P'S')(A + B\,P\,C)^t \sum_{\tau=0}^{t-1} (A + B\,P\,C)^{-(\tau+1)} \cdot$$

$$\cdot B\Delta P C (A + B\,P\,C)^\tau x_0 \Big\} + \varepsilon \sum_{t=0}^{\infty} \operatorname{tr}\Big\{ (A' + C'P'B')^t \cdot$$

$$\cdot \ (C'\Delta P'R \ P \ C + C'P'R\Delta P \ C)(A + B \ P \ C)^{t}x_0 \Big\} \qquad (E.9)$$

Using Kleinman's lemma (see F.I)

$$\left.\frac{\partial J}{\partial P}\right|_{P^*} = 2 \sum_{t=1}^{\infty} \sum_{\tau=0}^{t-1} \Big\{ C(A + B \ P^*C)^{\tau}x_0(A' + C'P^{*'}B')^{t} \cdot$$

$$\cdot (Q + C'P^{*'}R \ P^*C + S \ P^*C + C'P^{*'}S')(A + B \ P^*C)^{t-(\tau+1)}B \Big\}$$

$$+ \ 2 \sum_{t=0}^{\infty} \Big\{ C(A + B \ P^*C)^{t}x_0(A' + C'P^{*'}B')^{t}(C'P^{*'}R + S) \Big\}$$

$$\qquad (E.10)$$

which is equation (3.41).

APPENDIX F

On the stability of a plant under

a control law designed for an aggregated model

Following Aoki [A1, p. 251] it can be shown that once the aggregated model:

$$\dot{z}(t) = F\, z(t) + D\, u(t) \qquad\qquad (F.1)$$

is stable under a control law of the form:

$$u(t) = P\, z(t), \qquad\qquad (F.2)$$

then the true model:

$$\dot{x}(t) = A\, x(t) + B\, u(t) \qquad\qquad (F.3)$$

is stable under the same control law if:

i)   A is diagonal and the eigenvalues not included in the aggregated model are stable

ii) the aggregated transformation is of the form:

$$H = [\overline{H} \;\vdots\; 0] \qquad\qquad (F.4)$$

where $\overline{H}$ is square and full rank.

Proof:

The eigenvalues of the closed loop system (F.2)-(F.3) are given by the roots of the following equation in λ:

$$\det[\lambda I - (A + B\, P\, H)] = \det[\lambda I - \overline{A}] = 0 \qquad\qquad (F.5)$$

where:

$$\overline{A} = \begin{pmatrix} J_1 & & & 0 \\ & \ddots & & \\ & & J_j & \\ & & & \ddots \\ 0 & & & \ddots J_i \end{pmatrix} + \begin{pmatrix} & \vdots & \\ B\ P\ \overline{H} & \vdots & 0 \\ & \vdots & \\ & \vdots & \end{pmatrix} \qquad \text{(F.6)}$$

where J_i is the ith Jordan Block. The Jordan Blocks from J_{j+1} to J_i

represent the "unimportant" very stable modes of the true model.

Using (F.4)

$$\begin{pmatrix} J & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & J_{j+1} \\ & & \ddots J_i \end{pmatrix} + \begin{pmatrix} (I \vdots 0)B\ P\ \overline{H} & \vdots & 0 \\ \cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ (0 \vdots I)B\ P\ \overline{H} & \vdots & 0 \end{pmatrix} \qquad \text{(F.7)}$$

Therefore (F.5) becomes:

$$\det\left[\lambda I - \begin{pmatrix} J + (I \vdots 0)B\ P\ \overline{H} & \vdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ (0 \vdots I)B\ P\ \overline{H} & \vdots & \begin{matrix} J_{j+1} & 0 \\ & \ddots \\ 0 & J_i \end{matrix} \end{pmatrix} \right] = 0 \qquad \text{(F.8)}$$

which can be reduced to:

$$\det\left[\lambda I - \left(J + (I \vdots 0)B\ P\ \overline{H} \right) \right] \cdot \det\left[\lambda I - \begin{pmatrix} J_{j+1} & & 0 \\ & \ddots & \\ 0 & & J_i \end{pmatrix} \right] = 0$$

$$\text{(F.9)}$$

The roots of the equation:

$$\det\left[\lambda I - \begin{pmatrix} J_{j+1} & & 0 \\ & \ddots & \\ 0 & & J_i \end{pmatrix} \right] = 0 \qquad \text{(F.10)}$$

are all in the L.H.S. of the imaginary plane by assumption. Therefore

the stability test reduces to investigate the nature of the roots of:

$$\det[\lambda I - (J + (I \; \vdots \; 0)B\ P\ \overline{H})] \qquad\qquad (F.11)$$

