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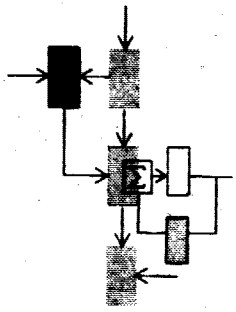

EXACT SOLUTION OF SOME LINEAR MATRIX EQUATIONS USING ALGEBRAIC METHODS

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EXACT SOLUTION OF SOME LINEAR MATRIX EQUATIONS USING ALGEBRAIC METHODS

Abstract

Let PA+BP = -C be a linear matrix equation where A is an n×n matrix, B an m×m matrix and C an m×n matrix, all three matrices taken over the reals. Let R[x,y] be the ring of polynomials in two indeterminants x and y over the reals R, and $\mathcal{MN}$ the set of all m×n matrices over the reals. Let $\Psi = (\phi_2(x), \psi_2(y))$ be the ideal in R[x,y] generated by $\phi_2(x)$ and $\psi_2(y)$, where $\phi_2(x)$ is the characteristic polynomial of A and $\psi_2(y)$ the characteristic polynomial of B. The elements of the quotient ring $R[x,y]/\Psi$ are denoted by $\Psi + a(x,y) = \{h(x,y) + a(x,y) \mid h(x,y) \in \Psi\}$.

Define the action $f_{BA}: R[x,y] \times \mathcal{MN} \to \mathcal{MN}$ in the following manner:

$$f_{BA}(h(x,y),M) = \sum_{jk} h_{jk} B^j \cdot M \cdot A^k$$

where $h(x,y) = \sum_{jk} h_{jk} x^k y^j$ is an element in R[x,y],

M is an element in $\mathcal{MN}$

The action f_{BA} allows for the interpretation of $\mathcal{MN}$ as a module over $R[x,y]/\Psi$ with multiplication (*) of elements in $R[x,y]/\Psi$ with elements in $\mathcal{MN}$ given by:

$$(\Psi + a(x,y)) * M = f_{BA}(a(x,y) \bmod \Psi, M)$$

where $a(x,y) \bmod \Psi$ is the element of minimal degree in $\Psi + a(x,y)$.

The polynomial x+y is in the coset $\Psi + (x+y)$. In the event that $\Psi + (x+y)$ has an "inverse" $\Psi + q_u(x,y)$ in $R[x,y]/\Psi$ such that $(\Psi + (x+y)) \cdot (\Psi + q_u(x,y)) = \Psi + u$ where $\Psi + u$ is the coset containing the real number $u \neq 0$ we can write:

$$(\Psi + (x+y))*P = PA + BP = -C$$

$$u \cdot P = (\Psi + q_u(x,y))*(-C) = f_{BA}(q_u(x,y) \bmod \Psi, -C)$$

$$P = \frac{1}{u} \cdot f_{BA}(q_u(x,y) \bmod \Psi, -C).$$

1. Introduction

In the past fifteen years or so there has been impressive progress in the theoretical understanding of the structure, representation and control of linear multivariable systems. In contrast, workers in the field have paid little attention to the computational aspects of systems problems. This does not mean that algorithms for the solution of systems problems have not been developed. But most of the algorithms that have been proposed have never been seriously studied as far as stability convergence and similar issues are concerned. Even the LQG problem, bulwark of the so called "modern control theory" seems to be little understood from the computational point of view.

In this paper we undertake a study of solution methods for Linear Matrix Equations including Lyapunov's equation

$$PA + A'P = -Q \tag{1.1}$$

using methods of modern algebra. The emphasis is on the use of finite algebraic procedures which are easily implemented on a digital computer and which lead to an explicit solution to the problem.

It is well known that the Lyapunov equation is important in the study of stability of linear finite dimensional time-invariant systems. If Q is symmetric and positive definite and if A is a stability matrix then the unique solution to (1.1) is given by the convergent integral

$$P = \int_0^\infty e^{A't} Q e^{At} dt$$

(cf. BROCKETT).

However, the solution requires the evaluation of an integral over an

infinite time interval. Other methods of solution do exist all of which have the basic drawback of giving an approximate solution. This becomes frustrating when the problem is ill-conditioned.

