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VARIATIONAL ENERGY PRINCIPLE FOR COMPRESSIBLE, BAROCLINIC FLOW
I. FIRST AND SECOND VARIATIONS OF TOTAL KINETIC ACTION

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ABSTRACT

As a preliminary to the construction of the effective total free-energy functional for general inviscid flow that is carried out in the following paper, the special case of a cold gas in the absence of external force fields is considered. Since the only energy involved is kinetic energy, the total kinetic action (i.e., the space-time integral of the kinetic energy density) should serve as the total free-energy functional in this case, and as such should be a local minimum for all possible fluctuations about stable flow. This conjecture is tested by calculating explicit, manifestly covariant expressions for the first and second variations of the total kinetic action in the context of Lagrangian kinematics. The first variation vanishes if the motion of every particle is rectilinear, and the second variation depends on the velocity shear. If the Lagrange surfaces are parameterized so as to correspond to incompressible flow, a generalization of Kelvin's Energy Theorem results, in the sense that the total kinetic action is shown to be a minimum not only for potential flow, but also for steady Beltrami flow and for unsteady flow that satisfies Euler's equation with spatially homogeneous Bernoulli constant. Which of these cases arises depends on the imposed boundary and terminal conditions, which are more general than those of Kelvin's Theorem. Finally, the general question of the correlation between physical stability and the convexity of any action integral that can be interpreted as the total free-energy functional of the flow is discussed and illustrated for the cases of rectilinear and rotating shearing flows.

The type of instability in question is characterized by an evolution that is consistent with the preservation of a specified average-velocity profile for the flow, and so corresponds to the onset of ordered turbulent motion within a given large-scale flow.

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VARIATIONAL ENERGY PRINCIPLE FOR COMPRESSIBLE, BAROCLINIC FLOW

I. FIRST AND SECOND VARIATIONS OF TOTAL KINETIC ACTION

I. INTRODUCTION AND SUMMARY

It is the purpose of this paper and the one that follows (which will be referred to as Paper II) to construct a variational principle for inviscid, compressible, baroclinic flow in a gravitational field that can serve as a basis for direct solution of the kinds of problems encountered in dynamic meteorology. By "direct solution" is meant the trial-and-error procedure in which the desired solution is expressed in terms of parameterized trial functions, and the numerical values of the parameters are determined by minimizing the action integral of the variational principle.

In Paper II this objective will be accomplished by constructing an action integral $\mathscr{A}$ whose first variation vanishes when the known equations of motion are satisfied, and whose second variation has the form of the time-average change in the total free-energy of the system that is produced by arbitrary fluctuations in the different variables of the problem about the extremal[1] trial functions that satisfy the equations of motion. Thus $\mathscr{A}$ can be interpreted as the time-average total free-energy of the flow, and its construction amounts to the extension of the formalism of thermodynamic potential functions to include a fluid in arbitrary motion. A stable flow corresponds to a local minimum (in parameter space) of this total free-energy, whereas an unstable flow corresponds to a saddle point whose properties are correlated to the physical nature of the instability.

The various thermodynamical questions will be left to Paper II. The present paper deals with the kinematics of the problem, and the only energy discussed will be the kinetic energy. The kinetic action integral $\mathscr{W}$ is the time-average of the total kinetic energy, and the variational principle based on $\mathscr{W}$ can be regarded as the fluid generalization of the Least Action Principle for particle motion. It is shown in Section IV A that the flow for which $\mathscr{W}$ is an extremum is pure rectilinear motion of the individual particles. This corresponds to the motion of a cold compressible gas in the absence of any external field.

In Section V it is shown that, if the trial functions are parameterized so that the fluctuations are density-preserving, then the extremal flow is no longer limited to rectilinear motion. Rather, it is the flow that satisfies Euler's equation for spatially uniform Bernoulli constant, a condition that can result from irreversible turbulent mixing. In other words, the extremal flow is of the kind that results after mixing has caused the fluid to "forget" its initial energy distribution. For time-independence, the extremum corresponds to Beltrami flow, which includes potential flow as a special case. For density-preserving fluctuations, the extremum is always a minimum. Thus the variational principle based on $\mathscr{W}$ with density-preserving fluctuations is a generalization of Kelvin's Energy Theorem,[2,10] which says that the total kinetic energy is less for potential flow than for any other incompressible flow having the same normal velocity component at the boundary. The reason that the extremal flow discussed in Section V is more general than the steady potential flow of Kelvin's Theorem is that the imposed boundary conditions are correspondingly more general.

Section VI A is a discussion of the relation between physical stability and the second variation of any action integral $\mathscr{A}$ which, like the one discussed in Paper II, can be interpreted as the total free-energy of the flow. (The action integral $\mathscr{W}$, whose extremal flows are restricted by the complete absence of any potential energy, is a rather uninteresting special case.) It is shown that the requirement that $\delta^{(2)} \mathscr{A} > 0$ for all possible fluctuations about the extremal flow constitutes a sufficient, but not necessary, condition for stability. The explicit expression for $\delta^{(2)} \mathscr{W}$ that is given in Section IV A is applied in Section VI B to derive sufficient criteria for stability of parallel shearing and rotating flows.

It is pointed out in Sections IV C, D and VI A that any stability criterion derived from a variational energy principle refers to a different kind of instability from that which is predicted by a normal-mode analysis of the linearized inviscid differential equations of motion. This latter variety, which is called "laminar instability" in this paper, is typified either by the onset of a self-excited growing sinuous oscillation, or else by the transition to an entirely different mode, such as the onset of convection. In both cases, the original velocity profile of the base flow is destroyed by the instability. In the case of the criterion $\delta^{(2)} \mathscr{A} > 0$, however, the instability, which is called "internal instability," is constrained to preserve the average-velocity profile of the base flow. The jet streams in the upper troposphere provide an example of both types of instability. A large-scale undulation, which completely changes the original velocity field of the jet is an example of "laminar instability," whereas the high-level clear-air turbulence that is encountered near the jet maximum, and consists of eddies embedded in the jet wind and carried with it, is an example of "internal instability." In Section VI B 2, it is noted that observations of high-level clear-air turbulence confirm that, for this type of instability, the internal stability criterion is indeed the appropriate one to use. In Section IV D it is pointed out that both kinds of instability are necessary for a complete description of turbulent flow.

The fact that a stability analysis based on a variational energy principle involves different modes of instability from those predicted by an analysis of the differential equations of motion is intimately related to the fact that, in order to solve the differential equations, it is necessary to specify the **initial** velocity of every particle, whereas in the case of a variational principle, it is effectively the **time-average** velocity that must be specified. This means, as explained in Section IV B, that a variational analysis corresponds to an eigenvalue problem which in general does not have a **single** unique solution as in the case of the differential-equation analysis, but rather has a whole **ensemble** of dynamically acceptable solutions. This difference, and its physical interpretation, is elaborated in Sections IV C and D.

At the end of Section VI A, it is pointed out that one way of partially bridging the gap between the extreme idealizations represented by the "laminar" and "internal" stability criteria is to relax the terminal and boundary conditions imposed on a variational analysis. A simple example involving the vertical mixing of different layers in a horizontal wind shear is given in Section VI C.

In Section VII it is shown that the first variation of $\mathscr{W}$ is invariant under a Galilean transformation, but not the second variation. This means that the convexity of $\mathscr{W}$ (or of any action integral of which it is a part) can be interpreted as a stability criterion only if the fluid velocity is referred to an appropriately chosen Galilean frame which must be identified with the "momentum reservoir" with which the fluid system under study is in equilibrium. The interpretation of the terminal and boundary conditions of the variational principle in terms of a momentum reservoir in interaction with the flow system under study is made in Section IV D.

The central analytical result of this paper is the explicit expression for $\delta^{(1\,+\,2)}\mathscr{W}$ that is given in Equation (48) of Section IV A. As a preliminary to deriving this expression, the formalism of Lagrangian kinematics is developed in Section II. It is absolutely necessary that the Lagrange, rather than the Euler, description be used. The physical reason for this is that, in order to find a functional that behaves like the total free-energy of the flow, it is necessary that the fluctuations of the trial functions imitate the physical fluctuations that actually occur in an inviscid fluid. First and foremost, this means that the fluctuations must be such that the conservation equation is **identically** satisfied for **all trial** functions, not just for the set of extremal functions. This is possible in the Lagrange formalism, which uses the three families of Lagrange surfaces $\Lambda^A(\boldsymbol{x}, t)$, $(A = 1, 2, 3)$, as the basic kinematical variables, but it is not possible in the Euler description, which treats the **four** functions $n(\boldsymbol{x}, t)$ and $V^j(\boldsymbol{x}, t)$, $(j = 1, 2, 3)$, as primitive and **independent**. In the Lagrange formalism n and **V** are secondary functions that are expressed in terms of the derivatives of Λ^A in such a way that the conservation equation is **identically** satisfied for arbitrary $\Lambda^A(\boldsymbol{x}, t)$.

In Section III the necessary functional differentials are derived. It is emphasized that, in order to obtain second-order functional differentials that can be assigned an **absolute** physical meaning in the sense of a stability criterion, the expressions must be manifestly covariant with respect to transformations both of the inertial coordinate system and of the system of Lagrange surfaces. The derivation can be done more straight-forwardly in an inertial frame than in the convected frame. However, in applying the variational principle to the numerical solution of a problem, only an evaluation of the **total action integral** is involved (rather than the derivation of a local functional differential), and this can be done as conveniently in the convected frame as in an inertial frame (much more so, in fact, if a free boundary is involved).

II. KINEMATICS OF LAGRANGE DESCRIPTION

A. Reciprocal Tripod Fields

The entire kinematics of the Lagrange description is generated either by the three functions $\Lambda^A(\boldsymbol{x}, t)$, or by the three functions $X^j(\Lambda^A, t)$ (or the single vector function $\mathbf{X}(\Lambda, t)$). The respective spatial derivatives of these functions generate two reciprocal tripod fields $\mathbf{G}^A$ and $\mathbf{G}_A$:

$$\mathbf{G}^A \equiv \mathbf{e}^j(\partial\Lambda^A/\partial x^j)_t \equiv \nabla\Lambda^A \; ; \tag{1}$$

$$\mathbf{G}_A \equiv (\partial\mathbf{X}/\partial\Lambda^A)_t \equiv \partial_A\mathbf{X} \; . \tag{2}$$

These tripod fields satisfy the following identities:

$$\mathbf{G}_A \cdot \mathbf{G}^B = \delta^B_A ; \; \mathbf{G}_A\mathbf{G}^A = \mathbf{G}^A\mathbf{G}_A = \mathcal{I} \tag{3a, b}$$

where the summation convention with respect to the index A is used, and δ^B_A and $\mathcal{I}$ are respectively the Kronecker delta and the idem dyadic. The tripod field $\mathbf{G}_A$ can be expressed directly in terms of the vectors $\mathbf{G}^B$, and hence in terms of derivatives of $\Lambda^A(\boldsymbol{x}, t)$, rather than those of $\mathbf{X}(\Lambda, t)$, by means of the following relation:

$$\mathbf{G}_A = \mathbf{G}^B \times \mathbf{G}^C / J^\Lambda_x \tag{4}$$

where J^Λ_x is the functional determinant (Jacobian) whose elements are the derivatives $G^A_j \equiv \partial\Lambda^A/\partial x^j$. A similar relation expresses the $\mathbf{G}^A$ tripod directly in terms of the derivatives $\partial X^j/\partial\Lambda^A$.

These tripod fields suffice to generate all of the mathematical quantities encountered in curvilinear tensor analysis.[3] For example, the metric tensor g_{AB} and the Christoffel symbol Γ^C_{AB} are given by

$$G_{AB} = \mathbf{G}_A \cdot \mathbf{G}_B \; , \tag{5}$$

$$\Gamma^C_{AB} = (\partial_A \mathbf{G}_B) \cdot \mathbf{G}^C = (\partial_A \partial_B \mathbf{X}) \cdot \mathbf{G}^C \; . \tag{6}$$

Thus the option exists either to work directly with the tripod fields, or else to use G_{AB} and Γ^{C}_{AC} in the conventional Riemannian formalism. In general, the first course is the simpler.

If U is an arbitrary vector, its contravariant form U^A in the convected frame is constructed by projection onto the G^A tripod, and the covariant form U_A by projection onto the G_A tripod:

$$U^A \equiv \mathbf{U} \cdot \mathbf{G}^A \; ; \; U_A \equiv \mathbf{U} \cdot \mathbf{G}_A \; . \tag{7a, b}$$

Although any vector can always be cast into either contravariant or covariant forms, it turns out that every physical vector has a preferred or "natural" form in terms of which the equations of motion assume their simplest form. For example, in the case of fluid dynamics the natural form of the velocity vector is V^A whereas the natural form of the molar momentum vector is MV_A. It is typical that canonically conjugate pairs such as these have natural forms that are contragredient so that a direct contraction yields a scalar.

The definition of the Lagrangian surfaces Λ^A at the initial time t_I is arbitrary and so also is the resulting Jacobian $(J^{\Lambda}_{x})_I$. Thus the normalization of the Lagrangian surfaces must be specified. In this paper and in Paper II they will be normalized so that at t_I the volume contained in every parallelepiped defined by neighboring pairs of Λ^A–surfaces that differ by unity is equal to the molar volume $\hat{V}$ of the fluid. Because these surfaces move with the fluid, this normalization will be maintained for all later time. Because $\hat{V}^{-1} = n$, the mole density, it follows that for the specified normalization

$$\hat{V} = J^{X}_{\Lambda} \equiv \det\left(\frac{\partial X^j}{\partial \Lambda^A}\right) , \tag{8a}$$

$$n = J^{\Lambda}_{x} = (\nabla\Lambda^{\underline{1}}) \cdot (\nabla\Lambda^{\underline{2}} \times \nabla\Lambda^{\underline{3}}) \tag{8b}$$

(where the indices have been underlined in order to indicate that they are numerical values of A rather than of j).

The following identities are very useful:

$$\partial\hat{V}/\partial G_A = \hat{V}G^A; \;\; \partial n/\partial G^A = nG_A; \tag{9a, b}$$

$$\partial_A(\hat{V}G^A) = 0; \;\; \nabla\cdot(nG_A) = 0; \tag{10a, b}$$

$$\partial G_A/\partial G^B = -G_B G_A; \;\; \partial G^A/\partial G_B = -G^B G^A; \tag{11a, b}$$

$$D_t G_A = G_A \cdot \nabla V; \;\; D_t G^A = -(\nabla V)\cdot G^A \tag{12a, b}$$

where D_t is the material time derivative (defined in (13) and (45c) below).

B. Convected Frame

In the convected frame of the fluid the independent coordinates are (Λ^A, t) and the kinematics is generated by the vector function $\mathbf{X}(\Lambda^A, t)$ which gives the time-dependent position vector $\mathbf{X}$ (referred to an inertial frame) of the particle that is identified by the Lagrange parameters Λ^A. The molar volume $\hat{V}$ is given in terms of $\mathbf{X}(\Lambda, t)$ by (8a), and the velocity $\mathbf{V}$ is given by

$$\mathbf{V} = (\partial_t \mathbf{X})_\Lambda \equiv D_t \mathbf{X}. \tag{13}$$

Because of the normalization chosen for the Λ-surfaces, the volume element $d^3(\Lambda)$ in the convected frame can be identified with the infinitesimal mole number dN.

Thus the total instantaneous kinetic energy W is given by

$$\mathsf{W} = \int_{(\Lambda)} \tfrac{1}{2}M(\partial_t \mathbf{X})_\Lambda \cdot (\partial_t \mathbf{X})_\Lambda \, d^3(\Lambda) = \int_N W dN \tag{14}$$

where M is the molar mass, and $W = \tfrac{1}{2}M\mathbf{V}\cdot\mathbf{V}$ is the molar kinetic energy, and N is the total number of moles of fluid contained in the region of integration. The total kinetic action $\mathcal{W}$ is defined by

$$\mathcal{W} \equiv \int_{t_I}^{t_F} \mathsf{W} dt = \int_{(\Delta t,\, N)} \tfrac{1}{2}M(\partial_t \mathbf{X})_\Lambda \cdot (\partial_t \mathbf{X})_\Lambda \, dt\, d^3(\Lambda) \tag{15}$$

where $\Delta t \equiv t_F - t_I$ is the time range of integration. Note that except for the factor Δt, $\mathcal{W}$ can be interpreted as the time average of the total kinetic energy:

$$\langle W \rangle_t \equiv \frac{1}{(t_F - t_I)} \int_{t_I}^{t_F} W dt = \mathcal{W}/\Delta t \,. \tag{16}$$

In fact, the time unit can be chosen so that $\Delta t = 1$, in which case $\mathcal{W}$ and $\langle W \rangle_t$ are numerically equal, and the two can be regarded as conceptual equivalents.

