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SPANWISE VARIATION OF POTENTIAL FORM DRAG

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SUMMARY

A new method has been developed for calculating the spanwise variation of potential form drag due to thickness and lift for a nonplanar wing of arbitrary planform in subsonic and supersonic potential flow. A computer program has been developed to perform the numerical calculations.

The configuration is subdivided into a large number of panels, each of which contains an aerodynamic singularity distribution. Linearly-varying chordwise source and spanwise vortex distributions are used to represent thickness and lift. The normal components of the velocity induced at specified control points by each singularity distribution are calculated and make up the coefficients of a system of linear equations relating the strengths of the singularities to the magnitude of the normal velocities. The singularity strengths which satisfy the boundary conditions of tangential flow at the control points are determined by solving this system of equations.

The potential form drag due to thickness and lift is determined from the strengths of the singularities and the panel drag influence equations obtained by analytically integrating the tangential and normal velocity perturbations over the panel surface respectively.

Several examples of the spanwise variation of drag calculated by the program are presented and compared with available analytic results. Good correlation was obtained in each case.

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INTRODUCTION

Prior to the present study, no procedure existed for computing the spanwise variation of potential form drag at transonic and supersonic speeds. Also, at subsonic speeds the calculation of the spanwise variation of drag was limited to planar wings represented by a vortex lattice with even spacing in the spanwise direction. Due to the necessity of determing the leading edge force in order to obtain the nonlinear vortex lift and drag on nonstraight leading edge wings (using the Polhamus leading edge analogy), a solution to this problem is essential. Also, in order to predict the contribution of horizontal surfaces to such stability derivatives as C_{n_β}, C_{n_p}, and C_{n_r} and the pivot moments due to the outer panel of a variable sweep wing, the spanwise distribution of leading edge force must be computed[1].

In addition to the requirements for the spanwise distribution of leading edge force for the prediction of loads, the analysis is also valuable in optimizing the planform shape to minimize three dimensional induced effects. For instance, the occurrence of local wave drag, shocks, vortex formation, and flow separation induced by the planform shape can be minimized by shaping the planform to have a constant value of leading edge force across the span.

In order to compute the section potential form drag for wings of arbitrary shape, a finite element must be used which is capable of representing the waves produced by supersonic leading and trailing edges and which has a spanwise variation of vorticity that is continuous and goes to zero at the tips of the element. The most elemetrary lifting finite element which meets these requirements is one in which the bound vorticity varies linearly in the spanwise direction, such that the vorticity at the tips of the element is zero, and is constant in the chordwise direction. The influence equations for this type of panel are in terms of the same quantities as needed in the calculation of the influence of a constant pressure panel. Therefore, the linearly varying pressure panel should cost the same as the constant pressure panel to use. However, due to the lower order singularity at the tips, the normal velocity can be analytically integrated over the complete element to obtain a finite value for the potential form drag due to itself or another element. A finite element with a spanwise constant and chordwise linearly varying source distribution was chosen to represent the contribution due to wing thickness. The chordwise linear variation was necessary to avoid the infinite drag associated with constant source panels having sonic leading edges.

This report describes the mathematical development of a method for calculating the spanwise variation of potential form drag of a wing at subsonic and supersonic speeds using these linearly varying panels. The wing may be of arbitrary planform and nonplanar provided the wing panels are all parallel to the aircraft axis.

LIST OF SYMBOLS

b	Span
B	$\frac{1}{T}\sqrt{T^2+\beta^2}$
c	Chord
c_{avg}	Reference chord such that $b \times c_{avg} = S_{ref}$
C_D	Drag coefficient, Drag/$q_\infty S_{ref}$
$C_{D_{ij}}$	Near field drag coefficient giving the drag singularity j induces on area i for unit values of pressure coefficient
C_P	Pressure coefficient $(P-P_\infty)/\frac{1}{2}\rho_\infty U_\infty^2$
ΔC_P	Net lift pressure coefficient $C_{P_{lower}}-C_{P_{upper}}$
$C_T(y)$	Leading edge thrust coefficient, $\dfrac{\text{Thrust/unit length}}{q_\infty}$
f_{ij}^k	Basic functions integrated in Appendix C
$\vec{f}_{ij}$	Vector of basic functions f_{ij}^k (See Appendix D)
k	Coefficient = 1 if $M_\infty < 1.$ or 2 if $M_\infty > 1.$
M_∞	Free stream Mach number
q_∞	Free stream dynamic pressure $\frac{1}{2}\rho_\infty U_\infty^2 = \frac{1}{2}\gamma P_\infty M_\infty^2$
R	Either $\sqrt{x^2+\beta^2(y^2+z^2)}$ or $\sqrt{x^2-\beta^2(y^2+z^2)}$, as indicated
S_{ref}	Wing planform area
T	The tangent of the sweep angle
(u,v,w)	x, y and z components of velocity
V_r, V_θ	Radial and circumferential velocity components in polar coordinates
β^2	Either $M_\infty^2 - 1$ or $1 - M_\infty^2$
ξ	Usually $x - Ty$
η	Usually Ty
ζ	Usually Tz
Φ	Perturbation velocity potential

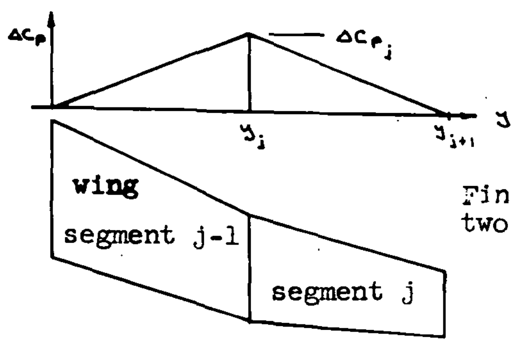
THEORY

FORM DRAG DUE TO LIFT

General

It will be shown that an exact solution for calculating the form drag due to lift can be obtained by integrating the product of the local induced normal velocity times the local ΔCp over the planform area. If this integration is performed analytically, the drag calculation for that pressure distribution is obtained exactly, and the drag as calculated in the near and far fields must agree. This drag includes the leading edge suction forces since all momentum losses have been accounted for in the far field analysis. To be suitable for analytic integration, the spanwise distribution of vorticity must be continuous over the planform, so as to avoid trailing vorticies of finite strength and the resulting infinite drag. Any method which does not employ a continuous spanwise distribution of vorticity can only calculate the drag approximately and the value as calculated in the near and far field need not agree.

The method employed in this effort uses the most elementary method of obtaining a continuous spanwise distribution of vorticity. The planform is composed of quadrilateral panels each having a linearly varying vorticity distribution in the spanwise direction and a constant vorticity distribution in the chordwise direction. Two adjacent panels are combined into a finite element having a vorticity distribution which is zero at the outer edges of the combined panels and varies linearly in between, reaching a maximum value at the junction of the panels.



Finite element composed of two adjacent quadrilateral panels

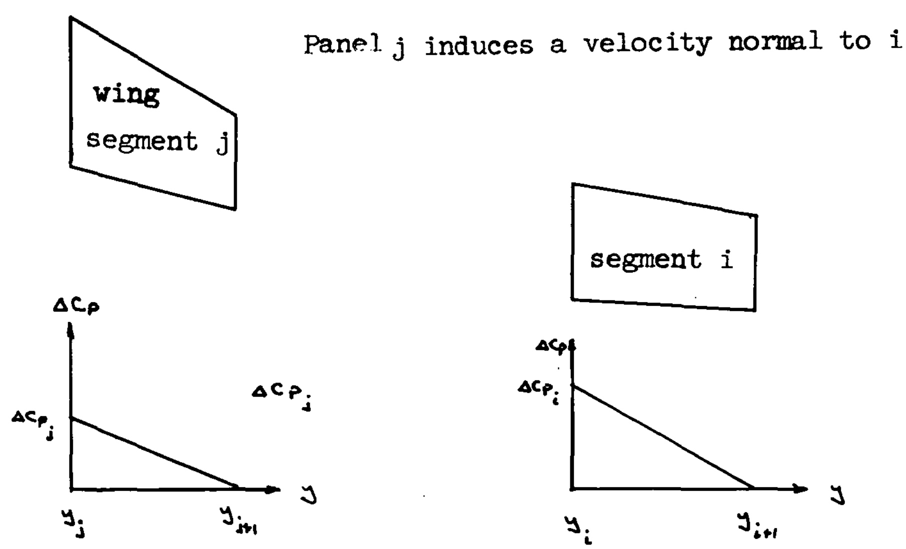
The entire vorticity distribution can be described by knowing the value of ΔCp at the junction. A finite element extends over two span segments of the wing. Therefore each span segment has a distribution of ΔCp which is the sum of the ΔCp from the left panel of one finite element and from the right half of another finite element. On span segment i

$$\Delta C_p = \Delta C_{p_{i+1}} \frac{(y-y_i)}{(y_{i+1}-y_i)} + \Delta C_{p_i} \frac{(y_{i+1}-y)}{(y_{i+1}-y_i)}$$

The influence equations for the type of panels required for this distribution are developed in the next section. The values of C_p for each singularity are obtained by satisfying the boundary conditions for normal velocity at a set of control points distributed over the planform. The control points are located at the same places used for constant vorticity panels.

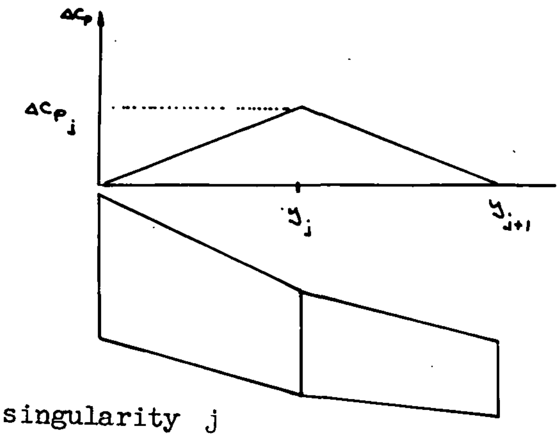
The section after that describes how the normal velocity induced by one of the panels making up such a panel pair is integrated over a panel of another panel pair after multiplying by the local ΔC_p. Since the integration is performed exactly the result is the drag which one panel induces on the other. Since the drag is proportional to the product of the ΔC_p's at the panel edges, a drag influence coefficient $C_{D_{ij}}$ may be defined such that the drag which panel j induces on panel i is

$$C_{D_{ij}} \Delta C_{p_i} \Delta C_{p_j}$$

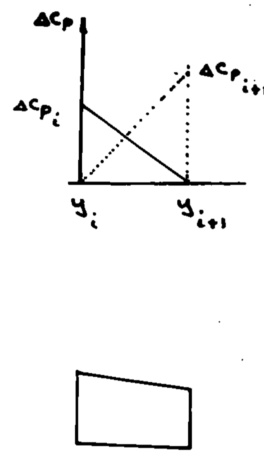


The integral of induced downwash times ΔC_p is $= C_{D_{ij}} \Delta C_{p_i} \Delta C_{p_j}$

Since a finite element is composed of two adjacent panels, the drag which a finite element (composed of two panels) induces on a panel located on a given span segment can also be written as $C_{D_{ij}}^{R} \Delta C_{p_i} \Delta C_{p_j}$



singularity j



Since a span segment is composed of a ΔCp from two singularities, the drag induced by singularity j on span segment i can be witten

$$\left\{ C_{D_{ij}}^{R} \Delta C_{p_i} + C_{D_{i+1,j}}^{L} \Delta C_{p_{i+1}} \right\} \Delta C_{p_j}$$

The superscripts L and R refer to the left or right half of singularity pressure distributions on segment i. The drag induced on a panel spanning such a span segment is obtained by summing over all panels j.

The leading edge suction force is obtained by subtracting the near field drag, as described above, from the zero suction drag. On a given panel the zero suction drag is calculated by integrating the product of the local ΔCp and the normal velocity as given by the camber shape. This normal velocity is only equal to the induced velocity at the control point and the difference accounts for the leading edge suction.

Spanwise Linearly Varying Vorticity Finite Element

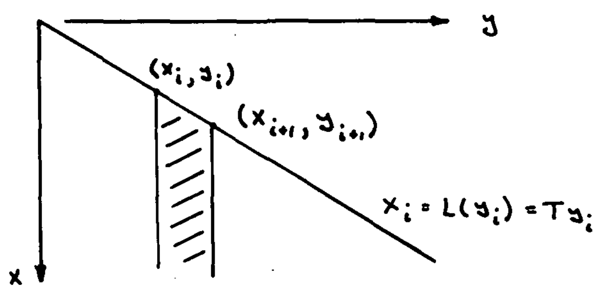
A constant pressure or constant vorticity panel with a quadrilateral shape can be constructed by adding or subtracting four semi-infinite triangular shaped panels. These semi-infinite triangles, each determined by a corner of the quadrilateral, can be assumed to induce a velocity potential everywhere in the flow. However, each corner represents only an integration limit, and all four corners must be included to make any sense. If it is kept in mind that four corners must be included, one of these triangles, having unit ΔC_p, lying in the $z=0$ plane, and having sides determined by $y=0$ and $x-Ty=0$, induces the following velocity potential[2]

$$\phi_{oo}(x,y,z) = \frac{k}{8\pi}\left\{(x-Ty)\tan^{-1}\frac{TzR}{Ty(x-Ty)-T^2z^2} + Tz\frac{1}{2}\log\frac{R+x}{R-x}\right.$$

$$\left. - z\sqrt{T^2+\beta^2}\,\frac{1}{2}\log\frac{\sqrt{T^2+\beta^2}\,R+(Tx+\beta^2y)}{\sqrt{T^2+\beta^2}\,R-(Tx+\beta^2y)} - (2-k)\left[(x-Ty)\tan^{-1}\frac{y}{z} + Tz\frac{1}{2}\log(y^2+z^2) - Tz\right]\right\}$$

where $\beta^2 = 1-M_\infty^2$ $\qquad R^2 = x^2+\beta^2(y^2+z^2)$ $\qquad k = \begin{cases} 1 & \beta^2>0 \\ 2 & \beta^2<0 \end{cases}$

To obtain a panel with a spanwise variation of ΔC_p we can combine panels composed of thin chordwise strips, each with its own ΔC_p, and then take the limit as the strip width goes to zero.



The resulting potential is

$$\phi(x,y,z) = \lim_{\max \Delta y_i \to 0} \sum_i \left[\phi_{oo}(x-x_i, y-y_i, z) - \phi_{oo}(x-x_{i+1}, y-y_{i+1}, z)\right]\Delta C_{p_i}$$

$$= -\int_0^{y_{MAX}} \frac{\partial}{\partial \tilde{y}}\phi_{oo}[x-L(\tilde{y}), y-\tilde{y}, z]\,C_p(\tilde{y})\,d\tilde{y}$$

$$= C_p(\tilde{y})\,\phi_{oo}[x-L(\tilde{y}), y-\tilde{y}, z]\Big|_{y_{MAX}}^{0} + \int_0^{y_{MAX}} C_p'(\tilde{y})\,\phi_{oo}[x-L(\tilde{y}), y-\tilde{y}, z]\,d\tilde{y}$$

In our case we will use a linearly varying $C_p(\tilde{y})$ and a straight leading edge such that

$$C_p(\tilde{y}) = a + b\tilde{y} \qquad L(\tilde{y}) = T\tilde{y} \qquad C_p'(\tilde{y}) = b$$

Therefore

$$\begin{aligned}\phi(x,y,z) = {} & a\,\phi_{00}(x,y,z) \\ & - (a + b y_{max})\,\phi_{00}\left[(x-Ty)+T(y-y_{max}),\ y-y_{max},\ z\right] \\ & + b\,\phi_{01}\left[(x-Ty)+T(y-\tilde{y}),\ y-\tilde{y},\ z\right]\Big|_{y_{max}}^{0}\end{aligned}$$

where, if we let

$$\xi = x - Ty$$

$$\tilde{\eta} = T(y-\tilde{y}) \qquad\qquad B = \frac{\sqrt{T^2+\beta^2}}{T}$$

$$\zeta = Tz$$

then $$\phi_{01}\left[(x-Ty)+T(y-\tilde{y}),(y-\tilde{y}),\ z\right]\Big|_{y_{max}}^{0} = \int_0^{y_{max}} \phi_{00}\left[(x-Ty)+T(y-\tilde{y}),\ y-\tilde{y},\ z\right] d\tilde{y}$$

$$= \frac{1}{T}\int_{T(y-y_{max})}^{Ty} \phi_{00}\left(\xi+\tilde{\eta},\ \frac{1}{T}\tilde{\eta},\ \frac{1}{T}\zeta\right) d\tilde{\eta}$$

$$= \Phi_{01}(\xi,\tilde{\eta},\zeta,T)\Big|_{T(y-y_{max})}^{Ty}$$

$$= \Phi_{01}\left[(x-Ty),\ T(y-\tilde{y}),\ Tz,\ T\right]\Big|_{\tilde{y}=y_{max}}^{\tilde{y}=0}$$

$$= \phi_{01}(x,y,z) - \phi_{01}\left[(x-Ty)+T(y-y_{max}),\ y-y_{max},\ z\right]$$

However, the integration over η can be performed more easily by integrating the velocities rather than the velocity potential. In our case

$$\phi_{00}(x,y,z) = \phi_{00}(\xi,\eta,\zeta)$$

$$= \frac{k}{8\pi}\left\{ \zeta\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \zeta B\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR+(\xi+B^2\eta)} \right.$$

$$\left. + \xi\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - (2-k)\left[\xi\tan^{-1}\frac{\eta}{\zeta} + \zeta\frac{1}{2}\log(\eta^2+\zeta^2) - \zeta\right]\right\}$$

and

$$u_{00} = \frac{\partial\phi_0}{\partial x} = \frac{\partial\phi_0}{\partial\xi} = \frac{k}{8\pi}\left\{\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - (2-k)\tan^{-1}\frac{\eta}{\zeta}\right\}$$

$$v_{00} = \frac{\partial\phi_0}{\partial y} = T\left[\frac{\partial}{\partial\eta} - \frac{\partial}{\partial\xi}\right]\phi_0 = \frac{-kT}{8\pi}\left\{\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \frac{\zeta R}{\eta^2+\zeta^2} \right.$$

$$\left. - (2-k)\left[\tan^{-1}\frac{\eta}{\zeta} - \frac{\zeta(\xi+\eta)}{\eta^2+\zeta^2}\right]\right\}$$

$$w_{00} = \frac{\partial\phi_0}{\partial z} = T\frac{\partial\phi_0}{\partial\zeta} = \frac{kT}{8\pi}\left\{\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{\eta R}{\eta^2+\zeta^2}\right.$$

$$- B\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$\left. + (2-k)\left[\frac{\eta(\xi+\eta)}{\eta^2+\zeta^2} - \frac{1}{2}\log(\eta^2+\zeta^2)\right]\right\}$$

where

$$R^2 = (\xi+\eta)^2 + (B^2-1)(\eta^2+\zeta^2)$$

If we define

$$u_{01}(\xi,\eta,\zeta,B) = \int u_{00}(\xi,\eta,\zeta,B)\,d\eta$$

$$V_{01}(\xi,\eta,\zeta,B) = \int V_{00}(\xi,\eta,\zeta,B)\,d\eta$$

$$W_{01}(\xi,\eta,\zeta,B) = \int W_{00}(\xi,\eta,\zeta,B)\,d\eta$$

then

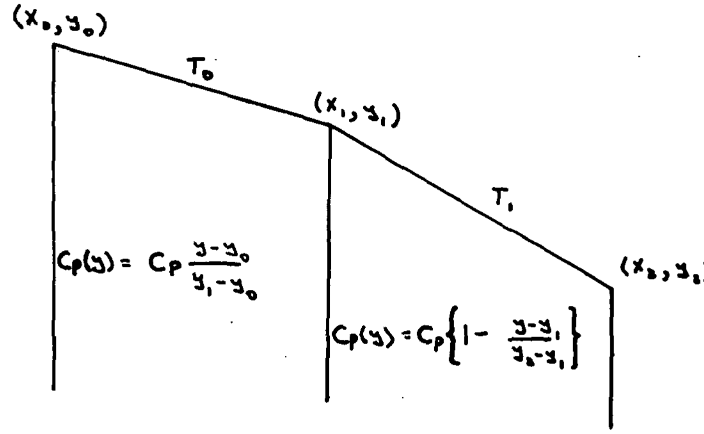
$$u_{01}(\xi,\eta,\zeta,B) = \frac{k}{8\pi}\Big\{ \eta \tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \zeta\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\ \zeta\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - (2-k)\Big[\eta\tan^{-1}\frac{\eta}{\zeta} - \zeta\,\frac{1}{2}\log(\eta^2+\zeta^2)\Big]\Big\}$$

$$V_{01}(\xi,\eta,\zeta,B) = -\frac{kT}{8\pi}\Big\{ (\eta-\xi)\tan^{-1}\frac{\zeta R}{\eta\xi-\zeta^2} + \zeta(B^2+1)\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\ 2\zeta\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - (2-k)\Big[(\eta-\xi)\tan^{-1}\frac{\eta}{\zeta} - 2\zeta\,\frac{1}{2}\log(\eta^2+\zeta^2)\Big]\Big\}$$

$$W_{01}(\xi,\eta,\zeta,B) = \frac{kT}{8\pi}\Big\{ (\eta-\xi)\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + (\xi-B^2\eta)\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\ 2\zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + 2R$$
$$-\ (2-k)\Big[(\eta-\xi)\,\frac{1}{2}\log(\eta^2+\zeta^2) + 2\zeta\tan^{-1}\frac{\eta}{\zeta} - 2(\xi+\eta)\Big]\Big\}$$

These velocities can be thought of as being induced by a semi-infinite vorticity panel with $C_p(y) = Ty = \eta$

For triangular loading, consider two adjacent panels in the z=0 plane.



Then for any velocity component, say w, we can combine terms to obtain

$$
\begin{aligned}
w(x,y,z) = C_p \Big\{ & -w_{00}(x-x_1, y-y_1, z, T_0) + w_{00}(x-x_1, y-y_1, z, T_1) \\
& + \frac{1}{T_0}\frac{1}{(y_1-y_0)}\left[w_{01}(x-x_0, y-y_0, z, T_0) - w_{01}(x-x_1, y-y_1, z, T_0)\right] \\
& - \frac{1}{T_1}\frac{1}{(y_2-y_1)}\left[w_{01}(x-x_1, y-y_1, z, T_1) - w_{01}(x-x_2, y-y_2, z, T_1)\right]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{T_0}\frac{C_p}{(y_1-y_0)}\left\{ w_{01}(x-x_0, y-y_0, z, T_0) - w_{01}(x-x_1, y-y_1, z, T_0) - T_0(y_1-y_0)w_{00}(x-x_1, y-y_1, z, T_0)\right\} \\
& + \frac{1}{T_1}\frac{C_p}{(y_2-y_1)}\left\{ w_{01}(x-x_2, y-y_2, z, T_1) - w_{01}(x-x_1, y-y_1, z, T_1) + T_1(y_2-y_1)w_{00}(x-x_1, y-y_1, z, T_1)\right\}
\end{aligned}
$$

where w_{00} is the velocity induced by a constant vorticity panel and w_{01} is the velocity induced by a spanwise linearly varying vorticity panel.

Consider the following definitions

$$\xi = x - Ty$$

$$\eta = Ty$$

$$\zeta = Tz$$

$$B^2 = \frac{1}{T^2}(T^2+\beta^2) \qquad\qquad R^2 = (\xi+\eta)^2 + (B^2-1)(\eta^2+\zeta^2)$$

Then, considering only the near field terms, the velocities induced by each corner of a constant vorticity panel are :

$$u_{00}(\xi,\eta,\zeta,B) = \frac{k c_p}{8\pi} \tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

$$v_{00}(\xi,\eta,\zeta,B) = -\frac{k c_p T}{8\pi}\left\{\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \frac{\zeta R}{\eta^2+\zeta^2}\right\}$$

$$w_{00}(\xi,\eta,\zeta,B) = \frac{k c_p T}{8\pi}\left\{\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \frac{\eta R}{\eta^2+\zeta^2}\right\}$$

$$k = \left\{\begin{array}{lll} 1 & \beta^2 > 0 & , M_\infty < 1 \\ 2 & \beta^2 < 0 & , M_\infty > 1 \end{array}\right\}$$

Now define for each velocity component

$$\frac{\partial}{\partial\xi} w_{ij}(\xi,\eta,\zeta,B) = w_{i-1,j}(\xi,\eta,\zeta,B)$$

$$\frac{\partial}{\partial\eta} w_{ij}(\xi,\eta,\zeta,B) = w_{i,j-1}(\xi,\eta,\zeta,B)$$

A discussion of the calculation of these velocity components is given in Appendix C

Using this notation the velocities induced by a single corner of a spanwise linearly varying panel would be :

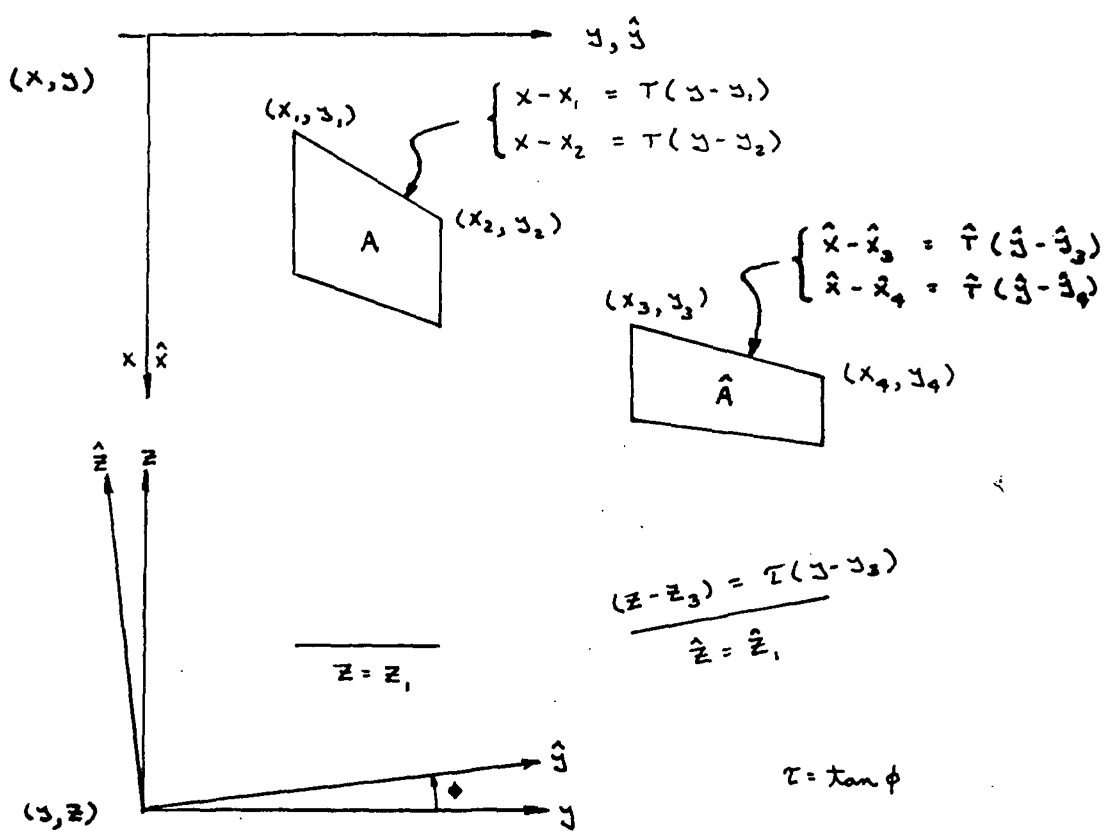
$$\frac{1}{T}\, u_{01}(\xi,\eta,\zeta,B) \quad , \quad \frac{1}{T}\, v_{01}(\xi,\eta,\zeta,B) \quad , \quad \frac{1}{T}\, w_{01}(\xi,\eta,\zeta,B)$$

because

$$\frac{\partial}{\partial\eta}\, u_{01} = u_{00} \qquad \frac{\partial}{\partial\eta}\, v_{01} = v_{00} \qquad \frac{\partial}{\partial\eta}\, w_{01} = w_{00}$$

The factor $\frac{1}{T}$ is necessary because the integration is over η and not y.

Using the above definitions the interference drag between two panels can be derived. Consider two panels A and $\hat{A}$ having leading edge sweeps T and $\hat{T}$, and coordinate systems such that A lies in the plane Z = constant and $\hat{A}$ in the plane $\hat{Z}$ = constant, and the coordinate axes x and $\hat{x}$ coincide.



The coordinate transformation between the two coordinate systems can be written in the following form:

$$\hat{x} = x \qquad\qquad x = \hat{x}$$

$$\hat{y} = \frac{1}{\sqrt{(1+\tau^2)}}(y + \tau z) \qquad\qquad y = \frac{1}{\sqrt{(1+\tau^2)}}(\hat{y} - \tau\hat{z})$$

$$\hat{z} = \frac{1}{\sqrt{(1+\tau^2)}}(-\tau y + z) \qquad\qquad z = \frac{1}{\sqrt{(1+\tau^2)}}(\tau\hat{y} + \hat{z})$$

If we define

$$(\ell+1) = \frac{\hat{T}\sqrt{1+\tau^2}}{T} \qquad\qquad (\hat{\ell}+1) = \frac{T\sqrt{1+\tau^2}}{\hat{T}}$$

$$\hat{\tau} = -\tau$$

then the following relations, which will later prove to be useful, can be derived :

$$(\ell+1)(\hat{\ell}+1) = (1+\tau^2)$$

$$\frac{(B^2-1)}{(\ell+1)} = \frac{(\hat{B}^2-1)}{(\hat{\ell}+1)}$$

$$\ell = -\frac{(\hat{\ell}-\hat{\tau}^2)}{(\hat{\ell}+1)}$$

$$\frac{\ell}{(\ell+1)} = -\frac{(\hat{\ell}-\hat{\tau}^2)}{(1+\hat{\tau}^2)}$$

$$(\ell-\tau^2) = -\frac{\hat{\ell}(1+\tau^2)}{(\hat{\ell}+1)}$$

$$\frac{(\ell+1)}{(\ell^2+\tau^2 B^2)} = \frac{(\hat{\ell}+1)}{(\hat{\ell}^2+\hat{\tau}^2\hat{B}^2)}$$

In addition

$$\hat{\xi} = \hat{x} - \hat{T}\hat{y}$$

$$\hat{\eta} = \hat{T}\hat{y} = \hat{T}\frac{1}{\sqrt{1+\tau^2}}(y+\tau z) = \frac{(\ell+1)}{(1+\tau^2)}(\eta+\tau\zeta)$$

$$\hat{\zeta} = \hat{T}\hat{z} = \hat{T}\frac{1}{\sqrt{1+\tau^2}}(z-\tau y) = \frac{(\ell+1)}{(1+\tau^2)}(\zeta-\tau\eta)$$

and

$$(\hat{\xi}+\hat{\eta}) = (\xi+\eta) = \hat{x} = x$$

$$(\ell+1)(\hat{\eta}^2+\hat{\zeta}^2) = (\ell+1)(\eta^2+\zeta^2)$$

$$R^2 = (\xi+\eta)^2 + (B^2-1)(\eta^2+\zeta^2) = (\hat{\xi}+\hat{\eta})^2 + (\hat{B}^2-1)(\hat{\eta}^2+\hat{\zeta}^2)$$

Assume each panel has a vorticity distribution

$$\Delta C_p = C_p(y-y_1) \qquad y_1 < y < y_2 \qquad \text{on } A$$

$$\Delta C_p = \hat{C}_p(\hat{y}-\hat{y}_3) \qquad \hat{y}_3 < \hat{y} < \hat{y}_4 \qquad \text{on } \hat{A}$$

Then the velocities induced by the leading edge of panel A at any point (ξ,η,ζ) are:

$$u = \frac{C_p}{T}\left\{ u_{01}(\xi-\xi_1,\eta-\eta_1,\zeta-\zeta_1,B) - u_{01}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B) - (\eta_2-\eta_1)u_{00}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B)\right\}$$

$$v = \frac{C_p}{T}\left\{ v_{01}(\xi-\xi_1,\eta-\eta_1,\zeta-\zeta_1,B) - v_{01}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B) - (\eta_2-\eta_1)v_{00}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B)\right\}$$

$$w = \frac{C_p}{T}\left\{ w_{01}(\xi-\xi_1,\eta-\eta_1,\zeta-\zeta_1,B) - w_{01}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B) - (\eta_2-\eta_1)w_{00}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B)\right\}$$

Note that $\xi_1 = \xi_2$, because $(x_1 - Ty_1) = (x_2 - Ty_2)$ on the leading edge, and $\zeta_1 = \zeta_2$.