The need for solving the Lyapunov equation also arises when one uses Newton's Method to solve the Algebraic Riccati equation (cf. KLEINMAN)

$$(A - BR^{-1} B'P)'P + P(A - BR^{-1} B'P) = -C'C - PBR^{-1}B'P.$$

Here a finite number of Lyapunov equations have to be solved.

This paper has been inspired by an important paper by KALMAN. Kalman's concern was the characterization of polynomials whose zeroes lie in certain algebraic domains (and the unification of the ideas of Hermite are Lyapunov). In this paper we show that the same ideas lead to finite algorithms for the solution of linear matrix equations.

This paper is divided into five sections. In section 2 we introduce the action f_{BA} and prove a Basic Lemma. In section 3 we deal with the equation $PA + BP = -C$. In section 4 we analyze the Lyapunov equation, give algorithms for its solution and comment on the arithmetic complexity. In section 5 we deal with the equation $P - A'PA = Q$ and in section 6 we present numerical examples.

2. The action f_{BA}

Let A be an n×n matrix and B an m×m matrix both over the reals. Let R[x,y] be the ring of polynomials in two indeterminate x and y over the real numbers R. Let $\Psi = (\phi_2(x), \psi_2(y))$ be the ideal in R[x,y] generated by $\phi_2(x)$ the characteristic polynomial of A, and $\psi_2(y)$ the characteristic polynomial of B. Elements of the quotient ring $R[x,y]/\Psi$ are cosets denoted by $\Psi + a(x,y)$.

Define the action $f_{BA}: R[x,y] \times MN \to MN$ in the following manner:

$$f_{BA}(h(x,y),M) = \sum_{jk} h_{jk} B^j \cdot M \cdot A^k$$

where $h(x,y) = \sum_{jk} h_{jk} x^k y^j$, is an element in R[x,y] and M an element in MN.

It can be shown [2] that f_{BA} has the following properties:

i) $f_{BA}(u,M) = M$ where u is a real number

ii) $f_{BA}(g(x,y)+h(x,y),M) = f_{BA}(g(x,y),M) + f_{BA}(h(x,y),M))$

iii) $f_{BA}(g(x,y)\cdot h(x,y),M) = f_{BA}(g(x,y),f_{BA}(h(x,y),M))$

$= f_{BA}(h(x,y),f_{BA}(g(x,y),M))$

iv) Let $g(x,y) \bmod \Psi$ denote the polynomial of minimal degree in $\Psi + g(x,y)$ (which can be found by first dividing g(x,y) by $\phi_2(x)$, obtaining the remainder $R_x(x,y)$ and in turn dividing $R_x(x,y)$ by $\psi_2(y)$ and picking its remainder).

Then: $f_{BA}(g(x,y),M) = f_{BA}(g(x,y) \bmod \Psi, M)$

v) $f_{BA}(g(x,y), M+N) = f_{BA}(g(x,y),M) + f_{BA}(g(x,y), N)$

for all g(x,y) in R[x,y] and M,N in MN.

The definition of f_{BA} allows for the interpretation of MN as an

$R[x,y]/\Psi$- module.

Basic Lemma. The set MN of $m \times n$ matrices with real entries is a module over the quotient ring $R[x,y]/\Psi$.

Proof of Lemma: The set of $m \times n$ matrices under addition is an abelian group. Define multiplication (*) of cosets $\Psi + h(x,y)$ and $m \times n$ matrices M in the following manner:

$$(\Psi + h(x,y)) * M = f_{BA}(h(x,y) \bmod \Psi, M).$$

The multiplication is well defined and satisfies the properties:

1) $(\Psi + h(x,y)) * (M + N) = (\Psi + h(x,y)) * M + (\Psi + h(x,y)) * N$

2) $(\Psi + h(x,y)) * [(\Psi + g(x,y)) * M] = [(\Psi + h(x,y)) \cdot (\Psi + g(x,u))] * M$

3) $[(\Psi + h(x,y)) + (\Psi + g(x,y))] * M = (\Psi + h(x,y)) * M + (\Psi + g(x,y)) * M$

4) $(\Psi + 1) * M = M$

for all M,N in MN and all $\Psi + h(x,y), \Psi + g(x,y)$ in $R[x,y]/\Psi$, with $\Psi+1$ being the multiplicative identity in $R[x,y]/\Psi$.