Before proceeding to analyze (F.11), let's expand the equation (4.4)

using (F.4):

$$F\ H = H\ A$$

$$[F\ \overline{H} \; \vdots \; 0] = [\overline{H}\ J \; \vdots \; 0]$$

therefore:

$$F = \overline{H}\ J\ \overline{H}^{-1} \qquad\qquad (F.12)$$

In order to obtain (F.11) in terms of F, (F.12) is used:

$$\det\left[\overline{H}\left(\lambda I - (J + (I \; \vdots \; 0)B\ P\ \overline{H})\right)\overline{H}^{-1}\right] \qquad\qquad (F.13)$$

$$= \det\left[\lambda I - (F + (\overline{H} \; \vdots \; 0)B\ P)\right] = 0 \qquad\qquad (F.14)$$

Using (4.5) and (F.4), (F.14) becomes:

$$\det[\lambda I - (F + D\ P)] = 0 \qquad\qquad (F.15)$$

which gives the eigenvalues of the aggregated model (F.1) under the

feedback (F.2). It is clear that the roots of (F.15) are in the

L.H.S. of the imaginary plane by assumption.

APPENDIX G

Method to obtain the aggregated matrix H given

the structure of the aggregated model

In this method it is fundamental to have a controllable true

model as well as a controllable aggregated model.

Method:

Since the following controllability matrix:

$$\left[D \vdots F\,D \vdots \ldots F^{p-1}D \right] \tag{G.1}$$

is full rank and the following two relations must hold:

$$D = H\,B \tag{G.2}$$

$$F\,H = H\,A \tag{G.3}$$

Then, augmenting the (G.1) to the nth column* and using (G.2) and

(G.3):

$$H\left[B \vdots A\,B \vdots \ldots A^{n-1}B \right] = \left[D \vdots F\,D \vdots \ldots F^{n-1}D \right] \tag{G.4}$$

Notice that in the L.H.S. of (G.4) the controllability matrix of the

pair (A,B) is full rank and square.*

Therefore:

---

*We are assuming that rank B = 1, otherwise we shall use the Penrose
Inverse in (G.4) for solving for H and then check if (4.6) is
satisfied.

$$H = \left[D \quad \vdots \quad F\,D \quad \vdots \quad \ldots \quad F^{n-1} \right] \left[B \quad \vdots \quad A\,B \quad \vdots \quad A^{n-1}B \right]^{-1}$$

(G.5)

In the above derivation the following fact was used:

$$F \cdot F \cdot F \cdots F\,D = F \cdot F \cdot F \cdots F\,H\,B$$

$$= \cdots F \cdots F\,H\,A \cdot B$$

$$= H\,A \cdot A \cdots A \cdot B$$

(G.6)

REFERENCES

[A1]   M. Aoki, "Control of Large-Scale Dynamic Systems by Aggrega-
       tion", IEEE Trans. Auto. Control, Vol. AC-13, No. 3, June 1968,
       pp. 246-253.

[A2]   M. Aoki, Optimization of Stochastic Systems, New York: Academic
       Press, 1967.

[A3]   M. Athans, "The Matrix Minimum Principle", Information and Con-
       trol, Vol. 11, December 1967, pp. 592-606.

[A4]   M. Athans and P.L. Falb, Optimal Control, New York: McGraw-
       Hill, Inc., 1966.

[B1]   R. Bellman, Introduction to Matrix Analysis, New York: McGraw-
       Hill, Inc., second edition, 1970.

[B2]   P.J. Blanvillain, "Optimal Compensator Structure for Linear
       Time-Invariant Plant with Inaccessible States", Report ESL-R-546,
       M.I.T., May 1974.

[B3]   P.J. Blanvillain and T.L. Johnson, "Invariants of Optimal
       Minimal-Order Observer-Base Compensators", to be published.

[B4]   R.W. Brockett, Finite Dimensional Linear Systems, New York:
       John Wiley and Sons, Inc., 1970.

[B5]   A.E. Bryson, Jr. and Y. Ho, Applied Optimal Control, New York:
       John Wiley and Sons, second edition, 1975.

[C1]   P.E. Caines and D.Q. Mayne, "On the Discrete-Time Matrix
       Riccati Equation of Optimal Control", Int. J. Control, Vol. 12,
       No. 5, 1970, pp. 785-794.

[D1]   P. Dorato and A.H. Levis, "Optimal Linear Regulators: the.
       Discrete-Time Case", IEEE Trans. Auto. Control, Vol. AC-16,
       No. 6, December 1971, pp. 613-620.

[F1]   T.E. Fortmann and D. Williamson, "Design of Low Order Observers
       for Linear Feedback Control Laws", IEEE Trans. Auto. Control,
       Vol. AC-17, No. 3, June 1972, pp. 301-308.