The induced drag of panel A on panel $\hat{A}$ is obtained by integrating the normal velocity $(1+\tau^2)^{-\frac{1}{2}}(\omega-\tau v)$ times the vorticity strength $\Delta C_p = (\hat{y}-\hat{y}_3)$ over panel $\hat{A}$. (See page 34.) At a fixed value of η the integral in the x direction may be peformed at once. Since $\frac{\partial}{\partial \xi}\omega_{11} = \omega_{01}$ we can write for the integrals of u, v, and ω:

$$\frac{C_p}{T}\left\{ u_{11}(\xi-\xi_1,\eta-\eta_1,\zeta-\zeta_1,B) - u_{11}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B) - (\eta_2-\eta_1)u_{10}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B)\right\}_{\xi_{T.E.}}^{\xi_{L.E.}}$$

$$\frac{C_p}{T}\left\{ v_{11}(\xi-\xi_1,\eta-\eta_1,\zeta-\zeta_1,B) - v_{11}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B) - (\eta_2-\eta_1)v_{10}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B)\right\}_{\xi_{T.E.}}^{\xi_{L.E.}}$$

and

$$\frac{C_p}{T}\left\{ \omega_{11}(\xi-\xi_1,\eta-\eta_1,\zeta-\zeta_1,B) - \omega_{11}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B) - (\eta_2-\eta_1)\omega_{10}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,B)\right\}_{\xi_{T.E.}}^{\xi_{L.E.}}$$

where $\xi_{L.E.}$ and $\xi_{T.E.}$ refer to the values of ξ at the leading and trailing edge of $\hat{A}$ for some value of η

Since the procedure is identical for determining the location of both leading and trailing edges, only the leading edge will be considered. Along this edge:

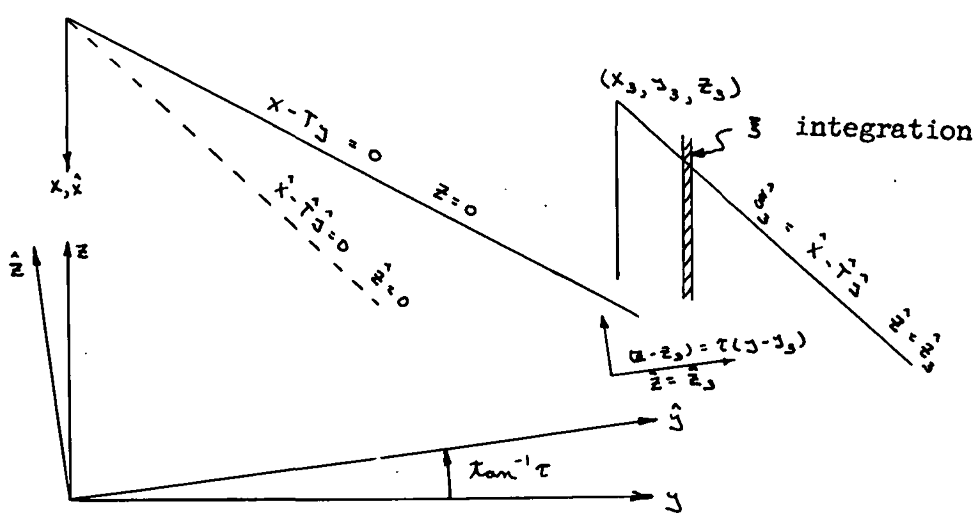
$$(x-x_3) = \hat{T}(\hat{y}-\hat{y}_3) = \hat{T}\sqrt{1+\tau^2}\,(y-y_3)$$

$$(z-z_2) = (z-z_3)+(z_3-z_2) = \tau(y-y_3)+(z_3-z_2)$$

or

$$\xi = x-Ty = x_3-Ty_3+\left[\sqrt{1+\tau^2}\,\hat{T}-T\right](y-y_3) = \xi_3+\ell(\eta-\eta_3)$$

$$\zeta = (\zeta_3-\zeta_2)+\tau(\eta-\eta_3)$$



To complete the integration over a quadrilateral panel we insert the ξ limits (which depend upon η) and integrate over η .

If we define $$\tilde{\omega}_{ij}(\xi,\eta,\zeta,B) = \frac{1}{\sqrt{1+\tau^2}}\left[\omega_{ij}(\xi,\eta,\zeta,B) - \tau V_{ij}(\xi,\eta,\zeta,B)\right]$$

the induced drag is then obtained from the following expression.

$$\frac{C_D S_{REF}}{C_p \hat{C}_p} = \frac{(1+\tau^2)}{T^2}\int_{\eta_3}^{\eta_4}(\eta-\eta_3)\,\tilde{\omega}_{11}\left[(\xi_3-\xi_1)+\lambda(\eta-\eta_3),\ \eta-\eta_1,\ (\zeta_3-\zeta_1)+\tau(\eta-\eta_3),\ B\right]d\eta$$

$$-\frac{(1+\tau^2)}{T^3}\int_{\eta_3}^{\eta_4}(\eta-\eta_3)\,\tilde{\omega}_{11}\left[(\xi_3-\xi_2)+\lambda(\eta-\eta_3),\ \eta-\eta_2,\ (\zeta_3-\zeta_2)+\tau(\eta-\eta_3),\ B\right]d\eta$$

$$-\frac{(1+\tau^2)}{T^3}(\eta_2-\eta_1)\int_{\eta_3}^{\eta_4}(\eta-\eta_3)\,\tilde{\omega}_{10}\left[(\xi_3-\xi_2)+\lambda(\eta-\eta_3),\ \eta-\eta_2,\ (\zeta_3-\zeta_2)+\tau(\eta-\eta_3),\ B\right]d\eta$$

These integrals require the following quantities of type W such that:

$$W_1' = \eta\,\tilde{\omega}_{11}(\xi,\eta,\zeta,B)$$

$$W_2' = \tilde{\omega}_{11}(\xi,\eta,\zeta,B)$$

$$W_3' = \eta\,\tilde{\omega}_{10}(\xi,\eta,\zeta,B)$$

$$W_4' = \tilde{\omega}_{10}(\xi,\eta,\zeta,B)$$

where

$$W' = \left[\lambda\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} + \tau\frac{\partial}{\partial\zeta}\right]W = (\lambda+1)\frac{\partial}{\partial\hat{\eta}}W$$

Integrals of this type are discussed in Appendix D

Since $\xi = \xi(\eta)$ and $\zeta = \zeta(\eta)$,

we will denote $\vec{f}_{ij}(\xi, \eta, \zeta, B)$ by $\vec{f}_{ij}(\eta)$

Therefore, since

$$\frac{1}{T}\,\tilde{\omega}_{ij}(\xi,\eta,\zeta,B) = \frac{1}{T}\,\frac{1}{\sqrt{1+\tau^2}}\left[\omega_{ij}(\xi,\eta,\zeta,B) - \tau\, V_{ij}(\xi,\eta,\zeta,B)\right]$$

$$= \frac{k}{8\pi}\,\frac{1}{\sqrt{1+\tau^2}}\;A\,\vec{f}_{ij}(\eta) \qquad \begin{cases} k = 1 & \beta^2 > 0 \\ k = 2 & \beta^2 < 0 \end{cases}$$

where

$$A = \begin{Bmatrix} 1 \\ -B^2 \\ 1 \\ \tau \\ \tau \end{Bmatrix}^*$$

we have

$$\frac{8\pi}{k}\,\frac{C_0 S_{REF}}{C_p \hat{C}_p} = \frac{\sqrt{(1+\tau^2)}}{T^2}\,A\left\{\int_{\eta_3}^{\eta_4}(\eta-\eta_3)\,\vec{f}_{11}(\eta-\eta_1)\,d\eta\right.$$

$$-\int_{\eta_3}^{\eta_4}(\eta-\eta_3)\,\vec{f}_{11}(\eta-\eta_2)\,d\eta$$

$$\left.-(\eta_2-\eta_1)\int_{\eta_3}^{\eta_4}(\eta-\eta_3)\,\vec{f}_{10}(\eta-\eta_2)\,d\eta\right\}$$

Or, using the results of Appendix D,

$$\frac{8\pi}{k}\,\frac{C_D S_{REF}}{C_p \hat{C}_p}$$

$$= \frac{\sqrt{1+\tau^2}}{T^2}\, A \Big\{ (\eta_4-\eta)E\left[\vec{f}_{21}(\eta_4-\eta) - F\vec{f}_{30}(\eta_4-\eta)\right] - E^2\vec{f}_{31}(\eta_4-\eta) + \left[EF+FE\right]E\vec{f}_{40}(\eta_4-\eta)$$

$$- (\eta_3-\eta)E\left[\vec{f}_{21}(\eta_3-\eta) - F\vec{f}_{30}(\eta_3-\eta)\right] + E^2\vec{f}_{31}(\eta_3-\eta) - \left[EF+FE\right]E\vec{f}_{40}(\eta_3-\eta)$$

$$- (\eta-\eta_3)E\left[\vec{f}_{21}(\eta_4-\eta) - F\vec{f}_{30}(\eta_4-\eta) - \vec{f}_{21}(\eta_3-\eta) + F\vec{f}_{30}(\eta_3-\eta)\right] \Big\}_{\eta=\eta_2}^{\eta=\eta_1}$$

$$- \frac{\sqrt{1+\tau^2}}{T^2}\, AE(\eta_2-\eta_1) \Big\{ (\eta_4-\eta_2)\vec{f}_{20}(\eta_4-\eta_2) - E\vec{f}_{30}(\eta_4-\eta_2)$$

$$- (\eta_3-\eta_2)\vec{f}_{20}(\eta_3-\eta_2) + E\vec{f}_{30}(\eta_3-\eta_2) + (\eta_2-\eta_3)\left[\vec{f}_{20}(\eta_4-\eta_2) - \vec{f}_{20}(\eta_3-\eta_2)\right] \Big\}$$

$$+ \frac{\sqrt{1+\tau^2}}{T^2}\,\frac{(\hat{\ell}+1)}{(1+\tau^2)}\, A \Big\{ (\eta_4-\eta)F^2T\vec{f}_{30}(\hat{\eta}_4-\hat{\eta}) - \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{f}_{31}(\hat{\eta}_4-\hat{\eta}) - \left[FE+EF\right]FT\vec{f}_{40}(\hat{\eta}_4-\hat{\eta})$$

$$- (\eta_3-\eta)F^2T\vec{f}_{30}(\hat{\eta}_3-\hat{\eta}) + \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{f}_{31}(\hat{\eta}_3-\hat{\eta}) + \left[FE+EF\right]FT\vec{f}_{40}(\hat{\eta}_4-\hat{\eta})$$

$$+ (\eta-\eta_3)\,F^2T\left[\vec{f}_{30}(\hat{\eta}_4-\hat{\eta}) - \vec{f}_{30}(\hat{\eta}_3-\hat{\eta})\right] \Big\}_{\hat{\eta}=\hat{\eta}_2,\ \eta=\eta_2}^{\hat{\eta}=\hat{\eta}_1,\ \eta=\eta_1}$$

$$- \frac{\sqrt{1+\tau^2}}{T^2}\,\frac{(\hat{\ell}+1)}{(1+\tau^2)}\,(\eta_2-\eta_1)A \Big\{ -(\eta_4-\eta_2)FT\vec{f}_{20}(\hat{\eta}_4-\hat{\eta}_2) + EFT\vec{f}_{30}(\hat{\eta}_4-\hat{\eta}_2) + \frac{(\hat{\ell}+1)}{(1+\tau^2)}FT\vec{f}_{21}(\hat{\eta}_4-\hat{\eta}_2)$$

$$+ (\eta_3-\eta_2)FT\vec{f}_{20}(\hat{\eta}_3-\hat{\eta}_2) - EFT\vec{f}_{30}(\hat{\eta}_3-\hat{\eta}_2) - \frac{(\hat{\ell}+1)}{(1+\tau^2)}FT\vec{f}_{21}(\hat{\eta}_3-\hat{\eta}_2)$$

$$- (\eta_2-\eta_3)\,FT\left[\vec{f}_{20}(\hat{\eta}_4-\hat{\eta}_2) - \vec{f}_{20}(\hat{\eta}_3-\hat{\eta}_2)\right] \Big\}$$

$$\frac{8\pi}{k} \; \frac{C_D S_{REF}}{C_p \hat{c}_p}$$

$$= \frac{\sqrt{1+\tau^2}}{T^2} A \Big\{ (EF+FE)E\vec{f}_{40}(\eta_4-\eta_1) - E^2\vec{f}_{31}(\eta_4-\eta_1) + (\eta_4-\eta_3)E\left[\vec{f}_{21}(\eta_4-\eta_1) - F\vec{f}_{30}(\eta_4-\eta_1)\right]$$

$$- (EF+FE)E\vec{f}_{40}(\eta_3-\eta_1) + E^2\vec{f}_{31}(\eta_3-\eta_1)$$

$$+ (EF+FE)E\vec{f}_{40}(\eta_3-\eta_2) - E^2\vec{f}_{31}(\eta_3-\eta_2) - (\eta_2-\eta_1)E^2\vec{f}_{30}(\eta_3-\eta_2)$$

$$- (EF+FE)E\vec{f}_{40}(\eta_4-\eta_2) + E\vec{f}_{31}(\eta_4-\eta_2) - (\eta_4-\eta_3)E\left[\vec{f}_{21}(\eta_4-\eta_2) - F\vec{f}_{30}(\eta_4-\eta_2)\right]$$

$$+ (\eta_2-\eta_1)E\left[E\vec{f}_{30}(\eta_4-\eta_2) - (\eta_4-\eta_3)\vec{f}_{20}(\eta_4-\eta_2)\right] \Big\}$$

$$- \frac{\sqrt{1+\tau^2}}{\hat{T}^2}\frac{(1+\tau^2)}{(\hat{\ell}+1)^3}\hat{A}T^{-1}\Big\{ (EF+FE)FT\vec{f}_{40}(\hat{\eta}_3-\hat{\eta}_2) + \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{f}_{31}(\hat{\eta}_3-\hat{\eta}_2) - (\eta_2-\eta_1)\left[\frac{(\hat{\ell}+1)}{(1+\tau^2)}FT\vec{f}_{21}(\hat{\eta}_3-\hat{\eta}_2) + EFT\vec{f}_{30}(\hat{\eta}_3-\hat{\eta}_2)\right]$$

$$- (EF+FE)FT\vec{f}_{40}(\hat{\eta}_3-\hat{\eta}_1) - \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{f}_{31}(\hat{\eta}_3-\hat{\eta}_1)$$

$$+ (EF+FE)FT\vec{f}_{40}(\hat{\eta}_4-\hat{\eta}_1) + \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{f}_{31}(\hat{\eta}_4-\hat{\eta}_1) - (\eta_4-\eta_3)F^2T\vec{f}_{30}(\hat{\eta}_4-\hat{\eta}_1)$$

$$- (EF+FE)FT\vec{f}_{40}(\hat{\eta}_4-\hat{\eta}_2) - \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{f}_{31}(\hat{\eta}_4-\hat{\eta}_2) + (\eta_2-\eta_1)\left[\frac{(\hat{\ell}+1)}{(1+\tau^2)}FT\vec{f}_{21}(\hat{\eta}_4-\hat{\eta}_2) + EFT\vec{f}_{30}(\hat{\eta}_4-\hat{\eta}_2)\right]$$

$$+ (\eta_4-\eta_3)F\left[FT\vec{f}_{30}(\hat{\eta}_4-\hat{\eta}_2) - (\eta_2-\eta_1)T\vec{f}_{20}(\hat{\eta}_4-\hat{\eta}_2)\right] \Big\}$$

Now define

$$\Delta S_{21} = (s_2 - s_1) = \frac{1}{T}(\eta_2-\eta_1) \qquad \Delta S_{43} = (\hat{s}_4 - \hat{s}_3) = \frac{(1+\tau^2)}{(\hat{\ell}+1)}\frac{1}{\hat{T}}(\eta_4-\eta_3)$$

and note

$$T^{-1}FT = -(\hat{\ell}+1)\hat{E} \qquad T^{-1}ET = -\frac{(\hat{\ell}+1)}{(1+\tau^2)}\hat{F}$$

$$T^{-1}F^2T = (\hat{\ell}+1)^2\hat{E}^2$$

$$T^{-1}EFT = \frac{(\hat{\ell}+1)^2}{(1+\tau^2)}\hat{F}\hat{E}$$

$$T^{-1}(EF+FE)FT = -\frac{(\hat{\ell}+1)^3}{(1+\tau^2)}(\hat{E}\hat{F}+\hat{F}\hat{E})\hat{E}$$

Therefore

$$\frac{8\pi}{k} \frac{C_D S_{REF}}{C_p \hat{C}_p}$$

$$= \frac{\sqrt{1+\tau^2}}{T^2} A \Big\{ (EF+FE)E \vec{f}_{40}(\eta_4-\eta_1) - E^2 \vec{f}_{31}(\eta_4-\eta_1) + \frac{T\Delta S_{43}}{\sqrt{1+\tau^2}} E\left[\vec{f}_{21}(\eta_4-\eta_1) - F\vec{f}_{30}(\eta_4-\eta_1)\right]$$

$$- (EF+FE)E\vec{f}_{40}(\eta_3-\eta_1) + E^2\vec{f}_{31}(\eta_3-\eta_1)$$

$$+ (EF+FE)E\vec{f}_{40}(\eta_3-\eta_2) - E^2\vec{f}_{31}(\eta_3-\eta_2) - T\Delta S_{21}E^2\vec{f}_{30}(\eta_3-\eta_2)$$

$$- (EF+FE)E\vec{f}_{40}(\eta_4-\eta_2) + E^2\vec{f}_{31}(\eta_4-\eta_2) - \frac{T\Delta S_{43}}{\sqrt{1+\tau^2}} E\left[\vec{f}_{21}(\eta_4-\eta_2) - F\vec{f}_{30}(\eta_4-\eta_2)\right]$$

$$+ T\Delta S_{21} E\left[E\vec{f}_{30}(\eta_4-\eta_2) - \frac{T\Delta S_{43}}{\sqrt{1+\tau^2}}\vec{f}_{20}(\eta_4-\eta_2)\right]\Big\}$$

$-$ $\Big\{$The same terms with the following replacements$\Big\}$

$$\xi \to \hat{\xi}$$

$$\eta \to \hat{\eta}$$

$$\zeta \to \hat{\zeta} \qquad 1 \to 3$$

$$B \to \hat{B} \qquad 2 \to 4$$

$$T \to \hat{T} \qquad 3 \to 1$$

$$\lambda \to \hat{\lambda} \qquad 4 \to 2$$

$$\tau \to \hat{\tau}$$

From this expression it can be seen that the total interference drag between two panels is zero. Interchanging the influencing and influenced panels changes the sign of the expression and the sum is zero.

There are four expressions (AE, AEE, AEF, and A(EF + FE)E required for the evaluation of the interference drag. These coefficients will now be evaluated.

$$AE = AH^{-1} = \frac{1}{(\ell^2+\tau^2B^2)} \begin{Bmatrix} (\ell-\tau^2) - \frac{1}{2}\tau^2(B^2-1) \\ -B^2(\ell-\tau^2) \\ (\ell-\tau^2) \\ \tau(B^2+\ell) \\ \tau(\ell+1) \end{Bmatrix}^* \qquad \begin{aligned} E &= H^{-1} \\ F &= H^{-1}G \end{aligned}$$

$$AF = AH^{-1}G = \frac{1}{(\ell^2+\tau^2B^2)} \begin{Bmatrix} (\ell-\tau^2) - \tau^2(\ell+1) - \frac{1}{2}\tau^2(B^2-1) \\ -B^2(\ell-\tau^2) + \tau^2(B^2+\ell) + \frac{1}{2}\ell\tau^2(B^2-1) \\ (\ell-\tau^2) - \tau^2(\ell+1) \\ \tau\left[(B^2+\ell) + (\ell-\tau^2) + \frac{1}{2}\tau^2(B^2-1)\right] \\ \tau\left[(\ell+1) + (\ell-\tau^2)\right] \end{Bmatrix}^*$$

$$AEE = AH^{-1}H^{-1} = \frac{1}{(\ell^2+\tau^2B^2)^2} \begin{Bmatrix} \ell(\ell-\tau^2) - \tau^2B^2(\ell+1) - \frac{1}{2}\tau^2(B^2-1) \\ -B^2\left[\ell(\ell-\tau^2) - \tau^2(B^2+\ell)\right] \\ \ell(\ell-\tau^2) - \tau^2(B^2+\ell) \\ \tau\left[\ell(B^2+\ell) + B^2(\ell-\tau^2)\right] \\ \tau\left[\ell(\ell+1) + (\ell-\tau^2) - \frac{1}{2}\tau^2(B^2-1)\right] \end{Bmatrix}^*$$

$$AFE = A H^{-1} G H^{-1} = \frac{1}{(\ell^2+\tau^2 B^2)^2} \left\{ \begin{array}{c} (\ell-\tau^2)^2 - \tau^2(\ell+1)\left[(\ell+1)+\frac{3}{2}(B^2-1)\right] \\ -(\ell-\tau^2)^2 B^2 + \tau^2(B^2+\ell)^2 + \frac{1}{2}\tau^2(\ell^2+\tau^2B^2)(B^2-1) \\ (\ell-\tau^2)^2 - \tau^2\left[(\ell+1)^2 + (B^2-1) + \frac{1}{2}\tau^2(B^2-1)\right] \\ 2\tau(\ell-\tau^2)(B^2+\ell) \\ 2\tau(\ell-\tau^2)(\ell+1) - \frac{1}{2}\tau^3(B^2-1) \end{array} \right\}^{*}$$

$$AEF = A H^{-1} H^{-1} G = \frac{1}{(\ell^2+\tau^2 B^2)^2} \left\{ \begin{array}{c} (\ell-\tau^2)^2 - \tau^2\left[(\ell+1)^2 + \left\{1+(\ell+1)-\frac{1}{2}(1+\tau^2)\right\}(B^2-1)\right] \\ -B^2(\ell-\tau^2)^2 + \tau^2\left\{(B^2+\ell)^2 - \ell\left[1-\frac{1}{2}(\ell+1)\right](B^2-1)\right\} \\ (\ell-\tau^2)^2 - \tau^2\left[(\ell+1)^2 + (B^2-1) - \frac{1}{2}\tau^2(B^2-1)\right] \\ 2\tau(\ell-\tau^2)(B^2+\ell) - \tau^3(B^2-1)\left[1-\frac{1}{2}(\ell+1)\right] \\ 2\tau(\ell-\tau^2)(\ell+1) - \frac{3}{2}\tau^3(B^2-1) \end{array} \right\}^{*}$$

$$A[FE+EF]E = A\left[H^{-1}GH^{-1} + H^{-1}H^{-1}G\right]H^{-1}$$

$$= \frac{2}{(\ell^2+\tau^2B^2)^3} \left\{ \begin{array}{l} (\ell-\tau^2)^3 + \tau^2\left[(1+\tau^2)^2 - (\ell+1)^3 - \frac{1}{4}\left\{[9+4(\ell+1)](\ell-\tau^2) + 5\ell(\ell+1) - 2\tau^2(B^2-1)\right\}(B^2-1)\right] \\ -\ell(\ell-\tau^2)^2B^2 + \tau^2\ell(B^2+\ell)^2 + 2\tau^2B^2(\ell-\tau^2)(B^2+\ell) - \frac{1}{2}\left[\frac{3}{2}-(\ell+1)\right](\ell^2+\tau^2B^2)\tau^2(B^2-1) \\ (\ell-\tau^2)^3 + \tau^2\left[(1+\tau^2)^2 - (\ell+1)^3 - \left\{\ell + 2(\ell-\tau^2) - \frac{1}{2}\tau^2\left[1-\frac{1}{2}(\ell+1)\right]\right\}(B^2-1)\right] \\ \tau\left\{(\ell-\tau^2)^2B^2 - \tau^2(B^2+\ell)^2 + 2\ell(\ell-\tau^2)(B^2+\ell) - \frac{1}{4}(\ell^2+\tau^2B^2)\tau^2(B^2-1)\right\} \\ \tau\left\{(\ell-\tau^2)^2 - \tau^2(\ell+1)^2 + 2\ell(\ell-\tau^2)(\ell+1) + \frac{1}{2}\left[1-\frac{3}{2}(\ell+1)+\frac{1}{2}(1+\tau^2)\right]\tau^2(B^2-1)\right\} \end{array} \right\}^{*}$$

In terms of more familiar quantities

$$\ell = \frac{1}{T\cos\phi}(\hat{T} - T\cos\phi), \qquad B^2 = \frac{1}{T^2}(T^2+\beta^2), \qquad \tau = \frac{\sin\phi}{\cos\phi}$$

$$(\ell^2+\tau^2B^2) = \frac{1}{T^2\cos^2\phi}\left\{(\hat{T}-T\cos\phi)^2 + \sin^2\phi\,(T^2+\beta^2)\right\}$$

$$= \frac{1}{T^2\cos^2\phi}\left\{(\hat{T}-T)^2 + 2(1-\cos\phi)T\hat{T} + \beta^2\sin^2\phi\right\}$$

$$(\ell-\tau^2) = \frac{1}{T\cos^2\phi}\left\{(\hat{T}-T\cos\phi)\cos\phi - T\sin^2\phi\right\} = \frac{1}{T\cos^2\phi}(\hat{T}\cos\phi - T)$$

$$(B^2+\ell) = \frac{1}{T^2\cos\phi}\left\{(T^2+\beta^2)\cos\phi + T(\hat{T}-T\cos\phi)\right\} = \frac{1}{T^2\cos\phi}\left\{T\hat{T} + \beta^2\cos\phi\right\}$$

$$\tau(\ell+1) = \frac{1}{T\cos^2\phi}\hat{T}\sin\phi \qquad (\ell+1) = \frac{\hat{T}}{T\cos\phi}$$

and letting $g = (\hat{T}-T)^2 + 2(1-\cos\phi)\hat{T}T + \beta^2\sin^2\phi$

$$AH^{-1} = AE = \frac{1}{(\ell^2+\tau^2B^2)}\begin{Bmatrix} (\ell-\tau^2) - \frac{1}{2}\tau^2(B^2-1) \\ -B^2(\ell-\tau^2) \\ (\ell-\tau^2) \\ \tau(B^2+\ell) \\ \tau(\ell+1) \end{Bmatrix}^* = \frac{1}{g}\begin{Bmatrix} T(\hat{T}\cos\phi - T) - \frac{1}{2}\beta^2\sin^2\phi \\ -\frac{1}{T}(T^2+\beta^2)(\hat{T}\cos\phi - T) \\ T(\hat{T}\cos\phi - T) \\ \sin\phi\,(T\hat{T}+\beta^2\cos\phi) \\ T\hat{T}\sin\phi \end{Bmatrix}^*$$

$$\frac{\sqrt{1+\tau^2}}{T}\,AE^2 = \frac{1}{g^2}\left\{\begin{array}{l} T(\hat{T}-T\cos\phi)(\hat{T}\cos\phi-T)-\sin^2\phi\left[\hat{T}(T^2+\beta^2)+\frac{1}{2}T\beta^2\cos\phi\right] \\ -\frac{1}{T}(T^2+\beta^2)\left[(\hat{T}-T\cos\phi)(\hat{T}\cos\phi-T)-\sin^2\phi\,(T\hat{T}+\beta^2\cos\phi)\right] \\ T\left[(\hat{T}-T\cos\phi)(\hat{T}\cos\phi-T)-\sin^2\phi\,(T\hat{T}+\beta^2\cos\phi)\right] \\ \sin\phi\left[(\hat{T}-T\cos\phi)(T\hat{T}+\beta^2\cos\phi)+(\hat{T}\cos\phi-T)(T^2+\beta^2)\right] \\ T\sin\phi\left[(\hat{T}-T)(\hat{T}+T)-\frac{1}{2}\beta^2\sin^2\phi\right] \end{array}\right\}^{*}$$

$$\frac{1}{T}\,AEF = \frac{1}{g^2}\left\{\begin{array}{l} T(\hat{T}\cos\phi-T)^2-\sin^2\phi\left\{T\hat{T}^2+\beta^2\left[\cos\phi\,(T\cos\phi+\hat{T})-\frac{1}{2}T\right]\right\} \\ -\frac{1}{T}\left\{(T^2+\beta^2)(\hat{T}\cos\phi-T)^2-\sin^2\phi\left[(T\hat{T}+\beta^2\cos\phi)^2-\beta^2(\hat{T}-T\cos\phi)(T\cos\phi-\frac{1}{2}\hat{T})\right]\right\} \\ T\left\{(\hat{T}\cos\phi-T)^2-\sin^2\phi\left[\hat{T}^2+\beta^2\left(1-\frac{3}{2}\sin^2\phi\right)\right]\right\} \\ 2\sin\phi\left\{(\hat{T}\cos\phi-T)(T\hat{T}+\beta^2\cos\phi)-\frac{1}{2}\beta^2\sin^2\phi\,(T\cos\phi-\frac{1}{2}\hat{T})\right\} \\ 2T\sin\phi\left\{(\hat{T}\cos\phi-T)\hat{T}-\frac{3}{4}\beta^2\sin^2\phi\cos\phi\right\} \end{array}\right\}^{*}$$

$$
\frac{\sqrt{1+\tau^2}}{T^2} A[EF+FE]E = \frac{2}{g^3}
\left\{
\begin{array}{l}
T(\hat{T}-T\cos\phi)(\hat{T}\cos\phi-T)^2+\sin^2\phi\Big\{T\hat{T}\big[(T-\hat{T})(T+\hat{T})-T(\hat{T}\cos\phi-T)\big] \\
\quad -\frac{1}{4}\beta^2\big[(9T\cos\phi+4\hat{T})(\hat{T}\cos\phi-T)+5\hat{T}(\hat{T}-T\cos\phi)\cos\phi-2\beta^2\sin^2\phi\cos\phi\big]\Big\} \\
\\
-\frac{1}{T}(T^2+\beta^2)(\hat{T}-T\cos\phi)(\hat{T}\cos\phi-T)^2+\frac{1}{T}\sin^2\phi\Big\{(T\hat{T}+\beta^2\cos\phi)\big[(\hat{T}-T\cos\phi)(T\hat{T}+\beta^2\cos\phi) \\
\quad +2(T^2+\beta^2)(\hat{T}\cos\phi-T)\big]-\frac{1}{2}\beta^2\big[(\hat{T}-T)^2+2(1-\cos\phi)T\hat{T}+\beta^2\sin^2\phi\big](\frac{3}{2}T\cos\phi-\hat{T})\Big\} \\
\\
T(\hat{T}-T\cos\phi)(\hat{T}\cos\phi-T)^2+T\sin^2\phi\Big\{\hat{T}\big[(T-\hat{T})(T+\hat{T})-T(\hat{T}\cos\phi-T)\big] \\
\quad -\beta^2\big[(\hat{T}-T\cos\phi)\cos^2\phi+2(\hat{T}\cos\phi-T)\cos\phi-\frac{1}{2}(T\cos\phi-\frac{1}{2}\hat{T})\sin^2\phi\big]\Big\} \\
\\
\sin\phi\Big\{(T^2+\beta^2)(\hat{T}\cos\phi-T)^2-(T\hat{T}+\beta^2\cos\phi)\big[(T\hat{T}+\beta^2\cos\phi)\sin^2\phi \\
\quad -2(\hat{T}-T\cos\phi)(\hat{T}\cos\phi-T)\big]-\frac{1}{4}\beta^2\sin^2\phi\big[(\hat{T}-T)^2+2(1-\cos\phi)T\hat{T}+\beta^2\sin^2\phi\big]\Big\} \\
\\
T\sin\phi\Big\{T(\hat{T}\cos\phi-T)^2+2\hat{T}(\hat{T}-T\cos\phi)(\hat{T}\cos\phi-T) \\
\quad -\sin^2\phi\big[T\hat{T}^2-\frac{1}{2}\beta^2\{\cos\phi(T\cos\phi-\frac{9}{2}\hat{T})+\frac{1}{2}T\}\big]\Big\}
\end{array}
\right\}^{*}
$$