Property v) of the action guarantees 1. Property iii) ensures the validity of 2. Property ii) makes certain that 3 holds. Property i) ensures the correctness of 4.

3. The equation PA + BP = -C

The Basic Lemma provides the groundwork for the construction of a method for obtaining the solution P of the equation

$$PA + BP = -C \tag{3.1}$$

whenever a unique solution does exist.

Equation (3.1) can be written as

$$f_{BA}(x+y,P) = PA + BP = -C.$$

Suppose that there exists a coset $\Psi + q_u(x,y)$ such that

$$(\Psi + q_u(x,y))\cdot(\Psi + (x+y)) = (\Psi + u) \tag{3.2}$$

where $\Psi + u$ is a coset which contains the real number $u \neq 0$. Let $q_u(x,y) \bmod \Psi$ be the polynomial of minimal degree in $\Psi + q_u(x,y)$. We then have:

$$(\Psi + (x+y))*P = PA + BP = -C$$

$$[(\Psi + q_u(x,y))\cdot(\Psi + (x+y))]*P = (\Psi + q_u(x,y))*(-C)$$

$$(\Psi + u)*P = (\Psi + q_u(x,y))*(-C)$$

$$uP = (\Psi + q_u(x,y))*(-C)$$

$$P = \frac{1}{u} f_{BA}(q_u(x,y) \bmod \Psi, -C)$$

The idea therefore is to ensure that for $\Psi + (x+y)$ condition (3.2) holds and to then construct such a polynomial $q_u(x,y) \bmod \Psi$.

Proposition 1. The coset $\Psi + (x+y)$ contains the polynomial $x+y$. There exists a coset $\Psi + q_u(x,y)$ for which we have

$$(\Psi + q_u(x,y))\cdot(\Psi + (x+y)) = \Psi + u \tag{3.3}$$

where $\Psi + u$ is a coset containing a real number $u \neq 0$ if and only if $\lambda_i + \mu_j \neq 0$ where $\lambda_i, i \leq i \leq n$ are the eigenvalues of $\phi_2(x) = \det(Ix-A)$ and $\mu_j, 1 \leq j \leq m$ are the eigenvalues of $\phi_2(y) = \det(Iy-B)$.

Proof of Proposition: We prove this Proposition by first showing that $\lambda_i + \mu_j \neq 0$ for all i,j iff $\psi_1(x) = \psi_2(-x)$ and $\phi_2(x)$ are relatively prime. Assume that $\psi_1(x)$ and $\phi_2(x)$ are relatively prime. Suppose then that there exist λ_i, μ_j such that $\lambda_i + \mu_j = 0$. This means that $\lambda_i = -\mu_j$

which implies that $\psi_1(x)$ and $\phi_2(x)$ have at least one root in common. This in turn implies that $\psi_1(x)$, $\phi_2(x)$ have a non-trivial common divisor which is a contradiction. Assume on the other hand that $\lambda_i + \mu_j \neq 0$ for all i,j. Suppose then that there exists a k(x) of degree greater than or equal to one such that $k(x) | \psi_1(x)$ $k(x) | \phi_2(x)$. This would imply that $\psi_1(x)$ and $\phi_2(x)$ have at least one root in common which contradicts our assumption.

It can be shown [2] that

$$x+y \mid \phi_2(x)\psi_2(y) - \phi_1(y)\psi_1(x).$$

$$\text{Let } P_{\psi\phi}(x,y) = \frac{\phi_2(x)\psi_2(y) - \phi_1(y)\psi_1(x)}{x + y}. \tag{3.4}$$

We now prove the Proposition.

Assume that $\lambda_i + \mu_j \neq 0$. We then have that $\phi_2(x)$ and $\psi_1(x)$ are relatively prime, which implies that there exist polynomials $\lambda_e(x)$, $\mu_e(x)$, $\lambda_e'(x)$,$\mu_e'(x)$ such that

$$\begin{aligned} \lambda_e(x)\psi_1(x) + \mu_e(x)\phi_2(x) &= e \\ \lambda_e'(x)\psi_2(x) + \mu_e'(x)\phi_1(x) &= e \end{aligned} \tag{3.5}$$

for some element $e \neq 0$ in R.