-139-

[G1] F.R. Gantmacher, The Theory of Matrices, Vol. I, New York: Chelsea, 1959.

[G2] B. Gopinath, "On the Control of Linear Multiple Input-Output Systems", The Bell System Tech. J., Vol. 50, No. 3, March 1971, pp. 1063-1081.

[J1] T.L. Johnson and M. Athans, "On the Design of Optimal Constrained Dynamic Compensators for Linear Constant Systems", IEEE Trans. Auto. Control, Vol. AC-15, No. 6, December 1970, pp. 658-660.

[K1] R.E. Kalman, "Contributions to the Theory of Optimal Control", Boletín de la Sociedad Matemática Mexicana, Vol. 5, 1960, pp. 102-119.

[K2] R.E. Kalman, "Mathematical Description of Linear Dynamical Systems", SIAM Journal of Control, Vol. 1, No. 2, 1963, pp. 152-192.

[K3] R.E. Kalman, P.L. Falb, and M. Arbib, Topics in Mathematical System Theory , New York: McGraw-Hill, 1969.

[K4] D.L. Kleinman, "Suboptimal Design of Linear Regulator Systems Subject to Computer Storage Limitations", M.I.T. Report ESL-R-297, February 1967.

[K5] V. Kucera, "A Contribution to Matrix Quadratic Equations", IEEE Trans. Auto. Control, Vol. AC-17, June 1972, pp. 344-347.

[K6] H. Kwakernaak and R. Sivan, Linear Optimal Control Systems , New York: John Wiley and Sons, Inc., 1972.

[L1] W.S. Levine, "Optimal Output-Feedback Controllers for Linear Systems", M.I.T., Report ESL-R-374, February 1969.

[L2] W.S. Levine, T.L. Johnson and M. Athans, "Optimal Limited State Variable Feedback Controllers for Linear Systems", IEEE Trans. Auto. Control, Vol. AC-16, No. 6, December 1971, pp. 785, 793.

[L3] D.G. Luenberger, "An Introduction to Observers", IEEE Trans. Auto. Control, Vol. AC-16, No. 6, December 1971, pp. 596-602.

[M1] R.A. Miller, "Specific Optimal Control of the Linear Regulator using a Minimal Order Observer", Int. J. Control, Vol. 18, No. 1, 1973, pp. 139-159.

[M2]   A. Motazedi, "Specific Optimal Control of the Linear Regulator
       Using a Low Order Observer", B.S. thesis, Department of Elec-
       trical Engineering and Computer Science, M.I.T., Cambridge,
       Mass., February 1976.

[N1]   M.M. Newmann, "Specific Optimal Control of the Linear Regulator
       using a Dynamical Controller based on the Minimal-Order Luen-
       berger Observer", Int. J. Control, Vol. 12, July 1970,
       pp. 33-48.

[O1]   J. O'Reilly and M.M. Newmann, "On the Design of Discrete Time
       Optimal Dynamical Controllers using a Minimal-Order Observer",
       Int. J. Control, Vol. 23, No. 2, 1976, pp. 257-275.

[R1]   G. Roberts, "Considerations of Computer Limitations in Imple-
       menting On-Line Controls", M.I.T., Report ESL-R-665, 1976.

[R2]   D.B. Rom and P.E. Sarachik, "The Design of Optimal Compensators
       for Linear Constant Systems with Inaccessible States", IEEE
       Trans. Auto. Control, Vol. AC-18, No. 5, October 1973, pp. 509-
       512.

[R3]   D.B. Rom and P.E. Sarachick, "Further Results on the Design of
       Optimal Compensators Using a Minimal Order Observer", Int. J.
       Control, Vol. 18, No. 4, 1973, pp. 695-704.

[S1]   M.K. Sain, "On the Control Applications of a Determinant
       Equality Related to Eigenvalue Computation", IEEE Trans. Auto.
       Control, Vol. 11, No. 1, January 1966, pp. 109-111.

[S2]   H.R. Sirisena and S.S. Choi, "Optimal Pole Placement in Linear
       Multivariable Systems Using Dynamic Output Feedback", Int. J.
       Control, Vol. 21, No. 4, 1975, pp. 661-671.

[W1]   W.M. Wonham, "On a Matrix Riccati Equation of Stochastic
       Control", SIAM Journal Control, Vol. 6, No. 4, 1968, pp. 681-
       697.

[Y1]   Y.O. Yüksel and J.J. Bongiorno, Jr., "Observers for Linear
       Multivariable Systems with Applications", IEEE Trans. Auto.
       Control, Vol. AC-16, No. 6, December 1971, pp. 603-613.