If $\sin\phi = 0$, these reduce to

$$
AE = \frac{1}{(\hat{T}-T)}
\left\{
\begin{array}{c}
T \\
-\frac{1}{T}(T^2+\beta^2) \\
T \\
0 \\
0
\end{array}
\right\}^{*}
\qquad
\begin{aligned}
\frac{\sqrt{1+\tau^2}}{T} AE^2 &= \frac{1}{(\hat{T}-T)} AE \\
\frac{1}{T} AEF &= \frac{1}{(\hat{T}-T)} AE \\
\frac{\sqrt{1+\tau^2}}{T^2} A[EF+FE] &= \frac{2}{(\hat{T}-T)} AE
\end{aligned}
$$

For the case where $\lambda = 0$ or $T = \hat{T}$ and z is constant, then,

$$\omega_{10} \equiv \omega_{11}'$$

$$\omega_{11} \equiv \omega_{12}'$$

$$\eta\,\omega_{10} \equiv [\eta\,\omega_{11} - \omega_{12}]'$$

$$\eta\,\omega_{11} \equiv [\eta\,\omega_{12} - \omega_{13}]'$$

and

$$\begin{aligned}\frac{C_D\,S_{REF}}{C_p\,\hat{C}_p} = \frac{1}{T^3}\Big\{ &(\eta_4-\eta_3)\,\omega_{12}(\xi_4-\xi_1,\ \eta_4-\eta_1,\ \zeta) - \omega_{13}(\xi_4-\xi_1,\ \eta_4-\eta_1,\ \zeta) \\ &+ \omega_{13}(\xi_3-\xi_1,\ \eta_3-\eta_1,\ \zeta) \\ &- \big[\,\omega_{13}(\xi_3-\xi_2,\ \eta_3-\eta_2,\ \zeta) + (\eta_2-\eta_1)\,\omega_{12}(\xi_3-\xi_2,\ \eta_3-\eta_2,\ \zeta)\big] \\ &+ \big[\,\omega_{13}(\xi_4-\xi_2,\ \eta_4-\eta_2,\ \zeta) - (\eta_4-\eta_3)\,\omega_{12}(\xi_4-\xi_2,\ \eta_4-\eta_2,\ \zeta)\big] \\ &+ (\eta_2-\eta_1)\big[\,\omega_{12}(\xi_4-\xi_2,\ \eta_4-\eta_2,\ \zeta) - (\eta_4-\eta_3)\,\omega_{11}(\xi_4-\xi_2,\ \eta_4-\eta_2,\ \zeta)\big]\Big\}\end{aligned}$$

The expression for C_D with $\hat{T} \cdot T$ can be shown to be equivalent to this in the limit as $T \rightarrow \hat{T}$. This can be demonstrated by using the fact that

$$\frac{\partial}{\partial T}\left\{\frac{1}{T^j}\,\omega_{ij}(\xi,\eta,\zeta,\beta)\right\} = -(j+1)\,\frac{1}{T^{j+1}}\,\omega_{i-1,j+1}(\xi,\eta,\zeta,\beta)$$

where $\frac{\partial}{\partial T}$ involves holding x, y, z and β constant

FORM DRAG DUE TO THICKNESS

Chordwise Linearly Varying Source Finite Element

The influence functions for a constant source panel are[2]

$$u_{00} = -\frac{k}{2\pi}\;\frac{1}{\sqrt{T^2+\beta^2}}\;\frac{1}{2}\log\frac{\sqrt{T^2+\beta^2}\,R + (Tx+\beta^2 y)}{\sqrt{T^2+\beta^2}\,R - (Tx+\beta^2 y)}$$

$$V_{00} = -\frac{k}{2\pi}\left\{\frac{1}{2}\log\frac{R+x}{R-x} - \frac{T}{\sqrt{T^2+\beta^2}}\;\frac{1}{2}\log\frac{\sqrt{T^2+\beta^2}\,R - (Tx+\beta^2 y)}{\sqrt{T^2+\beta^2}\,R \quad (Tx+\beta^2 y)}\right\}$$

$$W_{00} = \frac{-k}{2\pi}\tan^{-1}\frac{zR}{xy - T(y^2+z^2)}$$

Integrating in the chordwise direction gives the influence equations for a chordwise linearly varying source panel

$$u_{10} = -\frac{k}{2\pi}\left\{y\,\frac{1}{2}\log\frac{R+x}{R-x} + \frac{(x-Ty)}{\sqrt{T^2+\beta^2}}\;\frac{1}{2}\log\frac{\sqrt{T^2+\beta^2}\,R + (Tx+\beta^2 y)}{\sqrt{T^2+\beta^2}\,R - (Tx+\beta^2 y)} + z\tan^{-1}\frac{zR}{xy - T(y^2+z^2)}\right\}$$

$$V_{10} = -\frac{k}{2\pi}\left\{(x-Ty)\frac{1}{2}\log\frac{R+x}{R-x} - \frac{T(x-Ty)}{\sqrt{T^2+\beta^2}}\;\frac{1}{2}\log\frac{\sqrt{T^2+\beta^2}\,R + (Tx+\beta^2 y)}{\sqrt{T^2+\beta^2}\,R - (Tx+\beta^2 y)} - Tz\tan^{-1}\frac{zR}{xy - T(y^2+z^2)} - R\right\}$$

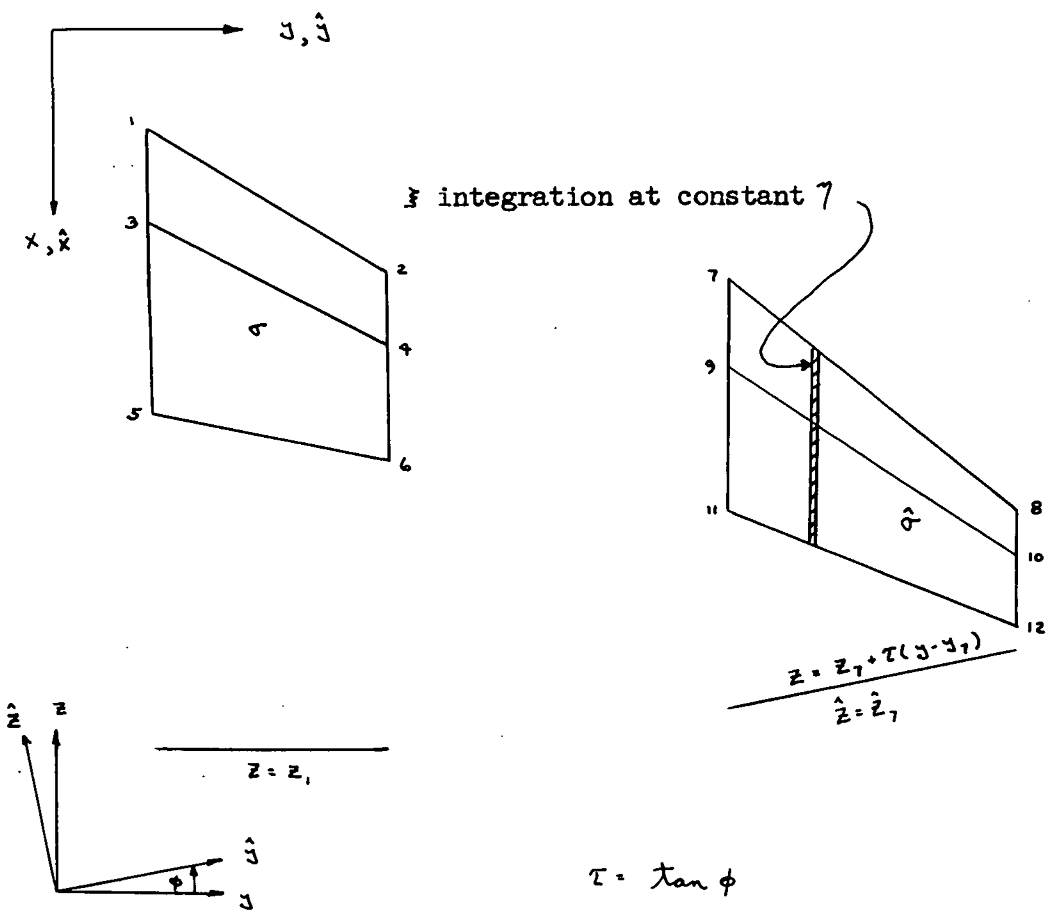
$$W_{10} = -\frac{k}{2\pi}\left\{(x-Ty)\tan^{-1}\frac{zR}{xy - T(y^2+z^2)} + Tz\,\frac{1}{2}\log\frac{R+x}{R-x} - z\sqrt{T^2+\beta^2}\;\frac{1}{2}\log\frac{\sqrt{T^2+\beta^2}\,R + (Tx+\beta^2 y)}{\sqrt{T^2+\beta^2}\,R - (Tx+\beta^2 y)}\right\}$$

Again, let

$$\xi = x - Ty$$
$$\eta = Ty$$
$$\zeta = Tz$$
$$B^2 = \frac{1}{T^2}(T^2+\beta^2)$$

The source panel singularities have the following properties:

1. All panel leading and trailing edges are at constant $(\frac{\xi}{c})$; side edges are at constant y
2. Each source finite element is composed of a panel pair
3. Source strength varies linearly with chord measured from the leading edge of a panel pair, i.e. the maximum value of the source strength is proportional to the local chord and attains this maximum on the panel edge joining the panel pair



Define $$\Delta_{31} = \left(\frac{x}{c}\right)_3 - \left(\frac{x}{c}\right)_1 = \left(\frac{x}{c}\right)_4 - \left(\frac{x}{c}\right)_2$$

$$\Delta_{53} = \left(\frac{x}{c}\right)_5 - \left(\frac{x}{c}\right)_3 = \left(\frac{x}{c}\right)_6 - \left(\frac{x}{c}\right)_4$$

The panel pair induces the following perturbation velocities

$$u(\xi,\eta,\zeta) = \frac{\sigma}{\Delta_{31}}\Big[u_{10}(\xi-\xi_1,\eta-\eta_1,\zeta-\zeta_1,T_1) - u_{10}(\xi-\xi_3,\eta-\eta_3,\zeta-\zeta_3,T_3)$$
$$+ u_{10}(\xi-\xi_2,\eta-\eta_2,\zeta-\zeta_2,T_2) - u_{10}(\xi-\xi_4,\eta-\eta_4,\zeta-\zeta_4,T_4)\Big]$$
$$+ \frac{\sigma}{\Delta_{53}}\Big[u_{10}(\xi-\xi_5,\eta-\eta_5,\zeta-\zeta_5,T_5) - u_{10}(\xi-\xi_3,\eta-\eta_3,\zeta-\zeta_3,T_3)$$
$$+ u_{10}(\xi-\xi_6,\eta-\eta_6,\zeta-\zeta_6,T_6) - u_{10}(\xi-\xi_4,\eta-\eta_4,\zeta-\zeta_4,T_4)\Big]$$

Results using these equations are compared with an exact solution for a biconvex airfoil in figure L. The only difficulties encountered were due to the fact that a chordwise linearly varying source panel must have zero thickness slope at the leading and trailing edges. Therefore the thickness variation could only be satisfied approximately in these regions.

Interference Drag Between Two Chordwise Linearly Varying Source Panels

To obtain the interference drag, the u velocity induced by one finite element must be multiplied by the local slope on another finite element and integrated over the area.

$$\frac{C_D S_{REF}}{\sigma\hat{\sigma}} = \iint u(\xi,\eta,\zeta)\,\Delta\omega\,dS$$

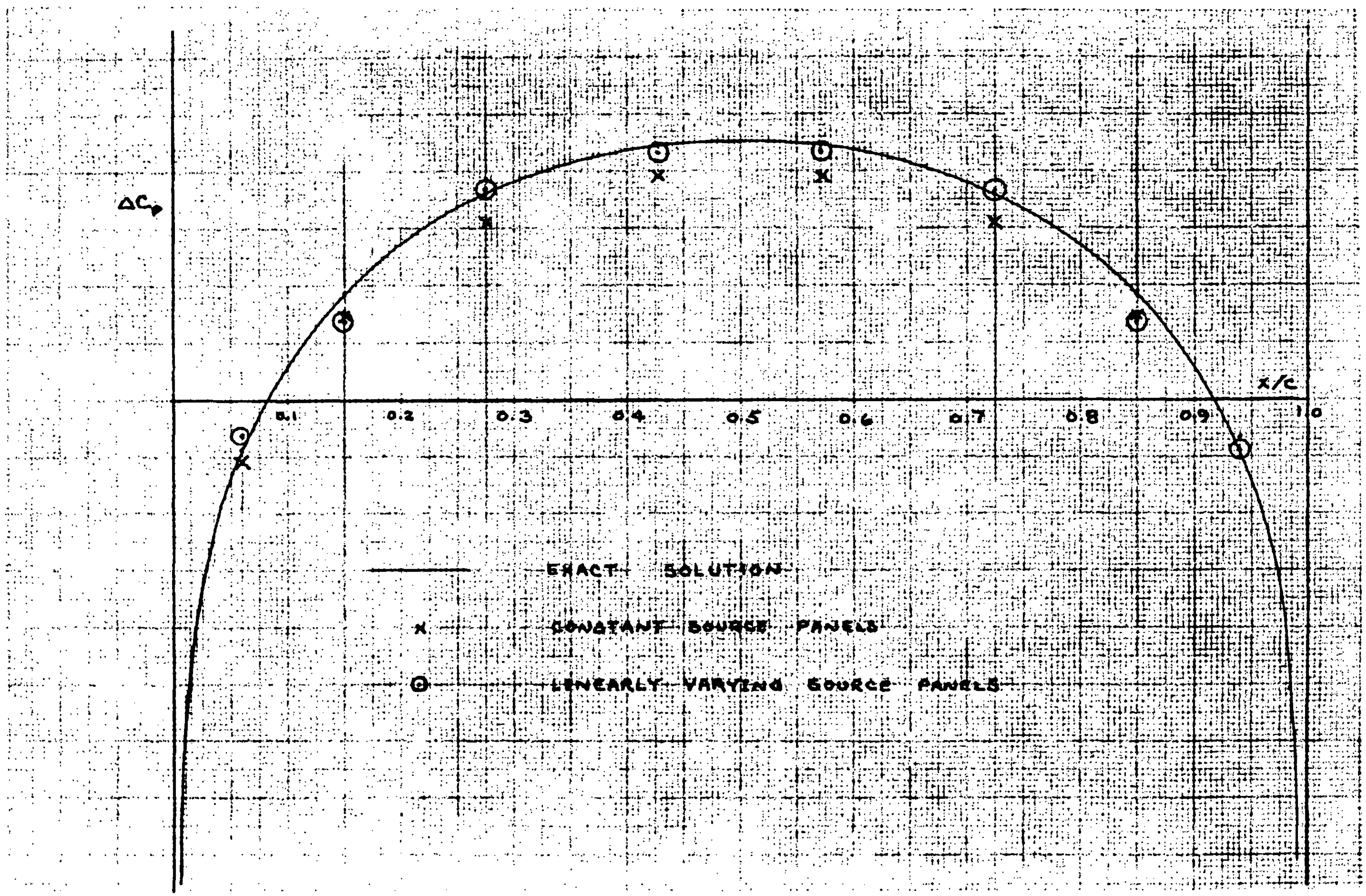


Figure 1. Rectangular Biconvex Wing AR = 5 Pressures at Mid Span

Consider first the integral over ξ of a typical term

$$\frac{1}{\Delta_{9,7}} \int_{\xi_7(\eta)}^{\xi_9(\eta)} [\xi - \xi_7(\eta)]\, u_{10}[\xi-\xi_1, \eta-\eta_1, \zeta-\zeta_1, T_1]\, d\xi$$

$$- \frac{1}{\Delta_{11,9}} \int_{\xi_9(\eta)}^{\xi_{11}(\eta)} [\xi - \xi_{11}(\eta)]\, u_{10}[\xi-\xi_1, \eta-\eta_1, \zeta-\zeta_1, T_1]\, d\xi$$

$$= \frac{1}{\Delta_{9,7}} \Big\{ [\xi - \xi_7]\, u_{20}[\xi-\xi_1, \eta-\eta_1, \zeta-\zeta_1, T_1] - u_{30}[\xi-\xi_1, \eta-\eta_1, \zeta-\zeta_1, T_1] \Big\}_{\xi_7}^{\xi_9}$$

$$- \frac{1}{\Delta_{11,9}} \Big\{ [\xi - \xi_{11}]\, u_{20}[\xi-\xi_1, \eta-\eta_1, \zeta-\zeta_1, T_1] - u_{30}[\xi-\xi_1, \eta-\eta_1, \zeta-\zeta_1, T_1] \Big\}_{\xi_9}^{\xi_{11}}$$

$$= \frac{1}{\Delta_{9,7}} \Big\{ u_{30}[\xi_7-\xi_1, \eta_7-\eta_1, \zeta_7-\zeta_1, T_1] - u_{30}[\xi_9-\xi_1, \eta_9-\eta_1, \zeta_9-\zeta_1, T_1] \Big\}$$

$$+ \frac{1}{\Delta_{11,9}} \Big\{ u_{30}[\xi_{11}-\xi_1, \eta_{11}-\eta_1, \zeta_{11}-\zeta_1, T_1] - u_{30}[\xi_{11}-\xi_1, \eta_{11}-\eta_1, \zeta_{11}-\zeta_1, T_1] \Big\}$$

because $$\frac{(\xi_9 - \xi_7)}{\Delta_{9,7}} = \frac{(x_9 - x_7)}{\Delta_{9,7}} = \frac{(\xi_{11} - \xi_9)}{\Delta_{11,9}} = \frac{(x_{11} - x_9)}{\Delta_{11,9}}$$

However, analogous to the vorticity case,

$$u_{ij} = \frac{k}{2\pi T} A \vec{f}_{ij} \qquad \text{where} \qquad A = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}^{*}$$

and $$A\,T = A = \hat{A}$$

Performing the integration over η of a typical term (where $\xi = \xi(\eta)$, $\zeta = \zeta(\eta)$]

$$\frac{k}{2\pi T^2}\sqrt{1+\tau^2}\, A \int_{\eta_7}^{\eta_8} \vec{f}_{30}(\xi-\xi_1, \eta-\eta_1, \zeta-\zeta_1, T_1)\, d\eta$$

$$= \frac{k}{2\pi T^2}\sqrt{1+\tau^2}\, A \left\{ E_{17}\vec{f}_{40}(\xi-\xi_1, \eta-\eta_1, \zeta-\zeta_1, T_1) - \frac{(\hat{\ell}+1)}{(1+\tau^2)} F_{17} T\, \vec{f}_{40}(\hat{\xi}-\hat{\xi}_1, \hat{\eta}-\hat{\eta}_1, \hat{\zeta}-\hat{\zeta}_1, T_7) \right\}_{\eta_7}^{\eta_8}$$

$$\frac{1}{T_1^2}\frac{(\hat{\ell}+1)}{(1+\tau^2)} A F T = \frac{1}{T_7^2}\frac{T_7^2}{T_1^2}\frac{(\ell_{71}+1)}{(1+\tau^2)} A T\, T^{-1} F_{17} T$$

$$= -\frac{1}{T_7^2}\frac{(\ell_{17}+1)^2}{(1+\tau^2)}\frac{(\ell_{71}+1)}{(1+\tau^2)}(\ell_{71}+1)\, A E_{71} = -\frac{1}{T_7^2} A E_{71}$$

where $\quad (\ell_{71}+1) = \dfrac{T_1\sqrt{1+\tau^2}}{T_7}, \quad (\ell_{17}+1) = \dfrac{T_7\sqrt{1+\tau^2}}{T_1}$

and $\quad f_{40}[-\xi, -\eta, -\zeta, T] = -f_{40}(\xi, \eta, \zeta, T)$

Therefore, one typical term for the drag induced by one panel on another is:

$$\frac{C_D S_{REF}}{\sigma \hat{\sigma}} = \frac{2k}{\pi T_1^2}\sqrt{1+\tau^2}\,\frac{1}{\Delta_{31}}\frac{1}{\Delta_{97}}\, A E_{17}\vec{f}_{40}(\xi_7-\xi_1, \eta_7-\eta_1, \zeta_7-\zeta_1, T_1)$$

- the same term with the indicies 1 and 7 interchanged and all variables based in the $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ coordinate system.

The complete drag coefficient requires performing this calculation for all panel corner pairs and adding the result after mutliplying by the appropriate sign.

From this expression it can be seen that the total interference drag between two panels is zero. Interchanging the influencing and influenced panels changes the sign of the expression and the sum is zero.

In the foregoing

$$\frac{\sqrt{1+\tau^2}}{T^2} AE = \frac{\sqrt{1+\tau^2}}{T^2} \frac{1}{(\ell^2+\tau^2 B^2)} \left\{ \begin{array}{c} 0 \\ \ell \\ 0 \\ -\tau \\ 0 \end{array} \right\}$$

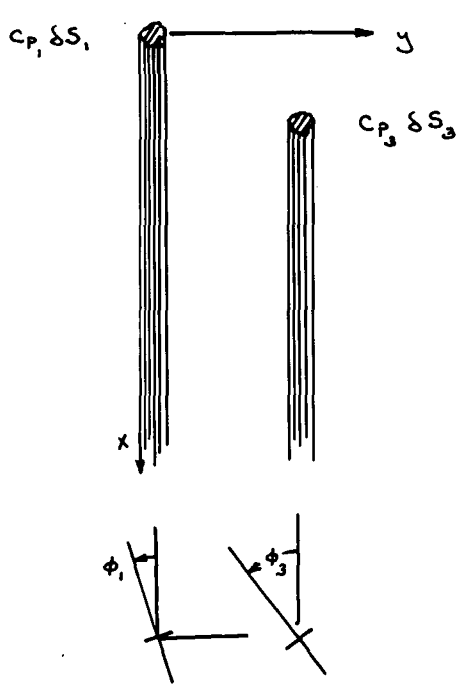
$$= \frac{1}{\left[(\hat{T}-T)^2 + 2(1-\cos\phi)T\hat{T} + \beta^2 \sin^2\phi\right]} \left\{ \begin{array}{c} 0 \\ \frac{1}{T}\left[\hat{T} - T\cos\phi\right] \\ 0 \\ -\sin\phi \\ 0 \end{array} \right\}$$

NEAR FIELD AND FAR FIELD INTERFERENCE DRAG EQUIVALENCE

This section will show the equivalence between the drag force as calculated in the near field and as calculated in the far field. The near field drag induced between two singularities is calculated by multiplying the induced velocity normal to one vortex panel by the local C_P and integrating over the area. The far field drag is calculated in two parts by integrating perturbation velocities over the appropriate areas on surfaces far removed from the singularities.

The singularites chosen to show the equivalance are two infinitesimal areas of constant pressure panels. It will be shown that the total interference drag between these two singularities is the same whether calculated in the near field or far field. The velocities induced by these singularities may be found by differentiating the velocities induced by a constant pressure panel with respect to x and y. These singularities actually represent elementary horseshoe vorticies. Any singularity field or pressure distribution on any set of surfaces may be constructed by integrating together the appropriate distribution of these elementary elements.

Consider two infinitesimal area vorticity distributions and their associated trailing vorticity. Place an (x, y, z) or (x, r, ϕ) coordinate system inside the upstream area with the y axis passing through the trailing vorticity. Then later in this section, the following results are derived for supersonic flow.



$$\Delta C_P = C_{P_L} - C_{P_U} = -C_P$$

The near field drag is

$$C_{D_{13}} S_{REF_{N.F.}} = - \frac{[C_{P_1} \delta S_1][C_{P_3} \delta S_3]\, x}{4\pi r^2} \left\{ \cos\phi_3 \cos\phi_1 \frac{1}{R} - \sin\phi_3 \sin\phi_1 \frac{1}{R^3} [R^2 - \beta^2 r^2] \right\}$$

the far field wave drag is

$$C_{D_{13_{WAVE}}} S_{REF} = - \frac{\beta^2 [C_{P_1} \delta S_1][C_{P_3} \delta S_3]}{4\pi} \left\{ \cos\phi_1 \cos\phi_3 \frac{x}{R^3} - \cos(\phi_1+\phi_3) \left[\frac{x}{R^3} - \frac{x-R}{\beta^2 r^2 R} \right] \right\}$$

the far field vortex drag is

$$C_{D_{13_{VORTEX}}} S_{REF} = - \frac{[C_{P_1} \delta S_1][C_{P_3} \delta S_3]}{4\pi r^2} \cos(\phi_3 + \phi_1)$$

From these results it is easily seen that the total far field drag is

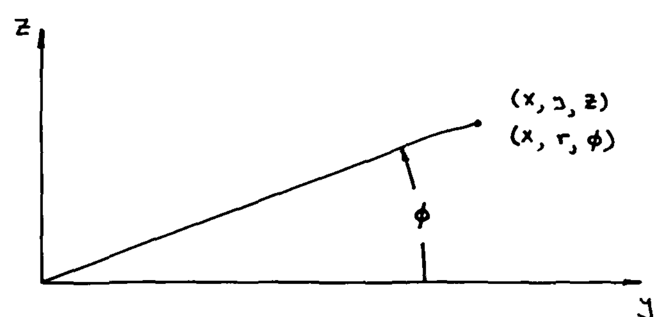
$$C_{D_{13_{TOTAL}}} = C_{D_{13_{WAVE}}} + C_{D_{13_{VORTEX}}} = C_{D_{13_{NEAR\ FIELD}}}$$

For supersonic flow only the near field terms of the influence equations were considered. For subsonic flow the near field drag as calculated using these terms will cancel since the value of x will change sign when we interchange the influenced and influencing vorticity distributions. For any distribution of vorticity this interference drag will cancel exactly if the integration is performed exactly. Therefore in subsonic flow the only contribution to the total drag comes from the far field terms of the influence equations and this is identical to the vortex drag.

Induced Velocities - Supersonic

To derive the above equations consider a constant vorticity panel located in the Z = 0 plane with one of its corners at x = 0, y = 0. Consider also a polar coordinate system with the following definitions

$$y = r\cos\phi$$
$$z = r\sin\phi$$
$$r^2 = y^2 + z^2$$
$$R^2 = x^2 - \beta^2 r^2$$
$$\beta^2 = M_\infty^2 - 1$$



The velocities induced by an infinitesimal streamwise-running strip of this constant vorticity panel are found by differentiating the velocities induced by the edge of the panel at y = 0 with respect to y. The resulting strip is actually a line doublet which varies linearly until the end of the vortex strip and is constant thereafter. The velocities induced by the end of the strip starting at x = 0 are:

$$u = -\frac{\partial}{\partial y} u_{oo} = -\frac{C_p \delta y}{4\pi} \frac{xz}{r^2 R} = -\frac{C_p \delta y}{4\pi} \frac{x}{rR} \sin\phi$$

$$V = -\frac{\partial}{\partial y} V_{oo} = -\frac{C_p \delta y}{4\pi} \frac{yz}{r^4 R} \left[2R^2 + \beta^2 r^2 \right] = -\frac{C_p \delta y}{4\pi} \frac{\sin\phi \cos\phi}{r^2 R} \left[2R^2 + \beta^2 r^2 \right]$$

$$\omega = -\frac{\partial}{\partial y} \omega_{oo} = -\frac{C_p \delta y}{4\pi} \left\{ \frac{z^2}{r^4 R} \left[2R^2 + \beta^2 r^2 \right] - \frac{R}{r^2} \right\} = -\frac{C_p \delta y}{4\pi r^2} \left\{ \frac{\sin^2\phi}{R} \left[2R^2 + \beta^2 r^2 \right] - R \right\}$$

and radial and tangential velocities in polar coordinates may be defined

$$V_r = V\cos\phi + W\sin\phi = -\frac{C_p \delta y}{4\pi} \frac{x^2 \sin\phi}{r^2 R}$$

$$V_\phi = -V\sin\phi + W\cos\phi = \frac{C_p \delta y}{4\pi} \frac{R\cos\phi}{r^2}$$

The perturbation velocities are found by evaluating the functions at the start and end of the strip using a coordinate system located at the respective end points.

The velocities induced by an elementary horseshoe vortex or line doublet are found by differentiating these velocities with respect to x and multiplying by -1.

$$u = -\frac{C_p \delta S}{4\pi} \frac{\beta^2 r \sin\phi}{4\pi R^3}$$

$$V_\phi = -\frac{C_p \delta S}{4\pi} \frac{x\cos\phi}{r^2 R}$$

$$V_r = \frac{C_p \delta S}{4\pi} \frac{x\sin\phi}{r^2 R^3} \left\{ R^2 - \beta^2 r^2 \right\}$$

$$\delta S = \delta x \, \delta y$$

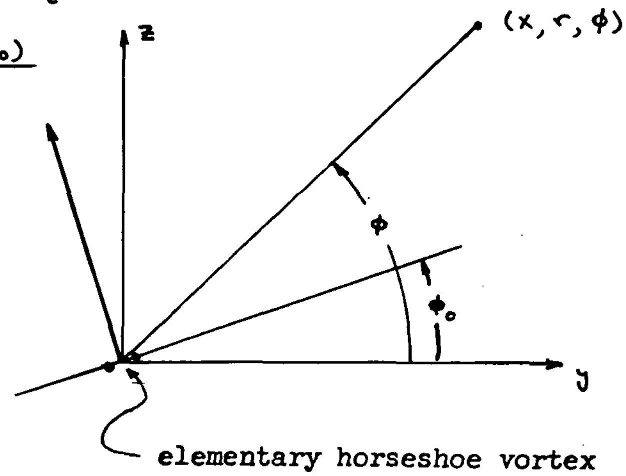
Near Field Drag

Consider an elementary horseshoe vortex with its local z axis inclined at an angle ϕ_0 with respect to the vertical. Then the velocities induced are

$$V_{r_0} = \frac{C_{p_0}\delta S_0}{4\pi}\,\frac{x\sin(\phi-\phi_0)}{r^2R^3}\left\{R^2-\beta^2 r^2\right\}$$

$$V_{\theta_0} = -\frac{C_{p_0}\delta S_0}{4\pi}\,\frac{x\cos(\phi-\phi_0)}{r^2R}$$

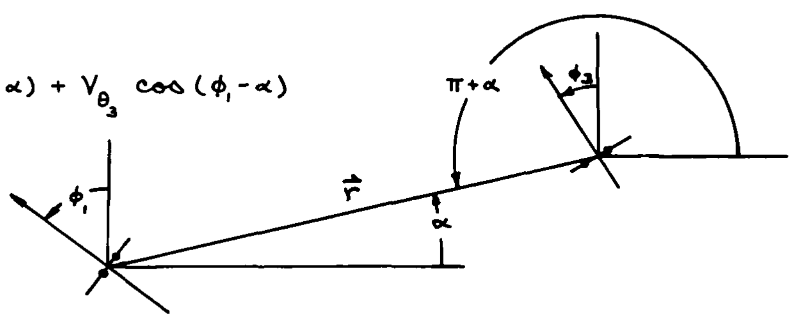
$$R^2 = x^2-\beta^2 r^2$$



If we consider two singularities, denoted by subscripts 1 and 3, then the component of velocity of vortex 3 normal to horseshoe vortex 1 is

$$V_{N_1} = -V_{r_3}\sin(\phi_1-\alpha)+V_{\theta_3}\cos(\phi_1-\alpha)$$

$$\Delta C_{p_1}\delta S_1 = -\left[C_{p_1}\delta S_1\right]$$



$$[\Delta C_{p_1}\delta S_1]\,V_{N_1} = \frac{[C_{p_1}\delta S_1][C_{p_3}\delta S_3]\,x}{4\pi r^2}\left\{\sin(\pi+\alpha-\phi_3)\sin(\phi_1-\alpha)\frac{1}{R^3}\left[R^2-\beta^2r^2\right] + \cos(\pi+\alpha-\phi_3)\cos(\phi_1-\alpha)\frac{1}{R}\right\}$$

$$= -\frac{[C_{p_1}\delta S_1][C_{p_3}\delta S_3]\,x}{4\pi r^2}\left\{\cos(\phi_3-\alpha)\cos(\phi_1-\alpha)\frac{1}{R} - \sin(\phi_3-\alpha)\sin(\phi_1-\alpha)\frac{1}{R^3}\left[R^2-\beta^2r^2\right]\right\}$$

Since only one singularity can lie in the downstream Mach cone of the other the near field drag is given by this expression on page 35.