Let $q_u(x,y) = \lambda_e(x)\mu_e'(y)P_{\psi\phi}(x,y)$.

Since

$$\begin{aligned} (x+y)\cdot q_u(x,y) &= \lambda_e(x)\mu_e'(y)P_{\psi\phi}(x,y) \\ &= \lambda_e(x)\mu_e'(y)\phi_2(x)\psi_2(y) + e\lambda_e'(y)\psi_2(y) \\ &\quad + e\mu_e(x)\phi_2(x) - \mu_e(x)\lambda_e'(y)\phi_2(x)\psi_2(y) - e^2. \end{aligned}$$

we must have ($u = -e^2$)

$$(\Psi + (x+y)) \cdot (\Psi + q_u(x,y)) = \Psi + u.$$

Assume on the other hand that there exists a coset $\Psi + q_u(x,y)$ such that $(\Psi + q_u(x,y)) \cdot (\Psi + (x+y)) = \Psi + u$ where $\Psi + u$ contains the real number $u \neq 0$. Show that $\lambda_i + \mu_j \neq 0$ for all i,j.

We have that

$$q_u(x,y) \cdot (x+y) = a(x,y)\phi_2(x) + b(x,y)\psi_2(y) + u. \tag{3.6}$$

Suppose that there exist $i = i'$ and $j = j'$ such that

$$\lambda_{i'} = -\mu_{j'}.$$

Evaluating (3.6) at $x = \lambda_{i'}$ and $y = \mu_{j'}$ we have that

$$0 = u$$

which is a contradiction. This completes the proof of Proposition 1.

As can be seen from the proof of Proposition 1 the polynomial $q_u(x,y)$ can be constructed and this prescribes an algorithm for the solution of equation (3.1).

Algorithm for solving the Linear matrix equation PA + BP = -C.

A1) Obtain $\phi_2(x)$, $\psi_2(x)$ the characteristic polynomials of matrices A and B respectively.

A2) Set $P_{\psi\phi} = \dfrac{\phi_2(x)\psi_2(y) - \phi_1(y)\psi_1(x)}{x + y}$.

A3) Using the Extended Euclidean algorithm or an equivalent method obtain the polynomials $\lambda_e(x)$, $\mu_e'(x)$ and e.

A4) Find $q_u(x,y) = \lambda_e(x)\mu_e'(y)\ P_{\psi\phi}(x,y)$.

A5) Form $P_u = f_{BA}\ (q_u(x,y) \bmod \Psi, -C)$.

A6) Set $P = \frac{1}{u} P_u$, $u = -e^2$.

4. The Lyapunov equation A'P + PA = -Q

Suppose that B = A', C = Q = Q' with A a stability matrix (one which has the real parts of its eigenvalues in the left half complex plane). This is the special case of equation (3.1) known as the Lyapunov equation. Because of its importance we study it separately.

In this case where B = A' let us denote the action f_{BA} by f_A. Let $\Phi = (\phi_2(x), \phi_2(y))$ be the ideal in R[x,y] generated by $\phi_2(x)$ and $\phi_2(y)$ where as previously $\phi_2(x)$ is the characteristic polynomial of A. We denote by $\Phi + g(x,y)$ the cosets in $R[x,y]/\Phi$. We then have the following corollary to Proposition 1.

Corollary 1: The coset $\Phi + (x+y)$ contains the polynomial x+y. There exists a coset $\Phi + q_u(x,y)$ for which

$$(\Phi + q_u(x,y)) \cdot (\Phi + (x+y)) = \Phi + u$$

where $\Phi + u$ is a coset containing the real number $u \neq 0$, if and only if $\lambda_i + \lambda_j \neq 0$ for $1 \leq i, j \leq n$ where $\lambda_i, 1 \leq i \leq n$ are the eigenvalues of $\phi_2(x)$.