C. Inertial Frame

In the inertial (i.e., laboratory) frame the independent coordinates are $(\boldsymbol{x}, t)$. (Note that $\boldsymbol{x}$ is used to denote the position vector when it serves as the aggregate of **independent** spatial coordinates whereas $\mathbf{X}$ is used to denote the **dependent** variable $\mathbf{X}(\Lambda, t)$ that gives the position vector of a given particle. The two are, of course, numerically equal, i.e., $\boldsymbol{x} = \mathbf{X}$.) The position vector $\boldsymbol{x}$ will be taken to be expressed in Cartesian coordinates so that the volume element $d^3(x)$ is equal to the physical differential volume element dV.

The kinematics is generated by the three families of Lagrange surfaces $\Lambda^A(\boldsymbol{x}, t)$. The mole density n is given in terms of $\nabla\Lambda^A$ by (8b). As indicated in (7), the velocity $\mathbf{V}$ can be written in the form

$$\mathbf{V} = V^A \mathbf{G}_A ; \quad V^A \equiv \mathbf{V} \cdot \mathbf{G}^A \,. \tag{17}$$

Because $\mathbf{V}$, and hence V^A, is not a primitive variable (as in the Euler description) it is necessary to express V^A in terms of $\Lambda^A(\boldsymbol{x}, t)$. Such a relation results from substitution of $d\Lambda^A = 0$ and $d\mathbf{X} = \mathbf{V}\, dt$ into $d\Lambda^A = \mathbf{G}^A \cdot d\mathbf{X} + (\partial_t \Lambda^A)_x\, dt$. Making use of (17), the result is

$$V^A = -(\partial_t \Lambda^A)_x \; ; \quad \mathbf{V} = -(\partial_t \Lambda^A)_x \, \mathbf{G}_A \,. \tag{18}$$

The vectors $\mathbf{G}_A$ are expressed in terms of $\nabla\Lambda^B$ by means of (4). Thus the expression to be used for $\mathbf{V}$ in Lagrangian kinematics in an inertial frame is

$$V = - \sum_{A=1}^{3} (\partial_t \Lambda^A)_x (\nabla\Lambda^B \times \nabla\Lambda^C)/n; \quad (A, B, C \text{ cyclic}) \tag{19}$$

where n is given by (8b). When n and V are given by (8b) and (19) respectively, the conservation equation is **identically** satisfied:

$$(\partial_t n)_x + \nabla \cdot (nV) = 0. \quad \text{(identity)} \tag{20}$$

This means that, for an arbitrary choice for the three functions $\Lambda^A(\boldsymbol{x}, t)$, the Lagrangian kinematics will always satisfy conservation of matter. This is **not** the case for the Eulerian description in which $V(\boldsymbol{x}, t)$ and $n(\boldsymbol{x}, t)$ are primitive and independent variables. This is the reason why fluctuations in the functional forms of $\Lambda^A(\boldsymbol{x}, t)$ may be regarded as generating valid descriptions of the physical fluctuations in a turbulent fluid, whereas fluctuations in the functional forms of $V(x, t)$ and $n(\boldsymbol{x}, t)$ may not be so regarded.

In the inertial frame, the total kinetic action $\mathscr{W}$ is given by

$$\mathscr{W} = \int_{(V, \Delta t)} \tfrac{1}{2} MV \cdot V\, n\, d^3(x)\, dt \equiv \int_{(V, \Delta t)} w\, d^3(x)\, dt \tag{21}$$

where V and n are given by (19) and (8b) respectively, and V is the total instantaneous volume of integration, and $w \equiv nW$ is the instantaneous kinetic energy density.

D. Steady Flow

A limited class of steady flows can be treated very easily in the Lagrangian formalism by eliminating the time-dependent functions $\Lambda^A(\boldsymbol{x}, t)$ in favor of the time-independent functions $\lambda^A(\boldsymbol{x})$ by means of the relations

$$\Lambda^A(\boldsymbol{x}, t) = \lambda^A(\boldsymbol{x}) - V^A t \tag{22}$$

where the V^A are three given constants. Then

$$n \equiv J_x^\Lambda = J_x^\lambda = (\nabla\lambda^1) \cdot (\nabla\lambda^2 \times \nabla\lambda^3), \tag{23}$$

$$G^A = (\nabla\Lambda^A)_t = \nabla\lambda^A \equiv g^A, \tag{24}$$

$$G_A = G^A \times G^B/n = g^A \times g^B/n \equiv g_A\,, \quad (A, B, C \text{ cyclic}) \tag{25}$$

$$\begin{aligned} V &= -(\partial_t \Lambda^A)_x G_A = V^A g_A \\ &= \sum_{A=1}^{3} V^A (\nabla\lambda^B \times \nabla\lambda^C)/n\,. \quad (A, B, C \text{ cyclic}) \end{aligned} \tag{26}$$

It follows directly from (26) and the constancy of the V^A that

$$\nabla \cdot (nV) = 0\,. \quad \text{(identity)} \tag{27}$$

From (23) and (26) it is evident that the total kinetic energy W is

$$W = \int_V \tfrac{1}{2}MV \cdot V\, n\, d^3(x) = \int_V \tfrac{1}{2}M(g_{AB} V^A V^B)\, J_x^\lambda\, d^3(x)\,, \tag{28}$$

where $g_{AB} \equiv g_A \cdot g_B$ and g_A is given by (25).

The relation of the λ-formalism to the usual stream function formalism can be shown by considering the special case $V^A = (0, 0, 1)$. Then (26) becomes

$$nV = \nabla\lambda^1 \times \nabla\lambda^2 \tag{29}$$

which is the standard way of expressing the flux density nV in terms of two stream functions $\lambda^1(\boldsymbol{x})$ and $\lambda^2(\boldsymbol{x})$. The λ-formalism differs, however, from the stream function formalism in that the former is based on the three functions $\lambda^1(\boldsymbol{x})$, $\lambda^2(\boldsymbol{x})$, and $\lambda^3(\boldsymbol{x})$ whereas the latter is based on the three functions $\lambda^1(\boldsymbol{x})$, $\lambda^2(\boldsymbol{x})$, and $n(\boldsymbol{x})$ (where n is set equal to a constant for an incompressible fluid). The fact that the stream function formalism suppresses one of the three $\lambda^A(\boldsymbol{x})$ in favor of $n(\boldsymbol{x})$ means that the complete mapping necessary to transform from the $\boldsymbol{x}$-frame to the λ-frame is not available, so the stream function formalism can only be applied in the $\boldsymbol{x}$-frame whereas the λ-formalism can be applied in both the $\boldsymbol{x}$-frame and the λ-frame.

In the λ-frame (28) becomes

$$W = \int_N \tfrac{1}{2} M V^A V^B \left(\frac{\partial \boldsymbol{x}}{\partial \lambda^A}\right) \cdot \left(\frac{\partial \boldsymbol{x}}{\partial \lambda^B}\right) d^3(\lambda)\,. \tag{30}$$

For the special case $V^A = (0, 0, 1)$ this becomes

$$W = \int_N \tfrac{1}{2} M \left(\frac{\partial \boldsymbol{x}}{\partial \lambda^{\underline{3}}}\right) \cdot \left(\frac{\partial \boldsymbol{x}}{\partial \lambda^{\underline{3}}}\right) d^3(\lambda) \tag{31}$$

and it is evident that $\lambda^{\underline{3}}$ has taken over the role played by t in the time-dependent formalism.

Only a limited class of steady flows can be represented in the form (22). It will now be shown that the physical nature of this limitation consists of the fact that for closed (i.e., re-entrant) steady flows represented by (22) the average of the covariant strain-rate tensor taken over any closed streamline is zero. The same is true of many open-ended flows. This means that only steady flows that are shearless in this average sense can be represented by (22). The proof starts with the observation that the strain-rate tensor can be referred to the "natural" coordinate system defined by the λ-surfaces as follows:

$$e_j(\partial^j V^k + \partial^k V^j)\, e_k = g_A (\nabla^A V^B + \nabla^B V^A)\, g_B \tag{32}$$

where $\nabla^A V^B$ is the covariant gradient of the velocity defined by

$$\nabla^A V^B = g^{AC}\left(\frac{\partial V^B}{\partial \lambda^C} + \Gamma^B_{CD}\, V^D\right) \tag{33}$$

where Γ^B_{CD} is the Christoffel symbol defined on the tripods g_A and g^A in the manner analogous to that given in (6). For the flow under consideration $\partial V^B/\partial \lambda^C = 0$ and application of Gauss' theorem leads to the following result:

$$\int_V (\nabla^A V^B + \nabla^B V^A)\, n\, d^3(x) = -\int_{\mathscr{S}} g^{AB}\, nV \cdot d\mathscr{S}. \tag{34}$$

When the volume and surface integrals involved are taken over the volume and surface of a very fine elemental flow tube, the desired proof results. For simplicity, it will be stated for the special case $V^A = (0, 0, V^3)$. Then (34) becomes

$$\begin{aligned}\langle \nabla^A V^B + \nabla^B V^A \rangle_{\lambda^3} &\equiv \frac{1}{(\lambda_F^3 - \lambda_I^3)} \int_{\lambda_I^3}^{\lambda_F^3} (\nabla^A V^B + \nabla^B V^A)\, d\lambda^3 \\ &= -V^3 (g_F^{AB} - g_I^{AB})/(\lambda_F^3 - \lambda_I^3)\end{aligned} \tag{35}$$

where λ_I^3 and λ_F^3 are respectively the values of λ^3 at which the fluid enters and leaves the elemental flow tube. Thus the tube-average of the covariant strain-rate tensor is proportional to the difference between the values of the metric tensor g^{AB} at the two ends of the tube. If the flow is closed, this difference vanishes, and so too does the tube-average of the strain-rate tensor. The same is true for open-ended flow for which the geometry defined by the λ-surfaces is the same at the entry and exit surfaces of the flow. In this sense, we can say that (22) describes steady flow that on the average is shearless.

When the flow is independent of the λ^3-coordinate, the covariant strain-rate tensor vanishes at every point in the flow. This is the case for axially-symmetric rotational flow in which λ^3 is the azimuth angle. In this case V^3 is the angular velocity, and the fact that it is a constant means that the fluid moves as if in rigid rotation. Thus in this case (22) is unable to represent those non-rigid vortices, such as the potential vortex, that involve non-vanishing velocity shear.

It is possible to generalize the representation given in (22) so that the V^A are no longer constant and hence restricted to shearless flows. In fact, besides admitting arbitrary shear, both V^A and λ^A can be made time-dependent. This formalism, which combines the advantages of the Euler and Lagrange representations, will not be developed in this paper.

It should be emphasized that the general time-dependent Λ-formalism, which will be employed throughout the remainder of this paper, does not share any of the restrictions discussed above in connection with (22). The general Λ-formalism is perfectly capable of describing either steady or time-dependent shearing flow. However, steady shearing flow in the Λ-formalism is characterized by the fact that one or more of the G_A fields is time-dependent, even though the velocity field as seen in the inertial frame is time-independent.

III. FIRST AND SECOND FUNCTIONAL DIFFERENTIALS

A. Calculational Methods

If the objective were merely to calculate $\delta^{(1 + 2)} \mathscr{W}$, i.e., the first and second variations of the total kinetic action, then this could be done either in the convected frame using the expression for $\mathscr{W}$ given in (15), or in the inertial frame using (21). In either case the result would be expressions that would show the sensitivity of the **total** kinetic action to changes in the functional dependence of the three functions that describe the kinematics.

If, however, the objective is to obtain a **local** second-order expression that can be interpreted as a stability criterion, then the desired second-order expression must have an **absolute** significance that is independent of the choice of coordinates and of the definition of the Λ^A. This means that both the independent coordinates and the displacements (i.e., variations) appearing in the expression must be referred to an **inertial** frame. The reason for this is that second-order virtual changes in kinetic energy are being used as a measure of stability, and kinetic energy must be defined with respect to an inertial frame. Correspondingly, the Prinicple of Virtual work can be related to changes in kinetic energy only if the displacements involved are the particle displacements $\delta \mathbf{X}$. Then the virtual work done on the particle can be related to its change in kinetic energy. Thus it is necessary to derive an expression involving $\delta \mathbf{X}(\boldsymbol{x}, t)$ whose coefficients are also functions of $(\boldsymbol{x}, t)$.

In the convected frame the second of these requirements is violated because the independent variables are (Λ^A, t), and in the inertial frame the first requirement is violated because the variations involved are $\delta \Lambda^A (\boldsymbol{x}, t)$ rather than $\delta \mathbf{X}(\boldsymbol{x}, t)$. Thus in either case it would be necessary to transform the original expression for the variation in order to give it absolute significance. It turns out to be more straight-forward to perform the calculation in the inertial frame, and then eliminate $\delta\Lambda^A(\boldsymbol{x}, t)$ in favor of $\delta X(\boldsymbol{x}, t)$. It can be shown that these two displacements are related by

$$-\delta X(\boldsymbol{x}, t) = \overline{G}_A \, \delta\Lambda^A + \tfrac{1}{2}\overline{G}_C \, \overline{\Gamma}^C_{AB} \, \delta\Lambda^A \delta\Lambda^B + O^3(\delta\Lambda) \tag{36}$$

where $\overline{G}_A$ and $\overline{\Gamma}^C_{AB}$ are calculated from $\overline{\Lambda}^B(\boldsymbol{x}, t)$ by means of (4) and (6) respectively. The overhead bar is used to indicate the particular arbitrarily chosen functional dependence of $\Lambda^B(\boldsymbol{x}, t)$ about which the expansion is carried out. Thus $\delta\Lambda^A \equiv \Lambda^A(\boldsymbol{x}, t) - \overline{\Lambda}^A(\boldsymbol{x}, t)$ where the functional form of $\Lambda^A(\boldsymbol{x}, t)$ differs slightly from that of $\overline{\Lambda}^A(\boldsymbol{x}, t)$. The minus sign on the left side of (36) results from the fact that, if the fluid is carrying particles toward a region of **larger** Λ^A-values, then the Λ^A-values that flow past an observer fixed in the inertial frame appear to **decrease** in value.

Except for the minus sign, the first-order part of (36) is just the statement of the standard transformation law of a vector. The extra second-order contribution results from the fact that $-\delta X$ and $\delta\Lambda^A$ are not really different expressions for the same vector, but rather they are two physically different vectors. The vector δX is the displacement of a **single** particle as it traverses **many Cartesian coordinate planes**, whereas the $\delta\Lambda^A$ are the changes in Lagrange parameters of **many different** particles that flow past an observer fixed at the intersection of a **single given set of Cartesian planes.**

It is evident from (36) that an expression that is of first order in δX will appear to be of both first **and second** order in $\delta\Lambda^A$, and correspondingly for a first order expression in $\delta\Lambda^A$ that is converted into one involving δX. Thus, when the first-order variation of $\mathcal{W}$ expressed in terms of $\delta\Lambda^A$ is converted into an expression in δX, this expression will contain second-order terms which will contribute to the stability criterion.

Note that none of these subtleties enter into a first-order calculation of the variation, or a direct application of the Euler–Lagrange equations[4] to find the differential equations that must be satisfied if the first variation of the action integral is to vanish. What little attention the second-variation has received in the classical mathematical literature has concentrated on the problem (Jacobi's Accessory Problem[5]) of establishing a sufficient condition for a positive-definite second variation (in the form of the existence of solutions to a certain set of partial differential equations that can be derived from the action integral). From this point of view, the question of the **absolute significance** of the variation never arises.