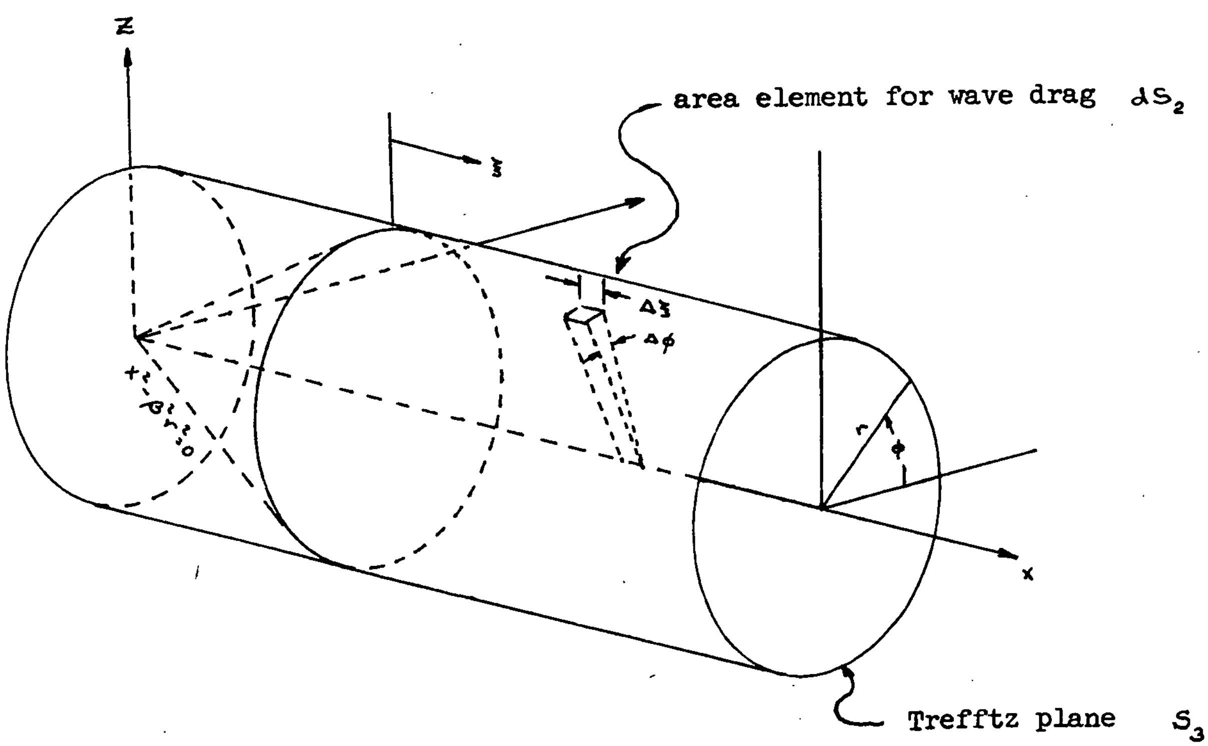
Far Field Drag

The far field drag is claculated by integrating over two surfaces at infinity.[3] The wave drag involves an integral over a cylinder whose axis is the x axis and whose radius approaches infinity. The vortex drag is found by integrating over the end of the cylinder as $x \to \infty$ (Trefftz plane).

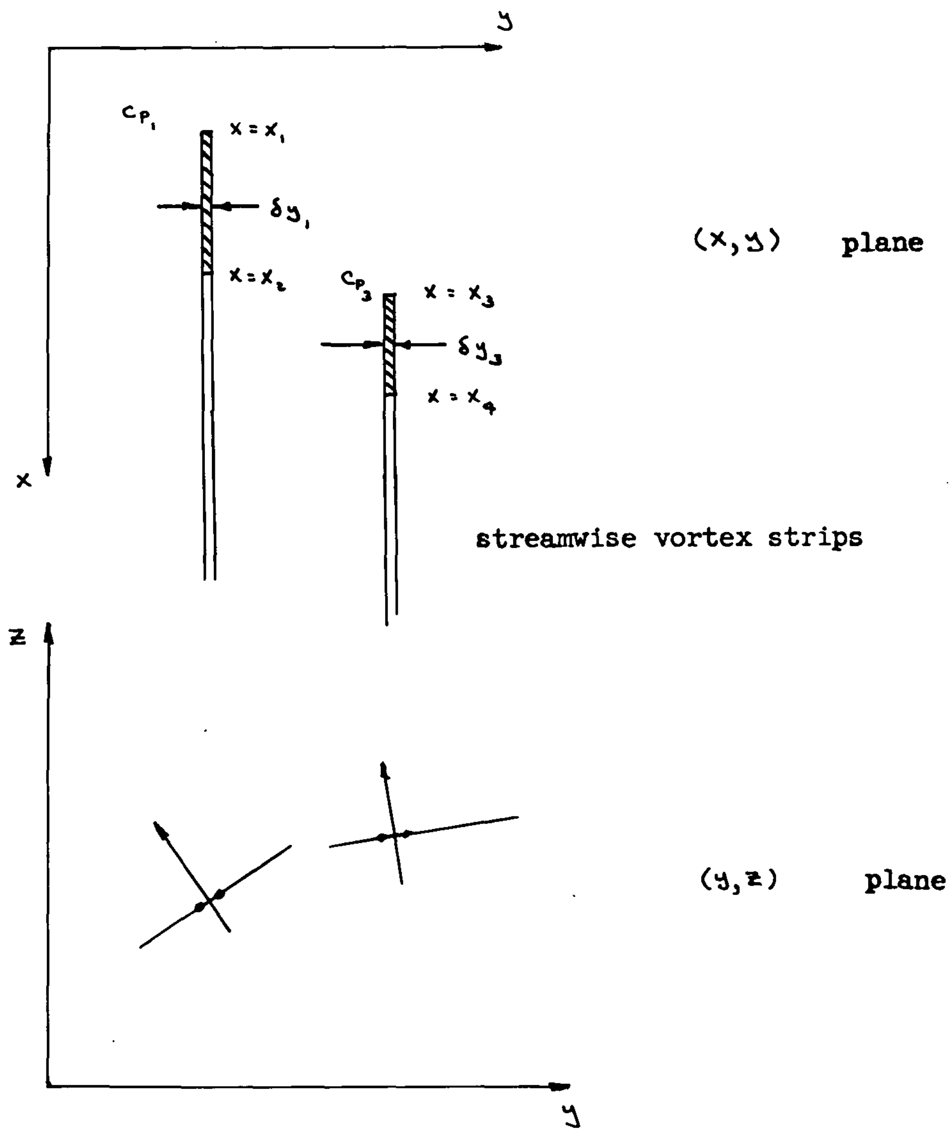
$$C_{D_{WAVE}} S_{REF} = -2 \iint_{S_2} \Phi_x \Phi_r \, dS_2$$

$$C_{D_{VORTEX}} S_{REF} = \iint_{S_3} (\Phi_y^2 + \Phi_z^2) \, dS_3$$

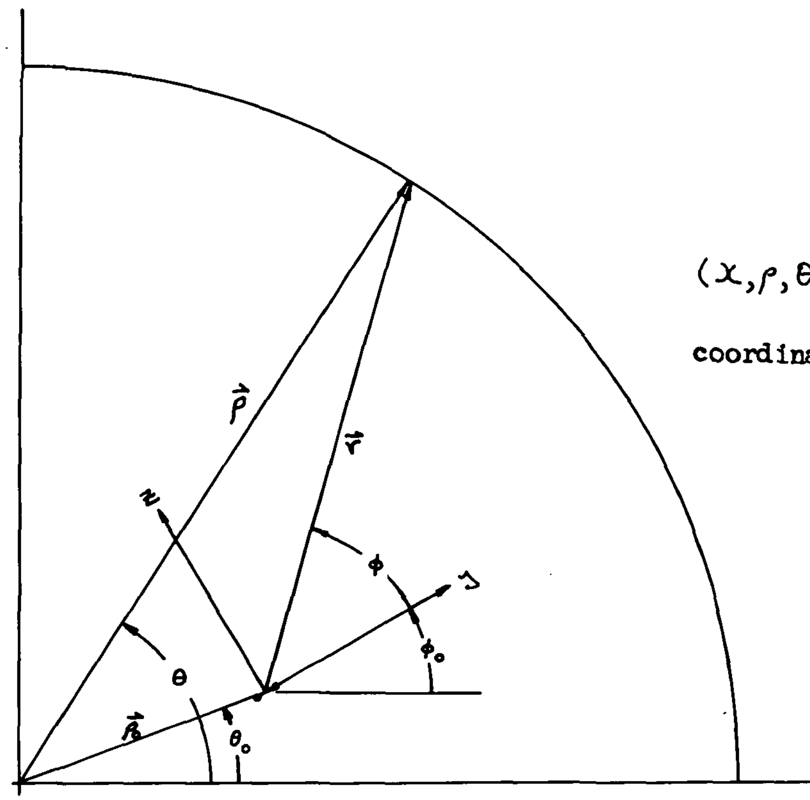
Φ = velocity potential



Wave drag-to evaluate the wave drag we must perform an integration over a large cylinder whose radius approaches infinity. To avoid complications in the integrations we will calculate the interference drag between two streamwise vortex strips of finite length. The wave drag between the elementary horseshoe vorticies may then be obtained by differentrating the result. The velocities induced by these finite length vortex strips are given by the equations on the top of page 36.



Consider a coordinate system located at the center of the large cylinder. One end of one of the vortex strips will be located at (x_0, ρ_0, θ_0).



(χ, ρ, θ)

coordinate system

Then

$$x = \chi - x_0$$

$$r^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\theta - \theta_0)$$

$$\rho \cos\theta = \rho_0 \cos\theta_0 + r\cos(\phi + \phi_0)$$

$$\rho \sin\theta = \rho_0 \sin\theta_0 + r\sin(\phi + \phi_0)$$

The perturbation velocities induced by this end of the strip are:

$$u = -\frac{c_{p_0}\,\delta y_0\,(\chi - x_0)\sin\phi}{4\pi r\ R}$$

$$V_r = -\frac{c_{p_0}\,\delta y_0\,(\chi - x_0)^2 \sin\phi}{4\pi r^2 R}$$

Since there are no perturbation velocities ahead of the Mach cone centered on the (x, y, z) system we will define a new variable $\xi = x - \beta\rho$. On a cylinder with large ρ the values of ξ will be of order one in the region where the Mach cone intersects the cylinder.

$$
\begin{aligned}
R^2 &= x^2 - \beta^2 r^2 \\
&= (\xi - x_0 + \beta\rho)^2 - \beta^2 r^2 \\
&= (\xi - x_0)^2 + 2\beta\rho(\xi - x_0) + \beta^2\rho^2 - \beta^2[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\theta - \theta_0)] \\
&= 2\beta\rho[(\xi - x_0) + \beta\rho_0 \cos(\theta - \theta_0)] + (\xi - x_0)^2 - \beta^2\rho_0^2
\end{aligned}
$$

As $\rho \to \infty$ the following limiting values are reached for finite values of ξ

$$\phi \to \theta - \phi_0$$

$$r \to \rho$$

$$R \to \sqrt{2\beta\rho[(\xi - x_0) + \beta\rho_0 \cos(\theta - \theta_0)]}$$

and

$$u = -\frac{c_{p_0}\delta y_0 (\xi - x_0 + \beta\rho) \sin\phi}{4\pi\rho R} \to -\frac{c_{p_0}\delta y_0}{4\pi} \frac{\beta \sin(\theta - \phi_0)}{\sqrt{2\beta\rho[(\xi - x_0) + \beta\rho_0 \cos(\theta - \theta_0)]}}$$

$$V_r = -\frac{c_{p_0}\delta y_0 (\xi - x_0 + \beta\rho)^2 \sin\phi}{4\pi\rho^2 R} \to -\frac{c_{p_0}\delta y_0}{4\pi} \frac{\beta^2 \sin(\theta - \phi_0)}{\sqrt{2\beta\rho[(\xi - x_0) + \beta\rho_0 \cos(\theta - \theta_0)]}}$$

These functions must be evaluated at the start and end of the vortex strip (linearly varying line doublet).

To find the interference wave drag between two strips we must integrate the product of u from one strip times v_r from the other strip and multiply by the element of area over the infinite cylinder. Assume

strip 1	begins at	$X = x_1$	and ends at	$X = x_2$
strip 2	begins at	$X = x_3$	" " "	$X = x_4$

The associated values of $(\rho_o, \theta_o, \phi_o)$ will be $(\rho_1, \theta_1, \phi_1)$, $(\rho_3, \theta_3, \phi_3)$

Or

$$u_1 = -\frac{C_{p_1}\delta y_1}{4\pi} \frac{\beta \sin(\theta - \phi_1)}{\sqrt{2\beta\rho\left[(\xi - x_1) + \beta\rho_1\cos(\theta - \theta_1)\right]}}$$

$$V_{r_1} = -\frac{C_{p_1}\delta y_1}{4\pi} \frac{\beta^2 \sin(\theta - \phi_1)}{\sqrt{2\beta\rho\left[(\xi - x_1) + \beta\rho_1\cos(\theta - \theta_1)\right]}}$$

with corresponding values for ends 2, 3, and 4

The wave drag is evaluated by integrating

$$2\left[u_1 V_{r_3} + u_3 V_{r_1}\right] - 2\left[u_1 V_{r_4} + u_4 V_{r_1}\right]$$
$$- 2\left[u_2 V_{r_3} + u_3 V_{r_2}\right] + 2\left[u_2 V_{r_4} + u_4 V_{r_2}\right]$$

over the large cylinder, which has the differential surface area

$$dS_2 = \rho \, d\xi \, d\theta$$
$$-\infty < \xi < \infty$$
$$0 \le \theta \le 2\pi$$

Considering only a single term of the above and letting

$$a_1 = x_1 - \beta\rho_1 \cos(\theta - \theta_1)$$

$$a_3 = x_3 - \beta\rho_3 \cos(\theta - \theta_3)$$

$$\rho[u_1 v_{r_3} + u_3 v_{r_1}] = \frac{[c_{p_1} \delta y_1][c_{p_3} \delta y_3]}{(4\pi)^2} \quad \frac{\beta^2 \sin(\theta - \phi_1)\sin(\theta - \phi_3)}{\sqrt{(\xi - a_1)(\xi - a_3)}}$$

we must evaluate

$$C_{D_{13}} S_{REF} = -\frac{[c_{p_1}\delta y_1][c_{p_3}\delta y_3]}{(4\pi)^2} \; 2\beta^2 \int_0^{2\pi} \sin(\theta - \phi_1)\sin(\theta - \phi_3) \int_{a_3}^{\infty} \frac{d\xi}{\sqrt{(\xi - a_1)(\xi - a_3)}} \, d\theta$$

Now $$\int_{a_3}^{\infty} \frac{d\xi}{\sqrt{(\xi - a_1)(\xi - a_3)}} = -\log(a_3 - a_1)$$

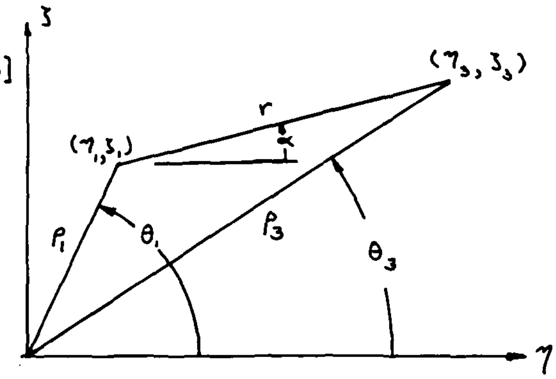
plus a term which cancels when all four ends are considered.

Also,

$$(a_3 - a_1) = (x_3 - x_1) - \beta[\rho_3 \cos(\theta - \theta_3) - \rho_1 \cos(\theta - \theta_1)]$$

and

$$\begin{aligned} \rho_3 \cos(\theta - \theta_3) - \rho_1 \cos(\theta - \theta_1) &= \rho_3 [\cos\theta\cos\theta_3 + \sin\theta\sin\theta_3] \\ &\quad - \rho_1 [\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1] \\ &= (\eta_3 - \eta_1)\cos\theta + (\zeta_3 - \zeta_1)\sin\theta \\ &= r\cos(\theta - \alpha) \end{aligned}$$



$$r^2 = (\eta_3 - \eta_1)^2 + (\zeta_3 - \zeta_1)^2$$

$$\alpha = \tan^{-1} \frac{(\zeta_3 - \zeta_1)}{(\eta_3 - \eta_1)}$$

$$x = x_3 - x_1$$

So

$$(a_3 - a_1) = x - \beta r \cos(\theta - \alpha)$$

Therefore,

$$C_{D_{13}} S_{REF} = \frac{[C_{P_1} \delta y_1][C_{P_3} \delta y_3]}{(4\pi)^2} \; 2\beta^2 \int_0^{2\pi} \sin(\theta-\phi_1)\sin(\theta-\phi_3)\log[X-\beta r\cos(\theta-\alpha)]d\theta$$

$$= \frac{[C_{P_1} \delta y_1][C_{P_3} \delta y_3]}{(4\pi)^2} \; 2\beta^2 \int_0^{2\pi} \sin[\theta-(\phi_1-\alpha)]\sin[\theta-(\phi_3-\alpha)]\log[X-\beta r\cos\theta]d\theta$$

where

$$\sin[\theta-(\phi_1-\alpha)]\sin[\theta-(\phi_3-\alpha)] = \frac{1}{2}\cos[\phi_3-\phi_1] - \frac{1}{2}\cos[2\theta-(\phi_1-\alpha)-(\phi_3-\alpha)]$$

$$= \frac{1}{2}\cos(\phi_3-\phi_1) - \frac{1}{2}\cos 2\theta\cos[(\phi_1-\alpha)+(\phi_3-\alpha)] - \frac{1}{2}\sin 2\theta\sin[(\phi_1-\alpha)+(\phi_3-\alpha)]$$

$$= \frac{1}{2}\cos(\phi_3-\phi_1) - \frac{1}{2}(2\cos^2\theta-1)\cos[(\phi_1-\alpha)+(\phi_3-\alpha)]$$
$$- \sin\theta\cos\theta\sin[(\phi_1-\alpha)+(\phi_3-\alpha)]$$

$$= \cos(\phi_1-\alpha)\cos(\phi_3-\alpha) - \cos^2\theta\cos[(\phi_1-\alpha)+(\phi_3-\alpha)]$$
$$- \sin\theta\cos\theta\sin[(\phi_1-\alpha)+(\phi_3-\alpha)]$$

Therefore, since

$$\int_0^{2\pi} \sin\theta\cos\theta\log(X-\beta r\cos\theta)d\theta = 0$$

we have

$$C_{D_{13}} S_{REF} = \frac{[C_{P_1} \delta y_1][C_{P_3} \delta y_3]}{(4\pi)^2} \; 2\beta^2 \Big\{ \cos(\phi_1-\alpha)\cos(\phi_3-\alpha)\int_0^{2\pi}\log(X-\beta r\cos\theta)d\theta$$
$$- \cos[(\phi_1-\alpha)+(\phi_3-\alpha)]\int_0^{2\pi}\cos^2\theta\log(X-\beta r\cos\theta)d\theta \Big\}$$

The integrals over Θ may be evaluated as follows

$$F = \int_0^{2\pi} \log(x - \beta r \cos\theta)\, d\theta$$

$$\frac{\partial F}{\partial x} = \int_0^{2\pi} \frac{d\theta}{x - \beta r \cos\theta} = \frac{2\pi}{R} \quad \text{where} \quad R^2 = x^2 - \beta^2 r^2$$

$$G = \int_0^{2\pi} \cos^2\theta \log(x - \beta r \cos\theta)\, d\theta$$

$$\frac{\partial G}{\partial x} = \int_0^{2\pi} \frac{1 - \sin^2\theta}{x - \beta r \cos\theta}\, d\theta = 2\int_0^{\pi} \frac{1 - \sin^2\theta}{x + \beta r \cos\theta}\, d\theta$$

$$= 2\pi \left\{ \frac{1}{R} - \frac{1}{x+R} \right\} = 2\pi \left\{ \frac{1}{R} - \frac{x-R}{\beta^2 r^2} \right\}$$

The interference wave drag between two strips is

$$C_{D_{13_{wave}}} S_{REF} = S_{REF} \left[C_{D_{13_{wave}}} - C_{D_{23_{wave}}} - C_{D_{14_{wave}}} + C_{D_{24_{wave}}} \right]$$

Therefore, if

$$(x_2 - x_1) = \delta x_1 \rightarrow 0$$

$$(x_4 - x_3) = \delta x_3 \rightarrow 0 \qquad \delta S_1 = \delta y_1 \delta x_1 \qquad \delta S_3 = \delta y_3 \delta x_1$$

the interference wave drag between two elementary horseshoe vorticies is

$$C_{D_{WAVE}} S_{REF} = \frac{[C_{P_1} \delta S_1][C_{P_3} \delta S_3]}{(4\pi)^2} 2\beta^2 \left\{ \cos(\phi_1 - \alpha)\cos(\phi_3 - \alpha) \frac{\partial^2 F}{\partial x^2} - \cos[(\phi_1 - \alpha) + (\phi_3 - \alpha)] \frac{\partial^2 G}{\partial x^2} \right\}$$

$$= -\frac{\beta^2 [C_{P_1} \delta S_1][C_{P_3} \delta S_3]}{4\pi} \left\{ \cos(\phi_1 - \alpha)\cos(\phi_3 - \alpha) \frac{x}{R^3} - \cos[(\phi_1 - \alpha) + (\phi_3 - \alpha)] \left[\frac{x}{R^3} - \frac{x-R}{R\beta^2 r^2} \right] \right\}$$

This is the result given on page 35.

Vortex Drag.- To evaluate the vortex drag the following integral must be evaluated in the Trefftz plane.

$$C_{D_v} S_{REF} = \int (v^2 + w^2)\, dS_3$$

But, since $v = \dfrac{\partial \Phi}{\partial y}$ and $w = \dfrac{\partial \Phi}{\partial z}$, where Φ is the velocity potential

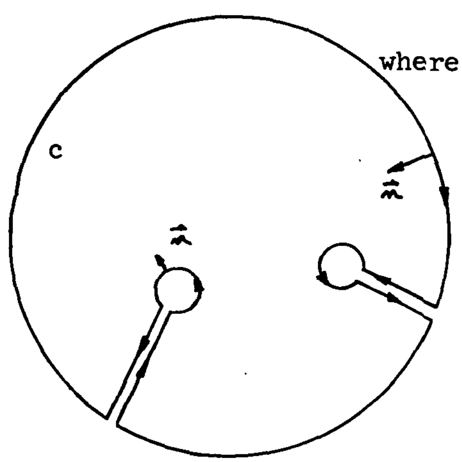
and $$\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

we can write $$v^2 + w^2 = \frac{\partial}{\partial y}\left(\Phi \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Phi \frac{\partial \Phi}{\partial z}\right)$$

Then, using Stokes theorem, the surface integral may be reduced to a series of line integrals around the singularities:

$$C_{D_v} S_{REF} = \iint \left\{ \frac{\partial}{\partial y}\left(\Phi \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Phi \frac{\partial \Phi}{\partial z}\right) \right\} dS = - \int_C \Phi \frac{\partial \Phi}{\partial n}\, d\ell$$

where n is the normal to the contour c.



The velocities induced by an elementary horseshoe vortex in the Trefftz plane are found by taking the limit as $x \longrightarrow \infty$

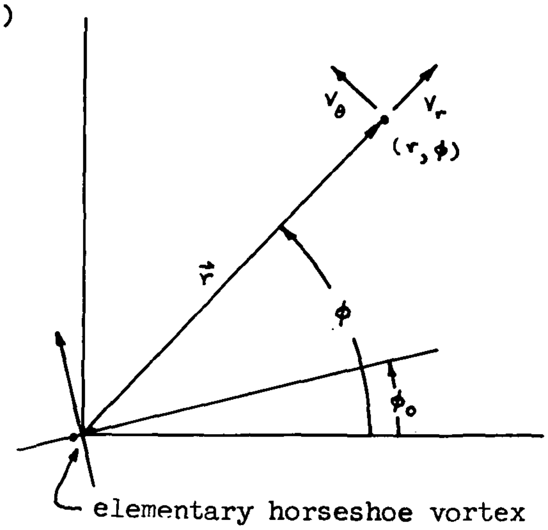
$$V_r = \lim_{x \to \infty} \frac{C_{P_0}\,\delta S_0}{4\pi} \; \frac{x \sin(\phi-\phi_0)}{r^2 R^3}\left[R^2 - \beta^2 r^2 \right]$$

$$= \frac{C_{P_0}\,\delta S_0}{4\pi r^2} \sin(\phi - \phi_0)$$

and also

$$V_{\theta_0} = -\frac{C_p\,\delta S_0}{4\pi r^2} \cos(\phi - \phi_0)$$

$$\Phi_0 = -\frac{C_p\,\delta S_0}{4\pi r} \sin(\phi - \phi_0)$$



For two elementary horseshoe vorticies (doublets) the velocity potential is the sum of the individual velocity potentials

$$\Phi = \Phi_1 + \Phi_3$$

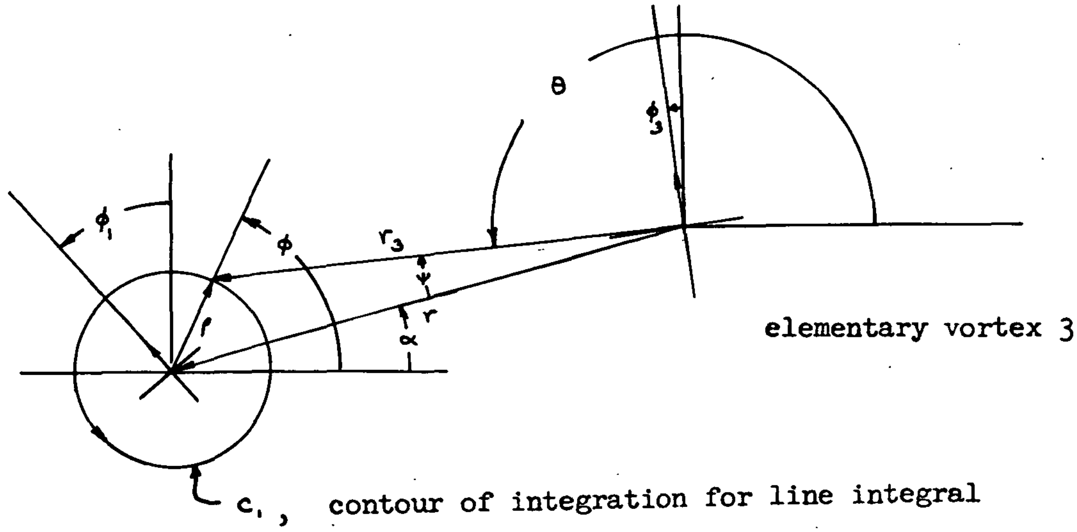
$$C_{D_v} S_{REF} = -\int_c (\Phi_1 + \Phi_3)\left\{ \frac{\partial \Phi_1}{\partial n} + \frac{\partial \Phi_3}{\partial n} \right\} d\ell$$

The interference drag between doublet 1, and doublet 3 is given by

$$C_{D_{v_{13}}} S_{REF} = -\int_c \left\{ \Phi_1 \frac{\partial \Phi_3}{\partial n} + \Phi_3 \frac{\partial \Phi_1}{\partial n} \right\} d\ell$$

The contour c will be chosen to be a small circle around each doublet.

Consider the integral around elementary horseshoe vortex 1 for $\rho << r$



$$\Phi_1 = -\frac{C_{P_1}\,\delta S_1}{4\pi\rho}\sin(\phi-\phi_1) \qquad \Phi_3 = -\frac{C_{P_3}\,\delta S_3}{4\pi r_3}\sin(\theta-\phi_3)$$

$$V_{r_1} = \frac{C_{P_1}\,\delta S_1}{4\pi\rho^2}\sin(\phi-\phi_1) \qquad V_{r_3} = \frac{C_{P_3}\,\delta S_3}{4\pi r_3^2}\sin(\theta-\phi_3)$$

$$V_{\theta_3} = -\frac{C_{P_3}\,\delta S_3}{4\pi r_3^2}\cos(\theta-\phi_3)$$

On c_1
$$\lim_{\rho\to 0}\frac{\partial\Phi_3}{\partial n} = \frac{C_{P_3}\,\delta S_3}{4\pi r^2}\left\{\sin(\pi+\alpha-\phi_3)\cos(\phi-\pi-\alpha) - \cos(\pi+\alpha-\phi_3)\sin(\phi-\pi-\alpha)\right\}$$

$$= -\frac{C_{P_3}\,\delta S_3}{4\pi r^2}\sin\left[(\phi-\alpha)+(\phi_3-\alpha)\right]$$

Therefore,

$$\lim_{\rho\to 0}\int_{c_1}\Phi_1\frac{\partial\Phi_3}{\partial n}\,dl_1 = \frac{[C_{P_1}\delta S_1][C_{P_3}\delta S_3]}{(4\pi)^2 r^2}\int_0^{2\pi}\sin(\phi-\phi_1)\sin\left[(\phi-\alpha)+(\phi_3-\alpha)\right]d\phi$$

$$= \frac{[C_{P_1}\,\delta S_1][C_{P_3}\,\delta S_3]}{16\pi r^2}\cos\left[(\phi_1-\alpha)+(\phi_3-\alpha)\right]$$

To evaluate $\displaystyle\int_{C_1} \Phi_3 \frac{\partial \Phi_1}{\partial n}\, d\ell = \int_0^{2\pi} \rho\, \Phi_3\, V_{r_1}\, d\phi$

we again choose a contour by letting $\rho \to 0$

Note that

$$\rho^2 + r^2 = r_3^2 + 2\rho r \cos(\phi - \alpha)$$

$$\frac{\sin\psi}{\rho} = \frac{\sin[\pi - (\phi-\alpha) - \psi]}{r}$$

As $\rho \to 0$:

$$r_3 \to r\left[1 - \frac{\rho}{r}\cos(\phi - \alpha)\right]$$

$$\sin(\theta - \phi_3) = \sin(\pi + \alpha - \phi_3 - \psi) \to \sin(\pi + \alpha - \phi_3) - \rho\cos(\pi + \alpha - \phi_3)\left[\frac{\partial \psi}{\partial \rho}\right]_{\rho=0}$$

$$\to \sin(\pi + \alpha - \phi_3) - \frac{\rho}{r}\cos(\pi + \alpha - \phi_3)\sin(\pi - \phi + \alpha)$$

$$= \sin(\phi_3 - \alpha) + \frac{\rho}{r}\cos(\phi_3 - \alpha)\sin(\phi - \alpha)$$

$$\Phi_3 \to -\frac{C_{P_3}\,\delta S_3}{4\pi r}\left[1 + \frac{\rho}{r}\cos(\phi-\alpha)\right]\left[\sin(\phi_3 - \alpha) + \frac{\rho}{r}\cos(\phi_3-\alpha)\sin(\phi-\alpha)\right]$$

$$\to -\frac{C_{P_3}\,\delta S_3}{4\pi r}\left\{\sin(\phi_3-\alpha) + \frac{\rho}{r}\left[\cos(\phi-\alpha)\sin(\phi_3-\alpha) + \sin(\phi-\alpha)\cos(\phi_3-\alpha)\right]\right\}$$

$$= -\frac{C_{P_3}\,\delta S_3}{4\pi r}\left\{\sin(\phi_3-\alpha) + \frac{\rho}{r}\sin[(\phi-\alpha) + (\phi_3-\alpha)]\right\}$$

Therefore as $\rho \to 0$

$$\int_{C_1} \Phi_3 \frac{\partial \Phi_1}{\partial n}\, d\ell \to -\frac{[C_{P_1}\,\delta S_1][C_{P_3}\,\delta S_3]}{(4\pi r)^2}\int_0^{2\pi} \sin[(\phi-\alpha) + (\phi_3-\alpha)]\sin(\phi - \phi_1)\, d\phi$$

$$= -\frac{[C_{P_1}\,\delta S_1][C_{P_3}\,\delta S_3]}{16\pi r^2}\cos[(\phi_1-\alpha) + (\phi_3-\alpha)]$$

The vortex drag is thus

$$C_{D_V} S_{REF} = - \int_{C_1+C_3} \left\{ \bar{\Phi}_1 \frac{\partial \bar{\Phi}_3}{\partial n} + \bar{\Phi}_3 \frac{\partial \bar{\Phi}_1}{\partial n} \right\} d\ell$$

$$= - \frac{[C_{P_1} \delta s_1][C_{P_3} \delta s_3]}{4\pi r^2} \cos\left[(\phi_3 - \alpha) + (\phi_1 - \alpha) \right]$$

which is the result given on page 35.