In this case we have

$$\begin{aligned} &\phi_1(x) = \phi_2(-x) \\ &\tau_e(x)\phi_1(x) + \lambda_e(x)\phi_2(x) = e \end{aligned} \tag{4.1}$$

for a real number $e \neq 0$

$$P_\phi(x,y) = \frac{\phi_2(x)\phi_2(y) - \phi_1(x)\phi_1(y)}{x + y} \tag{4.2}$$

$$q_u(x,y) = \tau_e(x)\tau_e(y)P_\phi(x,y). \tag{4.3}$$

Algorithm for solving the Lyapunov equation $A'P + PA = -Q$.

R_1) Obtain $\phi_2(x)$ the characteristic polynomial of A

R_2) Set $P_\phi(x,y) = \dfrac{\phi_2(x)\phi_2(y) - \phi_1(y)\phi_1(x)}{x + y}$.

R_3) Using the Extended Euclidean algorithm or an equivalent method obtain $\tau_e(x)$ and e.

R_4) Form $q_u(x,y) = \tau_e(x)\tau_e(y)P_\phi(x,y)$.

R_5) Find $P_u = f_A(q_u(x,y) \bmod \Phi, -Q)$.

R_6) Set $P = \frac{1}{u} \cdot P_u$, $u = -e^2$.

Computer Implementation

Since we are interested in an exact computer solution we restrict the field of interest to that of the rational numbers F. The algorithm is fully implementable, using the remarkable facilities provided by the computer programming system MACSYMA available at M.I.T. MACSYMA is a large computer programming system used for performing symbolic as well as numerical computations.

Three versions of the algorithm have been constructed and programmed on MACSYMA. They are the Rational algorithm, the Integer Algorithm and the Modular Algorithm having names indicative of the mode in which arithmetic operations are carried out.

The Rational Algorithm

It consists of carrying out steps R_1 through R_6 in rational arithmetic.

The Integer Algorithm

Suppose that the matrices A and Q only contained integer entries. The polynomials $\phi_2(x)$, $P\phi(x)$ then have integer coefficients. Define S to be the n×n matrix.

$$S = \begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ & & & \vdots & & & & \\ a_{2n-1} & a_{2n-2} & \cdots & & & & & a_n \end{bmatrix}$$

where $\phi_2(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ and $a_k = 0$ for $k > n$. Since $\phi_2(x)$ is a stability polynomial det $S > 0$ [1], and a_i $0 \leq i \leq n$ are positive integers. If we let $e = 2\det S$ the linear system

$$S \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \frac{e}{2} \end{bmatrix}$$

has an integer solution and there exists a polynomial $\tau_e(x) = d_1x^{n-1} + d_2x^{n-2} \ldots + d_n$ with $d_i = (-1)^{n-i}c_i$ which satisfies

$$\tau_e(x)\phi_1(x) + \lambda_\epsilon(x)\phi_2(x) = e.$$

This means that $q_u(x,y)$ as in (4.3) has integer coefficients. The polynomial $q_u(x,y) \bmod \Phi$ also has integer coefficients which implies that

$$P_u = f_A(q_u(x,y),-Q)$$

is a matrix with integer entries.

The algorithm proceeds as follows.

I_1) Find $\phi_2(x)$ the characteristic polynomial of A.

I_2) Set $P_\phi(x,y) = \dfrac{\phi_2(x)\phi_2(y) - \phi_1(x)\phi_1(y)}{x + y}$.

I_3) Find $\tau_e(x)$ and e.

I_4) Form $q_u(x,y) = \tau_e(x)\tau_e(y)P_\phi(x,y)$.

I_5) Find $P_u = f_A(q_u(x,y) \bmod \Phi, -Q)$.

I_6) Set $P = \frac{1}{u} \cdot P_u$, $u = -e^2$.

The Modular Algorithm

The integer algorithm paves the way for a modular approach to the solution.