It should be noted that, except for the sign on the left side of (36), this relation is identical to the one used in Riemannian geometry to introduce geodesic coordinates.[6] In the present context, the introduction of geodesic coordinates amounts to redefining the Lagrange parameters Λ^A in such a way that at a given instant t_0 (the moment at which the variations are being calculated) the redefined Lagrange surfaces coincide everywhere with the Cartesian planes of the inertial frame. This coincidence will not hold for $t \neq t_0$, but at $t = t_0$ the Christoffel symbol for the geodesic coordinates (i.e., the redefined Lagrange parameters) is everywhere zero. This fact can be used to simplify the calculation of $\delta^{(1+2)} \mathscr{W}$ at t_0, and because t_0 is arbitrarily chosen, the expressions that are derived can be applied at any instant of time.

The results given below in Section III B were calculated by the geodesic method just outlined, and in addition by an entirely independent, somewhat less abstract method. This second method involves using (36) directly to replace $\delta\Lambda^A$ by $\delta\mathbf{X}$. It also involves taking into account that the tripod vectors $\mathbf{G}^A$ and $\mathbf{G}_A$ fluctuate when the functional forms of $\Lambda^A(\boldsymbol{x}, t)$ fluctuate. In constructing an absolute vector, the components of the vector

after the change in the functional forms of $\Lambda^A(\boldsymbol{x}, t)$ must be contracted with the appropriate tripod vectors as calculated also **after** the change. This change in tripod vectors is most easily expressed in terms of δX by making use of several simple identities. If $\overline{X}$ and $X = \overline{X} + \delta X$ are the old and new particle positions respectively, and $(\overline{G}_A, \overline{G}^A)$ and (G_A, G^A) are the old and new tripods, then

$$\overline{G}_A \equiv (\partial\overline{X}/\partial\Lambda^A) = \partial(X - \delta X)/\partial\Lambda^A = G_A - \partial_A \delta X . \tag{37}$$

Contracting this with G^A and using (36), the desired identity results:

$$G^A\overline{G}_A = \mathcal{I} - \nabla\delta X \tag{38}$$

which shows that $G^A\overline{G}_A$ can be regarded as a generalization of the idem dyadic $\mathcal{I} = \overline{G}^A\overline{G}_A$. The reciprocal of (38) gives another identity:

$$\overline{G}^A G_A = (G^B\overline{G}_B)^{-1} = \mathcal{I} + \nabla\delta X + (\nabla\delta X) \cdot (\nabla\delta X) + O^3(\delta X) . \tag{39}$$

Similar identities can be found that involve $\delta\Lambda^A$ instead of δX:

$$G^A\overline{G}_A = \mathcal{I} + (\nabla\delta\Lambda^A)\overline{G}_A \tag{40a}$$

$$= \mathcal{I} + (\nabla\delta\Lambda^A) G_A + (\nabla\delta\Lambda^A)(\partial_A \delta\Lambda^B) G_B + O^3(\delta\Lambda) ; \tag{40b}$$

$$\overline{G}^A G_A = \mathcal{I} - (\nabla\delta\Lambda^A)\overline{G}_A + (\nabla\delta\Lambda^A)(\overline{\partial}_A \delta\Lambda^B)\overline{G}_B + O^3(\delta\Lambda) \tag{41a}$$

$$= \mathcal{I} - (\nabla\delta\Lambda^A) G_A \tag{41b}$$

where $\partial_A \equiv G_A \cdot \nabla$ and $\overline{\partial}_A \equiv \overline{G}_A \cdot \nabla$. Expressions for $\delta^{(1+2)}G_A$ and $\delta^{(1+2)}G^A$ result from the contraction of (39) and (38) with $\overline{G}_A$ and $\overline{G}^A$ respectively:

$$\delta^{(1+2)}G_A \equiv G_A - \overline{G}_A = \overline{G}_A \cdot \nabla\delta X + \overline{G}_A \cdot (\nabla\delta X) \cdot (\nabla\delta X) \tag{42a}$$

$$\delta^{(1+2)}G^A \equiv G^A - \overline{G}^A = -(\nabla\delta X) \cdot \overline{G}^A . \tag{42b}$$

As an example of the transformation of an expression from one involving $\delta\Lambda^A$ into one involving δX, consider the expression for $\delta^{(1+2)} S(\Lambda^A, t) \equiv S(\Lambda^A, t) - S(\overline{\Lambda}^A, t)$ where $S(\Lambda^A, t)$ is a specified functional relation. In this case it is easy to show that it is necessary to use nothing more than (6) and (36) in order to change the expression for

$\delta^{(1\,+\,2)}\,S$ from one involving $\delta\Lambda^A$ into one involving δX:

$$\delta^{(1\,+\,2)}\,S(\Lambda, t) = \delta\Lambda^A\,\overline{(\partial_A S)} + \tfrac{1}{2}\delta\Lambda^A\,\delta\Lambda^B\,\overline{(\partial_A\,\partial_B\,S)} \tag{43a}$$

$$= -\delta X \cdot \nabla\overline{S} + \tfrac{1}{2}(\delta X\delta X : \nabla\nabla\overline{S})\,. \tag{43b}$$

B. Functional Differentials of n, V, and w

The functional differential of the mole density n results from substitution of $\Lambda^A = \overline{\Lambda}^A + \delta\Lambda^A$ into $n \equiv \det\,(\partial\Lambda^A/\partial x^j)$ and subsequent conversion of the resulting expression in $\delta\Lambda^A$ into one involving δX. This was done by both of the methods outlined above, with the following results:

$$\delta^{(1)}\,n = -\,\nabla\cdot\,(\overline{n}\,\delta X)\,; \tag{44a}$$

$$\delta^{(1\,+\,2)}\,n = -\,\nabla\cdot\,\delta^{(1\,+\,2)}\,\mathbf{n} \tag{44b}$$

where

$$\delta^{(1\,+\,2)}\,\mathbf{n} = \overline{n}(\delta X + \delta X \cdot \nabla\delta X) - \tfrac{1}{2}\nabla \cdot (\delta X\,\overline{n}\,\delta X) \tag{44c}$$

or

$$\delta^{(1\,+\,2)}\,\mathbf{n} = (\overline{n} + \tfrac{1}{2}\delta^{(1)}\,n)\,(\delta X + \tfrac{1}{2}\delta X \cdot \nabla\delta X)\,. \tag{44d}$$

From (44b) it follows that $\delta^{(1\,+\,2)}\,n$ may be regarded as being caused by a density flux $\delta^{(1\,+\,2)}\,\mathbf{n}$ produced by the field of particle displacements δX.

The functional differential of V results from substitution of $\Lambda^A = \overline{\Lambda}^A + \delta\Lambda^A$ into (19) and subsequent conversion of the resulting expression in $\delta\Lambda^A$ into one involving δX. This was done by both the geodesic coordinate method, and by the more direct method involving the identities given in (37)-(42), and the following results were obtained:

$$\delta^{(1)}\,V = \overline{D}_t\,\delta X - \delta X \cdot \nabla\overline{V}\,; \tag{45a}$$

$$\delta^{(1\,+\,2)}\,V = \overline{D}_t(\delta X + \delta X \cdot \nabla\delta X) - \delta X \cdot \nabla(\overline{V} + \overline{D}_t\,\delta X) \tag{45b}$$

where

$$\overline{D}_t \equiv (\partial_t)_x + \overline{V} \cdot \nabla\,. \tag{45c}$$

Since $\overline{V} = \overline{D_t X}$, it follows that $\overline{D}_t\, \delta X$ is just the change in V that is required by the change δX. The term $-\delta X \cdot \nabla\overline{V}$ in (45a) can be interpreted as the **advected** change in V as follows: The matter that is located at the observation point $\boldsymbol{x}$ after the displacement δX was originally located at the point $\boldsymbol{x} - \delta X$ before the displacement. If velocity behaved like an embedded property of the matter, then $-\delta X \cdot \nabla\overline{V}$ would be the change in velocity that would result simply because the matter at the observation point had been changed.

The expression for $\delta^{(1+2)} w$ results simply from substitution of $n = \overline{n} + \delta^{(1+2)} n$ and $V = \overline{V} + \delta^{(1+2)} V$ into $w = \frac{1}{2} nMV \cdot V$ where $\delta^{(1+2)} n$ and $\delta^{(1+2)} V$ are given by (44b) and (45b) respectively:

$$\begin{aligned} \delta^{(1+2)} w = & - \nabla \cdot [\delta^{(1+2)} n(\overline{W} - \tfrac{1}{2}\delta X \cdot \nabla\overline{W} + M\overline{V} \cdot \overline{D}_t\, \delta X)] \\ & + \overline{n}M\overline{V} \cdot \overline{D}_t(\delta X + \delta X \cdot \nabla\delta X) + \tfrac{1}{2}\overline{n}M(\overline{D}_t\, \delta X) \cdot (\overline{D}_t\, \delta X) \qquad (46) \\ & - \tfrac{1}{2}\overline{n}M(\delta X\delta X : \nabla\nabla\overline{V}) \cdot \overline{V}\,. \end{aligned}$$

All of the terms on the right have simple intuitive interpretations. The first term says that the effective flux of kinetic energy density is just equal to the density flux $\delta^{(1+2)} n$ multiplied by the molar kinetic energy $\overline{W}$, corrected for the average advected molar kinetic energy $-\frac{1}{2}\delta X \cdot \nabla\overline{W}$ and the increase $M\overline{V} \cdot \overline{D}_t\, \delta X$ that results from the velocity increase $\overline{D}_t\, \delta X$. The second and third terms on the right give the expected increase in kinetic energy that results from the velocity change caused by δX. The last term on the right side of (46) can be interpreted in terms of advection as follows: If V were an embedded fluid property like the function S in (43), then the second-order change in it produced by δX would be $\delta_x^{(2)} V \equiv \frac{1}{2}(\delta X\delta X : \nabla\nabla\overline{V})$ and the corresponding change in molar kinetic energy would be $\delta_x^{(2)} W = \frac{1}{2}M(\delta X\delta X : \nabla\nabla\overline{V}) \cdot \overline{V}$. Thus the last three terms in (46) can be interpreted as the mole density $\overline{n}$ multiplied by the increase in molar kinetic energy at the observation point that exceeds the second-order energy that is advected to the observation point.

IV. VARIATIONAL ENERGY PRINCIPLE BASED ON TOTAL KINETIC ACTION

A. General Expression for $\delta^{(1+2)}\mathscr{W}$

The expression for $\delta^{(1+2)}\mathscr{W}$ follows directly from (21) and (46). By means of partial integration the expression can be cast into the form presented below. This will be written in terms of $[\delta^{(1+2)}\mathscr{W}]_T$, $[\delta^{(1+2)}\mathscr{W}]_S$, and $[\delta^{(1+2)}\mathscr{W}]_V$ which are respectively the terminal, surface, and volume contributions, the last of these being represented as the sum of five contributions;

$$[\delta^{(1+2)}\mathscr{W}]_V = [\delta^{(1)}\mathscr{W}]_V + \sum_{N=1}^{4} [\delta^{(2)}\mathscr{W}]_V^{(N)} . \tag{47}$$

The general expression for $\delta^{(1+2)}\mathscr{W}$, which in no way restricts the choice of trial functions (indicated by an overhead bar) about which the fluctuations take place, is the following:

$$\begin{aligned}\delta^{(1+2)}\mathscr{W} &= \int_{(V,\Delta t)} [\delta^{(1+2)}w]\, d^3(x)\, dt \\ &= [\delta^{(1+2)}\mathscr{W}]_T + [\delta^{(1+2)}\mathscr{W}]_S + [\delta^{(1)}\mathscr{W}]_V + \sum_{N=1}^{4} [\delta^{(2)}\mathscr{W}]_V^{(N)}\end{aligned} \tag{48a}$$

where

$$\begin{aligned}[\delta^{(1+2)}\mathscr{W}]_T &= \int_{V_F} [\bar{n} M\bar{V} \cdot (\delta X + \delta X \cdot \nabla \delta X)]_{t=t_F}\, d^3(x) \\ &\quad - \int_{V_I} [\bar{n} M\bar{V} \cdot (\delta X + \delta X \cdot \nabla \delta X)]_{t=t_I}\, d^3(x) ;\end{aligned} \tag{48b}$$

$$[\delta^{(1+2)}\mathscr{W}]_S = -\int_{(\mathscr{S},\Delta t)} (\bar{W} - \tfrac{1}{2}\delta X \cdot \nabla\bar{W} + M\bar{V} \cdot \bar{D}_t \delta X + M\bar{A} \cdot \delta X)\, \delta^{(1+2)}\mathbf{n} \cdot d\mathscr{S}\, dt ; \tag{48c}$$

$$[\delta^{(1)}\mathscr{W}]_V = -\int_{(V,\Delta t)} M\bar{A} \cdot \delta X \bar{n}\, d^3(x)dt; \tag{48d}$$

$$[\delta^{(2)}\mathscr{W}]_V^{(1)} = \int_{(V,\Delta t)} M[-(\delta^{(1)}n)\,\bar{A} \cdot \delta X + \tfrac{1}{2}\bar{n}(\delta X\delta X : \nabla\bar{A})]\, d^3(x)\, dt\, ; \tag{48e}$$

$$[\delta^{(2)}\mathscr{W}]_V^{(2)} = \int_{(V,\Delta t)} \tfrac{1}{2}M(\bar{D}_t\delta X) \cdot (\bar{D}_t\delta X)\,\bar{n}\, d^3(x)\, dt\, ; \tag{48f}$$

$$[\delta^{(2)}\mathscr{W}]_V^{(3)} = \int_{(V,\Delta t)} \tfrac{1}{2}M(\delta X\delta X : \nabla\bar{A})\,\bar{n}\, d^3(x)\, dt\, ; \tag{48g}$$

$$[\delta^{(2)}\mathscr{W}]_V^{(4)} = -\int_{(V,\Delta t)} \tfrac{1}{2}M(\delta X\delta X : \nabla\nabla\bar{V}) \cdot \bar{V}\,\bar{n}\, d^3(x)\, dt \tag{48h}$$

where $\bar{A} \equiv \bar{D}_t\bar{V} = \partial_t\bar{V} + \bar{V} \cdot \nabla\bar{V}$ is the local acceleration.

Because the displacement field $\delta X(x, t)$ is arbitrary, it follows that a necessary condition for $\delta^{(1)}\mathscr{W} = 0$ is that

$$\tilde{A} \equiv \tilde{D}_t\tilde{V} \equiv (\partial_t\tilde{V})_x + \tilde{V} \cdot \nabla\tilde{V} = 0 \tag{49}$$

throughout the space–time volume (V, Δt). The tilde, which has replaced the overhead bar, indicates quantities that are appropriate to **extremal** flow, i.e., for which the condition $\delta^{(1)}\mathscr{W} = 0$ is satisfied. The condition (49) is necessary for $\delta^{(1)}\mathscr{W} = 0$, but is not sufficient because the first-order variation of the terminal and surface integrals given in (48b) and (48c) must also be made to vanish. These integrals will be discussed in Section IV B.

The necessary condition for the first-order vanishing of $\delta\,\mathscr{W}$ that is given in (49) is the Euler–Lagrange equation[4] for a variational principle for which the Lagrangian density is just the kinetic energy density expressed in terms of Lagrangian kinematics. The extremal flow described by (49) is **rectilinear** in the sense that every particle trajectory in space–time is a straight-line. Note, however, that this does not require that the trajectories be parallel.

If (15) is written in the form

$$\mathscr{W} = \int_N \left[\int_{t_I}^{t_F} w\, dt \right] dN \tag{50}$$

it is evident that the variational principle based on $\mathscr{W}$ is just the fluid generalization of the Least Action Principle of Maupertuis and Euler.[7] The particle action is simply replaced by the molar action, and the total action $\mathscr{W}$ is the sum of the individual molar actions.