RESULTS

Analysis of the spanwise variation of leading edge thrust using the linearly varying vortex finite element code of Appendix E is presented in this section. The numerical solution has been compared to analytic results, where available.

The calculated spanwise distribution of thrust for a highly swept cranked wing at incompressible speeds is presented on figure 2. A strong suction peak exists at the leading edge discontinuity. Weakening of the singularity requires cambering of the wing in this region or elimination of the break in the planform.

The calculated spanwise distribution of thrust for an aspect ratio 2.5 wing with a leading edge sweep of 63.4° and a taper ratio of .23 is presented on figure 3 for several Mach numbers. Transonic and supersonic conditions at an angle of attack of 1 radian are shown. The $M_\infty = 2.0$ case corresponds to a 3.4° subsonic leading edge. A comparison of the numerical result with the exact analytic solution[4] is presented at supersonic speeds.

$$\bar{C}_T(y) = \frac{C_T(y)c}{C_{AVG}} = \alpha^2 \frac{\pi\sqrt{1-m^2}}{[E'(m)]^2}\left[\frac{y}{C_{AVG}}\right] = \frac{1}{2}\alpha^2 AR \frac{\pi\sqrt{1-m^2}}{[E'(m)]^2}\eta$$

where $m = \dfrac{\beta}{T}$ $\qquad \eta = \dfrac{y}{\frac{1}{2}b}$

$$\beta^2 = M_\infty^2 - 1 \qquad T = 2$$

To illustrate the entirely different leading edge suction distribution on a planform with forward sweep, the results for the same planform in reverse flow (referred to as $M_\infty = -0.90$) are presented in the same figure. Although the suction values obtained by the program are qualitatively correct, better agreement is desired. The descrepancies are felt to be due to the problems associated with the pressure distributions obtained with the spanwise linearly varying panel. In most cases the boundary conditions can be satisfied exactly at all control points only at the expense of a considerable degree of instability in the spanwise pressure distribution.

The first attempt to alleviate the spanwise instability in the pressure distribution involved the use of spanwise constraint functions. This method sacrificed the boundary condition at the outer control point. The results, although they were a vast improvement, were still not satisfactory. Thus, additional methods were tried.

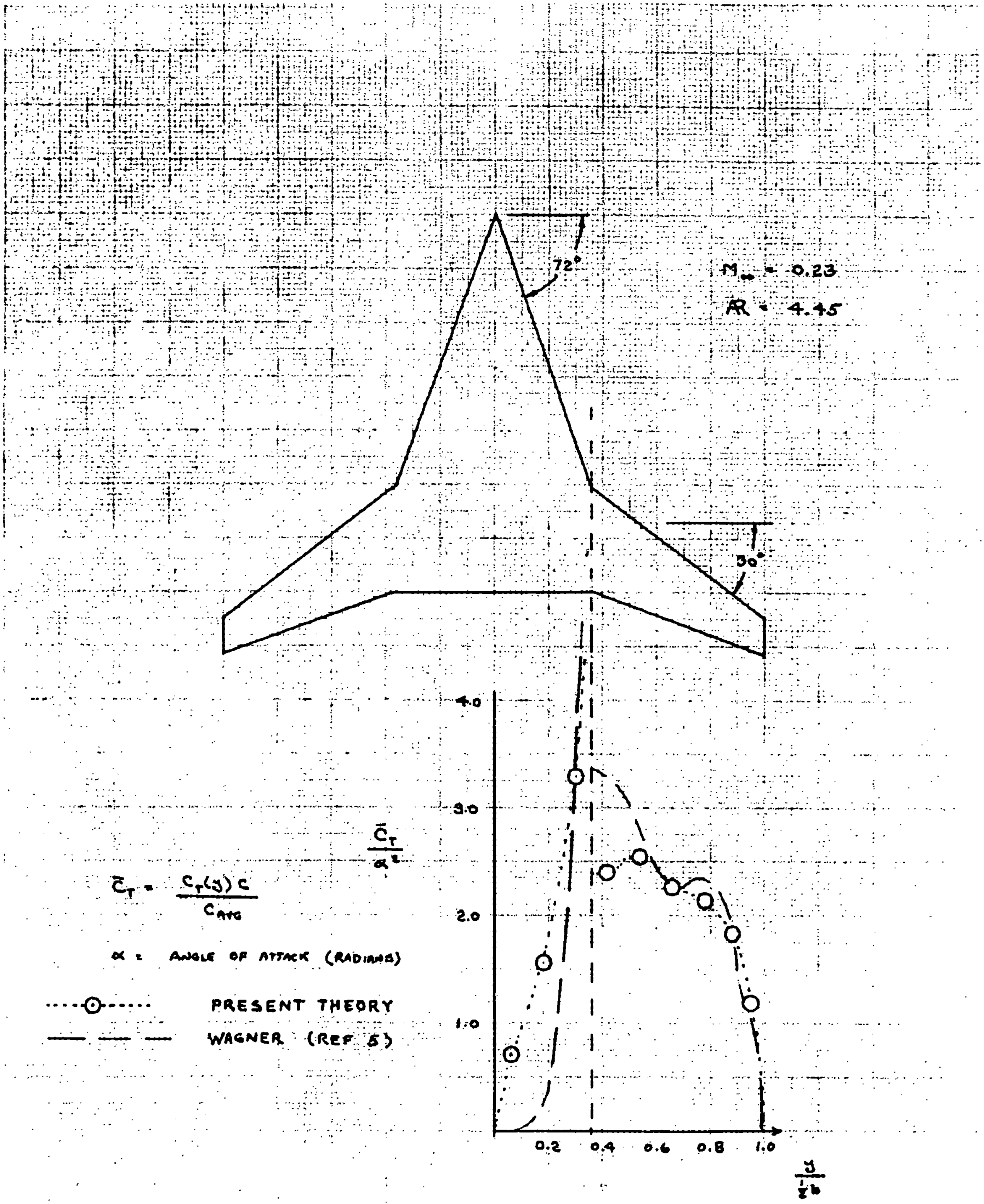


Figure 2. Numerical Calculation of Incompressible Spanwise Variation of Leading Edge Thrust for a High Swept Cranked Wing

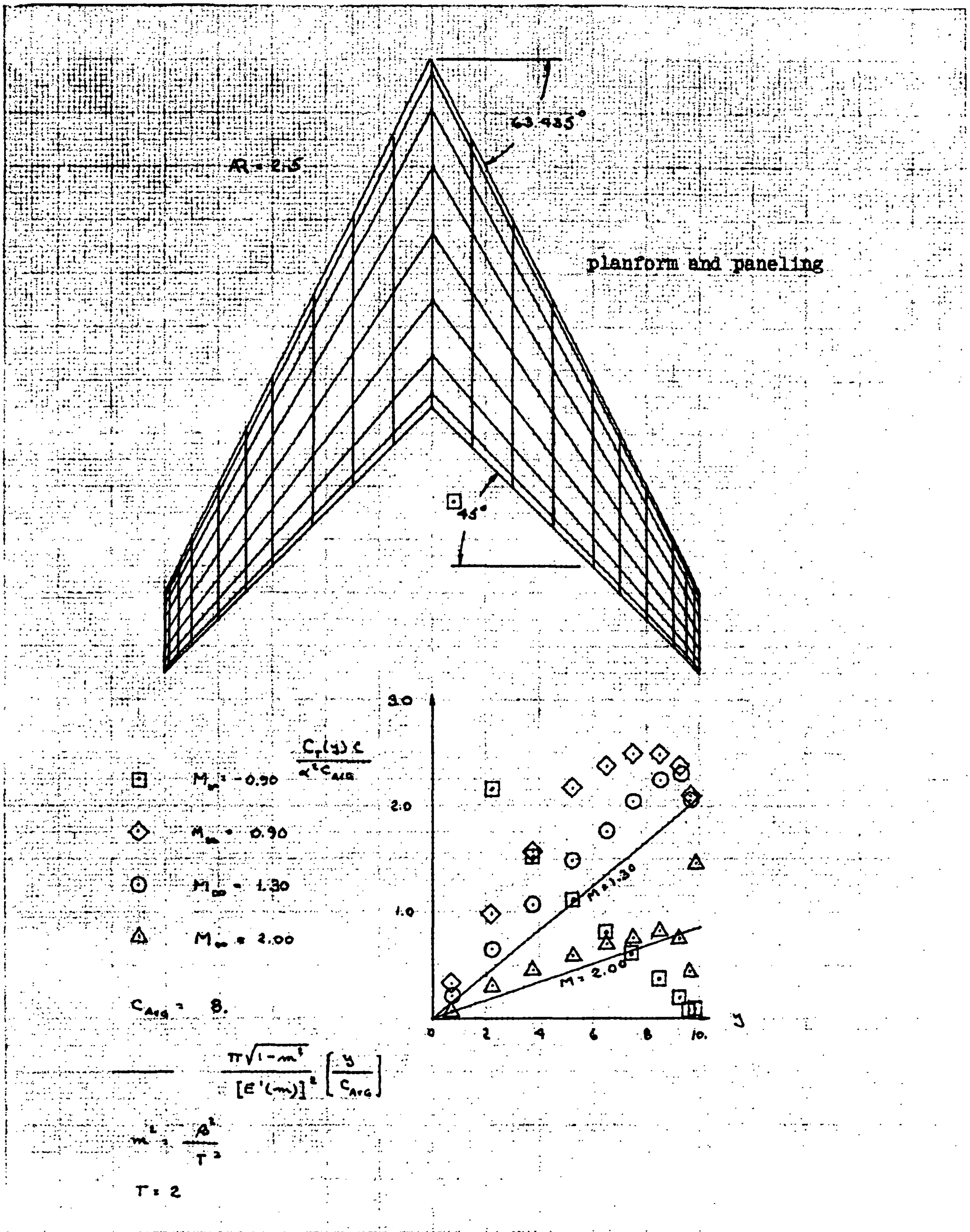


Figure 3. Comparison of Numerical and Analytic Spanwise Variation of Leading Edge Thrust for a Swept Trapezoidal Planform

The first method involves using the pressure distributions obtained from a constant vorticity finite element. Since the calculation of the near field drag requires only a pressure distribution to be given, no theoretical problems arise. The second method uses the linearly varying finite element to obtain the pressure distributions, but places the control point at a small fraction of the spanwise extent of each panel. The pressures are not strongly dependent on the spanwise location and a value of 0.15 was found to give satisfactory results.

There still remain some problems with each method. For planforms with large sweeps near the kink, the constant pressure panel gives pressure distributions which are felt to be too high near the leading edge. This results in negative values of computed leading edge suction in this region. The linearly varying panel, with the control point at 0.15 span, yields unsatisfactory pressure distributions near cranks. This problem can be overcome somewhat by moving the control point nearest the crank to a point near the midpoint of the panel.

At present the program allows the user to select either constant pressure panels or linearly varying panels to obtain the pressure distributions. For the linearly varying panels the fraction of the span for the control point may be specified and this may be overridden at any span station by specifying the spanwise location of the control point. Typical results for each method, when used on the planform shown in figure 3, are presented as the sample cases in Appendix E.

CONCLUSIONS

The technique presented in this report permits for the first time the exact analytical calculation of near field drag at all Mach numbers under the constraints of linearized theory. The technique permits the calculation of an influence matrix with elements representing the drag which one finite element induces on another. The drag may be calculated when the singularity strengths are known. The method is applicable to either source or vortex finite elements and can be used on planforms of arbitrary shape which may be nonplanar in the spanwise direction.

In addition, the method guarantees the equivalence of the drag as calculated in the near and far field, because the form of the influence coefficient shows that the near field drag term changes sign as the influenced and influencing panel are interchanged.

Although the technique involves analytic integrations, a large number of calculations are required. Since the computer code was not completely optimized the computation time required to compute the drag influence matrix is on the order of ten times the time required to compute the aerodynamic influence matrix. However once calculated the matrix may be stored and need not be recalculated if only the twist, camber and thickness distributions are changed.

Additional problems remain with the instability produced in the pressure distributions derived with the spanwise linearly varying lifting element. Although these problems have been largely overcome by relocating the control point or using pressure distributions obtained from constant pressure panels, some problems still remain.

APPENDIX A

TABLE OF DERIVATIVES

$$R^2 = (\xi+\eta)^2 + (B^2-1)(\eta^2+\zeta^2) \qquad B = \text{constant}$$

$$R\frac{\partial}{\partial\xi}R = (\xi+\eta)$$

$$\frac{\partial}{\partial\xi}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} = \frac{1}{R}$$

$$\frac{\partial}{\partial\xi}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} = \frac{\zeta^2-\xi\eta}{\xi^2+B^2\zeta^2}\frac{1}{R}$$

$$\frac{\partial}{\partial\xi}\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} = -\frac{\zeta(\xi+B^2\eta)}{(\xi^2+B^2\zeta^2)}\frac{1}{R}$$

$$R\frac{\partial}{\partial\eta}R = (\xi+B^2\eta)$$

$$\frac{\partial}{\partial\eta}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} = \frac{(\zeta^2-\xi\eta)}{(\eta^2+\zeta^2)}\frac{1}{R}$$

$$\frac{\partial}{\partial\eta}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} = \frac{1}{R}$$

$$\frac{\partial}{\partial\eta}\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} = -\frac{\zeta(\xi+\eta)}{(\eta^2+\zeta^2)}\frac{1}{R}$$

$$R\frac{\partial}{\partial\zeta}R = (B^2-1)\zeta$$

$$\frac{\partial}{\partial\zeta}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} = -\frac{\zeta(\xi+\eta)}{(\eta^2+\zeta^2)}\frac{1}{R}$$

$$\frac{\partial}{\partial\zeta}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} = -\frac{\zeta(\xi+B^2\eta)}{(\xi^2+B^2\zeta^2)}\frac{1}{R}$$

$$\frac{\partial}{\partial\zeta}\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} = \frac{\eta(\xi+\eta)}{(\eta^2+\zeta^2)}\frac{1}{R} + \frac{\xi(\xi+B^2\eta)}{(\xi^2+B^2\zeta^2)}\frac{1}{R}$$

$$T\frac{\partial}{\partial T}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} = \frac{\zeta R}{B(\xi^2+B^2\zeta^2)}$$

$$T\frac{\partial}{\partial T}\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} = \frac{\zeta R}{(\xi^2+B^2\zeta^2)}$$

APPENDIX B

TABLE OF INTEGRALS

$$\xi = x - T y$$

$$\eta = T y$$

$$\zeta = T z$$

$$R^2 = (\xi + \eta)^2 + (B^2 - 1)(\eta^2 + \zeta^2)$$

$$B^2 = \frac{1}{T^2}(T^2 + \beta^2)$$

$$\int \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)}\, d\xi \quad = \quad (\xi+\eta)\, \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)} - R$$

$$\int (\xi+\eta)\, \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)}\, d\xi = \frac{1}{2} \left\{ \left[(\xi+\eta)^2 + \frac{1}{2}(B^2-1)(\eta^2+\zeta^2) \right] \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{1}{2}(\xi+\eta) R \right\}$$

$$\int (\xi+\eta)^2 \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)}\, d\xi = \frac{1}{3} \left\{ (\xi+\eta)^3 \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{1}{3} R^3 + (B^2-1)(\eta^2+\zeta^2) R \right\}$$

$$\int (\xi+\eta)^3 \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)} = \frac{1}{4} \left\{ \left[(\xi+\eta)^4 - \frac{3}{8} (\eta^2+\zeta^2)^2 (B^2-1)^2 \right] \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)} \right.$$

$$\left. - \frac{1}{4}(\xi+\eta) \left[(\xi+\eta)^2 - \frac{3}{2}(B^2-1)(\eta^2+\zeta^2) \right] R \right\}$$

$$\int R\, d\xi \quad = \frac{1}{2} \left\{ (\xi+\eta) R + (B^2-1)(\eta^2+\zeta^2)\, \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)} \right\}$$

$$\int (\xi+\eta) R\, d\xi \quad = \frac{1}{3} R^3$$

$$\int (\xi+\eta)^2 R\, d\xi \quad = \frac{1}{4} \left\{ (\xi+\eta) R^3 - \frac{1}{2}(B^2-1)(\eta^2+\zeta^2) \left[(\xi+\eta) R + (B^2-1)(\eta^2+\zeta^2) \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)} \right] \right\}$$

$$\int (\xi+\eta)^3 R\, d\xi \quad = \frac{1}{5} R^5 - \frac{1}{3}(B^2-1)(\eta^2+\zeta^2) R^3$$

$$\int R^3 d\xi = \frac{1}{4}(\xi+\eta) R^3 + \frac{3}{8}(B^2-1)(\eta^2+\zeta^2)(\xi+\eta) R + \frac{3}{8}(B^2-1)^2 (\eta^2+\zeta^2)^2 \frac{1}{2} \log \frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$\int \frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\,d\xi = \left\{ \eta\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \xi\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} \right\}$$

$$\int (\xi+\eta)\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\,d\xi = \frac{1}{2}\left\{ (\eta^2-\zeta^2)\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} \right.$$

$$\left. + \left[(\xi^2+B^2\zeta^2)+2\xi\eta\right]\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + 2\eta\zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \eta R \right\}$$

$$\int (\xi+\eta)^2\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\,d\xi = \frac{1}{3}\left\{ \eta\left[B^2\eta^2-3\zeta^2-\frac{3}{2}(B^2-1)(\eta^2+\zeta^2)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} \right.$$

$$+ \left[(\xi+\eta)^3-\eta^3+3B^2\eta\zeta^2\right]\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$\left. + \zeta(3\eta^2-B^2\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \left[\frac{1}{2}\eta(\xi+\eta)-(\eta^2+\zeta^2)+2\eta^2\right]R \right\}$$

$$\int (\xi+\eta)^3\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\,d\xi$$

$$= \left\{ \left[\frac{3}{8}(B^2-1)(\eta^2+\zeta^2)+\frac{1}{4}(\zeta^2-\eta^2)-\frac{3}{2}(B^2-1)\eta^2-\frac{3}{2}\eta^2\right](\eta^2+\zeta^2) + \left[(B^2-1)+2\right]\eta^4 \right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$+ \left\{ \left[\frac{1}{4}(\xi^2-B^2\zeta^2)+\eta\left(\xi+\frac{3}{2}\eta\right)\right](\xi^2+B^2\zeta^2) + \xi\eta(\eta^2-B^2\zeta^2) \right\}\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$- \left\{ \eta\zeta^2-\frac{1}{4}\eta(\eta^2+\zeta^2)+\frac{1}{8}(\xi+\eta)(\zeta^2-\eta^2)+\frac{1}{4}(B^2-1)\eta(\eta^2+2\zeta^2) \right\}R + \frac{1}{12}\eta R^3$$

$$+ \eta\zeta(\eta^2-B^2\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

$$\int \tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}\,d\xi = \left\{ \zeta\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - B^2\zeta\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \xi\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} \right\}$$

$$\int(\xi+\eta)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}\,d\xi = \frac{1}{2}\left\{ \eta\zeta(B^2+1)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - 2B^2\eta\zeta\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} \right.$$

$$\left. + (\xi^2+2\xi\eta+B^2\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \zeta R \right\}$$

$$\int(\xi+\eta)^2\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}\,d\xi = \frac{1}{3}\left\{ \zeta\left[3B^2\eta^2-\zeta^2-\frac{3}{2}(B^2-1)(\eta^2+\zeta^2)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} \right.$$

$$+ B^2\zeta(B^2\zeta^2-3\eta^2)\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \zeta\left[\frac{1}{2}(\xi+\eta)+(B^2+1)\eta\right]R$$

$$\left. + \left[(\xi+\eta)^3+\eta(3B^2\zeta^2-\eta^2)\right]\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} \right\}$$

$$\int(\xi+\eta)^3\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}\,d\xi =$$

$$\left\{ (\eta^2-B^2\zeta^2) - \frac{1}{4}B^2(B^2-1)(\eta^2+\zeta^2) + \frac{1}{4}(B^2-1)(3\eta^2-B^2\zeta^2) \right\}\eta\zeta\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$+ B^2\zeta\eta(B^2\zeta^2-\eta^2)\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$+ \left\{ \frac{1}{4}(\xi+\eta)^4 + \frac{1}{4}(\eta^2+B^2\zeta^2)(3\eta^2-B^2\zeta^2) + \eta^2(B^2\zeta^2-\eta^2) \right\}\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

$$+ \left\{ \eta(\xi+\eta)\left[1+\frac{1}{2}(B^2-1)\right] + (B^2-1)(\eta^2-\zeta^2) + (3\eta^2-B^2\zeta^2) \right\}\zeta R + \frac{1}{3}R^3$$

APPENDIX C

INTEGRALS FOR THE CALCULATION OF MUTUAL INTERFERENCE DRAG

All of the quantities required to compute the induced drag are composed of integrals and derivatives of the basic velocity components induced by constant source or constant vortex panels. On page 11 set of velocities was defined such that :

$$\omega_{ij}(\xi, \eta, \zeta, B) = \frac{\partial}{\partial \xi}\, \omega_{i+1,j}(\xi, \eta, \zeta, B)$$

$$\omega_{ij}(\xi, \eta, \zeta, B) = \frac{\partial}{\partial \eta}\, \omega_{i,j+1}(\xi, \eta, \zeta, B)$$

Only 5 basic terms occur in the expressions for the velocity components for the non-planar source and vortex panels. The functions will be denoted by f_{oo}^{i} where,

$$f_{oo}^{1} = \frac{1}{2} \log \frac{R + (\xi + \eta)}{R - (\xi + \eta)}$$

$$f_{oo}^{2} = \frac{1}{B}\,\frac{1}{2} \log \frac{BR + (\xi + B^2\eta)}{BR - (\xi + B^2\eta)}$$

$$f_{oo}^{3} = \frac{\eta R}{\eta^2 + \zeta^2}$$

$$f_{oo}^{4} = \tan^{-1} \frac{\zeta R}{\xi\eta - \zeta^2}$$

$$f_{oo}^{5} = \frac{\zeta R}{\eta^2 + \zeta^2}$$

with

$$R^2 = x^2 + \beta^2 (y^2 + z^2)$$

$$B^2 = \frac{1}{T^2}(T^2 + \beta^2)$$

$$\xi = x - Ty$$

$$\eta = Ty$$

$$\zeta = Tz$$

All of the expressions required for the calculation of drag are integrals of these functions. If we define

$$f_{ij}^{k} = f_{ij}^{k}(\xi, \eta, \zeta, B)$$

$$\frac{\partial}{\partial \xi} f_{ij}^{k} = f_{i-1,j}^{k}$$

$$\frac{\partial}{\partial \eta} f_{ij}^{k} = f_{i,j-1}^{k}$$

then all of the expressions which are required are such that $i+j \leq 4$

The integrals and derivatives were computed analytically and the results are presented in this Appendix. These analytical results were checked by performing the differentiations numerically in a computer program. Thus the velocity components corresponding to a vortex panel could be constructed from:

$$u_{ij} = \frac{k}{8\pi} f_{ij}^{4}$$

$$v_{ij} = -\frac{kT}{8\pi} \left\{ f_{ij}^{4} + f_{ij}^{5} \right\}$$

$$w_{ij} = \frac{kT}{8\pi} \left\{ f_{ij}^{1} - B^2 f_{ij}^{2} + f_{ij}^{3} \right\}$$

Using the derivatives with respect to T, given in Appendix A, we can show that

$$\frac{\partial}{\partial T} T^{-j} w_{ij} = -(j+1)\, w_{i-1,j+1}$$

$$\frac{\partial}{\partial T} T^{-j} v_{ij} = -(j+1)\, v_{i-1,j+1}$$

$$\frac{\partial}{\partial T} T^{-j} u_{ij} = -(j+1)\, u_{i-1,j+1}$$

These relations were all verified numerically.

$$\frac{\partial}{\partial \xi}\left[\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}\right]_{01} = \left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right]_{00}$$

$$\frac{\partial}{\partial \xi}\left[\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\right]_{10} = \left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right]_{00}$$

$$\left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right]_{10} = -\left[\frac{\xi R}{\eta^2+\xi^2}\right]_{01}$$

$$\frac{\partial}{\partial \xi}\left[\frac{\eta R}{\eta^2+\xi^2}\right]_{01} = \left[\frac{\xi R}{\eta^2+\xi^2}\right]_{00} + \left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right]_{00}$$

$$\frac{\partial}{\partial \xi}\left[\frac{\eta R}{\eta^2+\xi^2}\right]_{02} = \left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right]_{01} - \left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right]_{10}$$

$$\left[\frac{\eta R}{\eta^2+\xi^2}\right]_{01} = \left[\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}\right]_{01} - \left[\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}\right]_{10}$$

$$\left[\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}\right]_{01} = \left[\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\right]_{10}$$

$$\left[\frac{\eta R}{\eta^2+\xi^2}\right]_{01} = \xi\,\frac{\partial}{\partial \xi}\left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right]_{01} + \xi\left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right]_{00} - B^2\left[\frac{\partial}{\partial \xi}\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\right]_{01}$$

$$\frac{\partial}{\partial \xi}\left[\tan^{-1}\frac{\xi R}{\xi\eta-\xi^2}\right] = \left[\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}\right]_{00} - B^2\left[\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\right]_{00} + \left[\frac{\eta R}{\eta^2+\xi^2}\right]_{00}$$

INTEGRALS OF $f_{00}^{(1)} = \frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$

00 $$\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

01 $$\eta\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \xi\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

10 $$(\xi+\eta)\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - R$$

11 $$\left[\xi\eta+\frac{1}{2}(\eta^2+\zeta^2)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{2}(\xi^2-B^2\zeta^2)\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\ \xi\zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - \frac{1}{2}\eta R$$

20 $$\frac{1}{2}\left[(\xi+\eta)^2-\frac{1}{2}(B^2-1)(\eta^2+\zeta^2)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{3}{4}(\xi+\eta)R$$

30 $$\left[\frac{1}{6}(\xi+\eta)^3-\frac{1}{4}(B^2-1)(\eta^2+\zeta^2)(\xi+\eta)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{5}{12}(B^2-1)(\eta^2+\zeta^2)R - \frac{11}{36}R^3$$

40 $$\left[\frac{1}{24}(\xi+\eta)^4-\frac{1}{8}(B^2-1)(\eta^2+\zeta^2)(\xi+\eta)^2+\frac{1}{64}(B^2-1)^2(\eta^2+\zeta^2)^2\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\ \frac{35}{192}(B^2-1)(\eta^2+\zeta^2)(\xi+\eta)R - \frac{25}{288}(\xi+\eta)R^3$$

21 $$\frac{1}{2}\left\{\frac{1}{3}\left[1-\frac{1}{2}(B^2-1)\right]\eta^3+\xi\eta^2+\left[\xi^2-\frac{1}{2}(B^2-1)\zeta^2\right]\eta + \xi\zeta^2\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\ \frac{1}{2}\zeta\left(\xi^2-\frac{1}{3}B^2\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \frac{1}{2}\left(\frac{1}{3}\xi^3-B^2\zeta^2\xi\right)\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\left\{-\frac{5}{12}\eta(\xi+\eta)+\frac{1}{6}(\eta^2-\zeta^2)\right\}R$$

02 $$\frac{1}{2}(\eta^2-\zeta^2)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \left\{\frac{1}{2}\frac{1}{B^2}(\xi^2+B^2\zeta^2)+\xi\eta\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\eta\zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - \frac{1}{2}\frac{1}{B^2}\xi R$$

03 $$\left(\frac{1}{6}\eta^3-\frac{1}{2}\eta\zeta^2\right)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{2}\left(\eta^2-\frac{1}{3}\zeta^2\right)\zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\frac{1}{2}\left\{\xi\eta^2+\frac{1}{B^2}(\xi^2+B^2\zeta^2)\eta+\frac{1}{3}\frac{1}{B^4}\left[1-\frac{1}{2}(B^2-1)\right]\xi^3-\frac{1}{2}\frac{1}{B^2}(B^2-1)\zeta^2\xi\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\frac{1}{12}\frac{1}{B^2}\left[5\xi\eta+2\zeta^2+3\frac{1}{B^2}\xi^2\right]R$$

12 $$\left[\frac{1}{6}\eta^3+\frac{1}{2}\xi\eta^2+\frac{1}{2}\zeta^2\eta-\frac{1}{2}\xi\zeta^2\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\left[\frac{1}{6}\frac{1}{B^2}\xi^3+\frac{1}{2}\xi^2\eta+\frac{1}{2}\zeta^2\xi-\frac{1}{2}B^2\zeta^2\eta\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\zeta\left(\xi\eta+\frac{1}{3}\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \frac{1}{B^2}\left[\frac{1}{2}B^2\zeta^2-\frac{1}{6}\xi^2+\frac{1}{3}\xi\eta\right]R - \frac{1}{6}\frac{1}{B^2}R^3$$

13 $$\left[\frac{1}{24}\eta^4+\frac{1}{6}\xi\eta^3+\frac{1}{4}\zeta^2\eta^2-\frac{1}{2}\xi\zeta^2\eta-\frac{1}{8}\zeta^4\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\left\{\frac{1}{24}\frac{1}{B^4}\left[1-\frac{1}{2}(B^2-1)\right]\xi^4+\frac{1}{6}\frac{1}{B^2}\eta\xi^3+\frac{1}{4}\left[\eta^2-\frac{1}{2}\frac{1}{B^2}(B^2-1)\zeta^2\right]\xi^2+\frac{1}{2}\eta\zeta^2\xi+\zeta^2\left[\frac{1}{8}B^2(\zeta^2-2\eta^2)-\frac{1}{16}(B^2-1)\zeta^2\right]\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\zeta\left(\frac{1}{2}\xi\eta^2+\frac{1}{3}\zeta^2\eta-\frac{1}{6}\xi\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\Big[(\xi+B^2\eta)\Big\{18B^2\zeta^2-2(\xi^2+R^2)-\frac{1}{B^2}\left[(11\xi^2+5B^2\zeta^2)+3(B^2-1)(\xi^2+B^2\zeta^2)\right]\Big\}$$
$$+\xi\Big\{2\frac{1}{B^2}\left[(3\xi^2+B^2\zeta^2)-(B^2-1)(\xi^2+2B^2\zeta^2)\right]-20B^2\zeta^2+6R^2\Big\}\Big]\frac{1}{48}\frac{1}{B^4}R$$