Suppose p is a prime that does not divide $e = 2\det S$. If $A = (a_{ij})$ and $Q = (q_{ij})$ are matrices with integer entries let ${}_pQ = (q_{ij} \bmod p)$ and ${}_pA = (a_{ij} \bmod p)$ be considered as matrices over Z_p. A left subscript p on a polynomial b(x,y) written as ${}_pb(x,y)$ denotes coefficient reduction modulo p. Suppose that coefficient arithmetic is done modulo p. We then have

$$ {}_p\phi_2(x) = \det(Ix - {}_pA) $$

$$ {}_pP_\phi(x,y) = \frac{{}_p\phi_2(x)\,{}_p\phi_2(y) - {}_p\phi_1(x)\,{}_p\phi_1(y)}{x + y} $$

$$ {}_p\tau_e(x)\,{}_p\phi_1(x) + {}_p\lambda_e(x)\,{}_p\phi_2(x) = {}_pe $$

$$ {}_pq_u(x,y) = {}_p\tau_e(x)\,{}_p\tau_e(y)\,{}_pP_\phi(x,y). $$

Let ${}_pP_u = f_A({}_pq_u(x,y) \bmod_p \Phi, -{}_pQ)$ where all arithmetic is done modulo p and ${}_p\Phi = ({}_p\phi_2(x), {}_p\phi_2(y))$ in $Z_p[x,y]$. If ${}_pP_u$ and ${}_pu$ are obtained for a

sufficient number of primes, the Chinese Remainder Theorem can be used to find P_u and u making it possible to obtain the solution $P = \frac{1}{u} \cdot P_u$.

The algorithm is as follows:

M_1) Obtain ${}_pA$, ${}_pQ$.

M_2) Obtain ${}_p\phi_2(x) = \det(Ix - {}_pA)$.

M_3) Set ${}_pP_\phi(x,y) = \dfrac{{}_p\phi_2(x)\,{}_p\phi_2(y) - {}_p\phi_1(x)\,{}_p\phi_1(y)}{x + y}$.

M_4) Obtain ${}_pT_e(x)$, ${}_pe$.

M_5) Set ${}_pq_u(x,y) = {}_pT_e(x)\,{}_pT_e(y)\,{}_pP_\phi(x,y)$.

M_6) Obtain ${}_pP_u = f_A\,({}_pq_u(x,y) \bmod {}_p\Phi,\ -{}_pQ)$.

M_7) Repeat steps 1-6 for a sufficient number of primes and and using the Chinese Remainder Theorem find P_u and $u = -e^2$.

M_8) Set $P = \frac{1}{u} \cdot P_u$.

Since considerable coefficient growth takes place in intermediate computations of the Integer algorithm a lot of storage is being used up. In such cases it is advantageous to use the Modular Algorithm.

Arithmetic Complexity of the Integer Algorithm

We are concerned with the number of integer operations (addition, subtraction, multiplication, division) involved in running the Integer Algorithm when A and Q are n×n matrices, using classical operations.

Step I_1: There are several methods for obtaining the Characteristic polynomial $\phi_2(x)$ of a stability matrix. Evaluating $\phi_2(x)$ at n distinct points and then solving for the coefficients requires $0(n^4)$ operations. If n is in the range $n \leq 20$ evaluating $\phi_2(x)$ at x=1 where $\phi_2(1) = \Lambda$,

$\lambda = \lceil \log_{10} \Lambda \rceil$ and then at $x = 10^{\lambda}$ allows one to "read off" the coefficients of $\phi_2(x)$ from a large integer. This procedure requires only $0(n^3)$ operations.

Step I_2: This step can be done in $0(n^2)$ operations.

Step I_3: Solving a linear set of n equations simultaneously is an $0(n^3)$ operation.

Step I_4: Performing the multiplication as $\tau_e(x)[\tau_e(y) \cdot P_\phi(x,y)]$ requires $0(n^3)$ operations.

Step I_5: Obtaining $q_u(x,y) \bmod \Phi$ involves two polynomial divisions and can be done in $0(n^3)$ operations. To form $f_A(q_u(x,y) \bmod \Phi, -Q)$ we use $0(n^4)$ operations. In the event that the matrix Q is a product of vectors $Q = c \cdot c'$ this calculation can be done in $0(n^3)$ operations.

Step I_6: It can be done in $0(n^2)$ operations.

It can therefore be seen that the overall calculation requires $0(n^4)$ operations in general and $0(n^3)$ operations in the special cases mentioned.

Storage requirements are much harder to determine since the implementation is on a variable length word computer.