Such a variational principle can be given a geometric interpretation in the sense that it generates a geometry whose geodesics are determined by the Euler–Lagrange equation given in (49), which are just straight lines. The molar action $\int w\, dt$ plays the role of an arc–length. This interpretation is very well known[8] in the case of particle dynamics. This interpretation can be extended to a more general variational principle, such as the one developed in Paper II, in which the kinetic action $\mathscr{W}$ is replaced by a more general total action $\mathscr{A}$. The Euler–Lagrange equations generated by $\mathscr{A}$ determine the system of "geodescis" to be associated with $\mathscr{A}$. For a complete specification, however, it is necessary to specify the end–points $\mathbf{X}_I$ and $\mathbf{X}_F$ of each particle trajectory (geodesic) as well as the Euler–Lagrange equation that it obeys. The specification of these end–points for all particle trajectories that traverse the space–time region over which $\mathscr{W}$ or $\mathscr{A}$ is integrated constitutes the terminal conditions, and part of the boundary conditions, that must be imposed on the variational principle. Before discussing these in Section IV B, some observations will be made regarding $[\delta^{(2)}\mathscr{W}]_V$.

For the extremal trial functions that satisfy (49), it is obvious that the integrals given in (48e) and (48g) vanish, with the result that

$$[\delta^{(2)}\widetilde{\mathscr{W}}]_V = \int_{(V,\Delta t)} \tfrac{1}{2}M[(\widetilde{D}_t\delta X)^2 - (\delta X\delta X{:}\nabla\nabla\widetilde{V})\cdot\widetilde{V}]\widetilde{n}d^3(x)dt, \qquad (51)$$

where the overhead tilde indicates quantities appropriate to the extremal trial functions for which (49) is satisfied. The first term in the integrand of (51) is positive-definite regardless of the properties of the extremal flow, but the sign of the second term depends on the spatial dependence of $\widetilde{V}(\boldsymbol{x}, t)$, and it will be this term that determines the stability of parallel shearing flow, which will be discussed in Section VI B1. The positive-definite expression $(\widetilde{D}_t\delta X)^2$ can be made small by specifying $\delta X(\boldsymbol{x}, t)$ to have the weakest possible t-dependence. Because the end-points X_I and X_F must be held fixed during the variation, a certain time-dependence of $\delta X(\boldsymbol{x}, t)$ is unavoidable if the trajectory $X(\Lambda, t)$ is displaced from its extremal value $\widetilde{X}(\Lambda, t)$. However, this minimal time-dependence can be made arbitrarily small by making the time interval $\Delta t \equiv t_F - t_I$ arbitrarily large. For this reason, in calculating the stability of shearing flow, the first term in (51) will be neglected. It will be seen that this is a conservative procedure in finding a sufficient condition for stable flow.

Note that $\widetilde{n}\, d^3(x) = \widetilde{J}_x^{\Lambda}\, d^3(x) = d^3(\Lambda) = dN$ is just the infinitesimal mole number. Thus (51) could be expressed as an integral over mole number (and time).

B. Terminal and Boundary Conditions

It was noted above that the Euler–Lagrange equations of any action integral $\mathscr{A}$ could be regarded as the determining equations of a system of "geodesics" whose complete determination further requires that the end-points X_I and X_F of every geodesic (i.e., particle trajectory) be specified. If $\mathscr{A}$ or $\mathscr{W}$ is being discussed in the context of the convected frame this means that $X_I(\Lambda) \equiv X(\Lambda, t_I)$ and $X_F(\Lambda) \equiv X(\Lambda, t_F)$ are specified functions of Λ^A, and all trial functions $X(\Lambda, t)$ must be equal to these specified functions for $t = t_I$ or $t = t_F$. In the context of an analysis in an inertial frame $\Lambda_I^A(\boldsymbol{x}) \equiv \Lambda^A(\boldsymbol{x}, t_I)$ and

$\Lambda^A_F(\boldsymbol{x}) \equiv \Lambda^A(\boldsymbol{x}, t_F)$ are **specified** functions of $\boldsymbol{x}$, and all trial functions $\Lambda^A(\boldsymbol{x}, t)$ must be equal to these at t_I and t_F. In other words,

$$[\delta\Lambda^A]_{t = t_I, t_F} \equiv [\Lambda^A(\boldsymbol{x}, t) - \overline{\Lambda}^A(\boldsymbol{x}, t)]_{t = t_I, t_F} = 0 . \tag{52a}$$

The $\delta\mathbf{X}$ that appears in (48b) is determined by $\delta\Lambda^A$ by means of (36). Thus it follows from (52a) and (36) that

$$\delta\mathbf{X}_I = \delta\mathbf{X}_F = 0 . \tag{52b}$$

It follows from (52) and (48b) that

$$[\delta^{(1+2)}\mathscr{W}]_T = 0 \tag{53}$$

if all trial functions $\Lambda^A(\boldsymbol{x}, t)$ reduce to the specified functions $\Lambda^A_I(\boldsymbol{x})$ and $\Lambda^A_F(\boldsymbol{x})$ for $t = t_I$ and $t = t_F$ respectively. Note that (53) is valid even if the $\overline{\mathbf{V}}$ that appears in (48b) is not the extremal velocity field that satisfies the Euler–Lagrange equations.

In discussing the surface integral given in (48c), it is necessary to distinguish between two kinds of surfaces – **access** surfaces (designated by $\mathscr{S}_A$), and **bounding** surfaces (designated by $\mathscr{S}_B$). Access surfaces are those through which fluid passes. These are further subdivided into **entry** and **exit** surfaces depending on whether the fluid enters or leaves the integration volume V. Bounding surfaces, either **rigid** or **free**, are those through which fluid never passes.

The surface condition to be imposed on an access surface is really just an extension of the terminal conditions. This is illustrated in Figure 1. Figure 1A shows a case in which the space–time integration volume (V, Δt) has been chosen in such a way that no fluid enters or leaves V. Thus only bounding surfaces $\mathscr{S}_B$ are involved. The end-points of all the trajectories lie in the terminal volumes V_I and V_F. If, however, the space–time volume of integration were chosen as shown in Figure 1b, then **for the same set of trajectories** many of the end-points would fall on $\mathscr{S}_A$ rather than on either of the terminal volumes. It is still necessary that the end-points of the trajectories be specified, but now this requirement has been partially shifted from the terminal volumes to the access surface

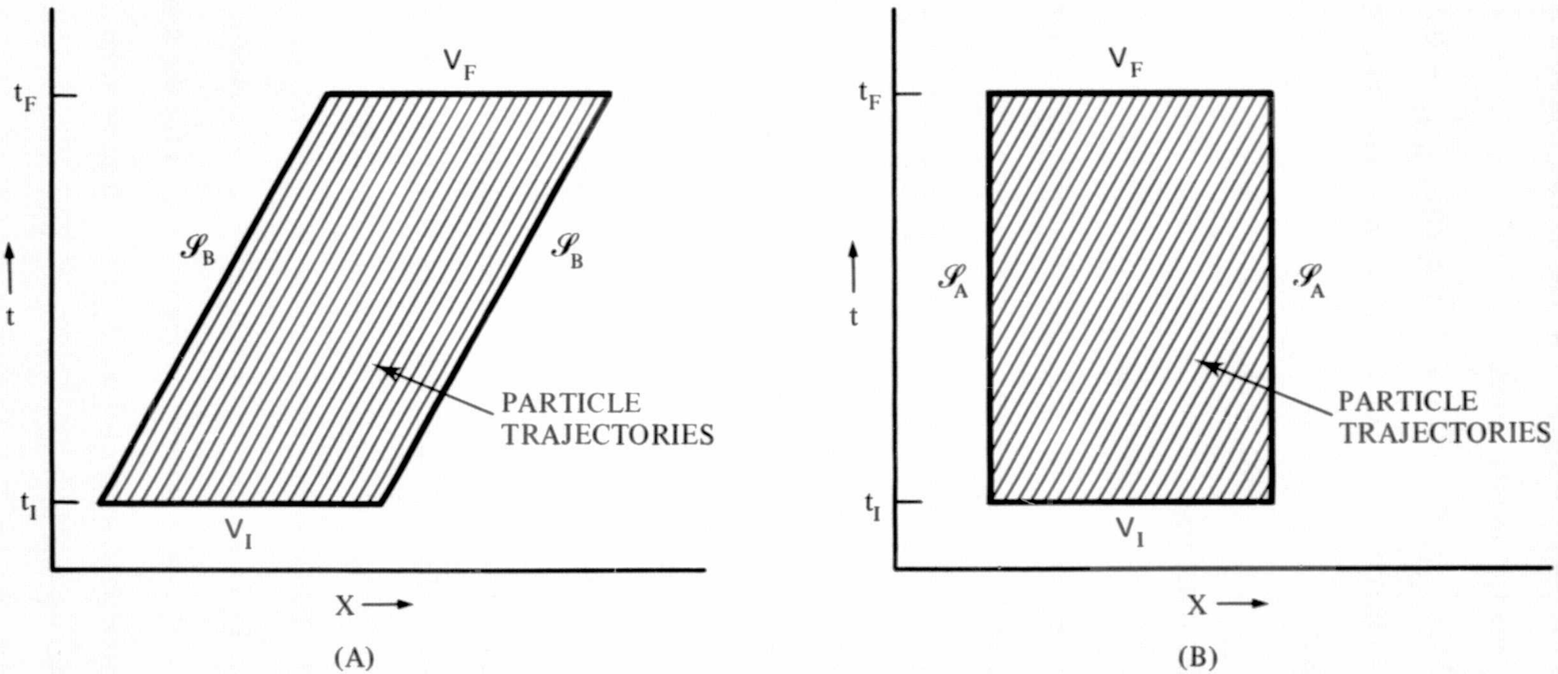

Figure 1. Space–Time Integration Volume. (A) Trajectories Cross Bounding Hypersurface Only at Terminal Volumes V_I and V_F but not at Bounding Surface $\mathscr{S}_B$. (B) Trajectories Also Cross Bounding Hypersurface at Access Surface $\mathscr{S}_A$.

$\mathscr{S}_A$. From the perspective of four-dimensional space-time, the terminal surfaces V_I and V_F and the totality of bounding and access surfaces constitute a single **hypersurface** that encloses the four-volume (V, Δt). The combined terminal and boundary conditions then require that the point at which any particle trajectory **crosses** the bounding hypersurface must be specified, and must be the same for all trial functions. Wherever a trajectory is **tangent** to the hypersurface, no condition is imposed except for the one that defines tangency:

$$\delta X \cdot d\mathscr{S} \equiv \delta X \cdot d\mathscr{S}_B = 0; \ (\delta X \times d\mathscr{S}_B) \text{ is arbitrary.} \tag{54a}$$

Wherever the condition $\delta X \cdot d\mathscr{S} = 0$ is not fulfilled, $d\mathscr{S}$ is by definition an element of an access surface and the requirement is

$$\delta X = 0 \text{ if } \delta X \cdot d\mathscr{S} \neq 0. \tag{54b}$$

The requirements stated in (54) together with (52b) constitute the conditions to be imposed on the trial functions on the bounding hypersurface.

It follows from (44d) that

$$\delta^{(1+2)}\mathbf{n} = 0 \text{ if } \delta X = 0. \tag{55a}$$

Moreover, it can be shown that

$$\delta^{(1+2)}\mathbf{n} \cdot d\mathscr{S} = 0 \text{ if } \delta X \cdot d\mathscr{S} = 0. \tag{55b}$$

From (48c), (54), and (55) it follows that

$$[\delta^{(1+2)}\mathscr{W}]_S = 0 \tag{56}$$

if the surface conditions (54) are satisfied. From (48a), (53), and (56) it is evident that

$$\delta^{(1+2)}\mathscr{W} = [\delta^{(1)}\mathscr{W}]_V + \sum_{N=1}^{4} [\delta^{(2)}\mathscr{W}]_V^{(N)} \tag{57}$$

if the combined hypersurface conditions of (52) and (54) are satisfied. This is true for fluctuations about any arbitrary set of trial functions. For fluctuations about the extremal set, the first-order term on the right-hand side of (57) vanishes (as do two of the four second-order terms).

The surface conditions (55) guarantee that the total matter N contained within V at every time instant is the same for all trial functions, i.e.

$$\delta^{(1+2)}N = 0 \text{ for } t_I \leqslant t \leqslant t_F \tag{58}$$

if the surface conditions stated in (55) are satisfied.

The terminal conditions (52) can be given an equally simple interpretation if it is noted that the time-average velocity $\langle V \rangle_t$ of a particle is given by

$$\langle V \rangle_t = (X_F - X_I) / (t_F - t_I). \tag{59}$$

(For trajectories traversing an access surface instead of a terminal volume, the $\langle V \rangle_t$ is given by an obvious modification of (59).) The point to be made is that the imposed hypersurface conditions guarantee that, regardless of the form that the trial functions might assume, the time-average velocity of every particle is always the same. Thus

$$\delta \langle V \rangle_t = 0 \tag{60}$$

for every given set of Λ^A values.

The fact that N and $\langle V \rangle_t$ are held constant constitutes a form of "normalization" of the problem that has the effect of bounding $\mathscr{W}$ from below. Thus the "Fluid Least Action Principle" based on $\mathscr{W}$ with the hypersurface conditions discussed above, has many features in common with the more familiar variational principle that can be constructed from any linear eigenvalue equation.[9] If H is an energy operator and E is one of its eigenvalues, and ϕ the corresponding eigenfunction, then the eigenvalue equation is $H\phi = E\phi$, and this can be solved by finding the stationary points of the functional $\langle \phi | H | \phi \rangle / \langle \phi | \phi \rangle$ where the brackets represent appropriate integrals. The normalization $\langle \phi | \phi \rangle = 1$ can be imposed as a constraint, in which case the action integral is simply $\langle \phi | H | \phi \rangle$. This integral is the analog of $\mathscr{W}$, and the normalization condition on ϕ is the analog of the hypersurface conditions (52) and (54). Both types of normalization bound the respective action integrals from below. Each is also necessary in order to remove an ambiguity: The normalization of ϕ is not determined by the linear eigenvalue equation, and so must be specified; and the Euler–Lagrange equations of $\mathscr{W}$ do not suffice for a unique specification of the trajectories until the end–points are specified.

This analogy indicates that the variational principle based on $\mathscr{W}$, or a generalization of it based on the total action $\mathscr{A}$, may be regarded as a complicated (nonlinear) eigenvalue problem. This way of thinking will now be used to make the point that, whereas the extremal flow for $\mathscr{W}$ with given hypersurface conditions is unique, this is not in general the case for a more complicated variational principle.

C. Non–Uniqueness of Extremal Flow

In the case of the variational principle based on $\mathscr{W}$, the fact that the extremal trajectories must be straight lines in space–time means that, once the end–points of these lines on the bounding hypersurface are specified, the solution is uniquely determined because there exist no other straight lines that can be passed through the same end–points.

In the case of a generalized variational principle of the kind that will be developed in Paper II, the extremal trajectories are no longer straight lines, and so it can no longer be asserted that a specification of the end–points on the bounding hypersurface, together with the Euler–Lagrange equations, suffices for a unique determination of the flow. In fact, the analogy with the eigenvalue problem discussed above indicates that this is **not** the case because an eigenvalue problem in general possesses a whole spectrum of solutions, each one having its own characteristic symmetry properties.

The point can be made in terms more appropriate to fluid dynamics as follows: If, in addition to the kinetic energy, various forms of potential energy are also included in the dynamics described by the action integral $\mathscr{A}$, then wave motion is possible. Thus, for given end–points on the hypersurface, in addition to the "smoothest possible flow" that connects these end–points and also obeys the Euler–Lagrange equations, it is also possible to obtain a dynamically acceptable flow by superimposing on the smooth flow one or more modes of wave motion. Besides the multiplicity that results from such superposition, there is the nonlinear "flip–flop" type of multiplicity that can involve a change in the topology of the flow. An example of this kind of multiplicity is the sudden appearance or disappearance of a vortex street or a bubble of back–flow embedded in the main flow.

From the point of view of using a variational principle to carry out a direct numerical solution of a flow problem, this multiplicity of solutions would be interpreted to mean that the variational problem had been incompletely specified. In addition to conditions imposed at the bounding hypersurface, it would be necessary to specify sufficient "internal boundary conditions" in order to make the extremum unique. This would mean specifying such things as the amplitude and mode of any wave motion to be superimposed on the smooth flow.

From a physical point of view, however, it must be recognized that such supplementary internal boundary conditions are not in general supplied by nature, and the multiplicity of solutions must be regarded as constituting an **ensemble of dynamically acceptable flows.** The role of this ensemble in the overall description of turbulent flow will be discussed in Subsection D below.