22 $$\Big\{ \tfrac{1}{24}\left[1-\tfrac{1}{2}(B^2-1)\right]\eta^4 + \tfrac{1}{6}\xi\eta^3 + \tfrac{1}{4}\left[\xi^2 - \tfrac{1}{2}(B^2-1)\zeta^2\right]\eta^2 + \tfrac{1}{2}\xi\eta\zeta^2$$
$$+ \zeta^2\left[\tfrac{1}{8}(B^2\zeta^2 - 2\xi^2) - \tfrac{1}{16}(B^2-1)\zeta^2\right]\Big\} \frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+ \left[\tfrac{1}{24}\tfrac{1}{B^2}\xi^4 + \tfrac{1}{6}\xi^3\eta + \tfrac{1}{4}\zeta^2\xi^2 - \tfrac{1}{2}B^2\zeta^2\xi\eta - \tfrac{1}{8}B^2\zeta^4\right]\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+ \zeta\left(\tfrac{1}{2}\xi^2\eta + \tfrac{1}{3}\zeta^2\xi - \tfrac{1}{6}B^2\zeta^2\eta\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+ (\xi+\eta)\Big\{\left(\tfrac{3}{8}\zeta^2 - \tfrac{1}{24}\eta^2\right) - \tfrac{1}{B^2}\left(\tfrac{11}{48}\eta^2 + \tfrac{5}{48}\zeta^2\right) - \tfrac{1}{16}\tfrac{1}{B^2}(B^2-1)(\eta^2+\zeta^2) - \tfrac{1}{24}\tfrac{1}{B^2}R^2\Big\}R$$
$$+ \eta\Big\{-\tfrac{5}{12}\zeta^2 + \tfrac{1}{24}\tfrac{1}{B^2}\left[(3\eta^2+\zeta^2) - (B^2-1)(\eta^2+2\zeta^2)\right] + \tfrac{1}{8}\tfrac{1}{B^2}R^2\Big\}R$$

31 $$\frac{1}{2}\Big\{\tfrac{1}{4}\left[\tfrac{1}{3} - \tfrac{1}{2}(B^2-1)\right]\eta^4 + \tfrac{1}{3}\left[1-\tfrac{1}{2}(B^2-1)\right]\xi\eta^3 + \tfrac{1}{2}\left[\xi^2 - \tfrac{1}{2}(B^2-1)\zeta^2\right]\eta^2 + \left[\tfrac{1}{3}\xi^2 - \tfrac{1}{2}(B^2-1)\zeta^2\right]\xi\eta$$
$$+ \zeta^2\left[\tfrac{1}{2}(\xi^2 - \tfrac{1}{6}\zeta^2) - \tfrac{1}{8}(B^2-1)\zeta^2\right]\Big\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{4}\left(\tfrac{1}{6}\xi^4 - B^2\zeta^2\xi^2 + \tfrac{1}{6}B^4\zeta^4\right)\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+ \tfrac{1}{6}\zeta\xi(\xi^2 - B^2\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - \Big\{\tfrac{7}{48}\xi\zeta^2 + \eta\left[\tfrac{1}{6}\xi^2 + \tfrac{3}{16}\xi\eta + \tfrac{1}{48}(3\eta^2+\zeta^2) - \tfrac{1}{24}(B^2-1)(\eta^2+2\zeta^2) + \tfrac{1}{72}R^2\right]\Big\}R$$

04 $$\frac{1}{4}\left[\tfrac{1}{6}\eta^4 - \zeta^2\eta^2 + \tfrac{1}{6}\zeta^4\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \tfrac{1}{6}\zeta\eta(\eta^2-\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+ \frac{1}{2}\frac{1}{B^2}\Big\{\tfrac{1}{4}\tfrac{1}{B^4}\left[\tfrac{1}{3} - \tfrac{1}{2}(B^2-1)\right]\xi^4 + \tfrac{1}{3}\tfrac{1}{B^2}\left[1 - \tfrac{1}{2}(B^2-1)\right]\eta\xi^3 + \tfrac{1}{2}\tfrac{1}{B^2}\left[B^2\eta^2 - \tfrac{1}{2}(B^2-1)\zeta^2\right]\xi^2$$
$$+ \left[\tfrac{1}{3}B^2\eta^2 - \tfrac{1}{2}(B^2-1)\zeta^2\right]\xi\eta + \zeta^2\left[\tfrac{1}{2}(B^2\eta^2 - \tfrac{1}{6}\zeta^2) - \tfrac{1}{8}(B^2-1)\zeta^2\right]\Big\}\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$- \frac{1}{B^4}\Big\{\tfrac{7}{48}B^2\eta\zeta^2 + \xi\left[\tfrac{1}{6}B^2\eta^2 + \tfrac{3}{16}\xi\eta + \tfrac{1}{48}\tfrac{1}{B^2}(3\xi^2 + B^2\zeta^2) - \tfrac{1}{24}\tfrac{1}{B^2}(B^2-1)(\xi^2 + 2B^2) + \tfrac{1}{72}R^2\right]\Big\}R$$

INTEGRALS OF $f^{(2)}_{00} = \frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$

00 $\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$

01 $\frac{1}{B^2}(\xi+B^2\eta)\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} - \frac{1}{B^2}R$

02 $\frac{1}{2}\frac{1}{B^4}\left[(\xi+B^2\eta)^2 - \frac{1}{2}(B^2-1)(\xi^2+B^2\zeta^2)\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} - \frac{3}{4}\frac{1}{B^4}(\xi+B^2\eta)R$

03 $\frac{1}{B^6}\left\{\left[\frac{1}{6}(\xi+B^2\eta)^3 - \frac{1}{4}(B^2-1)(\xi^2+B^2\zeta^2)(\xi+B^2\eta)\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \frac{5}{12}(B^2-1)(\xi^2+B^2\zeta^2)R - \frac{11}{36}B^2R^3\right\}$

04 $\frac{1}{B^8}\left\{\left[\frac{1}{24}(\xi+B^2\eta)^4 - \frac{1}{8}(B^2-1)(\xi^2+B^2\zeta^2)(\xi+B^2\eta)^2 + \frac{1}{64}(B^2-1)^2(\xi^2+B^2\zeta^2)^2\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}\right.$

$\left. + \frac{35}{192}(B^2-1)(\xi^2+B^2\zeta^2)(\xi+B^2\eta)R - \frac{25}{288}B^2(\xi+B^2\eta)R^3\right\}$

10 $\eta\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \xi\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$

20 $\left[\xi\eta + \frac{1}{2}(\eta^2+\zeta^2)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{2}(\xi^2-B^2\zeta^2)\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$

$+ \xi\zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - \frac{1}{2}\eta R$

30 $\frac{1}{2}\left\{\eta\xi^2 + (\eta^2+\zeta^2)\xi + \frac{1}{3}\left[1-\frac{1}{2}(B^2-1)\right]\eta^3 - \frac{1}{2}(B^2-1)\zeta^2\eta\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$

$+\left[\frac{1}{6}\xi^3 - \frac{1}{2}B^2\zeta^2\xi\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \frac{1}{2}\zeta(\xi^2 - \frac{1}{3}B^2\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - \frac{1}{12}\left[5\zeta\eta + 2\zeta^2 + 3\eta^2\right]R$

11 $$\frac{1}{2}(\eta^2-\zeta^2)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \left[\xi\eta + \frac{1}{2}\frac{1}{B^2}(\xi^2+B^2\zeta^2)\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\eta\zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - \frac{1}{2}\frac{1}{B^2}\xi R$$

12 $$\left(\frac{1}{6}\eta^3-\frac{1}{2}\eta\zeta^2\right)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{2}\left(\eta^2-\frac{1}{3}\zeta^2\right)\zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\frac{1}{2}\left\{\xi\eta^2+\frac{1}{B^2}(\xi^2+B^2\zeta^2)\eta+\frac{1}{3}\frac{1}{B^4}\left[1-\frac{1}{2}(B^2-1)\right]\xi^3-\frac{1}{2}\frac{1}{B^2}(B^2-1)\zeta^2\xi\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\frac{1}{12}\frac{1}{B^2}\left[5\xi\eta+2\zeta^2+3\frac{1}{B^2}\xi^2\right]R$$

13 $$\frac{1}{4}\left[\frac{1}{6}\eta^4-\zeta^2\eta^2+\frac{1}{6}\zeta^4\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{6}\zeta\eta(\eta^2-\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\frac{1}{2}\frac{1}{B^2}\left\{\frac{1}{4}\frac{1}{B^4}\left[\frac{1}{3}-\frac{1}{2}(B^2-1)\right]\xi^4+\frac{1}{3}\frac{1}{B^2}\left[1-\frac{1}{2}(B^2-1)\right]\eta\xi^3+\frac{1}{2}\frac{1}{B^2}\left[B^2\eta^2-\frac{1}{2}(B^2-1)\zeta^2\right]\xi^2\right.$$
$$\left.+\left[\frac{1}{3}B^2\eta^2-\frac{1}{2}(B^2-1)\zeta^2\right]\xi\eta + \zeta^2\left[\frac{1}{2}\left(B^2\eta^2-\frac{1}{6}\zeta^2\right)-\frac{1}{8}(B^2-1)\zeta^2\right]\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\frac{1}{B^4}\left\{\frac{7}{48}B^2\eta\zeta^2+\xi\left[\frac{1}{6}B^2\eta^2+\frac{3}{16}\xi\eta+\frac{1}{48}\frac{1}{B^2}(3\xi^2+B^2\zeta^2)-\frac{1}{24}\frac{1}{B^2}(B^2-1)(\zeta^2+2B^2)+\frac{1}{72}R^2\right]\right\}R$$

21 $$\left[\frac{1}{2}(\eta^2-\zeta^2)\xi+\frac{1}{2}\zeta^2\eta+\frac{1}{6}\eta^3\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \zeta\left(\xi\eta+\frac{1}{3}\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\left[\frac{1}{2}\frac{1}{B^2}\left(\frac{1}{3}\xi^3+B^2\zeta^2\xi\right)+\frac{1}{2}\xi^2\eta-\frac{1}{2}B^2\zeta^2\eta\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$\frac{1}{B^2}\left[\frac{1}{2}B^2\zeta^2-\frac{1}{6}\zeta^2-\frac{1}{3}\xi\eta\right]R-\frac{1}{6}\frac{1}{B^2}R^3$$

INTEGRALS OF $f^{(2)}_{00} = \frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$ (Concluded)

31 $$\left\{\frac{1}{4}(\eta^2-\zeta^2)\xi^2 + \left[\frac{1}{2}\zeta^2\eta+\frac{1}{6}\eta^3\right]\xi - \frac{1}{16}(B^2-1)(\eta^2+\zeta^2)^2 + \frac{1}{24}B^2(\eta^2+3\zeta^2)\zeta^2\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\left[\frac{1}{24}\frac{1}{B^2}\xi^4 + \frac{1}{6}\eta\xi^3 + \frac{1}{4}\zeta^2\xi^2 - \frac{1}{2}B^2\zeta^2\xi\eta - \frac{1}{8}\zeta^4\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\zeta\left(\frac{1}{2}\eta\xi^2+\frac{1}{3}\zeta^2\xi-\frac{1}{6}\eta\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

40 $$\frac{1}{2}\left\{\frac{1}{4}\left[\frac{1}{3}-\frac{1}{2}(B^2-1)\right]\eta^4+\frac{1}{3}\left[1-\frac{1}{2}(B^2-1)\right]\xi\eta^3+\frac{1}{2}\left[\xi^2-\frac{1}{2}(B^2-1)\zeta^2\right]\eta^2+\left[\frac{1}{3}\xi^2-\frac{1}{2}(B^2-1)\zeta^2\right]\xi\eta\right.$$
$$\left.+\zeta^2\left[\frac{1}{2}\left(\xi^2-\frac{1}{6}\zeta^2\right)-\frac{1}{8}(B^2-1)\zeta^2\right]\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{4}\left(\frac{1}{6}\xi^4-B^2\zeta^2\xi^2+\frac{1}{6}B^4\zeta^4\right)\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\frac{1}{6}\zeta\xi(\xi^2-B^2\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - \left\{\frac{7}{48}\xi\zeta^2+\eta\left[\frac{1}{6}\xi^2+\frac{3}{16}\xi\eta+\frac{1}{48}(3\eta^2+\zeta^2)-\frac{1}{24}(B^2-1)(\eta^2+2\zeta^2)+\frac{1}{72}R^2\right]\right\}R$$

22 $$\left[\frac{1}{24}\eta^4+\frac{1}{6}\xi\eta^3+\frac{1}{4}\zeta^2\eta^2-\frac{1}{2}\xi\zeta^2\eta-\frac{1}{8}\zeta^4\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\left\{\frac{1}{24}\frac{1}{B^4}\left[1-\frac{1}{2}(B^2-1)\right]\xi^4+\frac{1}{6}\frac{1}{B^2}\eta\xi^3+\frac{1}{4}\left[\eta^2-\frac{1}{2}\frac{1}{B^2}(B^2-1)\zeta^2\right]\xi^2+\frac{1}{2}\eta\zeta^2\xi+\zeta^2\left[\frac{1}{8}B^2(\zeta^2-2\eta^2)-\frac{1}{16}(B^2-1)\zeta^2\right]\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\zeta\left(\frac{1}{2}\xi\eta^2+\frac{1}{3}\zeta^2\eta-\frac{1}{6}\xi\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\left[(\xi+B^2\eta)\left\{18B^2\zeta^2-2(\xi^2+R^2)-\frac{1}{B^2}\left[(11\xi^2+5B^2\zeta^2)+3(B^2-1)(\xi^2+B^2\zeta^2)\right]\right\}\right.$$
$$\left.+\xi\left\{2\frac{1}{B^2}\left[(3\xi^2+B^2\zeta^2)-(B^2-1)(\xi^2+2B^2\zeta^2)\right]-20B^2\zeta^2+6R^2\right\}\right]\frac{1}{48}\frac{1}{B^4}R$$

INTEGRALS OF $f_{00}^{(3)} = \frac{\eta R}{(\eta^2+\zeta^2)}$

00 $$\frac{\eta R}{(\eta^2+\zeta^2)}$$

10 $$\frac{1}{2}(B^2-1)\eta \; \frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{2}\,\frac{\eta(\xi+\eta)}{(\eta^2+\zeta^2)}\,R$$

20 $$\frac{1}{2}(B^2-1)\eta(\xi+\eta)\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{1}{2}(B^2-1)\eta R + \frac{1}{6}\,\frac{\eta}{\eta^2+\zeta^2}\,R^3$$

30 $$\frac{1}{4}(B^2-1)\eta\left[(\xi+\eta)^2-\frac{1}{4}(B^2-1)(\eta^2+\zeta^2)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{5}{16}(B^2-1)\eta(\xi+\eta)R + \frac{1}{24}\,\frac{\eta(\xi+\eta)}{\eta^2+\zeta^2}\,R^3$$

40 $$\frac{1}{4}(B^2-1)\eta\left[\frac{1}{3}(\xi+\eta)^3-\frac{1}{4}(B^2-1)(\eta^2+\zeta^2)(\xi+\eta)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{5}{48}(B^2-1)\eta R^3 + \frac{1}{120}\,\frac{\eta}{\eta^2+\zeta^2}\,R^5$$

01 $$-\xi\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \xi\,\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \zeta\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + R$$

11 $$\left[-\frac{1}{2}\xi^2+\frac{1}{4}(B^2-1)(\eta^2+\zeta^2)+\frac{1}{2}\zeta^2\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \zeta\xi\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\frac{1}{2}(\xi^2-B^2\zeta^2)\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \frac{1}{4}(3\xi+\eta)R$$

12 $$\left\{\frac{1}{12}(B^2-1)\eta^3+\frac{1}{2}\left[\frac{1}{2}(B^2-1)\zeta^2+\zeta^2-\xi^2\right]\eta-\xi\zeta^2\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\left\{\frac{1}{2}(\xi^2-B^2\zeta^2)\eta-\frac{1}{6}\,\frac{1}{B^2}(B^2-1)\xi^3+\frac{1}{2}(B^2+1)\zeta^2\xi\right\}\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\zeta\left[\xi\eta-\frac{1}{2}\xi^2+\frac{1}{3}\left(\frac{1}{2}B^2+1\right)\zeta^2\right]\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \left\{\frac{1}{12}\eta^2+\frac{5}{12}\xi\eta-\frac{1}{6}\,\frac{1}{B^2}\xi^2+\frac{1}{2}\zeta^2\right\}R$$

INTEGRALS OF $\quad f_{oo}^{(3)} = \dfrac{\eta R}{\eta^2+\zeta^2}$ (Continued)

13 $$\left\{ \frac{1}{48}(B^2-1)\eta^4 + \frac{1}{4}\left[\frac{1}{2}(B^2-1)\zeta^2 + \zeta^2 - \xi^2\right]\eta^2 - \zeta^2\xi\eta - \frac{1}{4}\zeta^2\left[\frac{1}{4}(B^2-1)\zeta^2 + \zeta^2 - \zeta^2\right]\right\} \frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$+ \left\{ \frac{1}{4}(\xi^2 - B^2\zeta^2)\eta^2 + \left[\frac{1}{2}(B^2+1)\zeta^2 - \frac{1}{6}\frac{1}{B^2}(B^2-1)\xi^2\right]\xi\eta - \frac{1}{16}\frac{1}{B^4}(B^2-1)(\xi^2+B^2\zeta^2)^2 - \frac{1}{4}\zeta^2(\xi^2-B^2\zeta^2)\right\} \frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$+ \zeta\left\{ \frac{1}{2}\xi\eta^2 + \left[\frac{1}{3}\left(\frac{1}{2}B^2+1\right)\zeta^2 - \frac{1}{2}\xi^2\right]\eta - \frac{1}{2}\zeta^2\xi \right\} \tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

02 $$-(\xi\eta+\zeta^2)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \left[\xi\eta - \frac{1}{2}\frac{1}{B^2}(B^2-1)(\xi^2+B^2\zeta^2) + B^2\zeta^2\right] \frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$+ \zeta(\eta-\xi)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \frac{1}{2}\left(\eta - \frac{1}{B^2}\xi\right)R$$

21 $$\left\{ -\frac{1}{6}\xi^3 + \frac{1}{2}\left[\frac{1}{2}(B^2-1)(\eta^2+\zeta^2) + \zeta^2\right]\xi + \frac{1}{6}(B^2-1)\eta^3 \right\} \frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$+ \frac{1}{2}\left(\frac{1}{3}\xi^3 - B^2\zeta^2\xi\right)\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \frac{1}{2}\zeta\left(\xi^2 - \frac{1}{3}B^2\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

$$- \left\{ \frac{5}{12}\left[(B^2-1)(\eta^2+\zeta^2) + \eta(\xi+\eta)\right] - \frac{1}{6}(\eta^2-\zeta^2) \right\} R + \frac{11}{36}R^3$$

31 $$\left\{ -\frac{1}{24}\xi^4 + \frac{1}{4}\left[\frac{1}{2}(B^2-1)(\eta^2+\zeta^2)+\zeta^2\right]\xi^2 + \frac{1}{6}(B^2-1)\eta^3\xi + \frac{1}{4}(B^2-1)\left[\frac{1}{4}\eta^4 - \frac{1}{4}\zeta^4 - \frac{1}{16}(B^2-1)(\eta^2+\zeta^2)^2\right] - \frac{1}{24}\zeta^4 \right\} \frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$+ \frac{1}{4}\left(\frac{1}{6}\xi^4 - B^2\zeta^2\xi^2 + \frac{1}{6}B^4\zeta^4\right)\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \frac{1}{6}\zeta\xi(\xi^2-B^2\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

03
$$-\left(\tfrac{1}{2}\xi\eta^2+\zeta^2\eta-\tfrac{1}{2}\xi\zeta^2\right)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}+\zeta\left(\tfrac{1}{2}\eta^2-\xi\eta-\tfrac{1}{2}\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\left\{\frac{1}{2}\xi\eta^2-\frac{1}{2}\,\frac{1}{B^2}(B^2-1)(\xi^2+B^2\zeta^2)\eta+B^2\zeta^2\eta-\frac{1}{2}\xi\left[\frac{1}{2}\,\frac{1}{B^4}(B^2-1)(\xi^2+B^2\zeta^2)+\zeta^2\right]\right\}\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\frac{1}{4}\left[2\zeta^2+3\frac{1}{B^2}\xi\eta+\frac{1}{B^4}\xi^2\right]R+\frac{1}{6}\,\frac{1}{B^2}R^3$$

22
$$\left\{\frac{1}{24}(B^2-1)\eta^4+\frac{1}{12}(B^2-1)\xi\eta^3-\frac{1}{6}\xi^3\eta+\frac{1}{2}\left[1+\frac{1}{2}(B^2-1)\right]\xi\eta\zeta^2\right.$$
$$\left.+\zeta^2\left[\frac{1}{24}(3B^2+1)\zeta^2-\frac{1}{2}\xi^2\right]\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$-\left[\frac{1}{24}\,\frac{1}{B^2}(B^2-1)\xi^4-\frac{1}{6}\xi^3\eta-\frac{1}{4}(B^2+1)\zeta^2\xi^2+\frac{1}{2}B^2\zeta^2\xi\eta+\frac{1}{24}B^2\zeta^4(B^2+3)\right]\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\zeta\left[\frac{1}{2}\xi^2\eta+\frac{1}{3}\zeta^2\xi-\frac{1}{6}B^2\zeta^2\eta-\frac{1}{6}\xi^3+\frac{1}{6}B^2\zeta^2\xi\right]\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\left\{\frac{7}{48}\xi\zeta^2+\eta\left[\frac{1}{6}\xi^2+\frac{3}{16}\xi\eta+\frac{1}{48}(3\eta^2+\zeta^2)-\frac{1}{24}(B^2-1)(\eta^2-2\zeta^2)\right]+\frac{1}{72}\eta R^2\right\}R$$

INTEGRALS OF $f_{00}^{(4)} = \tan^{-1} \dfrac{\zeta R}{\xi\eta - \zeta^2}$

00 $$\tan^{-1} \frac{\zeta R}{\xi\eta - \zeta^2}$$

01 $$\eta \tan^{-1} \frac{\zeta R}{\xi\eta - \zeta^2} + \zeta \frac{1}{B} \frac{1}{2} \log \frac{BR + (\xi + B^2\eta)}{BR - (\xi + B^2\eta)} - \zeta \frac{1}{2} \log \frac{R + (\xi + \eta)}{R - (\xi + \eta)}$$

10 $$\xi \tan^{-1} \frac{\zeta R}{\xi\eta - \zeta^2} - B^2\zeta \frac{1}{B} \frac{1}{2} \log \frac{BR + (\xi + B^2\eta)}{BR\ \ (\xi + B^2\eta)} + \zeta \frac{1}{2} \log \frac{R + (\xi + \eta)}{R - (\xi + \eta)}$$

20 $$\frac{1}{2}(\xi^2 - B^2\zeta^2) \tan^{-1} \frac{\zeta R}{\xi\eta - \zeta^2} - B^2\zeta\xi \frac{1}{B} \frac{1}{2} \log \frac{BR + (\xi + B^2\eta)}{BR - (\xi + B^2\eta)} + \zeta\left[\xi - \frac{1}{2}(B^2 - 1)\eta\right] \frac{1}{2} \log \frac{R + (\xi + \eta)}{R - (\xi + \eta)} - \frac{1}{2}\zeta R$$

02 $$\frac{1}{2}(\eta^2 - \zeta^2) \tan^{-1} \frac{\zeta R}{\xi\eta - \zeta^2} + \zeta\left[\eta - \frac{1}{2}\frac{1}{B^2}(B^2 - 1)\xi\right] \frac{1}{B} \frac{1}{2} \log \frac{BR + (\xi + B^2\eta)}{BR - (\xi + B^2\eta)}$$
$$- \eta\zeta \frac{1}{2} \log \frac{R + (\xi + \eta)}{R - (\xi + \eta)} - \frac{1}{2} \frac{1}{B^2} \zeta R$$

03 $$\left[\frac{1}{6}\eta^3 - \frac{1}{2}\eta\zeta^2\right] \tan^{-1} \frac{\zeta R}{\xi\eta - \zeta^2} + \frac{1}{2}\zeta\left\{\eta^2 - \frac{1}{B^2}(B^2 - 1)\left[\xi\eta + \frac{1}{2}\frac{1}{B^2}(\xi^2 + B^2\zeta^2)\right] - \frac{1}{3}\frac{1}{B^2}\zeta^2\right\} \frac{1}{B} \frac{1}{2} \log \frac{BR + (\xi + B^2\eta)}{BR - (\xi + B^2\eta)}$$
$$- \zeta\left[\frac{1}{2}\eta^2 - \frac{1}{6}\zeta^2\right] \frac{1}{2} \log \frac{R + (\xi + \eta)}{R - (\xi + \eta)} + \frac{1}{12} \frac{1}{B^4} \zeta\left[(2B^2 - 3)\xi - 5B^2\eta\right] R$$

30 $$\left[\frac{1}{6}\xi^3 - \frac{1}{2}B^2\zeta^2\xi\right] \tan^{-1} \frac{\zeta R}{\xi\eta - \zeta^2} + B^2\zeta\left[\frac{1}{6}B^2\zeta^2 - \frac{1}{2}\xi^2\right] \frac{1}{B} \frac{1}{2} \log \frac{BR + (\xi + B^2\eta)}{BR - (\xi + B^2\eta)}$$
$$+ \frac{1}{2}\zeta\left\{\xi^2 - (B^2 - 1)\left[\xi\eta + \frac{1}{2}(\eta^2 + \zeta^2)\right] - \frac{1}{3}\zeta^2\right\} \frac{1}{2} \log \frac{R + (\xi + \eta)}{R - (\xi + \eta)} + \frac{1}{12}\zeta\left[(2B^2 - 3)\eta - 5\xi\right] R$$

40 $$\left[\tfrac{1}{24}\xi^4 - \tfrac{1}{4}B^2\zeta^2\xi^2 + \tfrac{1}{24}B^4\zeta^4\right]\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \tfrac{1}{6}B^2\zeta\xi\left[B^2\zeta^2-\xi^2\right]\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$+\tfrac{1}{2}\zeta\left\{\tfrac{1}{3}\xi^3 - \tfrac{1}{2}(B^2-1)\left[\eta\xi^2+(\eta^2+\zeta^2)\xi\right] - \tfrac{1}{3}\zeta^2\xi + \tfrac{1}{24}(B^2-1)\eta\left[(B^2-5)\eta^2+3(B^2-1)\zeta^2\right]\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$+\tfrac{1}{144}\zeta\left\{15(B^2-1)\eta^2 + 6(2B^2-1)\zeta^2 - 24\xi^2 - 2R^2 + 3(7B^2-9)\xi\eta\right\}R$$

04 $$\left[\tfrac{1}{24}\eta^4 - \tfrac{1}{4}\zeta^2\eta^2 + \tfrac{1}{24}\zeta^4\right]\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \frac{1}{6}\zeta\eta(\zeta^2-\eta^2)\,\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$+\tfrac{1}{2}\zeta\left\{\tfrac{1}{3}\eta^3 - \tfrac{1}{2}\,\tfrac{1}{B^4}(B^2-1)\eta\left[B^2\xi\eta+(\xi^2+B^2\zeta^2)\right] - \tfrac{1}{3}\,\tfrac{1}{B^2}\eta\zeta^2 + \tfrac{1}{24}\,\tfrac{1}{B^6}\left[B^2-1\right]\xi\left[(B^2-5)\xi^2+3(B^2-1)B^2\zeta^2\right]\right\}\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$+\frac{1}{144}\,\frac{1}{B^6}\,\zeta\left\{15(B^2-1)\xi^2 + 6(2B^2-1)B^2\zeta^2 - 24B^4\eta^2 - 2B^2R^2 + 3B^2(7B^2-9)\xi\eta\right\}R$$

11 $$(\xi\eta+\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \zeta(\xi-B^2\eta)\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \zeta(\eta-\xi)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \zeta R$$

12 $$\left(\tfrac{1}{2}\xi\eta^2 + \zeta^2\eta - \tfrac{1}{2}\xi\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \zeta\left(\tfrac{1}{2}\eta^2 - \xi\eta - \tfrac{1}{2}\zeta^2\right)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$-\zeta\left\{\tfrac{1}{2}B^2\eta^2 - \xi\eta + \tfrac{1}{2}\left[\tfrac{1}{2}\,\tfrac{1}{B^2}(B^2-1)(\xi^2+B^2\zeta^2) - B^2\zeta^2\right]\right\}\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$-\tfrac{1}{4}\zeta\left(\tfrac{1}{B^2}\xi - 3\eta\right)R$$

21 $$\left(\tfrac12\eta\xi^2+\zeta^2\xi-\tfrac12 B^2\zeta^2\eta\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}+\zeta\left(\tfrac12\xi^2-B^2\xi\eta-\tfrac12 B^2\zeta^2\right)\frac1B\,\frac12\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\zeta\left\{\tfrac12\xi^2-\xi\eta+\tfrac12\left[\tfrac12(B^2-1)(\eta^2+\zeta^2)-B^2\zeta^2\right]\right\}\frac12\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$-\tfrac14\zeta(\eta-3\xi)R$$

13 $$\left(\tfrac16\xi\eta^3+\tfrac12\zeta^2\eta^2-\tfrac12\xi\zeta^2\eta-\tfrac16\zeta^4\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}+\zeta\left(\tfrac16\eta^3-\tfrac12\xi\eta^2-\tfrac12\zeta^2\eta+\tfrac16\xi\zeta^2\right)\frac12\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$-\zeta\left\{\tfrac16 B^2\eta^3-\tfrac12\xi\eta^2+\tfrac12\left[\tfrac12\,\tfrac{1}{B^2}(B^2-1)(\xi^2+B^2\zeta^2)-B^2\zeta^2\right]\eta+\tfrac16\xi\left[\tfrac12\,\tfrac{1}{B^4}(B^2-1)(\xi^2+B^2\zeta^2)+\zeta^2\right]\right\}\frac1B\,\frac12\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\tfrac{1}{12}\zeta\left[\tfrac{1}{B^4}\xi^2+4\tfrac{1}{B^2}\xi\eta-3\eta^2+2\zeta^2\right]R+\tfrac{1}{18}\,\tfrac{1}{B^2}\zeta R^3$$

31 $$\left(\tfrac16\eta\xi^3+\tfrac12\zeta^2\xi^2-\tfrac12 B^2\zeta^2\xi\eta-\tfrac16 B^2\zeta^4\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}+\zeta\left(\tfrac16\xi^3-\tfrac12 B^2\eta\xi^2-\tfrac12 B^2\zeta^2\xi+\tfrac16 B^4\eta\zeta^2\right)\frac1B\,\frac12\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$-\zeta\left\{\tfrac16\xi^3-\tfrac12\eta\xi^2+\tfrac12\left[\tfrac12(B^2-1)(\eta^2+\zeta^2)-B^2\zeta^2\right]\xi+\tfrac16\eta\left[\tfrac12(B^2-1)(\eta^2+\zeta^2)+B^2\zeta^2\right]\right\}\frac12\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$-\tfrac{1}{12}\zeta\left[\eta^2-4\xi\eta-3\xi^2-2B^2\zeta^2\right]R+\tfrac{1}{18}\zeta R^3$$

INTEGRALS OF $f^{(5)}_{00} = \dfrac{\zeta R}{(\eta^2+\zeta^2)}$

00 $$\frac{\zeta R}{(\eta^2+\zeta^2)}$$

10 $$\frac{1}{2}(B^2-1)\zeta\ \frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + \frac{1}{2}\ \frac{\zeta(\xi+\eta)}{\eta^2+\zeta^2}\ R$$

20 $$\frac{1}{2}(B^2-1)\zeta(\xi+\eta)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{1}{2}(B^2-1)\zeta R + \frac{1}{6}\ \frac{\zeta}{\eta^2+\zeta^2}\ R^3$$