5. The equation $P - A'PA = Q$

The Basic Lemma provides the groundwork for the construction of a method for obtaining the solution P of the equation

$$P - A'PA = Q \tag{5.1}$$

whenever a unique solution exists.

Equation (5.1) can be written as

$$f_A(1 - xy, P) = P - A'PA = Q \tag{5.2}$$

where as in the Lyapunov equation, we are interested in the case when $B = A'$ and we denote f_{BA} by f_A. Let $\Phi = (\phi_2(x), \phi_2(y))$ with $\phi_2(x) = \det(Ix - A)$. We then have the following Proposition.

Proposition 2. The coset $\Phi + (1 - xy)$ contains the polynomial $1-xy$. There exists a coset $\Phi + q_u(x,y)$ for which

$$(\Phi + q_u(x,y))(\Phi + (1-xy)) = \Phi+u$$

where $\Phi+u$ is a coset containing the real number $u\neq 0$, if and only if $1-\lambda_i\lambda_j \neq 0$ for $1 \leq i,j \leq n$ where λ_i $1 \leq i \leq n$ are the eigenvalues of $\phi_2(x)$.

If we have that

$$\phi_2(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

then let

$$\phi_3(x) = x^n\phi_2(x^{-1}) = a_0x^n + a_1x^{n-1} + \ldots + a_n.$$

We can see that the roots of $\phi_3(x)$ are $\frac{1}{\lambda_i}$ $1 \leq i \leq n$.

Proof of Proposition 2: We will first show that $\phi_2(x)$ and $\phi_3(x)$ are relatively prime if and only if $1 - \lambda_i\lambda_j \neq 0$. Assume that $\phi_3(x),\phi_2(x)$ are relatively prime. Suppose then that there exist λ_i,λ_j such that $1 - \lambda_i\lambda_j = 0$. This means that $\lambda_i = \frac{1}{\lambda_j}$ which implies that $\phi_3(x)$ and $\phi_2(x)$ have at least one root in common. This implies in turn that $\phi_2(x),\phi_3(x)$ have a non-trivial common divisor which is a contradiction. Assume on the other hand that $1 - \lambda_i\lambda_j \neq 0$ for all i,j. Suppose that there exists a $k(x)$ of degree greater than or equal to one such that $k(x)|\phi_2(x)$ and $k(x)|\phi_3(x)$. This implies that $\phi_2(x)$, $\phi_3(x)$ have at least one root in common which contradicts

our assumption.

It can be shown that

$$1 - xy \mid \phi_2(x)\phi_2(y) - \phi_3(x)\phi_3(y).$$

Let

$$P_\phi(x,y) = \frac{\phi_2(x)\phi_2(y) - \phi_3(x)\phi_3(y)}{1-xy}.$$

We now prove the Proposition.

Assume that $1 - \lambda_i\lambda_j \neq 0$. We have that $\phi_2(x)$, $\phi_3(x)$ are relatively prime which implies that there exist polynomials $\tau_e(x)$, $\lambda_e(x)$ such that

$$\tau_e(x)\phi_3(x) + \lambda_e(x)\phi_2(x) = e$$

for some element $e \neq 0$ in R.

Let $q_u(x,y) = \tau_e(x)\tau_e(y)P_\phi(x,y)$.

Since

$$\begin{aligned} q_u(x,y)(1-xy) &= \tau_e(x)\tau_e(y)P_\phi(x,y)(1-xy) \\ &= \tau_e(x)\tau_e(y)\phi_2(x)\phi_2(y) \\ &\quad + e\lambda_e(y)\phi_2(y) + e\lambda_e(x)\phi_2(x) \\ &\quad - \lambda_e(x)\lambda_e(y)\phi_2(x)\phi_2(y) - e^2 \end{aligned}$$

we must have $(u = -e^2)$

$$(\phi + (1-xy))\cdot(\phi + q_u(x,y)) = \phi+u.$$

Assume on the other hand that there exists a coset $\phi + q_u(x,y)$ such that

$$(\phi + q_u(x,y))(\phi + (1-xy)) = \phi + u$$

where $\phi+u$ contains the real number $u \neq 0$. Show that $1 - \lambda_i\lambda_j \neq 0$ for all i,j.