The multiplicity of solutions that in general characterizes a variational principle stands in striking contrast to the uniqueness of the solution of the **initial–value** problem that is based on the same set of Euler–Lagrange equations. In the initial-value problem for a fluid, instead of specifying $\mathbf{X}_I$ and $\mathbf{X}_F$, or what amounts to the same thing, $\mathbf{X}_I$ and $\langle \mathbf{V} \rangle_t$, we specify $\mathbf{X}_I$ and the **initial** velocity $\mathbf{V}_I$. That is, the **average** velocity in the variational principle is replaced by the **initial** velocity in the differential–equation approach in which the Euler–Lagrange equations are solved directly. This replacement suffices to make the solution of the differential equations unique. In other words, although the same amount of information is specified in the two cases, the fact that it involves **average** velocity in one case as compared with **initial** velocity in the other has remarkable consequences. This difference is illustrated in Figure 2.

There is a simple physical explanation of this difference. In the initial–value problem, by specifying initial velocity and particle position, we are also specifying energy and momentum distribution, as well as all the spatial symmetries and topological features that characterize the flow at t_I. None of these things are specified, however, in the average–value (i.e., variational) approach. For example, modes of flow having very different total energies can satisfy the same conditions on $\mathbf{X}_I$ and $\langle \mathbf{V} \rangle_t$. Thus, in the **average–value** approach, we are effectively specifying an **ensemble of flows** that is characterized by given average conditions, whereas in the **initial–value** approach we are discussing the deterministic time–evolution of a completely specified **single flow**.

D. Physical Interpretation of Terminal and Boundary Conditions

The requirement that the average value of the velocity of every particle must be held constant can be explained in terms of a thermodynamic analogy. If a system is immersed in a heat bath, or, more exactly, an entropy reservoir, then the free exchange of entropy between system and reservoir maintains the temperature of the system (or, more exactly, the time–average of the temperature, since fluctuations are always present) equal to the

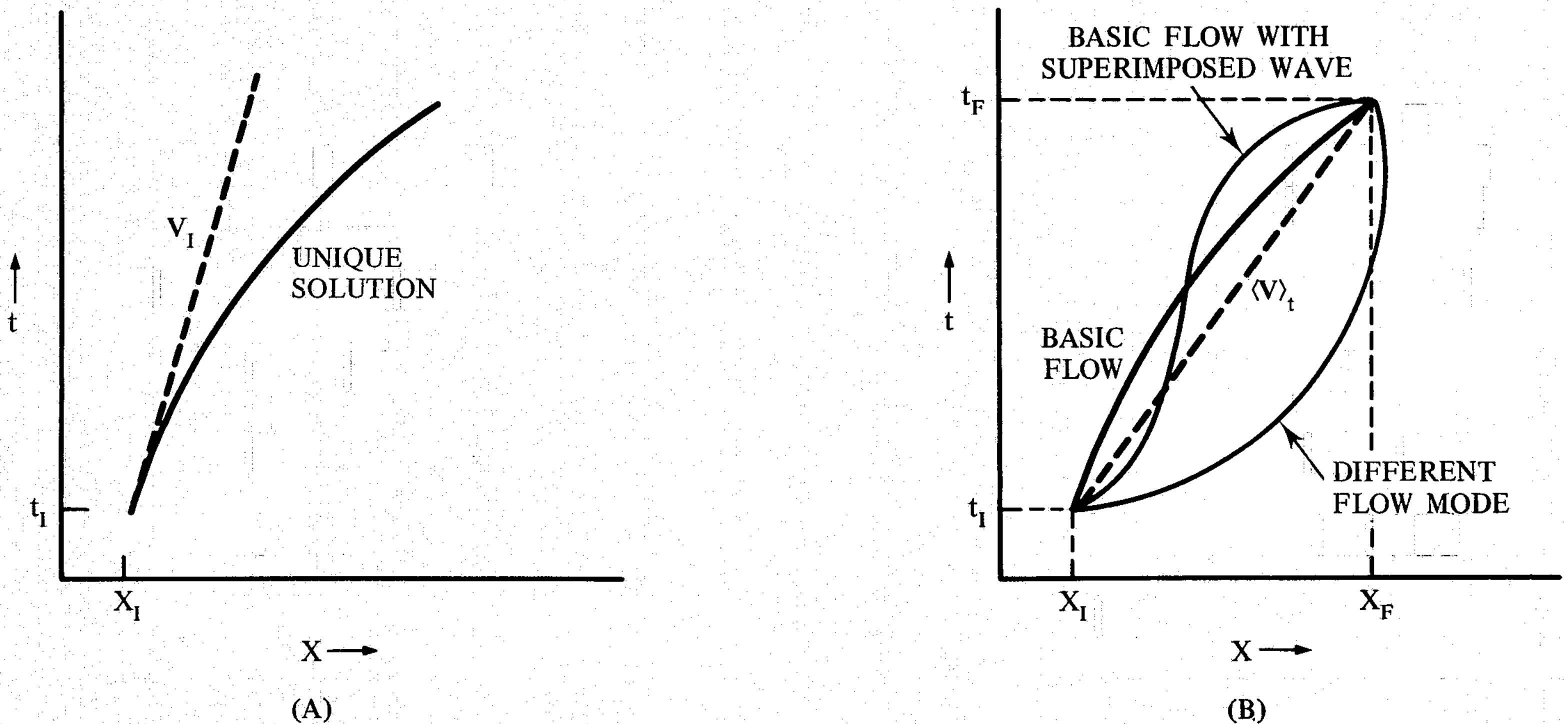

Figure 2. Dynamically Acceptable Flows. (A) Unique Solution of Differential Equations of Motion with Prescribed Initial Velocity V_I. (B) Ensemble of Extrema of Variational Principle with Prescribed Average Velocity $\langle V \rangle_t$

constant temperature of the reservoir. Temperature and entropy are conjugate variables, and the behavior of an entropy reservoir is typical of any thermodynamic reservoir: The extensive variable is freely exchanged between system and reservoir, and this keeps the average value of the corresponding intensive variable of the system equal to that of the reservoir. In the case of a moving system in contact with a momentum reservoir, this contact would keep the average value of the velocity of the system equal to the given velocity of the momentum reservoir. The same argument applies even if the reservoir velocity is a function of the spatial coordinates such as $\langle \mathbf{V} \rangle_t$. Thus the fluid described by the variational principle may be regarded as a system in contact with a momentum reservoir characterized by the spatially-dependent velocity $\langle \mathbf{V} \rangle_t$.

Within this system various dynamically acceptable modes of motion are possible, and these constitute an ensemble of "internal" fine-scale motions that are consistent with the imposed "mean" motion $\langle \mathbf{V} \rangle_t$ (which could be a slow function of **time** as well as a function of the spatial coordinates). This decomposition of the velocity field into a mean velocity and a fine-scale "internal" velocity corresponds exactly to the usual way of discussing turbulent flow in terms of a mean motion that satisfies the Euler equation (in the inviscid limit) to which a Reynolds force term has been added, and a superimposed fine-scale motion that determines the Reynolds force. This force is given by the divergence of the Reynolds stress tensor whose components are given by (minus) the local time averages $\langle (\Delta V^j)(\Delta V^k) \rangle_t$ where $\Delta \mathbf{V} \equiv \mathbf{V} - \langle \mathbf{V} \rangle_t$ is the deviation of the instantaneous velocity from the time-average velocity. Obviously, this stress tensor is different for the different modes of internal motion. The most stable mode will dominate. If this is "smooth," i.e., if the velocity deviation $\Delta \mathbf{V}$ and its derivatives are small, then the Reynolds force is small and the mean flow satisfies an equation that is nearly identical to the Euler equation, and so the flow is called "laminar." If, however, the smoothest possible internal mode is not the most stable one and the mode with the least free energy is considerably less smooth, then the Reynolds force is significant, and the equation of motion for the mean flow will differ significantly from the Euler equation. In this case the flow is called "turbulent." In such a case the flow problem could, in principle, be solved by the following iterative procedure: (1) The mean flow is solved using Euler's equation without the Reynolds force term. (2) Using the calculated mean flow to specify $\langle \mathbf{V} \rangle_t$, an appropriate variational principle is used to find the most stable fine-scale flow (which may or may not be smooth) that is consistent with the given $\langle \mathbf{V} \rangle_t$. (3) The Reynolds force field for this fine-scale flow is calculated. (4) This force field is added to the equation of motion for the mean flow, and this is solved again in order to determine $\langle \mathbf{V} \rangle_t$ more accurately. Step (2) is then repeated, etc.

This argument shows that whether the mean flow is laminar or turbulent is intimately related to whether or not the smoothest internal mode is stable. Sufficient conditions for the stability of several simple smooth flows will be given in Section VI.

The above argument also shows that a complete solution of turbulent flow involves **both** the initial-value approach (differential equations) and the average-value approach (variational principle). Instabilities can enter into either the mean-flow calculation or the fine-scale calculation. The first type is the familiar kind of stability problem that is much discussed in the literature, usually in the context of a normal-mode analysis. The fine-scale or internal stability, however, involves an ensemble and is necessarily a statistical concept. Ideally, it should be treated in a statistical context that assigns an appropriate statistical weight to every member of the ensemble. However, it can also be treated, albeit more crudely, in a thermodynamic context that uses the total free energy of each flow mode to determine the most stable mode of the ensemble which is then regarded as the unique solution of the internal flow problem.

In those problems in which the smoothest internal mode is stable, the Reynolds force is small and the **first** step of the iteration procedure outlined above suffices by itself to give a good solution of the large-scale flow. There exists another type of problem for which the **second** step is sufficient by itself. Dynamic meteorology provides many examples of this type in which the flow configuration of interest, such as a convection cell, is embedded in a much larger general flow whose velocity profile is given.

The thermodynamic approach to the determination of the most stable mode of an ensemble presupposes a knowledge of the functional that plays the role of the total free energy of the flow. In the case of incompressible flow, it is evident that this role must be played by the total time-average kinetic energy $\mathscr{W}$. The reason for this is that the incompressibility constraint decouples the compressive energy of the fluid from the kinetic energy, so only the distribution of kinetic energy is available to determine the flow. This special case will be discussed in the next section, and it will be shown that $\mathscr{W}$ does indeed possess the main characteristic of a free-energy functional, namely that it is a **minimum** for flow that is observed in nature, i.e., flow that satisfies the dynamical equations of motion.

V. GENERALIZATION OF KELVIN'S ENERGY THEOREM

Kelvin's Energy Theorem[2,10] states that, for incompressible flow of a fluid having a spatially uniform density, the total kinetic energy is less for potential flow than for any other flow having the same normal velocity field at the boundary. Kelvin actually proved this theorem for time-dependent flow that is driven by the motion of an impermeable boundary, but all the integrals involved in his proof are time-independent spatial integrals, so the proof is the same as one for steady flow. Thus, in terms of the present discussion, Kelvin's proof effectively involves a (non-Lagrangian) steady-flow treatment of the action integral in which the terminal volumes V_I and V_F have been replaced by access surfaces, and the specification of $\Lambda^A(\boldsymbol{x}, t)$ on these surfaces is constrained to be consistent with potential flow. The consequence of this constraint is the fact that only potential flow makes the first variation of the action integral vanish. If the specification of $\Lambda^A(\boldsymbol{x}, t)$ on the access surfaces were allowed to be completely arbitrary, then the class of extremal flows would be larger. It will be shown below that it becomes the class of steady Beltrami flows.[11] The treatment given below is still more general, in that the full time-dependence of the action integral is taken into account, with the result that the class of extremal flows is still further enlarged. The treatment is further generalized in that density stratification is admitted, although the fluctuations are still constrained to satisfy the time-independent conservation equation.

The nature of the implied constraint on the boundary conditions that is involved in Kelvin's proof can be clarified by constructing an **extended ensemble** of steady flows corresponding to all possible specifications of normal velocity on the access surfaces that are consistent with a given fixed **total flux** through the surfaces. Then it can be shown (proof not given below) that, of all the extremal (i.e., Beltrami) flows in the extended ensemble, potential flow has the least total kinetic energy.

The generalized Kelvin Theorem is obtained by specializing the general expression for $\delta^{(1+2)}\mathscr{W}$ given in (48) to the case for which the trial functions are parameterized in such a way that the condition

$$\delta^{(1)}n = \delta^{(2)}n = 0 \tag{61}$$

is identically satisfied. Without making any use of (61), the expression for $\delta^{(1+2)}w$ given in (46) can be cast into the following form:

$$\begin{aligned}\delta^{(1+2)}w = &- \nabla\cdot\left\{\delta^{(1+2)}n[2(\overline{W} + \tfrac{1}{2}\delta_x\overline{W}) + M\overline{V}\cdot\delta_t\overline{V} + MA\cdot\delta X]\right\}\\ &+ \bar{n}\overline{D}_t[M\overline{V}\cdot(\delta X + \delta X\cdot\nabla\delta X)] - \overline{W}\delta^{(1+2)}n\\ &- M(\bar{n}+\delta^{(1)}n)\,\delta X\cdot[(\partial_t\overline{V})_x - \overline{V}X(\nabla X\overline{V})]\\ &+ M\bar{n}\left\{\delta X\delta X : \nabla[(\partial_t\overline{V})_x - \overline{V}X(\nabla X\overline{V})]\right\}\\ &+ \tfrac{1}{2}M\bar{n}\,[(\delta_t\overline{V})\cdot(\delta_t\overline{V}) + (\delta_x\overline{V})\cdot(\delta_x\overline{V})]\end{aligned} \tag{62a}$$

where

$$\delta_t\overline{V} \equiv \overline{D}_t\delta X;\ \delta_x\overline{V} \equiv -\delta X\cdot\nabla\overline{V};\ \delta_x\overline{W} \equiv -\ \delta X\cdot\nabla\overline{W}. \tag{62b}$$

The desired expression is obtained by integrating (62a) over (V, Δt), making use of Gauss' Theorem to convert the first term on the right side of (62a) into a surface integral, and converting the second term into terminal integrals by integrating over t in the Λ-frame. Upon imposing the incompressibility condition (61), and the terminal and boundary conditions discussed in Section IV B, the following expressions result:

$$[\delta^{(1)}\mathscr{W}]_{\delta n=0} = -\int_{(V,\Delta t)} M[(\partial_t\overline{V})_x - \overline{V}X(\nabla X\overline{V})]\cdot\delta X\bar{n}d^3(x)dt, \tag{63a}$$

$$\begin{aligned}[\delta^{(2)}\mathscr{W}]_{\delta n=0} = &\int_{(V,\Delta t)} M\left\{\delta X\delta X : \nabla[(\partial_t\overline{V})_x - \overline{V}X(\nabla XV)]\right\}\ nd^3(x)dt\\ &+ \int_{(V,\Delta t)} \tfrac{1}{2}M[(\delta_t\overline{V})\cdot(\delta_t\overline{V}) + (\delta_x\overline{V})\cdot(\delta_x\overline{V})]\ \bar{n}d^3(x)dt.\end{aligned} \tag{63b}$$

At a given point in space the condition (61) represents a constraint on the spatial **derivatives** of δX, but not on δX itself. Thus, a necessary and sufficient condition that

$\delta^{(1)}\mathscr{W} = 0$ is

$$(\partial_t \tilde{V})_x - \tilde{V}X(\nabla X\tilde{V}) = 0 \tag{64}$$

throughout the space-time volume (V,Δt). The overhead bar has been replaced by a tilde in order to designate the extremal flow. Substitution of (64) into (63) yields

$$[\delta^{(1)}\tilde{\mathscr{W}}]_{\delta n=0} = 0; \; [\delta^{(2)}\tilde{\mathscr{W}}]_{\delta n=0} > 0. \tag{65a,b}$$

The fact that the second variation is positive-definite means that $\mathscr{W}$ is a **minimum** for the choice of trial functions $\tilde{\Lambda}^A(\boldsymbol{x}, t)$ that causes (64) to be satisfied. It is obvious that steady potential flow satisfies (64). However, even for steady flow, (64) admits the larger class of Beltrami flows.[11] (Non-potential Beltrami flow is characterized by streamlines that twist like the strands of a rope.)