30 $$\frac{1}{4}(B^2-1)\zeta\left[(\xi+\eta)^2-\frac{1}{4}(B^2-1)(\eta^2+\zeta^2)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{5}{16}(B^2-1)\zeta(\xi+\eta)R + \frac{1}{24}\ \frac{\zeta(\xi+\eta)}{(\eta^2+\zeta^2)}\ R^3$$

40 $$\frac{1}{4}(B^2-1)\zeta\left[\frac{1}{3}(\xi+\eta)^3-\frac{1}{4}(B^2-1)(\eta^2+\zeta^2)(\xi+\eta)\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \frac{5}{48}(B^2-1)\zeta R^3 + \frac{1}{120}\ \frac{\zeta}{\eta^2+\zeta^2}\ R^5$$

21 $$-\frac{1}{2}\zeta\left\{\xi^2-(B^2-1)\left[\xi\eta+\frac{1}{2}(\eta^2+\zeta^2)\right]-\frac{1}{3}\zeta^2\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\ B^2\zeta\left[\frac{1}{2}\xi^2-\frac{1}{6}B^2\zeta^2\right]\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} \quad -\left[\frac{1}{6}\xi^3-\frac{1}{2}B^2\zeta^2\xi\right]\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$\frac{1}{12}\ \zeta\left[5\xi+(3-2B^2)\eta\right]R$$

31 $$-\frac{1}{2}\zeta\left\{\frac{1}{3}\xi^3-\frac{1}{2}(B^2-1)\left[\eta\xi^2+(\eta^2+\zeta^2)\xi\right]-\frac{1}{3}\zeta^2\xi+\frac{1}{24}(B^2-1)\eta\left[(B^2-5)\eta^2+3(B^2-1)\zeta^2\right]\right\}\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$
$$+\ \frac{1}{6}B^2\xi\zeta(\xi^2-B^2\zeta^2)\frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} - \left[\frac{1}{24}\xi^4-\frac{1}{4}B^2\zeta^2\xi^2+\frac{1}{24}B^4\zeta^4\right]\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

01 $$-\zeta\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + B^2\zeta\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} - \xi\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

02 $$\zeta(\xi-\eta)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \zeta(\xi-B^2\eta)\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} - (\xi\eta+\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} - \zeta R$$

03 $$-\zeta\left(\frac{1}{2}\eta^2-\xi\eta-\frac{1}{2}\zeta^2\right)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \left(\frac{1}{2}\xi\eta^2+\zeta^2\eta-\frac{1}{2}\xi\zeta^2\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\zeta\left\{\frac{1}{2}B^2\eta^2-\xi\eta+\frac{1}{2}\left[\frac{1}{2}\frac{1}{B^2}(B^2-1)(\xi^2+B^2\zeta^2)-B^2\zeta^2\right]\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\frac{1}{4}\zeta\left(\frac{1}{B^2}\xi-3\eta\right)R$$

04 $$-\zeta\left(\frac{1}{6}\eta^3-\frac{1}{2}\xi\eta^2-\frac{1}{2}\zeta^2\eta+\frac{1}{6}\xi\zeta^2\right)\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \left(\frac{1}{6}\xi\eta^3+\frac{1}{2}\zeta^2\eta^2-\frac{1}{2}\xi\zeta^2\eta-\frac{1}{6}\zeta^4\right)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\zeta\left\{\frac{1}{6}B^2\eta^3-\frac{1}{2}\xi\eta^2+\frac{1}{2}\left[\frac{1}{2}\frac{1}{B^2}(B^2-1)(\xi^2+B^2\zeta^2)-B^2\zeta^2\right]\eta+\frac{1}{6}\xi\left[\frac{1}{2}\frac{1}{B^4}(B^2-1)(\xi^2+B^2\zeta^2)+\zeta^2\right]\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$
$$+\frac{1}{12}\zeta\left[\frac{1}{B^4}\xi^2+4\frac{1}{B^2}\xi\eta-3\eta^2+2\zeta^2\right]R - \frac{1}{18}\frac{1}{B^2}\zeta R^3$$

11 $$-\zeta\left[\xi-\frac{1}{2}(B^2-1)\eta\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} + B^2\zeta\xi\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} - \frac{1}{2}(\xi^2-B^2\zeta^2)\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2} + \frac{1}{2}\zeta R$$

12 $$\zeta\left[\frac{1}{4}(B^2-1)\eta^2-\xi\eta+\frac{1}{2}(\xi^2-B^2\zeta^2)+\frac{1}{4}(B^2-1)\zeta^2\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)} - \left[\frac{1}{2}(\xi^2-B^2\zeta^2)\eta+\zeta^2\xi\right]\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$
$$+\zeta\left[B^2\xi\eta-\frac{1}{2}(\xi^2-B^2\zeta^2)\right]\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)} + \frac{1}{4}\zeta(\eta-3\xi)R$$

13 $$\zeta\left[\frac{1}{12}(B^2-1)\eta^3-\frac{1}{2}\xi\eta^2+\left[\frac{1}{2}(\xi^2-B^2\zeta^2)+\frac{1}{4}(B^2-1)\zeta^2\right]\eta+\frac{1}{2}\zeta^2\xi\right]\frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$-\left\{\frac{1}{4}(\xi^2-B^2\zeta^2)\eta^2+\zeta^2\xi\eta-\left[\frac{1}{4}(\xi^2-B^2\zeta^2)+\frac{1}{6}(B^2-1)\zeta^2\right]\zeta^2\right\}\tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

$$+\zeta\left\{\frac{1}{2}B^2\xi\eta^2-\frac{1}{2}(\xi^2-B^2\zeta^2)\eta+\frac{1}{12}\frac{1}{B^2}(B^2-1)\xi^3-\frac{1}{4}(B^2+1)\zeta^2\xi\right\}\frac{1}{B}\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$-\frac{1}{12}\frac{1}{B^2}\zeta\left[(7B^2+2)\xi\eta+(4B^2-1)\zeta^2\right]R+\frac{1}{12}\frac{1}{B^2}\zeta R^3$$

INTEGRALS OF $\frac{1}{2}\log(\eta^2+\zeta^2)$

0-1 $\qquad \dfrac{\eta}{(\eta^2+\zeta^2)}$

00 $\qquad \frac{1}{2}\log(\eta^2+\zeta^2)$

01 $\qquad \eta\,\frac{1}{2}\log(\eta^2+\zeta^2) + \zeta\tan^{-1}\frac{\eta}{\zeta} - \eta$

02 $\qquad \frac{1}{2}(\eta^2-\zeta^2)\,\frac{1}{2}\log(\eta^2+\zeta^2) + \zeta\eta\tan^{-1}\frac{\eta}{\zeta} - \frac{3}{4}\eta^2$

03 $\qquad \eta\left[\frac{1}{6}\eta^2-\frac{1}{2}\zeta^2\right]\frac{1}{2}\log(\eta^2+\zeta^2) + \zeta\left[\frac{1}{2}\eta^2-\frac{1}{6}\zeta^2\right]\tan^{-1}\frac{\eta}{\zeta} - \frac{11}{36}\eta^3+\frac{1}{6}\eta\zeta^2$

04 $\qquad \left[\frac{1}{24}\eta^4-\frac{1}{4}\eta^2\zeta^2+\frac{1}{24}\zeta^4\right]\frac{1}{2}\log(\eta^2+\zeta^2)+\frac{1}{6}\eta\zeta(\eta^2-\zeta^2)\tan^{-1}\frac{\eta}{\zeta}+\frac{1}{288}\eta^2\left[42\zeta^2-25\eta^2\right]$

INTEGRALS OF $\tan^{-1}\frac{\eta}{\zeta}$

0-1 $\qquad \dfrac{\zeta}{(\eta^2+\zeta^2)}$

00 $\qquad \tan^{-1}\frac{\eta}{\zeta}$

01 $\qquad -\zeta\,\frac{1}{2}\log(\eta^2+\zeta^2) + \eta\tan^{-1}\frac{\eta}{\zeta}$

02 $\qquad -\zeta\eta\,\frac{1}{2}\log(\eta^2+\zeta^2) + \frac{1}{2}(\eta^2-\zeta^2)\tan^{-1}\frac{\eta}{\zeta} + \frac{1}{2}\zeta\eta$

03 $\qquad \zeta\left[\frac{1}{6}\zeta^2-\frac{1}{2}\eta^2\right]\frac{1}{2}\log(\eta^2+\zeta^2) + \eta\left[\frac{1}{6}\eta^2-\frac{1}{2}\zeta^2\right]\tan^{-1}\frac{\eta}{\zeta} + \frac{5}{12}\zeta\eta^2$

04 $\qquad \frac{1}{6}\eta\zeta(\zeta^2-\eta^2)\frac{1}{2}\log(\eta^2+\zeta^2)+\left[\frac{1}{24}\eta^4-\frac{1}{4}\eta^2\zeta^2+\frac{1}{24}\zeta^4\right]\tan^{-1}\frac{\eta}{\zeta}+\frac{1}{72}\eta\zeta\left[13\eta^2-3\zeta^2\right]$

APPENDIX D

SPANWISE INTEGRATION OF NONPLANAR FUNCTIONS

It is desired to integrate expressions of the form

$$\int_{\eta_a}^{\eta_b} f\,[\,\xi(\eta)\,,\,\eta\,,\,\zeta(\eta)\,,\,B\,]\;d\eta$$

where $\xi(\eta)$ is of the form $\xi_o + \ell\eta$

and $\zeta(\eta)$ is of the form $\zeta_o + \tau\eta$

If we define

$$f'(\xi,\eta,\zeta,B) = \left\{ \ell\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} + \tau\frac{\partial}{\partial\zeta} \right\} f(\xi,\eta,\zeta,B)$$

and we have a function g such that

$$g'(\xi,\eta,\zeta,B) = f(\xi,\eta,\zeta,B)$$

then

$$\int_{\eta_a}^{\eta_b} f\,[\,\xi(\eta),\,\eta,\,\zeta(\eta),\,B\,]\,d\eta = g\,[\,\xi(\eta),\,\eta,\,\zeta(\eta),\,B\,]\Big|_{\eta_a}^{\eta_b}$$

Since all of the desired integrals are related to five basic functions, the integration can be facilitated with the following scheme.

Let

$$f^{(1)}_{00}(\xi,\eta,\zeta,B) = \frac{1}{2}\log\frac{R+(\xi+\eta)}{R-(\xi+\eta)}$$

$$f^{(2)}_{00}(\xi,\eta,\zeta,B) = \frac{1}{B}\,\frac{1}{2}\log\frac{BR+(\xi+B^2\eta)}{BR-(\xi+B^2\eta)}$$

$$f^{(3)}_{00}(\xi,\eta,\zeta,B) = \frac{\eta R}{(\eta^2+\zeta^2)}$$

$$f^{(4)}_{00}(\xi,\eta,\zeta,B) = \tan^{-1}\frac{\zeta R}{\xi\eta-\zeta^2}$$

$$f^{(5)}_{00}(\xi,\eta,\zeta,B) = \frac{\zeta R}{(\eta^2+\zeta^2)}$$

and

$$\frac{\partial}{\partial\xi} f^{(k)}_{ij}(\xi,\eta,\zeta,B) = f^{(k)}_{i-1,j}(\xi,\eta,\zeta,B)$$

$$\frac{\partial}{\partial\eta} f^{(k)}_{ij}(\xi,\eta,\zeta,B) = f^{(k)}_{i,j-1}(\xi,\eta,\zeta,B)$$

$$\vec{f}_{ij}(\xi,\eta,\zeta,B) = \begin{Bmatrix} f^{(1)}_{ij} \\ f^{(2)}_{ij} \\ f^{(3)}_{ij} \\ f^{(4)}_{ij} \\ f^{(5)}_{ij} \end{Bmatrix} \qquad \frac{1}{T}\left[w_{ij}-\tau v_{ij}\right] = \frac{qk}{8\pi} A\vec{f}_{ij}$$

$$A = \begin{bmatrix} 1 & -B^2 & 1 & \tau & \tau \end{bmatrix}$$

Then, using the relations given in appendix C,

$$\frac{\partial}{\partial \zeta}\vec{f}_{ij} = \frac{\partial}{\partial \zeta}\begin{Bmatrix} f_{ij}^{(1)} \\ f_{ij}^{(2)} \\ f_{ij}^{(3)} \\ f_{ij}^{(4)} \\ f_{ij}^{(5)} \end{Bmatrix} = \begin{Bmatrix} f_{i,j-1}^{(4)} \\ f_{i-1,j}^{(4)} \\ -f_{i-1,j}^{(5)} + f_{i,j-1}^{(5)} \\ f_{i-1,j}^{(1)} - B^2 f_{i-1,j}^{(2)} + f_{i-1,j}^{(3)} \\ -f_{i,j-1}^{(1)} + B^2 f_{i,j-1}^{(2)} - f_{i,j-1}^{(3)} \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & B^2 & -1 & 0 & 0 \end{bmatrix} \vec{f}_{i,j-1} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & -B^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \vec{f}_{i-1,j}$$

Therefore, since $\vec{f}_{ij}' = \left\{ \ell \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \tau \frac{\partial}{\partial \zeta} \right\} \vec{f}_{ij}$

$$\vec{f}_{ij}' = \begin{bmatrix} 1 & 0 & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \tau \\ 0 & 0 & 0 & 1 & 0 \\ -\tau & \tau B^2 & -\tau & 0 & 1 \end{bmatrix} \vec{f}_{i,j-1} + \begin{bmatrix} \ell & 0 & 0 & 0 & 0 \\ 0 & \ell & 0 & \tau & 0 \\ 0 & 0 & \ell & 0 & -\tau \\ \tau & -\tau B^2 & \tau & \ell & 0 \\ 0 & 0 & 0 & 0 & \ell \end{bmatrix} \vec{f}_{i-1,j}$$

$$= G\,\vec{f}_{i,j-1} + H\,\vec{f}_{i-1,j}$$

However, the following relations exist between the functions.

$$f^{(1)}_{i-1,j} = f^{(2)}_{i,j-1}$$

$$f^{(3)}_{i-1,j} = -f^{(1)}_{i,j-1} + f^{(2)}_{i,j-1}$$

$$f^{(5)}_{i-1,j} = -f^{(4)}_{i,j-1}$$

$$f^{(1)}_{i,j-1} = f^{(1)}_{i-1,j} - f^{(3)}_{i-1,j}$$

$$f^{(2)}_{i,j-1} = f^{(1)}_{i-1,j}$$

$$f^{(4)}_{i,j-1} = -f^{(5)}_{i-1,j}$$

Therefore, for any a_1, a_3, a_5

$$a_1 f^{(1)}_{i-1,j} + a_3 f^{(3)}_{i-1,j} + a_5 f^{(5)}_{i-1,j}$$

$$= -a_3 f^{(1)}_{i,j-1} + (a_1+a_3) f^{(2)}_{i,j-1} - a_5 f^{(4)}_{i,j-1}$$

By using the proper choice for a_1, a_3, a_5 in each row of the matricies G and H, a different and more useful pair of G and H matricies may be derived.

With the changes we can write

$$
\vec{f}'_{ij} = \begin{bmatrix}
1 & \frac{-\frac{1}{2}\ell\tau^2(B^2+1)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & 0 & \frac{-\frac{1}{2}\tau^3(B^2+1)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & 0 \\
\frac{1}{\ell}\tau^2 & \frac{-\ell\tau^2}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]}+\frac{(\ell-\tau^2)}{\ell} & 0 & \frac{-\tau^3}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & 0 \\
\frac{1}{\ell}\tau^2B^2 & -\frac{1}{\ell}\tau^2B^2 & 1 & \tau & \tau \\
0 & \frac{\frac{1}{2}\tau^3(B^2-1)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & 0 & \frac{\ell(\ell-\tau^2)+\frac{1}{2}\tau^2(B^2-1)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & 0 \\
-\tau & \frac{\frac{1}{2}\tau\ell^2(B^2+1)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & -\tau & \frac{\frac{1}{2}\ell\tau^2(B^2+1)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & 1
\end{bmatrix} \vec{f}_{i,j-1}
$$

$$
+ \begin{bmatrix}
\frac{\ell(\ell^2+\tau^2B^2)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & 0 & 0 & 0 & \frac{-\tau(\ell^2+\tau^2B^2)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} \\
\frac{\ell\tau^2}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & \ell & \frac{1}{\ell}\tau^2 & \tau & \frac{-\tau^3}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} \\
0 & 0 & \frac{1}{\ell}(\ell^2+\tau^2B^2) & 0 & 0 \\
\frac{\tau\ell^2}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & -\tau B^2 & \tau & \ell & \frac{-\ell\tau^2}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} \\
\frac{\frac{1}{2}\tau(B^2-1)(\ell^2+\tau^2B^2)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]} & 0 & 0 & 0 & \frac{\ell(\ell^2+\tau^2B^2)}{[\ell^2+\frac{1}{2}\tau^2(B^2-1)]}
\end{bmatrix} \vec{f}_{i-1,j}
$$

$$
= G\,\vec{f}_{i,j-1} + H\,\vec{f}_{i-1,j}
$$

or $$\vec{f}_{i-1,j} = H^{-1}\vec{f}_{ij}' - H^{-1}G\,\vec{f}_{i,j-1}$$

$$H^{-1} = \frac{1}{(\ell^2+\tau^2B^2)}\begin{bmatrix} \ell & 0 & 0 & 0 & \tau \\ 0 & \ell & 0 & -\tau & 0 \\ 0 & 0 & \ell & 0 & 0 \\ -\tau & \tau B^2 & -\tau & \ell & 0 \\ -\frac{1}{2}\tau(B^2-1) & 0 & 0 & 0 & \ell \end{bmatrix}$$

$$\| H^{-1} \| = \ell\left[\ell^2 + \tfrac{1}{2}\tau^2(B^2-1)\right]$$

$$H^{-1}G = \frac{1}{(\ell^2+\tau^2B^2)}\begin{bmatrix} \ell-\tau^2 & 0 & -\tau^2 & 0 & \tau \\ \tau^2 & \ell-2\tau^2 & 0 & -\tau & 0 \\ \tau^2B^2 & -\tau^2B^2 & \ell & \tau\ell & \tau\ell \\ -\tau & \tau B^2 & -\tau & \ell-2\tau^2 & -\tau^2 \\ -\tau[\ell+\frac{1}{2}(B^2-1)] & \frac{1}{2}\tau\ell(B^2+1) & -\tau\ell & \frac{1}{2}\tau^2(B^2+1) & \ell \end{bmatrix}$$

A linear transformation exists between the vectors

$$\vec{f}_{i,-1}(\xi, \eta, \zeta, B) \quad \text{and} \quad \vec{f}_{i,-1}(\hat{\xi}, \hat{\eta}, \hat{\zeta}, \hat{B}) = \vec{f}_{i,-1}(\xi, \eta, \zeta, B)$$

such that

$$\vec{f}_{i,-1}(\xi, \eta, \zeta, B) = T\, \vec{f}_{i,-1}(\hat{\xi}, \hat{\eta}, \hat{\zeta}, \hat{B})$$

where

$$T = \begin{bmatrix} \frac{1}{(\hat{\ell}+1)} & \frac{\hat{\ell}}{(\hat{\ell}+1)} & 0 & \frac{\hat{\tau}}{(\hat{\ell}+1)} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{(\hat{\tau}^2-\hat{\ell})-2\hat{\tau}^2}{(\hat{\ell}+1)^2} & \frac{\hat{\tau}^2(\hat{B}^2-1)-(\hat{\tau}^2-\hat{\ell})+2\hat{\tau}^2}{(\hat{\ell}+1)^2} & \frac{2-(1+\tau^2)}{(\hat{\ell}+1)^2} & \hat{\tau}\,\frac{2-(\hat{\ell}+1)}{(\hat{\ell}+1)^2} & \frac{2\hat{\tau}}{(\hat{\ell}+1)^2} \\ -\frac{\hat{\tau}}{(\hat{\ell}+1)} & \frac{\hat{\tau}}{(\hat{\ell}+1)} & 0 & \frac{1}{(\hat{\ell}+1)} & 0 \\ \hat{\tau}\,\frac{(\hat{\ell}+1)-2}{(\hat{\ell}+1)^2} & \hat{\tau}\,\frac{(\hat{B}^2-1)-(\hat{\ell}+1)+2}{(\hat{\ell}+1)^2} & \frac{-2\hat{\tau}}{(\hat{\ell}+1)^2} & \frac{(\hat{\tau}^2-\hat{\ell})-2\hat{\tau}^2}{(\hat{\ell}+1)^2} & \frac{2-(1+\tau^2)}{(\hat{\ell}+1)^2} \end{bmatrix}$$

$$= \frac{(\ell+1)}{(1+\tau^2)} \begin{bmatrix} 1 & \frac{(\tau^2-\ell)}{(\ell+1)} & 0 & -\tau & 0 \\ 0 & \frac{(1+\tau^2)}{(\ell+1)} & 0 & 0 & 0 \\ \left\{\ell - \frac{2\tau^2(\ell+1)}{(1+\tau^2)}\right\} & \left\{\frac{\tau^2(B^2-1)}{(\ell+1)} - \ell + \frac{2\tau^2(\ell+1)}{(1+\tau^2)}\right\} & \left\{\frac{2(\ell+1)}{(1+\tau^2)} - (\ell+1)\right\} & -\tau\left\{\frac{2(\ell+1)}{(1+\tau^2)} - 1\right\} & -\frac{2\tau(\ell+1)}{(1+\tau^2)} \\ \tau & -\tau & 0 & 1 & 0 \\ -\tau\left\{1 - \frac{2(\ell+1)}{1+\tau^2}\right\} & -\tau\left\{\frac{B^2-1}{(\ell+1)} - 1 + \frac{2(\ell+1)}{(1+\tau^2)}\right\} & \frac{2\tau(\ell+1)}{(1+\tau^2)} & \ell - \frac{2\tau^2(\ell+1)}{(1+\tau^2)} & \frac{2(\ell+1)}{(1+\tau^2)} - (\ell+1) \end{bmatrix}$$

However, a transformation also exists between A and $\hat{A}$ and also between $H^{-1}G$, $H^{-1}$and $\hat{H}^{-1}\hat{G}$, $\hat{H}^{-1}$

$$
T^{-1}H^{-1} = \frac{1}{(\ell^2+\tau^2B^2)}\begin{bmatrix}
\frac{(\ell-\tau^2)}{(\ell+1)} & \frac{(\ell^2+\tau^2B^2)}{(\ell+1)} & \frac{-\tau^2}{(\ell+1)} & 0 & \frac{\tau}{(\ell+1)} \\
0 & \ell & 0 & -\tau & 0 \\
-\frac{(\ell^2+\tau^2B^2)}{(\ell+1)^2} & \frac{(\ell^2+\tau^2B^2)}{(\ell+1)^2} & \frac{(\ell-\tau^2)}{(\ell+1)^2} & -\frac{\tau(\ell^2+\tau^2B^2)}{(\ell+1)^2} & \frac{\tau(\ell-\tau^2)}{(\ell+1)^2} \\
-\frac{\tau(\ell+1)}{\ell+1} & \frac{\tau(B^2+\ell)}{(\ell+1)} & \frac{-\tau}{(\ell+1)} & \frac{(\ell-\tau^2)}{(\ell+1)} & -\frac{\tau^2}{(\ell+1)} \\
\frac{\tau[(\ell^2+\tau^2B^2)-\frac{1}{2}(1+\tau^2)(B^2-1)]}{(\ell+1)^2} & -\frac{\tau(\ell^2+\tau^2B^2)}{(\ell+1)^2} & -\frac{\tau(\ell-\tau^2)}{(\ell+1)^2} & -\frac{(\ell^2+\tau^2B^2)}{(\ell+1)^2} & \frac{(\ell-\tau^2)}{(\ell+1)^2}
\end{bmatrix}
$$

$$
T^{-1}H^{-1}T = \frac{1}{(\ell^2+\tau^2B^2)}\begin{bmatrix}
\ell & 0 & \frac{\tau^2(\ell+1)}{(1+\tau^2)} & 0 & \frac{\tau(\ell+1)}{(1+\tau^2)} \\
-\frac{\tau^2(\ell+1)}{(1+\tau^2)} & \ell+\frac{\tau^2(\ell+1)}{(1+\tau^2)} & 0 & -\frac{\tau(\ell+1)}{(1+\tau^2)} & 0 \\
-\frac{\tau^2(B^2-1)}{(\ell+1)}-\frac{\tau^2(\ell+1)}{(1+\tau^2)} & \frac{\tau^2(B-1)}{(\ell+1)}+\frac{\tau^2(\ell+1)}{(1+\tau^2)} & \frac{(\ell-\tau^2)}{(1+\tau^2)} & -\frac{\tau(\ell-\tau^2)}{(1+\tau^2)} & -\frac{\tau(\ell-\tau^2)}{(1+\tau^2)} \\
-\frac{\tau(\ell+1)}{(1+\tau^2)} & \frac{\tau(B^2-1)}{(\ell+1)}+\frac{\tau(\ell+1)}{(1+\tau^2)} & -\frac{\tau(\ell+1)}{(1+\tau^2)} & \ell+\frac{\tau^2(\ell+1)}{(1+\tau^2)} & \frac{\tau^2(\ell+1)}{(1+\tau^2)} \\
\frac{\tau(\ell-\tau^2)}{(1+\tau^2)}-\frac{\frac{1}{2}\tau(B^2-1)}{(\ell+1)} & -\tau(\ell-\tau^2)\left[\frac{1}{2}\frac{(B^2-1)}{(\ell+1)^2}+\frac{1}{(1+\tau^2)}\right] & \frac{\tau(\ell-\tau^2)}{(1+\tau^2)} & -\tau^2\left[\frac{(\ell+1)}{(1+\tau^2)}+\frac{1}{2}\frac{(B^2-1)}{(\ell+1)}\right] & \frac{(\ell-\tau^2)}{(1+\tau^2)}
\end{bmatrix}
$$

If we define

$$\hat{M}(\ell, \tau, B^2) = M\left[-\frac{(\ell-\tau^2)}{(\ell+1)}, -\tau, \frac{(B^2-1)}{(\ell+1)^2}(1+\tau^2)-1\right] = M(\hat{\ell}, \hat{\tau}, \hat{B}^2)$$

then

$$(\ell+1)\left[T^{-1}H^{-1}T\right]$$

$$= -\frac{1}{(\hat{\ell}^2+\hat{\tau}^2\hat{B}^2)}\begin{bmatrix} (\hat{\ell}-\hat{\tau}^2) & 0 & -\hat{\tau}^2 & 0 & \hat{\tau} \\ \hat{\tau}^2 & \hat{\ell}-2\hat{\tau}^2 & 0 & -\hat{\tau} & 0 \\ \hat{\tau}^2\hat{B}^2 & -\hat{\tau}^2\hat{B}^2 & \hat{\ell} & \hat{\tau}\hat{\ell} & \hat{\tau}\ell \\ -\hat{\tau} & \hat{\tau}\hat{B}^2 & -\hat{\tau} & \hat{\ell}-2\hat{\tau}^2 & -\hat{\tau}^2 \\ -\hat{\tau}[\hat{\ell}+\frac{1}{2}(\hat{B}^2-1)] & \frac{1}{2}\hat{\tau}\hat{\ell}(\hat{B}^2+1) & -\hat{\tau}\hat{\ell} & \frac{1}{2}\hat{\tau}^2(\hat{B}^2+1) & \hat{\ell} \end{bmatrix}$$

$$= \hat{H}^{-1}\hat{G}$$

Therefore, since $\hat{T} = T^{-1}$,

$$T^{-1}H^{-1}G\,T = -(\hat{\ell}+1)\hat{H}^{-1}$$

$$T^{-1}H^{-1}T = -\frac{(\hat{\ell}+1)}{(1+\tau^2)}\hat{H}^{-1}\hat{G}$$

Then the following relations may be used to perform the required integrations.

$$\frac{1}{T}\left(\omega_{ij} - \tau v_{ij}\right) = A\vec{f}_{ij}$$

$$A = \left[\,1 \quad -B^2 \quad 1 \quad \tau \quad \tau\,\right]$$

$$\vec{f}_{i-1,j} = H^{-1}\vec{f}_{ij}^{\,\prime} - H^{-1}G\,\vec{f}_{i,j-1} = E\vec{f}_{ij}^{\,\prime} - F\vec{f}_{i,j-1}$$

$$\vec{f}_{10} = E\vec{f}_{20}^{\,\prime} - F\vec{f}_{2,-1} = \left\{E\vec{f}_{20} - \frac{(\hat{\ell}+1)}{(1+\tau^2)}FT\vec{\hat{f}}_{20}\right\}^{\prime}$$

where
$$\hat{f}_{ij}(\xi,\eta,\zeta,B) = f_{ij}(\hat{\xi},\hat{\eta},\hat{\zeta},\hat{B})$$

$$\vec{f}_{11} = E\vec{f}_{21}^{\,\prime} - F\vec{f}_{20} = \left\{E\vec{f}_{21} - FE\vec{f}_{30} + \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{\hat{f}}_{30}\right\}^{\prime}$$

$$\eta\vec{f}_{10} = \left\{\eta\left[E\vec{f}_{20} - \frac{(\hat{\ell}+1)}{(1+\tau^2)}FT\vec{\hat{f}}_{20}\right]\right\}^{\prime} - \left[E\vec{f}_{20} - \frac{(\hat{\ell}+1)}{(1+\tau^2)}FT\vec{\hat{f}}_{20}\right]$$

$$= \left\{\eta\left[E\vec{f}_{20} - \frac{\hat{\ell}+1}{(1+\tau^2)}FT\vec{\hat{f}}_{20}\right] - E\left[\vec{f}_{30} - \frac{(\hat{\ell}+1)}{(1+\tau^2)}F\hat{T}\vec{\hat{f}}_{30}\right] + \frac{(\hat{\ell}+1)^2}{(1+\tau^2)^2}FT\vec{\hat{f}}_{21}\right\}^{\prime}$$

$$\eta\vec{f}_{11} = \left\{\eta\left[E\vec{f}_{21} - FE\vec{f}_{30} + \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{\hat{f}}_{30}\right]\right\}^{\prime}$$
$$- \left[E\vec{f}_{21} - FE\vec{f}_{30} + \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{\hat{f}}_{30}\right]$$

$$= \left\{\eta\left[E\vec{f}_{21} - FE\vec{f}_{30} + \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{\hat{f}}_{30}\right] - E\left[\vec{f}_{31} - FE\vec{f}_{40} + \frac{(\hat{\ell}+1)}{(1+\tau^2)}F^2T\vec{\hat{f}}_{40}\right]\right.$$
$$\left. + FE\left[\vec{f}_{40} - \frac{(\hat{\ell}+1)}{(1+\tau^2)}\vec{\hat{f}}_{40}\right] - \frac{(\hat{\ell}+1)^2}{(1+\tau^2)^2}F^2T\vec{\hat{f}}_{40}\right\}^{\prime}$$

Also,

$$A(\ell,\tau,B^2) = \begin{Bmatrix} 1 \\ -B^2 \\ 1 \\ \tau \\ \tau \end{Bmatrix}^* \qquad A(\hat{\ell},\hat{\tau},\hat{B}^2) = \hat{A}(\ell,\tau,B^2) = \begin{Bmatrix} 1 \\ -\frac{(B^2-1)}{(\ell+1)^2}(1+\tau^2) - 1 \\ 1 \\ -\tau \\ -\tau \end{Bmatrix}^*$$

$$AT = \begin{Bmatrix} \frac{(\ell+1)^2}{(1+\tau^2)} \\ -\frac{(\ell+1)^2}{(1+\tau^2)} - (B^2-1) \\ \frac{(\ell+1)^2}{(1+\tau^2)} \\ -\tau\frac{(\ell+1)^2}{(1+\tau^2)} \\ -\tau\frac{(\ell+1)^2}{(1+\tau^2)} \end{Bmatrix} = \frac{(1+\tau^2)}{(\hat{\ell}+1)^2}\ \hat{A}$$

APPENDIX E

POTENTIAL FORM DRAG PROGRAM

PROGRAM INFORMATION

A computer code has been written to perform the calculations needed to apply the method developed in this report to arbitrary wings. The code allows the user a number of options through the choice of inputs. For example, the pressure distribution can be calculated using the linearly varying vorticity panels or using constant pressure panels. Another option is in wing geometry; for simple planforms only leading-and trailing-edge sweeps, root chord and number of panels are needed as input, but for complicated wings the entire arrays of panel corners may be specified. The inputs required for the various options are described in the following section.