We have that

$$q_u(x,y)\cdot(1-xy) = a(x,y)\phi_2(x) + b(x,y)\phi_2(y) + u \qquad (5.3)$$

Suppose that there exists $i = i'$ and $j = j'$ such that $1 - \lambda_{i'}\lambda_{j'} = 0$. Evaluating (5.3) at $x = \lambda_{i'}$, $y = \lambda_{j'}$ we have

$$0 = u$$

which is a contradiction. This completes the proof of Proposition 2.

As can be seen from the proof of Proposition 2, the polynomial $q_u(x,y)$ can be constructed and this prescribes an algorithm for the solution of equation (5.1).

Algorithm for solving the linear matrix equation $P - A'PA = Q$.

B_1) Obtain $\phi_2(x) = \det(Ix - A)$.

B_2) Set $P_\phi(x,y) = \dfrac{\phi_2(x)\phi_2(y) - \phi_3(x)\phi_3(y)}{1 - xy}$.

B_3) Using the Extended Euclidean Algorithm or an equivalent method obtain $\tau_e(x)$, e.

B_4) Form $q_u(x,y) = \tau_e(x)\tau_e(y)P_\phi(x,y)$.

B_5) Form $P_u = f_A(q_u(x,y),Q)$.

B_6) Set $P = \frac{1}{u}\cdot P_u$.

6. Numerical Examples

We wish to compute

$$G = \int_0^\infty x'(t)\cdot Q\cdot x(t)\,dt$$

where $x(t)$ is a solution to

$$\dot{x}(t) = Ax(t) \qquad x(0) = c. \qquad (*)$$

The system modelled by (*) is of the form

where the number of blocks is finite.

Example 1: The number of blocks is 5 with ζ=1, K=1, M=10000. Listed are the corresponding A matrix, the Q matrix and the solution P to the equation PA + A'P = Q.

Example 2: The number of blocks is 2. Listed are the corresponding A matrix in parametric form, the Q matrix and the parametric solution P of the equation PA + A'P = Q. The parametric solution P is valid only for appropriate values of E, M, Z, (Z=ζ).

Example 1:

$$A=\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{5000} & -\frac{1}{5000} & \frac{1}{10000} & \frac{1}{10000} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{10000} & \frac{1}{10000} & -\frac{1}{5000} & -\frac{1}{5000} & \frac{1}{10000} & \frac{1}{10000} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{10000} & \frac{1}{10000} & -\frac{1}{5000} & -\frac{1}{5000} & \frac{1}{10000} & \frac{1}{10000} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{10000} & \frac{1}{10000} & -\frac{1}{5000} & -\frac{1}{5000} & \frac{1}{10000} & \frac{1}{10000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10000} & \frac{1}{10000} & -\frac{1}{5000} & -\frac{1}{5000}
\end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12500}{3} & 0 & -\frac{10000}{3} & 0 & -2500 & 0 & -\frac{5000}{3} & 0 & -\frac{2500}{3} \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{10000}{3} & 0 & -\frac{20000}{3} & 0 & -5000 & 0 & -\frac{10000}{3} & 0 & -\frac{5000}{3} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -2500 & 0 & -5000 & 0 & -7500 & 0 & -5000 & 0 & -2500 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{5000}{3} & 0 & -\frac{10000}{3} & 0 & -5000 & 0 & -\frac{20000}{3} & 0 & -\frac{10000}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{2500}{3} & 0 & -\frac{5000}{3} & 0 & -2500 & 0 & -\frac{10000}{3} & 0 & -\frac{12500}{3} \end{bmatrix}$$

Example 2.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2E}{M} & -\frac{2Z}{M} & \frac{E}{M} & \frac{Z}{M} \\ 0 & 0 & 0 & 1 \\ \frac{E}{M} & \frac{Z}{M} & -\frac{2E}{M} & -\frac{2Z}{M} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{E}{2Z} & 0 & 0 & 0 \\ 0 & -\frac{M}{3Z} & 0 & -\frac{M}{6Z} \\ 0 & 0 & -\frac{E}{2Z} & 0 \\ 0 & -\frac{M}{6Z} & 0 & -\frac{M}{3Z} \end{bmatrix}$$

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