The physical significance of time-dependent flows that satisfy (64) becomes apparent when it is noted that the Euler equation for a compressible (or incompressible) inviscid fluid moving in the presence of a gravitational field $\phi(\boldsymbol{x}, t)$ can be written in the form

$$M[(\partial_t V)_x - VX(\nabla XV)] = -\nabla(H + M\phi + W) + T\nabla S \tag{66}$$

where T is the absolute temperature and H, W, and S are respectively the molar enthalpy, molar kinetic energy, and molar entropy. (In the special case in which the given mole density $n(\boldsymbol{x}, t)$ has the constant value n_0, the right side of (66) reduces to $-\nabla B_0$ where $B_0 = p/n_0 + M\phi + W$ is the Bernoulli constant for unstratified incompressible flow.)

If S and the molar Bernoulli constant, $B = H + M\phi + W$, are both spatially uniform, the right side of (66) vanishes, and (66) reduces to (64). Thus the most general flow for which $\mathscr{W}$ is a minimum for density-preserving fluctuations of the trial functions is flow for which the characteristic thermodynamic quantities S and B have become spatially uniform. This can happen through turbulent mixing. Because this is an irreversible process, and because complete spatial homogeneity is this final stage of this process, the flow that satisfies (64) must be regarded as more stable, in the sense of the Second Law of Thermodynamics, than any other flow that satisfies the same boundary conditions but for which B and S are not spatially uniform. Thus maximum stability in the sense of the Second Law is equivalent to the minimum value of $\mathscr{W}$ that is consistent with the given boundary and terminal conditions. This equivalence is made self-evident if $\mathscr{W}/\Delta t$ is regarded as the time-average of the total **free energy** of the system, i.e., the energy that is convertible into useable forms. In classical quasi-static thermodynamics, the statement that stable equilibrium is characterized by a minimum value of the free energy of the system is one way of stating the Second Law. When time-dependence is admitted, it is evident that the instantaneous quantities of quasi-static thermodynamics must be replaced by suitable time-averages. In the case of an incompressible fluid, the thermal aspects of the flow are completely decoupled from the mechanical aspects, so it is legitimate to identify the free energy with the kinetic energy. The same is true for a compressible but cold fluid (i.e., $T = 0$), which is the case discussed in Section IV. Thus the variational energy principle based on the action integral $\mathscr{W}$ (with either $\delta n = 0$ or $\delta n \neq 0$) can be regarded as a statement of the Second Law of Thermodynamics in the special case of fluid flow in which the thermal aspects of the flow have either been suppressed or completely decoupled from the mechanical aspects.

The variational principles presented in this section and the previous one are both the same. The only difference is the respective classes of variations that are admitted in the two cases. It should be noted that the **smaller** class of **variations** (the density-preserving ones of this section) give rise to the **larger** class of **extremal flows.** (The class of flows satisfying (64) is larger than the rectilinear flows required by (49).) The reason for this is that each degree of freedom that is added to the class of variations gives rise to a new Euler-Lagrange equation which represents a new constraint on the extremal flow.

VI. INTERNAL STABILITY CRITERION

A. Convexity of Action Integral as Sufficient Condition for Internal Stability

The remarks of this subsection will apply to the variational principle of Paper II that is based on the action integral $\mathscr{A}$, as well as to the principle of this paper that is based on $\mathscr{W}$. In fact, they apply to any variational principle whose action integral can be identified with the appropriate total time-average thermodynamic potential function for the system under study. In such a case, **local stability** of an extremal flow is guaranteed if any conceivable variation about the extremal trial functions causes $\mathscr{A}$ to increase. (The term "local" refers to parameter space.) Thus, since $\delta^{(1)}\tilde{\mathscr{A}} > 0$, it can be asserted that

$$\delta^{(2)}\tilde{\mathscr{A}} > 0 \tag{67}$$

is **sufficient** to guarantee **local stability**. Such a case is illustrated in Figure 3A in which it is assumed that $\mathscr{A}$ is parameterized in terms of the two variational parameters a and b. Local stability does not guarantee absolute stability. If the flow in question corresponds to a point in parameter space that is at the bottom of a well that is connected to a deeper well by a very low "mountain pass," then sufficiently strong "jostling" of the system because of interaction with its surrounding momentum reservoir could cause the system to jump over into the deeper well. A complete statistical description would involve an ensemble in which each of the two extremal flows would be represented with appropriate statistical weights. The thermodynamical description given here and in Paper II is limited to a discussion of local stability of a single extremal flow.

The other three cases shown in Figure 3 do not satisfy (67). The failure to satisfy this condition does not, however, guarantee that the extremal flow in question is unstable. The reason for this is that the flow is unstable only if a **dynamically acceptable** alternative flow with lower $\mathscr{A}$ is available to it. This will be the case only if the "spillway" running

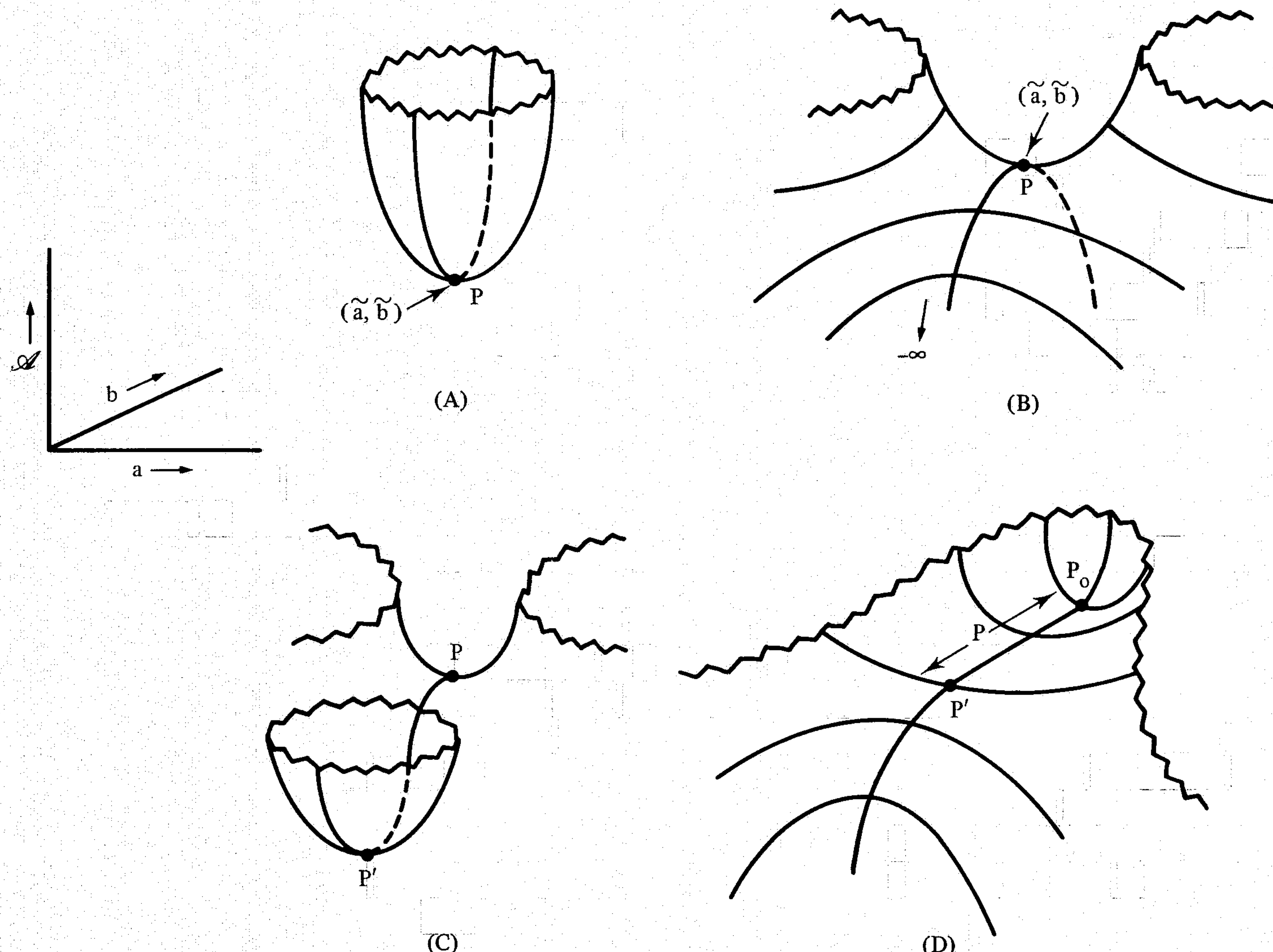

Figure 3. Various Types of Extrema (i.e., Stationary Points P) in Parameter Space

from the extremum in question leads to a lower extremum. Such a case is illustrated in Figure 3C. If no such neighboring extremum is available, however, a saddle-point in parameter space is stable "by default." Such a case is illustrated in Figure 3B. An example of such a case is provided by (51) when $\mathscr{W}$ is regarded as the total time-average free energy of the system. Because of the second term in the integrand of (51), the extremum can be a saddle point. (This is discussed in greater detail in Subsection B below.) But for the variational principle based on $\mathscr{W}$ the extremal trajectories are straight lines in space-time which are uniquely determined by the imposed terminal and boundary conditions. Thus, the extremum characterized by these rectilinear trajectories is unique, and so necessarily stable.

Figure 3D illustrates a case in which failure to satisfy (67) (at the point P_o) can indicate **neutral** stability (along the line $P_o - P'$) rather than instability. If a continuum of neighboring extrema (the line $P_o - P'$) exists, then because each must be tangent to a horizontal plane in parameter space, it follows that $\tilde{\mathscr{A}}$ is the same for all these extrema. Neutral wave motion is an illustration of such a situation. So long as the wave motion is characterized by strict equipartition between its kinetic energy and its convertible (i.e., free) potential energy the total time-average free energy (i.e., $\tilde{\mathscr{A}}$) is insensitive to the amplitude of the wave motion.

The conclusion to be drawn from these examples is that the failure to satisfy the condition (67) does not suffice to guarantee instability. Thus (67) is a **sufficient**, but **not necessary**, condition for **stability**. However, although instability is not guaranteed when fluctuations exist for which $\delta^{(2)}\tilde{\mathscr{A}} < 0$, it is nevertheless true that, in the case of an action integral that (unlike $\mathscr{W}$) includes potential as well as kinetic energy, the probability is very large indeed that when $\delta^{(2)}\tilde{\mathscr{A}} < 0$ there exists at least one neighboring dynamically acceptable extremum that has a lower value of $\mathscr{A}$, or one having the same value of $\mathscr{A}$ but larger kinetic energy. As to the second possibility, it should be noted that in a real fluid with viscosity a transition to a flow having larger kinetic energy would inevitably lead to one having a lower value of $\mathscr{A}$. Thus both types of neighboring flows can be regarded as being more stable than the original extremal flow. For this reason, it is justifiable to say that, when $\delta^{(2)}\widetilde{\mathscr{A}} < 0$, the extremal flow in question is "probably unstable," and this expressions will be used in which follows to indicate the violation of condition (67).

In asserting that (67) suffices to guarantee internal stability, an extremely important caveat must be observed, namely the fact that this is true only if the class of admissible flows is limited to those satisfying the imposed terminal and boundary conditions, i.e., those flows for which the time-average velocity of every particle is equal to a specified value. This makes physical sense if the average velocity is the same for all particles of the system, which corresponds to a system embedded in a constant-velocity momentum reservoir. If the reservoir velocity has shear, however, then turbulent mixing will tend, at least over short distances, to eliminate this shear. The inclusion of such effects would necessitate enlarging the comparison class of flows involved in the stability problem. That is, instead of comparing $\mathscr{A}$-values for the members of the single ensemble of extremal flows that corresponds to a given specification of the average velocity $\langle \mathbf{V} \rangle_t$ for each particle (i.e., a given specification of $\mathbf{X}_I$ and $\mathbf{X}_F$), it would be necessary to consider an **extended ensemble** consisting of the sum of the various individual ensembles corresponding to different specifications of $\mathbf{X}_I(\boldsymbol{x})$ and $\mathbf{X}_F(\boldsymbol{x})$. This extended ensemble will be defined so as to maintain the constancy of some total time-average quantity such as total flux or total angular momentum. A simple example of such an extended ensemble will be given in Subsection C below. For stability with respect to the extended ensemble, it is sufficient that (67) be satisfied for the **extended class of fluctuations** that results from dropping the requirement that $\delta \mathbf{X} = 0$ on the hypersurface bounding the integration volume $(V, \Delta t)$, and replacing it by the expression for $\delta \mathbf{X}(\boldsymbol{x}, t)$ on the hypersurface that defines the extended ensemble.

In Subsection B below, the laminar and internal stability criteria will be compared for the case of rectilinear flow with shear, and arbitrary axisymmetric azimuthal (zonal) flow. In making such a comparison, it is best to regard the two criteria as corresponding **to different modes of instability.** The "internal mode" maintains the time-average velocity profile of the base flow (i.e., the original extremal flow), whereas the "laminar mode" leads to the establishment of a new velocity profile. Thus a "laminar instability" is "macroscopically observable" in the sense that it either leads to growing sinuous oscillations that tend to eliminate the velocity shear that produced them, or else it leads to an entirely different stable mode, such as a convection roll, whose velocity profile is different from that of the base flow. An "internal instability," on the other hand, is necessarily a fine-scale phenomenon because it is constrained to maintain the given average velocity profile. The onset of such an instability would not be "macroscopically observable" in the sense of producing large-scale changes in the flow in the fashion of a laminar mode. Rather it would manifest itself in a sudden increase in the degree of turbulence present in the main flow. This turbulence would not be random, but would have the order of the first (i.e., most unstable) internal mode to be excited. If, however, the base flow were so unstable that many internal modes were excited, then the turbulence would appear to be effectively random.

These remarks must be tempered by the caveat that was emphasized above in introducing the extended ensemble. The introduction of an appropriate extended ensemble is a way of "introducing a bit of laminar instability" into a study of the internal stability criterion.

B. Comparison of Internal and Laminar Stability Criteria

1. Parallel Shearing Flow

It follows from (48) and (67) that if only variations in kinetic energy are involved (inertial instability) and the extremal flow is rectilinear ($\widetilde{\mathbf{A}} = 0$), then a **sufficient** condition for internal **stability** is

$$-(\delta X \delta X{:}\nabla\nabla\widetilde{V}) \cdot \widetilde{V} \geqslant 0 \tag{68}$$

throughout the space–time volume $(V, \Delta t)$. This is a conservative criterion in the sense that the stabilizing effect of the term $(\widetilde{D}_t \delta X)^2$ that appears in (51) has been ignored. The reason for this, as noted in the discussion following (51), is that this term can be made arbitrarily small by making Δt arbitrarily large.

As noted following (67), if $\mathscr{W}$ is used as the action integral (thereby excluding potential energy), the extremal flow $\widetilde{\mathbf{A}} = 0$ is always stable regardless of the sign of the left side of (68). The reason for this is that, for the given terminal end boundary conditions, no alternative extremum exists.