INPUT DATA

ALL DATA EXCEPT THE TITLE CARD IS INITIALLY INPUT INTO A SINGLE LARGE ARRAY. EACH DATA CARD HAS THE FORMAT I12,5F10.5. THE FIRST NUMBER ON EACH CARD IS AN INTEGER GIVING A LOCATION IN THE INPUT ARRAY; THE FOLLOWING NUMBERS ON THE CARD SPECIFY THE VALUES TO BE INPUT INTO THAT AND THE FOUR IMMEDIATELY SUCCEDING LOCATIONS. THESE NUMBERS ARE IN F FORMAT AND MUST INCLUDE A DECIMAL POINT. LOCATIONS LEFT BLANK WILL REMAIN UNCHANGED.
ALL LOCATIONS ARE INITIALLY SET EQUAL TO 0.

THE LAST CARD IN EACH CASE MUST HAVE A MINUS SIGN IN COLUMN 1.

PRECEEDING ALL DATA CARDS IN EACH RUN IS A SINGLE DATA CARD SPECIFYING THE ARRAY SIZES USED FOR ALL SUCCEEDING CASES. THE CARD CONTAINS TWO VALUES AND HAS THE FORMAT DESCRIBED ABOVE. THE LOCATION NUMBER (COLS 1-12) MUST BE 1.
THERE MUST BE A MINUS SIGN IN COLUMN 1
THE FIRST VALUE MUST BE AS GREAT AS THE MAXIMUM NUMBER OF OF BOUNDARY CONDITIONS IN ALL SUBSEQUENT CASES OF THIS RUN.
THE SECOND VALUE MUST BE AS GREAT AS THE MAXIMUM NUMBER OF PANELS IN ALL SUBSEQUENT CASES.

THE FIRST CARD IN EACH CASE CONTAINS THE TITLE

LOC	VARIABLE	DESCRIPTION
1		SET THIS LOCATION TO -1. AFTER THE LAST CASE TO TERMINATE THE RUN
2	N	THE NUMBER OF PANELS IN THE CHORDWISE DIRECTION
3	M	THE NUMBER OF PANELS IN THE SPANWISE DIRECTION
4	MC	THE NUMBER OF CONTROL POINTS IN THE SPANWISE DIRECTION IF 0. IT WILL BE SET EQUAL TO M
5	NDRG	=0. THE NEAR FIELD DRAG IS COMPUTED <0. NO NEAR FIELD DRAG IS COMPUTED
6	SYM	0. SYMMETRY ABOUT Y=0. IS ASSUMED .NE. 0. NO SYMMETRY
7	INFL	2 COMPUTE BOTH AERO AND DRAG MATRICIES AND STORE ON UNIT 11 1 COMPUTE NEW AERO MATRIX, READ DRAG MATRIX FROM UNIT 11, THEN STORE BOTH ON UNIT 11 0 COMPUTE NEW AERO AND DRAG MATRICIES -1 USE AERO AND DRAG MATRICIES FROM THE PREVIOUS CASE -2 READ AERO AND DRAG MATRICIES FROM UNIT 11 -3 COMPUTE AERO MATRIX AND READ DRAG MATRIX FROM UNIT 11

12	IOO	IF O. LINEARLY VARYING PANELS ARE USED TO COMPUTE CP'S
		IF >O CONSTANT CONSTANT PRESSURE PANELS ARE USED TO COMPUTE THE PRESSURE DISTRIBUTION
16	IWPRNT	IF O. THE DOWNWASH VALUES ARE NOT PRINTED
19	INTMED	VARIOUS ORDERS OF INTERMEDIATE PRINTOUT (-1. TO 2.)
		-1. LEAST PRINTOUT
		2. MOST PRINTOUT
20	INFPRT	≥1 PRINT THE DRAG INFLUENCE MATRIX
		=2 PRINT THE AERO INFLUENCE MATRIX
22	ND	THE NUMBER OF CHORDWISE LOCATIONS OF CAMBER INPUT
23	MD	THE NUMBER OF SPANWISE LOCATIONS OF CAMBER INPUT
101	CENT	1. CONTROL POINT AT PANEL CENTROID
		CONTROL POINT AT PANEL CENTER OTHERWISE
102	SPAN	SPAN, ANY CONSISTENT SET OF UNITS MAY BE USED. THIS VALUE IS NOT USED IF LOCATIONS 161 TO 161+M ARE USED
103	ALPHA	THE ANGLE OF ATTACK IN DEGREES. THIS IS IN ADDITION TO ANY CAMBER DISTRIBUTION
105	MACH	THE MACH NUMBER
108	CAVG	THE REFERENCE CHORD LENGTH (IF O. AN AVERAGE CHORD
109	SREF	THE REFERENCE AREA. IF O., SREF=CAVG*SPAN
110	SWEEPL	THE LEADING EDGE SWEEP IN DEGREES. THIS VALUE IS IGNORED IF 112 IS .LE. -1., WHICH MEANS A PLANFORRM SHAPE IS TO BE READ IN.
111	SWEEPT	THE TRAILING EDGE SWEEP IN DEGREES
112	ROOT	ROOT DIMENSION OR CHORD LENGTH ALONG THE SYMMETRY AXIS
		<O. THE PLANFORM SHAPE IS READ IN FROM 241-320
		=O. THE ROOT DIMENSION IS COMPUTED USING THE SPAN AND LEADING AND TRAILING EDGE SWEEPS TO MAKE CAVG=1.
		>O. THIS VALUE IS USED TO CALCULATE THE GEOMETRY
113	RATIO	THE CHORDWISE CONTROL POINT LOCATION. IF O. DEFAULT IS
		0.950 FOR MACH > 1.
		0.875 FOR MACH < 1.
114	RATIOY	THE SPANWISE CONTROL POINT LOCATION. IF O. A VALUE BASED ON LOCATION 101 IS USED.
117		STARTING CL FOR DRAG POLAR. DEFAULT = O.
118		ENDING CL FOR DRAG POLAR. DEFAULT = 1.
119		DELTA CL FOR DRAG POLAR. DEFAULT = .05
121	Z(J)	Z VALUES AT THE Y COORDINATES

```
161   Y(J)      Y COORDINATES.   (IF ((Y(2)-Y(1))**2+(Z(2)-Z(1))**2) IS
                  0.  THE SEMI SPAN IS BROKEN INTO M EQUAL SEGMENTS)
                  THERE MUST BE M+1 VALUES INPUT

201   YC(J)     Y COORDINATES FOR THE CONTROL POINTS.  THESE VALUES
                  MUST BE INPUT IF MC > M.  IF MC=M, NONZERO VALUES
                  WILL BE USED TO OVERRIDE VALUES BASED ON 114
241   XL(J)     THE LEADING EDGE COORDINATES AT Y(J)
                THESE VALUES ARE CONSIDERED ONLY IF 112 IS < 0
                XL(1)   CORRESPONDS TO THE COORDINATE ON THE AXIS
                XL(M+1) CORRESPONDS TO THE COORDINATE AT THE TIP
                 ANY VALUES WHICH ARE EXACTLY 0. WILL BE CHANGED
                 TO MAKE THE EDGE STRAIGHT BETWEEN THE TWO
                 SURROUNDING NONZERO VALUES.
281   XT(J)     THE TRAILING EDGE COORDINATES. SAME FORMAT AS 241-280

321   XD(I)     THE X/C VALUES WHERE THE CAMBER IS INPUT.  SEE 22
341   YD(J)     THE Y   VALUES WHERE THE CAMBER IS INPUT.  SEE 23

401             THE VALUES OF DZ/DX FOR THE CAMBER.
                  401     TO 400+ND     ARE FOR YD(1) AT XD(1) TO XD(ND)
                  401+ND TO 400+ND*2  ARE FOR YD(2) AT XD(1) TO XD(ND)
                THESE VALUES ARE INTERPOLATED TO OBTAIN THE BOUNDARY
                  CONDITIONS AT THE CONTROL POINTS.
```

SAMPLE CASES

This section present samples of the input and output of the computer program which is used to calculate the spanwise variation of drag due to lift. The planform used for the calculation had a straight leading edge with a sweep of 63.435 degrees and a straight trailing edge with a sweep of 45°. The Mach number was 1.30 and the paneling scheme used 10 panels spanwise and 8 panels chordwise. (see figure 3). The planform has no twist or camber and has an angle of attack of one radian.

There are two cases presented. The first uses spanwise linearly varying vorticity panels to compute the pressure distribution on the above planform while the second case uses constant pressure panels to compute the pressure distribution. The linearly varying panels have the control point placed at 0.15 of the span while the constant pressure panels have the downwash placed at the centroid. The pressures obtained by the constant pressure panels are interpolated to the panel edges using a curve fit subroutine.

A sample input for the first case is presented on pages 97 and 98. The program output begins on page 99.

Calculated geometric characteristics of the planform are presented on pages 100 and 101. The coordinates of the leading and trailing edges and the resulting chord dimensions are printed at the y coordinates of the panel edges and at the control points.

Page 102 presents the Cp values at the inside leading edge of each panel using linearly varying vorticity panels.

Page 103 presents the integrated loads at each span station (linearly varying panel analysis) as well as integrated loads for the entire planform where

Y	is the span coordinate of the panel midpoint
CN(Y)	is the CL*C/CAVG at the left edge of this row of panels
CL(Y)	is the CL*C/CAVG for this row of panels
XCP(Y)	is (x/c) value for the center of pressure at this span station
CD(Y)	is the CD*C/CAVG for this span station
CDO(Y)	is the zero suction drag for this span station obtained by multiplying the CP value times the control point downwash
CT(Y)	is the leading edge suction for this span station obtained from CDO(Y)-CD(Y)

Page 104 presents the calculated drag polar. A comparison of the calculation and the exact analytic results for the spanwise variation of leading edge thrust is presented on figure 3.

The results of a second sample case using the constant vorticity panel analysis on the same planform are presented on pages 105 through 109.

The downwash presented on page 107 represents the downwash obtained using a linearly varying panel influence matrix, with the control point located at the centroid, and the pressures obtained using the constant pressure panel with the control point also located at the centroid.

FORTRAN FIXED 10 DIGIT DECIMAL DATA

DECK NO. ______ PROGRAMMER ______ DATE ______ PAGE 1 of 2 JOB NO. ______

Col.	NUMBER	IDENTIFICATION	DESCRIPTION DO NOT KEY PUNCH
1	- 1	73–80	
13	2.		The maximum number of coundary conditions expected
25	200.		The maximum number of panels expected for all cases
37			
49			
61			
1	WING21.M130.	73–80	Title card
13	BIG01.DATA		
25			
37			
49			
61			
1	2	73–80	
13	8.		The number of panels in the chordwise direction
25	10.		The number of panels in the spanwise direction
37			
49			
61			
1	102	73–80	
13	20.		Span
25	57.2958		Angle of attack (degrees)
37			
49	1.30		Mach number
61			

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FORTRAN FIXED 10 DIGIT DECIMAL DATA

DECK NO. ______ PROGRAMMER ______ DATE ______ PAGE 2 of 2 JOB NO. ______

Col.	Sign	NUMBER	IDENTIFICATION (73–80)	DESCRIPTION DO NOT KEY PUNCH
1		110		
13		63.435		Leading edge sweep in degrees
25		45.		Trailing edge sweep in degrees
37		13.		Root chørd dimension
49				
61		0.15		Spanwise location of the control point
1		162		
13		1.5		Y Coordinate of the outer edge of panel #1
25		3.0		Y Coordinate of the outer edge of panel #2
37		4.5		
49		6.0		
61		7.0		
1	-	167		(- sign to indicate the last card of this case)
13		8.0		
25		9.0		
37		9.5		
49		9.8		
61		10.		Y Coordinate of the outer edge of panel #10
1	-	1		
13	-	1.		(This terminates the run)
25				
37				
49				
61				

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```
MAX    =  40001

IJMAX =  38200


WING21.M130.BIG01.DATA                                   05/09/77      18.02

INPUT DATA ARRAY


  1 0.0            0.800000E+01 0.100000E+02 0.0          0.0
  6 0.0           -0.300000E+01 0.0          0.0          0.0
 11 0.0            0.500000E+00 0.0          0.0          0.0
 16 0.200000E+01 0.0          0.0          0.100000E+01 0.0
101 0.100000E+01 0.200000E+02 0.572958E+02 0.0          0.130000E+01
106 0.0          0.0          0.0          0.0          0.634350E+02
111 0.450000E+02 0.130000E+02 0.0          0.150000E+00 0.0
161 0.0          0.150000E+01 0.300000E+01 0.450000E+01 0.600000E+01
166 0.700000E+01 0.800000E+01 0.900000E+01 0.950000E+01 0.980000E+01
171 0.100000E+02 0.0          0.0          0.0          0.0


 LEADING  EDGE SWEEP =  63.4350
 TRAILING EDGE SWEEP =  45.0000


 ANGLE OF ATTACK = 57.2958
 FLAP DEFLECTION ANGLE =  0.0
 ASPECT RATIO =   2.500
 SPAN  =  20.000
 MACH NUMBER  =   1.300
 BETA =  0.83066
 TOTAL NUMBER OF CONTROL POINTS =   80
 CONTROL POINT LOCATED AT 0.950 CHORD
 CONTROL POINT LOCATED AT 0.150  SPAN
 REFERENCE AREA  =   159.9997
```

Y	X-L.E.	X-T.E.	CHORD	Z
0.0	0.0	13.0000	13.0000	0.0
1.5000	3.0000	14.5000	11.5000	0.0
3.0000	6.0000	16.0000	10.0000	0.0
4.5000	9.0000	17.5000	8.5000	0.0
6.0000	12.0000	19.0000	7.0000	0.0
7.0000	14.0000	20.0000	6.0000	0.0
8.0000	16.0000	21.0000	5.0000	0.0
9.0000	18.0000	22.0000	4.0000	0.0
9.5000	19.0000	22.5000	3.5000	0.0
9.8000	19.6000	22.8000	3.2000	0.0
10.0000	20.0000	23.0000	3.0000	0.0

C-AVERAGE = 7.99998
SPAN LENGTH = 10.00000
REFERENCE AREA = 159.99969

Y	X-L.E.	X-T.E.	CHORD	Z	
0.2250	0.4500	13.2250	12.7750	0.0	1.0000
1.7250	3.4500	14.7250	11.2750	0.0	1.0000
3.2250	6.4500	16.2250	9.7750	0.0	1.0000
4.7250	9.4500	17.7250	8.2750	0.0	1.0000
6.1500	12.3000	19.1500	6.8500	0.0	1.0000
7.1500	14.3000	20.1500	5.8500	0.0	1.0000
8.1500	16.3000	21.1500	4.8500	0.0	1.0000
9.0750	18.1500	22.0750	3.9250	0.0	1.0000
9.5450	19.0900	22.5450	3.4550	0.0	1.0000
9.8300	19.6600	22.8300	3.1700	0.0	1.0000

TABLE OF CP VALUES

X/C	0.0	1.5000	3.0000	4.5000	6.0000
0.0	0.2191E+01	0.4807E+01	0.6809E+01	0.8739E+01	0.1097E+02
0.038	0.1493E+01	0.2644E+01	0.3614E+01	0.4579E+01	0.5716E+01
0.146	0.1544E+01	0.2043E+01	0.2580E+01	0.3175E+01	0.3887E+01
0.309	0.1641E+01	0.1864E+01	0.2188E+01	0.2596E+01	0.3099E+01
0.500	0.1686E+01	0.1805E+01	0.2019E+01	0.2338E+01	0.2615E+01
0.691	0.1706E+01	0.1781E+01	0.1946E+01	0.2165E+01	0.2088E+01
0.854	0.1719E+01	0.1753E+01	0.1829E+01	0.1590E+01	0.1501E+01
0.962	0.1853E+01	0.1195E+01	0.8923E+00	0.5797E+00	0.9696E+00

X/C	7.0000	8.0000	9.0000	9.5000	9.8000
0.0	0.1274E+02	0.1459E+02	0.1761E+02	0.1775E+02	0.2700E+02
0.038	0.6573E+01	0.7585E+01	0.8721E+01	0.1103E+02	0.8522E+01
0.146	0.4411E+01	0.4945E+01	0.6350E+01	0.6433E+01	0.1689E+01
0.309	0.3341E+01	0.4015E+01	0.5048E+01	0.1685E+01	0.3105E+00
0.500	0.2727E+01	0.3987E+01	0.2010E+01	0.2493E+00	-0.1081E+00
0.691	0.2651E+01	0.2937E+01	0.1473E+00	-0.3811E+00	-0.3818E+00
0.854	0.2231E+01	0.1251E+01	-0.6057E+00	-0.6679E+00	-0.3851E+00
0.962	0.1127E+01	-0.5013E-01	-0.5606E+00	-0.3471E+00	0.1019E+00

PLANFORM COMPOSED OF SPANWISE LINEARLY VARYING VORTICITY PANELS

Y	CN(Y)	CL(Y)	XCP(Y)	CD(Y)	CD0(Y)	CT(Y)
0.2250	2.7071	2.8220	0.4737	2.5430	2.8220	0.27904
1.7250	2.9143	2.9834	0.4258	2.3088	2.9834	0.67466
3.2250	3.0279	3.0321	0.3961	1.9732	3.0321	1.05891
4.7250	3.0106	2.9633	0.3710	1.5398	2.9633	1.42348
6.1500	2.8869	2.8743	0.3653	1.1054	2.8743	1.76891
7.1500	2.8415	2.7821	0.3598	0.8950	2.7821	1.88704
8.1500	2.7005	2.3254	0.2873	0.1681	2.3254	2.15733
9.0750	1.9665	1.6613	0.1767	-0.7539	1.6613	2.41522
9.5450	1.3727	1.1131	0.0871	-1.1310	1.1131	2.24409
9.8300	0.8657	0.4238	0.0233	-1.9142	0.4238	2.33802
		2.6932	12.6196	1.3617	2.6932	1.33159

CD(TOTAL) = 0.1362E+01
CD(VORTEX) = 0.9989E+00
CD(WAVE) = 0.3628E+00

CD/CL**2 = 0.1877E+00
CT/CL**2 = 0.1836E+00

CD(WAVE)/CL**2 = 0.5002E-01

E(VTX) = 0.9246E+00

PLANFORM COMPOSED OF SPANWISE LINEARLY VARYING VORTICITY PANELS

DRAG POLAR

		CAMBERED		UNCAMBERED	
CL	ALPHA	CD(0)	CD(100)	CD(0)	CD(100)
0.0	0.0	0.0	0.0	0.0	0.0
0.033	0.709	0.00039	0.00021	0.00039	0.00021
0.067	1.418	0.00156	0.00083	0.00156	0.00083
0.100	2.127	0.00351	0.00188	0.00351	0.00188
0.133	2.837	0.00623	0.00334	0.00623	0.00334
0.167	3.546	0.00974	0.00521	0.00974	0.00521
0.200	4.255	0.01402	0.00751	0.01402	0.00751
0.233	4.964	0.01909	0.01022	0.01909	0.01022
0.267	5.673	0.02493	0.01335	0.02493	0.01335
0.300	6.382	0.03155	0.01689	0.03155	0.01689
0.333	7.091	0.03895	0.02086	0.03895	0.02086
0.367	7.800	0.04713	0.02524	0.04713	0.02524
0.400	8.510	0.05609	0.03003	0.05609	0.03003
0.433	9.219	0.06583	0.03525	0.06583	0.03525
0.467	9.928	0.07635	0.04088	0.07635	0.04088
0.500	10.637	0.08765	0.04693	0.08765	0.04693
0.533	11.346	0.09972	0.05339	0.09972	0.05339
0.567	12.055	0.11258	0.06028	0.11258	0.06028
0.600	12.764	0.12621	0.06758	0.12621	0.06758
0.633	13.473	0.14062	0.07529	0.14062	0.07529
0.667	14.183	0.15581	0.08343	0.15581	0.08343
0.700	14.892	0.17179	0.09198	0.17179	0.09198
0.733	15.601	0.18854	0.10095	0.18854	0.10095
0.767	16.310	0.20607	0.11033	0.20607	0.11033
0.800	17.019	0.22437	0.12014	0.22437	0.12014
0.833	17.728	0.24346	0.13036	0.24346	0.13036
0.867	18.437	0.26333	0.14099	0.26333	0.14099
0.900	19.146	0.28397	0.15205	0.28397	0.15205
0.933	19.856	0.30540	0.16352	0.30540	0.16352
0.967	20.565	0.32760	0.17541	0.32760	0.17541
1.000	21.274	0.35058	0.18771	0.35058	0.18771

MAX = 40001

IJMAX = 38200

WING21.M130.BIG01.DATA 05/10/77 7.20

INPUT DATA ARRAY

1	0.0	0.800000E+01	0.100000E+02	0.0	0.0
6	0.0	-0.300000E+01	0.0	0.0	0.0
11	0.0	0.100000E+01	0.0	0.0	0.0
16	0.200000E+01	0.0	0.0	0.100000E+01	0.0
101	0.100000E+01	0.200000E+02	0.572958E+02	0.0	0.130000E+01
106	0.0	0.0	0.0	0.0	0.634350E+02
111	0.450000E+02	0.130000E+02	0.0	0.0	0.0
161	0.0	0.150000E+01	0.300000E+01	0.450000E+01	0.600000E+01
166	0.700000E+01	0.800000E+01	0.900000E+01	0.950000E+01	0.980000E+01
171	0.100000E+02	0.0	0.0	0.0	0.0

LEADING EDGE SWEEP = 63.4350
TRAILING EDGE SWEEP = 45.0000

TABLE OF CP VALUES

X/C	0.0	1.5000	3.0000	4.5000	6.0000
0.0	0.4553E+01	0.5727E+01	0.7841E+01	0.9861E+01	0.1194E+02
0.038	0.2499E+01	0.3035E+01	0.4043E+01	0.5046E+01	0.6080E+01
0.146	0.1956E+01	0.2226E+01	0.2767E+01	0.3338E+01	0.3984E+01
0.309	0.1846E+01	0.1990E+01	0.2310E+01	0.2666E+01	0.3104E+01
0.500	0.1805E+01	0.1886E+01	0.2075E+01	0.2304E+01	0.2600E+01
0.691	0.1784E+01	0.1833E+01	0.1933E+01	0.2044E+01	0.2200E+01
0.854	0.1746E+01	0.1718E+01	0.1640E+01	0.1550E+01	0.1526E+01
0.962	0.1349E+01	0.1107E+01	0.8029E+00	0.7268E+00	0.7294E+00

X/C	7.0000	8.0000	9.0000	9.5000	9.8000
0.0	0.1363E+02	0.1585E+02	0.1819E+02	0.1952E+02	0.1962E+02
0.038	0.6933E+01	0.8048E+01	0.9201E+01	0.9932E+01	0.9157E+01
0.146	0.4521E+01	0.5182E+01	0.5927E+01	0.6230E+01	0.4206E+01
0.309	0.3490E+01	0.3907E+01	0.4433E+01	0.3114E+01	0.1525E+01
0.500	0.2854E+01	0.3155E+01	0.2732E+01	0.1399E+01	0.5116E+00
0.691	0.2287E+01	0.2398E+01	0.1321E+01	0.3893E+00	0.3813E-02
0.854	0.1549E+01	0.1471E+01	0.3960E+00	-0.2140E+00	-0.3236E+00
0.962	0.7492E+00	0.5584E+00	-0.1375E+00	-0.3113E+00	-0.2042E+00

PLANFORM COMPOSED OF CONSTANT VORTICITY PANELS

CONTROL POINT AT PANEL CENTROID

TABLE OF DOWNWASH VALUES AT CONTROL POINTS

X/C	0.7347	2.2326	3.7297	5.2258	6.4872
0.036	0.1208E+01	0.1111E+01	0.1147E+01	0.1124E+01	0.1114E+01
0.141	0.1174E+01	0.1063E+01	0.1113E+01	0.1096E+01	0.1083E+01
0.301	0.1172E+01	0.1009E+01	0.1050E+01	0.1041E+01	0.1029E+01
0.490	0.1194E+01	0.1000E+01	0.1039E+01	0.1031E+01	0.1021E+01
0.682	0.1203E+01	0.9890E+00	0.1023E+01	0.1014E+01	0.9974E+00
0.845	0.1206E+01	0.9835E+00	0.1014E+01	0.1005E+01	0.9812E+00
0.957	0.1196E+01	0.9666E+00	0.9986E+00	0.9962E+00	0.9782E+00
0.998	0.1170E+01	0.9703E+00	0.1003E+01	0.9962E+00	0.9773E+00

X/C	7.4848	8.4815	9.2444	9.6478	9.8989
0.036	0.1158E+01	0.1122E+01	0.1044E+01	0.8728E+00	-0.5001E+01
0.141	0.1128E+01	0.1090E+01	0.1036E+01	0.5979E+00	-0.4294E+01
0.301	0.1058E+01	0.1039E+01	0.9872E+00	0.7020E+00	-0.3461E+01
0.490	0.1041E+01	0.1058E+01	0.6395E+00	0.9985E+00	-0.3136E+01
0.682	0.1038E+01	0.8528E+00	0.7533E+00	0.1237E+01	-0.2993E+01
0.845	0.1023E+01	0.7082E+00	0.8971E+00	0.1354E+01	-0.2866E+01
0.957	0.9982E+00	0.7337E+00	0.9851E+00	0.1394E+01	-0.2773E+01
0.998	0.9835E+00	0.7689E+00	0.1013E+01	0.1404E+01	-0.2789E+01

PLANFORM COMPOSED OF CONSTANT VORTICITY PANELS

CONTROL POINT AT PANEL CENTROID

Y	CN(Y)	CL(Y)	XCP(Y)	CD(Y)	CD0(Y)	CT(Y)
0.7500	3.2337	3.1861	0.4370	3.5792	3.1861	-0.39313
2.2500	3.1270	3.1666	0.4083	2.2504	3.1666	0.91623
3.7500	3.1830	3.1667	0.3796	1.9915	3.1667	1.17528
5.2500	3.1258	3.0651	0.3604	1.4487	3.0651	1.61641
6.5000	2.9758	2.9152	0.3491	0.9645	2.9152	1.95072
7.5000	2.8387	2.7528	0.3376	0.6525	2.7528	2.10036
8.5000	2.6482	2.3933	0.2996	-0.0147	2.3933	2.40802
9.2500	2.1368	1.8768	0.2386	-0.6775	1.8768	2.55430
9.6500	1.6283	1.3620	0.1741	-0.9361	1.3620	2.29810
9.9000	1.1075	0.5422	0.1235	-1.8578	0.5422	2.40000
		2.8394	12.3881	1.4516	2.8394	1.38779

CD(TOTAL) = 0.1452E+01
CD(VORTEX) = 0.1093E+01
CD(WAVE) = 0.3586E+00

CD/CL**2 = 0.1801E+00
CT/CL**2 = 0.1721E+00

CD(WAVE)/CL**2 = 0.4449E-01

E(VTX) = 0.9392E+00

PLANFORM COMPOSED OF CONSTANT VORTICITY PANELS

CONTROL POINT AT PANEL CENTROID

DRAG POLAR

		CAMBERED		UNCAMBERED	
CL	ALPHA	CD(0)	CD(100)	CD(0)	CD(100)
0.0	0.0	0.0	0.0	0.0	0.0
0.033	0.673	0.00038	0.00020	0.00038	0.00020
0.067	1.345	0.00151	0.00080	0.00151	0.00080
0.100	2.018	0.00341	0.00180	0.00341	0.00180
0.133	2.691	0.00606	0.00320	0.00606	0.00320
0.167	3.363	0.00947	0.00500	0.00947	0.00500
0.200	4.036	0.01363	0.00720	0.01363	0.00720
0.233	4.708	0.01855	0.00980	0.01855	0.00980
0.267	5.381	0.02423	0.01280	0.02423	0.01280
0.300	6.054	0.03067	0.01620	0.03067	0.01620
0.333	6.726	0.03786	0.02000	0.03786	0.02000
0.367	7.399	0.04581	0.02421	0.04581	0.02421
0.400	8.072	0.05452	0.02881	0.05452	0.02881
0.433	8.744	0.06398	0.03381	0.06398	0.03381
0.467	9.417	0.07421	0.03921	0.07421	0.03921
0.500	10.090	0.08519	0.04501	0.08519	0.04501
0.533	10.762	0.09692	0.05121	0.09692	0.05121
0.567	11.435	0.10942	0.05781	0.10942	0.05781
0.600	12.107	0.12267	0.06482	0.12267	0.06482
0.633	12.780	0.13668	0.07222	0.13668	0.07222
0.667	13.453	0.15144	0.08002	0.15144	0.08002
0.700	14.125	0.16696	0.08822	0.16696	0.08822
0.733	14.798	0.18324	0.09682	0.18324	0.09682
0.767	15.471	0.20028	0.10582	0.20028	0.10582
0.800	16.143	0.21807	0.11523	0.21807	0.11523
0.833	16.816	0.23663	0.12503	0.23663	0.12503
0.867	17.488	0.25593	0.13523	0.25593	0.13523
0.900	18.161	0.27600	0.14583	0.27600	0.14583
0.933	18.834	0.29682	0.15684	0.29682	0.15684
0.967	19.506	0.31840	0.16824	0.31840	0.16824
1.000	20.179	0.34074	0.18004	0.34074	0.18004

PROGRAM CODE

The complete program is available from COSMIC (Computer Software Management and Information Center, 112 Barrow Hall, University of Georgia, Athens, GA 30602) as Potential Form Drag Program, LAR-12236. The program subroutines, and their principal functions, are summarized in the following table in alphabetical order.

SUBROUTINE	FUNCTION
AAMAIN	Controls program flow
AIJKL	Calculates the vorticity panel influence coefficients
AMTX	Reads the main aerodynamic matricies on or off of the disk units
CAMBER	Interpolates the input camber to obtain the boundary conditions
COUNTV	Computes the elapsed CPU time for a given program step
DECRD	Subroutine used to read the input data
DISPLY	Prints computed pressures and downwash
DRGMTX	Controls the main flow of the drag influence coefficient calculation
DRAG01	Controls the near field drag influence coefficient calculation
FMTX	Computes various interpolation coefficients for camber etc.
F2F3	Computes functions F2 and F3 in the near field drag influence coefficient calculation
GØMTRY	Computes planform geometry
HSHLDR	Linear equation solution routine using a least square solution if there are more equations than unknowns
INTRP3	A third order curve fit subroutine
LATTCE	Subroutine for planform geometry printout
LIFT	Computes planform lift, moments and drag
MAIN	Sets up the structure and size of the main arrays
MATRXF	Routine used to print computed arrays of pressure coefficient, velocities, etc.
MTXMLT	Multiplies matricies
PLNFRM	Computes some program geometry

SUBROUTINE	FUNCTION
POLAR	Computes the 0 and 100 percent suction drag polars
RLØGX	Computes R and F1 between all panel corners for the near field drag computation
TIMERV	Computes the elapsed CPU time for a giver program step
TRAPZD	Obtains the Cp distribution using constrant pressure panels
VTXDRG	Computes the vortex drag using a Trefftz plane solution
VELCTY	Calculates F1, F2 and F3 for the vorticity panel influence coefficients
WVEVTX	Calls the routines to perform the drag calculations and stores the results on disk units

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3. Ashley, H. and M. T. Landahl, Aerodynamics of Wings and Bodies, Addison-Wesley Publishing Company, Inc., 1965.

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5. Wagner, S., "On the Singularity Method of Subsonic Lifting Surface Theory," AIAA Journal of Aircraft, Vol. 6, No. 6, pp. 549-558, Nov-Dec. 1969.