However, if we anticipate the results of Paper II, then it can be asserted that (68) is the internal stability criterion that is relevant to inertial stability of rectilinear flow of a compressible fluid (neglecting the possible stabilizing effects of the gravitational field). For parallel flow in the x-direction with velocity shear in the z-direction, (68) reduces to the following: The flow is stable if

$$\widetilde{V} \frac{d^2 \widetilde{V}}{dz^2} \leqslant 0 \tag{69}$$

everywhere. Figure 4 shows examples of stable velocity profiles that satisfy the stability condition given in (69). Figure 5 shows examples of "probably unstable" profiles that do not satisfy the sufficient condition given in (69). A comparison of Figure 4A and Figure 5A

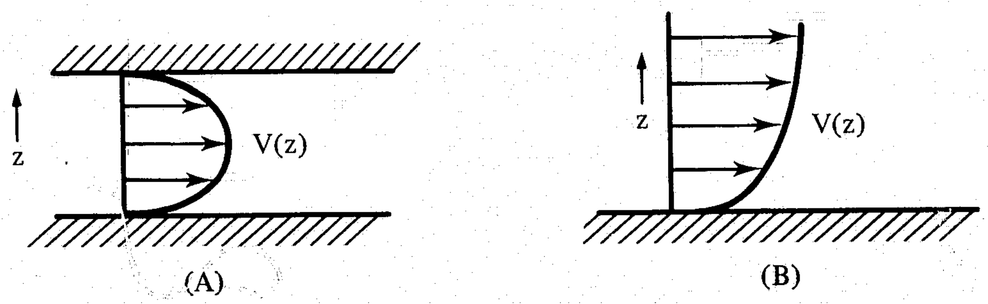

Figure 4. Sample Velocity Profiles that Satisfy the Internal Stability Criterion

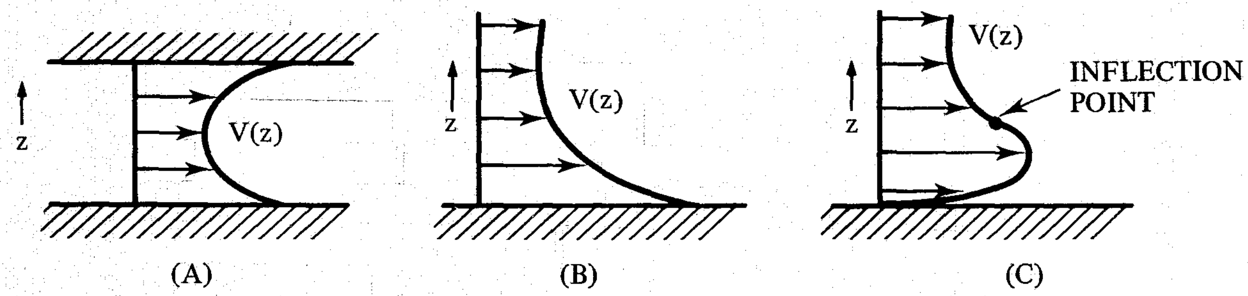

Figure 5. Sample Velocity Profiles that Violate the Internal Stability Criterion

shows that, whereas a jet-type profile is stable according to the internal stability criterion, a wake-type profile is "probable unstable." This suggests that turbulence along the center line of a wake in an inviscid fluid should be significantly greater than the turbulence at the center of a jet whose velocity profile has equal but opposite curvature. It should be noted, however, that this assertion is not invariant under velocity transformations parallel to the flow. That is, by adding a sufficiently large counter-velocity, the wake becomes a jet and the jet becomes a wake. This shows that, even though perfect slip is allowed at the wall, it is nevertheless necessary to regard the $\widetilde{V}$ in (69) as referring to the rest-frame of the walls. The physical reason for this will be discussed in Section VII.

The **laminar criterion** that corresponds to (69) is that the **absence** of an **inflection point** in the velocity profile is a **sufficient** condition for **stability.** This was shown by Rayleigh[12] in 1880. Tollmien[13] showed that the appearance of an inflection point in a symmetric profile (such as that of Figure 4A or Figure 5A) or a boundary layer profile (such as Figure 4B or Figure 5C) is sufficient to guarantee laminar instability. That is, for the profiles involved, Rayleigh's condition is not only sufficient, but also necessary, for stability. Taylor[14] pointed out that the derivation of Rayleigh's criterion involves imposing boundary conditions that amount to forbidding momentum exchange between the flow under study and the containing walls (or surrounding fluid in a free-boundary problem). The question of devising more realistic boundary conditions is the laminar-flow analog of the internal-stability problem of defining a suitable extended ensemble.

The literature relevant to Rayleigh's criterion is vast, but good reviews are available.[15-19] (The review article of Drazin and Howard,[18] which is limited to **inviscid** flow, is most relevant to the present discussion.)

According to the Rayleigh-Tollmien criterion, all of the profiles shown in Figures 4 and 5 with the exception of the one in Figure 5C are definitely stable against laminar instability because they have no inflection point. In comparison, the profiles of Figures 5A and 5B are "probably unstable" with respect to the onset of internal turbulence as indicated by the internal stability criterion given in (69). The two cases become unstable against laminar instability as well if the profiles are made more realistic by imposing no-slip conditions at the walls, since this introduces inflection points in the profiles. If this is done, the internal and laminar stability criteria coincide, which means that the onset of internal turbulence would coincide with the onset of the sinuous oscillations that characterize laminar instability in parallel shearing flow.

For the sake of making a comparison with the case of rotational flow treated in the next subsection, the internal stability criterion given in (68) will now be adapted to the case of **doubly-shearing flow** shown in Figure 6, i.e., flow in the x-direction that has different shear gradients in the y and z directions. The left side of (68) in such a case can be written as follows:

$$\begin{aligned} -\tfrac{1}{2}\widetilde{n}M(\delta X\delta X{:}\nabla\nabla\widetilde{V})\cdot\widetilde{V} &= \widetilde{w}a\left(\delta Z-\frac{b}{a}\,\delta Y\right)^2 \\ &\quad+\frac{\widetilde{w}}{a}\,(ac-b^2)(\delta Y)^2 \end{aligned} \tag{70a}$$

where

$$a = -\frac{1}{\widetilde{V}}\frac{\partial^2\widetilde{V}}{\partial z^2};\quad b = \frac{1}{\widetilde{V}}\frac{\partial^2\widetilde{V}}{\partial y\partial z};\quad c = -\frac{1}{\widetilde{V}}\frac{\partial^2\widetilde{V}}{\partial y^2} \tag{70b,c,d}$$

and $\widetilde{w}$ is the kinetic energy density of the extremal flow. It follows from (68) and (70) that the sufficient condition for internal stability of double-shearing parallel flow is

$$a \geqslant 0 \quad\text{and}\quad (ac-b^2) \geqslant 0. \tag{71a,b}$$

For $b = c = 0$ this reduces to the criterion given in (69).

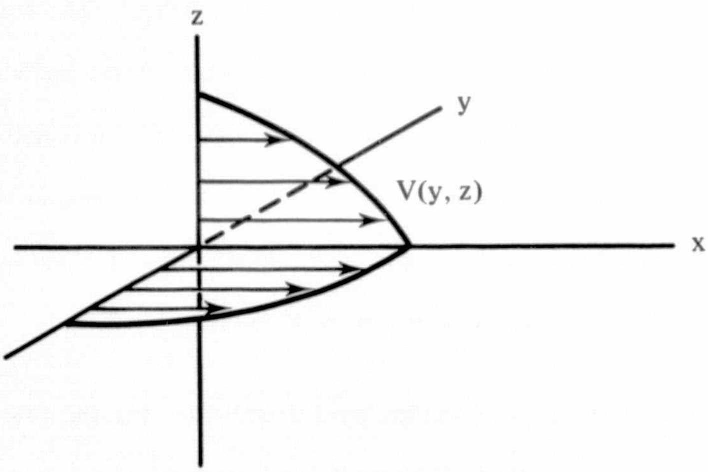

Figure 6. Doubly-Shearing Parallel Flow

2. Axisymmetric Zonal Flow

In this case $\widetilde{V} = \widetilde{V}(r, z)e_\varphi$ and $\widetilde{A} = -(\widetilde{V}^2/r)e_r$ where (r, φ, z) are cylindrical coordinates and (e_r, e_φ, e_z) are the corresponding unit vectors. Since $\widetilde{A} \neq 0$, (51) and (68) do not apply. In anticipation of the results of Paper II the sufficient condition for internal stability will be taken to be

$$\delta_X^{(2)}\widetilde{w} \equiv \tfrac{1}{2}\widetilde{n}M[(\delta X\delta X\!:\!\nabla\widetilde{A}) - (\delta X\delta X\!:\!\nabla\nabla\widetilde{V})\cdot\widetilde{V}] \geqslant 0 \tag{72}$$

everywhere in $(V, \Delta t)$. (The symbol $\delta_X^{(2)}\widetilde{w}$ has been introduced for convenience.) As in the case of parallel shearing flow, the stabilizing effect of $\tfrac{1}{2}M(\widetilde{D}_t\delta X)^2$ has been neglected. The integrand of $[\delta^{(2)}\mathscr{W}]_V^{(1)}$ given in (48e) has also been omitted. The reason for this is that it will be shown in Paper II that this term exactly cancels a term included in the expression for the second variation of the potential energy. If, moreover, $\widetilde{n}$ is taken to be the same throughout the fluid, then two terms in the expression for the second variation of the potential energy that depend on $\nabla\widetilde{n}$ also vanish, and the criterion for inertial internal stability of rotational flow of an unstratified fluid is given by (72).

This criterion can be written in cylindrical coordinates for the case of axisymmetric unstratified zonal (azimuthal) flow as follows:

$$[\delta_X^{(2)}\widetilde{w}/\widetilde{w}] = A\left[\frac{\delta Z}{r} - \frac{B}{A}\left(\frac{\delta R}{r}\right)\right]^2 + \frac{1}{A}(AC - B^2)\left(\frac{\delta R}{r}\right)^2 + D(\delta\Phi)^2 \geqslant 0 \tag{73a}$$

where

$$A = -\frac{r^2}{\widetilde{\widetilde{K}}}\frac{\partial^2\widetilde{K}}{\partial z^2}; \quad B = \frac{r^2}{\widetilde{\widetilde{K}}}\frac{\partial^2\widetilde{K}}{\partial r\partial z}; \tag{73b,c}$$

$$C = 1 - \frac{r^2}{\widetilde{\widetilde{K}}}\frac{\partial^2\widetilde{K}}{\partial r^2}; \quad D = 1 - \frac{r}{\widetilde{\widetilde{K}}}\frac{\partial\widetilde{K}}{\partial r}; \tag{73d,e}$$

where $\widetilde{K} = Mr\widetilde{V}$ is the molar angular momentum, and (δR, $\delta\Phi$, δZ) are the cylindrical components of the particle displacement $\delta\mathbf{X}$. A comparison of (70) and (73) shows the following correspondences: $(r, \varphi, z) \longleftrightarrow (y, x, z)$ and $(A, B, C) \longleftrightarrow (a, b, c)$. The term $D(\delta\Phi)^2$ in (73a) has no analog in (70a). The internal stability conditions arising from (73) are

$$A \geqslant 0; \quad AC - B^2 \geqslant 0; \quad D \geqslant 0 \tag{74a,b,c}$$

which include the extra condition $D \geqslant 0$ that has no analog in (71).

The comparison between (70) and (73) shows that it would be conceptually preferable to write (70) in terms of the molar momentum $M\widetilde{V}$ rather than the velocity $\widetilde{V}$. The linear and angular momentum, $M\widetilde{V}$ and $\widetilde{K}$, are the respective constants of motion of rectilinear and axisymmetric flow, and the spatial gradients of these constants of motion determine the stability of the flow. In the rectilinear case, turbulent mixing would tend to produce a uniform profile ($M\widetilde{V}$ = constant) for which internal instability is no longer possible. Similarly, in the case of rotational flow, mixing would tend to produce a vortex characterized by $\widetilde{K}$ = constant, which is just a **potential vortex**, i.e., one for which the velocity is expressible as the gradient of a scalar. ($\widetilde{V}(r) \propto r^{-1}$ in this case.) For $\widetilde{K}$ = constant, it follows from (73) that $A = B = 0$ and $C = D = 1$, which by (74) corresponds to internal stability.

It can be shown that, for vortices for which $\widetilde{K}(r)$ has a simple power-law dependence on r and no dependence on z, the vortex is internally stable according to (74) if

$$-0.62 \leqslant (d\,ln\widetilde{K}/dr) \leqslant 1.0. \tag{75}$$

For the potential and rigid vortices, $d\,ln\widetilde{K}/dr$ is equal to 0 and +2 respectively. Thus some subpotential vortices (for which $d\,ln\widetilde{K}/dr < 0$) are internally stable whereas the rigid vortex and all suprarigid vortices (for which $d\,ln\widetilde{K}/dr > 2$) are internally "probably unstable."

The lower limit in (75) comes from violation of the condition $C \geqslant 0$ whereas the upper limit comes from violation of $D \geqslant 0$. The condition on D can be made more transparent by writing it in terms of the molar kinetic energy $\widetilde{W} = \frac{1}{2}M\widetilde{V}^2$ instead of the molar angular momentum:

$$D = -\frac{r}{2\widetilde{W}}\frac{\partial\widetilde{W}}{\partial r} \geqslant 0. \tag{76}$$

This shows that the vortex becomes "probably unstable" when the molar kinetic energy increases with increasing r. If the criterion $D \geqslant 0$ is suppressed by imposing the constraint $\delta\Phi = 0$ so that only the condition $C \geqslant 0$ remains, the upper limit in (75) becomes 1.62 instead of 1.0.

These results for internal stability are to be contrasted with the Rayleigh criterion for rotational flow[15,16] which says that a sufficient condition for laminar stability of rotational flow against the onset of an axisymmetric perturbation (i.e., an axisymmetric convection roll) is that the magnitude of the molar angular momentum $|\widetilde{K}|$ **increase** with increasing r. Rayleigh[20] derived this criterion by a simple energy argument involving the interchange of rings of fluid subject to the constraint that each ring retain its angular momentum. (Contrast this constraint on angular momentum with a constraint on the average angular velocity, which is the specialization of (60) that is appropriate to the case of rotational flow.) Rayleigh's argument gives a **sufficient** condition for stability. Synge[21] showed that this condition is also **necessary**.

In contrasting the internal and laminar (Rayleigh) stability criteria for rotating flow, it should be recalled that they refer to the onset of different modes of instability – internal turbulence as compared with axisymmetric convection rolls. Subpotential vortices are unstable against the onset of axisymmetric convection rolls, but (if they are not too different from a potential vortex) are stable against the onset of internal turbulence. Rigid and suprarigid vortices, on the other band, are stable against such convection rolls, but "probably unstable" according to the internal stability criterion.

The fact that different modes of instability are involved is illustrated by the case of the eastward-flowing azimuthal streams of air (the westerly jet streams) in the upper troposphere. These are subject to the spontaneous onset of sinuous oscillations on a planetary or continental scale. Such instabilities clearly fall within the province of the laminar stability criterion. On the other hand, the clear-air turbulence that is observed near the jet streams is a spontaneous onset of internal turbulence that is swept along with the main winds of the jet stream and has no effect on the average velocity of these winds. This phenomenon clearly falls within the domain of the internal stability criterion. In fact, if the turbulent criterion given in (73) and (74) is written in terms of spherical instead of cylindrical coordinates and in terms of azimuthal velocity relative to earth's surface instead of the absolute molar angular momentum $\widetilde{K}$, then it can be shown that the **north** side of the jet maximum (in the northern hemisphere) is the region of internal **instability**, whereas the south side is stable. Observations[22] show, in fact, that clear-air turbulence occurs on the **north** side but not on the south side. This prediction of the internal criterion is to be contrasted with the prediction based on the Rayleigh criterion for rotational flow that the **south** side of the jet maximum is where any instability should occur[23] (if it occurs at all).

C. Flux-Averaging in Parallel Shearing Flow

The examples of internal stability criteria discussed above refer in every case to the **restricted ensemble** of flows that corresponds to imposing the condition that $\delta\mathbf{X} = 0$ on the hypersurface bounding the space-time volume $(V, \Delta t)$ (except for segments of rigid or free surfaces on which $\delta\mathbf{X} \cdot d\mathscr{S} = 0$). This restricted ensemble corresponds to the class of dynamically acceptable flows for which the time-average velocity $\langle \mathbf{V} \rangle_t$ for every particle equals the specified value. However, the fine-scale motions that appear at the onset of internal instability sometimes have the effect of mixing neighboring values of $\langle \mathbf{V} \rangle_t$ and replacing the individual values with a single average value. Whether or not such an averaging process must be taken into account is obviously a physical question that depends on the ratio of the characteristic amplitude of the internal motion to the distance over which a significant change in $\langle \mathbf{V} \rangle_t$ occurs. When the physics of the flow indicates that such mixing and consequent averaging of $\langle \mathbf{V} \rangle_t$ does in fact take place, then this effect can be introduced into the formalism by defining a suitable **extended ensemble** that is the sum of the individual restricted ensembles, each corresponding to a different specification of $\mathbf{X}(\boldsymbol{x}, t)$ on the bounding hypersurface. The expression for $\delta^{(1+2)}\mathcal{W}$ given in (48) is still valid, but now the terminal and surface integrals no longer vanish because $\delta\mathbf{X} \neq 0$ on the hypersurface. Moreover, the volume integrals are also affected because if, for example, $\delta\mathbf{X}_I = 0$ and $\delta\mathbf{X}_F \neq 0$, then choosing $t_I = 0$ it follows that $\delta\mathbf{X} \neq 0$ for $0 < t \leqslant t_F$. In fact, if we specify $\delta\mathbf{X} = \delta\mathbf{X}_F(t/t_F)$, then we simply add a constant velocity $\delta\mathbf{V} = \widetilde{D}_t\delta\mathbf{X} = \delta\mathbf{X}_F/t_F$ to the original velocity $\widetilde{\mathbf{V}}$.

The evaluation of $\delta^{(1+2)}\mathcal{W}$ for the specification of $\delta\mathbf{X}_F$ that is illustrated in Figure 7 (and $\delta\mathbf{X}_I \equiv 0$) will now be carried out. This specification corresponds to the flux (or momentum) averaging that results from "blob-exchange" between two layers in a horizontal wind with linear vertical shear. The layers are separated by the vertical distance δz. The problem is normalized so that each layer contains one mole of fluid and transfers half of this to the other layer. Thus the total amount of fluid actually moved in the exchange is one mole. The final velocities of the upper and lower layers ($\widetilde{V}'_+$ and $\widetilde{V}'_-$ respectively) are both equal to $\widetilde{V}_o$, the original velocity half-way between them. The original velocities of the upper and lower layers are respectively $\widetilde{V}_+ = \widetilde{V}_o + \frac{1}{2}\delta z\,(d\widetilde{V}/dz)$ and

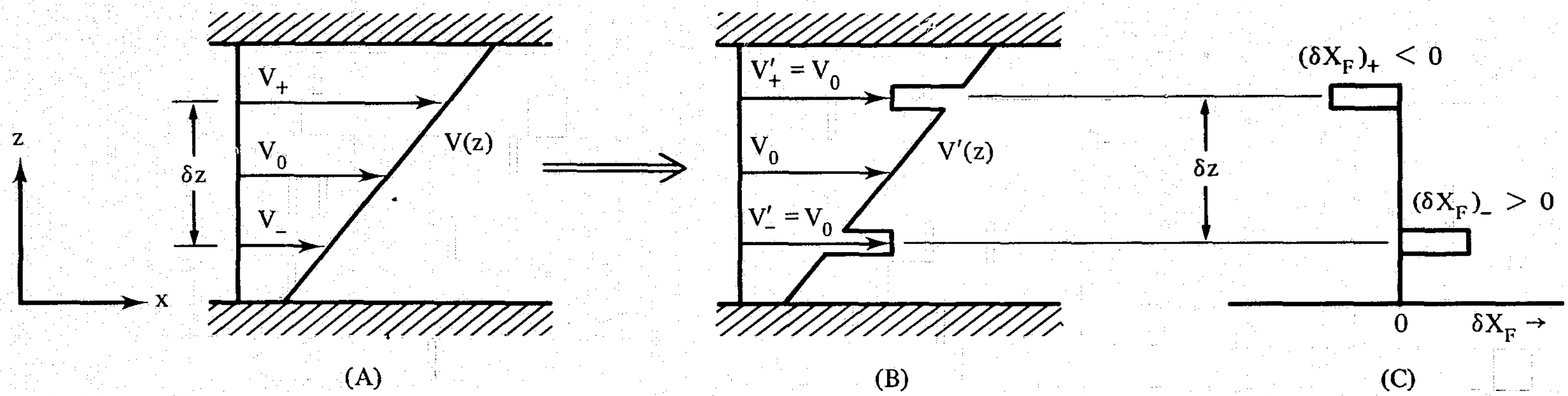

Figure 7. Flux-Conserving Mixing of Two Layers in a Shearing Flow. (A) Before Mixing. (B) After Mixing. (C) Change in Terminal Positions for Particles in Each Layer

$\widetilde{V}_- \doteq \widetilde{V}_0 - \tfrac{1}{2}\delta z(d\widetilde{V}/dz)$. The extended ensemble in this case consists of the two extremal flows pictured in Figure 7A and Figure 7B. (They are both extremal because they both satisfy the dynamical equation of motion, which in this case is just $\widetilde{\mathbf{A}} = 0$.) The change in $\delta\mathbf{X}$ on the hypersurface that changes flow A into flow B is $\delta\mathbf{X}_I = 0$ and

$$(\delta\mathbf{X}_F)_+ = t_F e_x(\widetilde{V}'_+ - \widetilde{V}) = -\, t_F e_x\left(\tfrac{1}{2}\delta z\frac{d\widetilde{V}}{dz}\right); \tag{77a}$$

$$(\delta\mathbf{X}_F)_- = t_F e_x(\widetilde{V}'_- - \widetilde{V}) = +\, t_F e_x\left(\tfrac{1}{2}\delta z\frac{d\widetilde{V}}{dz}\right) \tag{77b}$$

Substitution of (77) into (48) gives the difference $\mathscr{W}_B - \mathscr{W}_A$. For this problem the only non-zero integrals are (48b) and 48f), so

$$\widetilde{\mathscr{W}}_B - \widetilde{\mathscr{W}}_A = \int_N M\widetilde{\mathbf{V}}_F \cdot \delta\mathbf{X}_F\,dN + \int_{(N,\Delta t)} \tfrac{1}{2}M(\widetilde{D}_t\delta\mathbf{X})\cdot(\widetilde{D}_t\delta\mathbf{X})\,dN\,dt$$
$$= -\,[(1/8)M(d\widetilde{V}/dz)^2(\delta z)^2\,]\,t_F. \tag{78}$$

Since the flow is steady and $t_F = \Delta t$, it follows that

$$\widetilde{W}_B - \widetilde{W}_A = (\widetilde{\mathscr{W}}_B - \widetilde{\mathscr{W}}_A)/\Delta t = -(1/8)M(d\widetilde{V}/dz)^2(\delta z)^2, \tag{79}$$

a result that can also be obtained (more easily) by direct calculation of the total kinetic energy of flows A and B.

This example illustrates how (48) can be used to find the difference between the action of a given base flow $\mathscr{W}_A$ that is embedded in an extended ensemble and the action of a particular member of the ensemble $\mathscr{W}_B$ that corresponds to particular values of $\delta\mathbf{X} = \mathbf{X}_B - \mathbf{X}_A$ on the bounding hypersurface. This difference measures the relative stability of the two extremal flows. The minus sign in (78) means that flow B is more stable than flow A, which corresponds to the fact that the mixing involved in the blob exchange is an irreversible process.

If the flow of Figure 7A occurs in a statically stable stratified atmosphere in a gravitational field, then the destabilizing effect of the wind shear has to compete with the stabilizing influence of gravity. It will be shown in Paper II that when (78), which refers to a transfer of a total of one mole of fluid over a vertical distance δz, is combined with the corresponding expression for the change in potential (i.e., buoyancy) energy, the resulting expression constitutes a sufficient condition for stability of horizontal shearing wind in a gravitational field that is identical to a well–known criterion that results from the laminar (i.e., initial–value) approach to the stability problem.[24,25] This illustrates the remark made at the end of Section VI A that use of a suitable extended ensemble provides a means for "introducing a bit of laminar instability" into the internal stability problem.

Finally, it should be noted that the rectilinear flow of this example is a degenerate case in the sense that the averaging produced by the blob exchange may be regarded as an averaging of momentum, velocity, or flux. In general curvilinear flow these averages are all different, and it is necessary to decide which of them is appropriate to use in defining the extended ensemble.

VII. REFERENCE ENERGY AND VELOCITY

In open-ended flow problems in which V extends over an infinite range, the kinetic action integral $\mathscr{W}$ diverges. The flow in such a problem is usually of interest only in a limited region, as in the case of flow past an obstacle, and it would be desirable to eliminate the divergence by subtracting the known kinetic energy at large distances from the region of interest. This can often be done by subtracting from the kinetic energy W a **reference energy** W_R defined by

$$W_R \equiv W_R(\Lambda, t) \equiv W_R[\Lambda^A(x,t), t] \tag{80}$$

where the functional relation $W_R(\Lambda, t)$ is **specified** (but $\Lambda^A(x, t)$ is not). Thus the variation δW_R results entirely from the variation of its argument functions $\Lambda^A(x, t)$. If W_R has the form given in (80), it can easily be shown that

$$\delta^{(1+2)} \int_{(V,\Delta t)} W_R \, n d^3(x) dt = 0 \tag{81}$$

when the boundary conditions (54) are imposed. Thus, the introduction of W_R into the integrand of $\mathscr{W}$ has no effect on $\delta^{(1+2)}\mathscr{W}$, i.e.,

$$\begin{aligned} \delta^{(1+2)}\left[\mathscr{W} - \int_{(V,\Delta t)} W_R \, n d^3(x) dt\right] &= \delta^{(1+2)} \int_{(V,\Delta t)} [\tfrac{1}{2} M \mathbf{V} \cdot \mathbf{V} - W_R(\Lambda, t)] \, n d^3(x) \, dt \\ &= \delta^{(1+2)} \mathscr{W}. \end{aligned} \tag{82}$$

If it is possible to choose $W_R(\Lambda, t)$ so that it equals W at large distances from the region of interest, the integral $(\mathscr{W} - \mathscr{W}_R)$ will have a finite value whereas $\mathscr{W}$ diverges, and yet the two integrals will have the same first and second variations.

It is important to note that W_R must be a specified function of Λ^A and t, and **not** of $\boldsymbol{x}$ and t. A specified function of $\boldsymbol{x}$ and t would **not** satisfy (81).

Instead of introducing a reference **energy** W_R, it is possible to introduce a reference **velocity** $\mathbf{V}_R \equiv \mathbf{e}_j V_R^j(\Lambda)$ where the Cartesian components V_R^j are specified functions of Λ^A, but **not of t.** If $\mathbf{V}'$ is the relative velocity, then it follows from (18) that

$$\mathbf{V}' \equiv \mathbf{V} - \mathbf{V}_R = -(\partial_t \Lambda^A)_x \mathbf{G}_A - V_R^j[\Lambda^A(\boldsymbol{x}, t)]\,\mathbf{e}_j. \tag{83}$$

$\mathbf{V}_R$ is a rectilinear constant of motion, i.e., for a given particle it always has the same magnitude and direction (which would not be the case if the components of $\mathbf{V}_R$ were defined with respect to a curvilinear coordinate system, or if $V_R^j \equiv V_R^j(\Lambda, t)$ instead of $V_R^j(\Lambda)$). Because of this $D_t\mathbf{V}' = D_t\mathbf{V}$, so if $\mathbf{V}$ satisfies the equations of motion, so does $\mathbf{V}'$. As a consequence $\delta^{(1)}\mathcal{W}' = \delta^{(1)}\mathcal{W}$ where $\mathcal{W}'$ is the space–time integral of $nW' \equiv \frac{1}{2} nM\mathbf{V}' \cdot \mathbf{V}'$. The introduction of $\mathbf{V}_R$ does, however, change the second variation. It can be shown that, if the terminal and boundary conditions of (52) and (54) are imposed, then

$$\delta^{(1+2)}\mathcal{W}' - \delta^{(1+2)}\mathcal{W} = \int_{(V,\Delta t)} \tfrac{1}{2}M(\delta\mathbf{X}\delta\mathbf{X}:\nabla\nabla\overline{\mathbf{V}}) \cdot \overline{\mathbf{V}}_R\,\bar{n}d^3(x)dt. \tag{84}$$

The sign of the integrand depends on the choice made for $\mathbf{V}_R(\Lambda)$, and this is arbitrary.

The conclusion then is that the first variation of $\mathcal{W}$ (and hence the equation of motion) is invariant under a Galilean transformation of a very general kind, namely one that can involve a different velocity transformation for each particle. For example, $\mathbf{V}_R$ could be identified with the time–average velocity $\langle\mathbf{V}\rangle_t$, in which case W' would be the kinetic energy associated with the deviation of the instantaneous particle velocity from its time-average. The second variation, however, and hence the internal stability criterion, is not invariant under such a transformation,or even under the usual Galilean transformation for which $\mathbf{V}_R$ is a constant vector. This means that an absolute significance can be attached to the second variation of $\mathcal{W}$ only if a preferred Galilean frame is specified. This point was made already in Section VI B 1 in comparing the respective stabilities of the jet-type and wake-type profiles shown in Figures 4A and 5A, where it was pointed out that the preferred frame should be chosen to be the rest-frame of the rigid containing walls.

This choice can be explained very simply in thermodynamic terms. Free energy is the energy that can be converted into useful work during the transfer of some extensive quantity from the system to an **available** reservoir of this quantity by means of appropriate idealized engines. In the familiar thermodynamic example, entropy is transferred to the **coldest available** entropy reservoir by means of a Carnot engine. In the cases pictured in Figures 4A and 5A, the extensive quantity involved is momentum, and the **only** momentum reservoir available to the fluid consists of the channel walls and everything to which they are rigidly attached. Thus only the fluid velocity relative to the walls is significant in calculating the useful work that could be extracted from the fluid by transferring its momentum to the only available momentum reservoir, and the useful work so defined is the free energy whose magnitude serves as a measure of relative stability. Thus, the velocity of the walls plays a role analogous to that of the coldest available reservoir temperature in the case of heat engines. In the case of a planetary–scale analysis of the earth's atmosphere, the solid massive earth is the obvious momentum reservoir, and the velocity of its center of mass is the appropriate reference velocity. On a local scale, however, this need not be the case. For example, in the case of a convection cell embedded in a large mass of constant velocity air (no wind shear), the air mass serves as the only available momentum reservoir, i.e., the only one with which the convection cell can interact.

In summary, the internal stability criterion discussed in Section VI should be referred to that inertial frame in which the constant–velocity momentum reservoir **available to the system under study** appears to be a rest. In this frame the expressions as given in Section VI (without the inclusion of a reference velocity) represent valid stability criteria, because they represent fluctuations in the free energy that could ideally be extracted from the system in transferring momentum to the specified momentum reservoir.

VIII. CONCLUDING REMARKS

A necessary preliminary to the construction of a variational energy principle for fluid flow is a discussion of the first and second variations of the kinetic action $\mathcal{W}$, and of the terminal and boundary conditions that must be imposed on the trial functions, and this has been the main business of this paper.

The total kinetic action $\mathcal{W}$ can serve by itself as the action integral for either an incompressible fluid, or for a cold compressible fluid, both in the absence of any external forces. In these simple cases, $\mathcal{W}$ can be regarded as the total free-energy functional of the flow. In Paper II the appropriate free-energy functional that includes compressive and gravitational energy will be derived.

The same terminal and boundary conditions on the trial functions describing the flow kinematics that have been discussed in this paper will also apply to the variational energy principle derived in Paper II, and the various criteria for inertial stability of the internal modes that were derived in this paper will also apply to the principle of Paper II.

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FIGURE CAPTIONS

Figure 1. Space–Time Integration Volume. (A) Trajectories Cross Bounding Hypersurface Only at Terminal Volumes V_I and V_F but not at Bounding Surface $\mathscr{S}_B$. (B) Trajectories Also Cross Bounding Hypersurface at Access Surface $\mathscr{S}_A$

Figure 2. Dynamically Acceptable Flows. (A) Unique Solution of Differential Equations of Motion with Prescribed Initial Velocity $\mathbf{V}_I$. (B) Ensemble of Extrema of Variational Principle with Prescribed Average Velocity $\langle \mathbf{V} \rangle_t$

Figure 3. Various Types of Extrema (i.e., Stationary Points P) in Parameter Space

Figure 4. Sample Velocity Profiles that Satisfy the Internal Stability Criterion

Figure 5. Sample Velocity Profiles that Violate the Internal Stability Criterion

Figure 6. Doubly–Shearing Parallel Flow

Figure 7. Flux–Conserving Mixing of Two Layers in a Shearing Flow. (A) Before Mixing. (B) After Mixing. (C) Change in Terminal Positions for Particles in Each Layer