

General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.
- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.
- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.
- This document is paginated as submitted by the original source.
- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

GEOMETRIZATION OF THE DIRAC THEORY OF THE ELECTRON

V. Fock

Translation of "Geometrizierung der Dirachschen Theorie
des Elektrons," Zeitschrift für Physik, Vol. 57, 1929,
pp. 261-277.

(NASA-TM-75127) GEOMETRIZATION OF THE DIRAC
THEORY OF THE ELECTRON (Kanner (Leo)
Associates) 28 p HC A03/MF A01 CSCL 12/1

N77-25909

Unclass
H1/67 35879



1. Report No. NASA TM-75127	2. Government Accession No.	3. Recipient's Catalog No.	
4. Title and Subtitle GEOMETRIZATION OF THE DIRAC THEORY OF THE ELECTRON		5. Report Date June 1977	6. Performing Organization Code
		8. Performing Organization Report No.	10. Work Unit No.
7. Author(s) V. Fock, Leningrad		11. Contract or Grant No. NASW-2790	
		13. Type of Report and Period Covered Translation	
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, California 94063		14. Sponsoring Agency Code	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Washington, D.C. 20546			
15. Supplementary Notes Translation of "Geometrisierung der Dirachshen Theorie des Elektrons," Zeitschrift für Physik, Vol. 57, 1929, 261-277.			
16. Abstract Using the concept of parallel displacement of a half vector, the Dirac equations are generally written in invariant form. The energy tensor is formed and both the macroscopic and quantum mechanic equations of motion are set up. The former have the usual form: divergence of the energy tensor equals the Lorentz force and the latter are essentially identical with those of the geodesic line. The occurrence of the four-potential ϕ_1 together with the Ricci coefficient γ_{ikl} in the formula for parallel displacement on the one hand gives a simple geometrical reason for the occurrence of the expression $p_1 - \frac{e}{c}\phi_1$ in the wave equation, and on the other hand it shows that the ϕ_1 potentials, differing from Einstein's idea, have an independent place in the geometric world picture and must not, for example, be functions of γ_{ikl} .			
17. Key Words (Selected by Author(s))		18. Distribution Statement Unclassified - Unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages	22. Price

GEOMETRIZATION OF THE DIRAC THEORY OF THE ELECTRON

V. Fock
Leningrad

[Summary]

Using the concept of parallel displacement of a half vector, /261* the Dirac equations are generally written in invariant form. The energy tensor is formed and both the macroscopic and quantum mechanic equations of motion are set up. The former have the usual form: divergence of the energy tensor equals the Lorentz force and the latter are essentially identical with those of the geodesic line. The occurrence of the four-potential ϕ_1 together with the Ricci coefficient γ_{ikl} in the formula for parallel displacement on the one hand gives a simple geometrical reason for the occurrence of the expression $p_1 - \frac{e}{c}\phi_1$ in the wave equation, and on the other hand it shows that the ϕ_1 potentials, differing from Einstein's idea, have an independent place in the geometric world picture and must not, for example, be functions of γ_{ikl} .

In a work by D. Iwaneko and the author [7] it was suggested that the Dirac matrices have a purely geometric significance. In another work by these authors [6] the concept of parallel shift of a half vector, i.e. of a quadruple of quantities which are transformed like Dirac ψ functions, was proposed.

In another note [3] the author used this concept to set up the general relativity wave equation of the electron and derived the macroscopic equation of motion in the Einsteinian form.

The present work is a complete presentation of the observations made in the note cited above.

* Numbers in the margin indicate pagination in the foreign text.

1. The transformation properties of the Dirac ψ functions have been studied in detail by F. Möglich [9] and J. von Neumann [10]. The transformation equation has an especially simple form if /262 for the first three Dirac α matrices one chooses the expression

$$\alpha_1 = \sigma_1, \quad \alpha_2 = \rho_3 \sigma_2, \quad \alpha_3 = \sigma_3 \quad (1)$$

and for the fourth matrix one chooses

$$\alpha_4 = \rho_2 \sigma_2, \quad \alpha_5 = \rho_1 \sigma_2, \quad (1^*)$$

where $\rho_1, \rho_2, \rho_3; \sigma_1, \sigma_2, \sigma_3$ are the four-rowed matrices introduced by Dirac [2].

The the following transformation of the ψ functions corresponds to a general Lorentz transformation:

$$\left. \begin{aligned} \psi'_1 &= \alpha \psi_1 + \beta \psi_2; & \psi'_2 &= \bar{\alpha} \psi_2 + \bar{\beta} \psi_1, \\ \psi'_2 &= \gamma \psi_1 + \delta \psi_2; & \psi'_1 &= \bar{\gamma} \psi_2 + \bar{\delta} \psi_1. \end{aligned} \right\} \quad (2)$$

The complex quantities $\alpha, \beta, \gamma, \delta$ satisfy the condition

$$\alpha\delta - \beta\gamma = 1 \quad (3)$$

and in the case of a purely spatial rotation, are transformed into the usual parameters of Cayley and Klein.

If α_0 is used to designate the unit matrix, then the quantities

$$A_i = \bar{\psi} \alpha_i \psi \quad (i = 0, 1, 2, 3) \quad (4)$$

form the components of a four-vector, and the quantities

$$A_i = \bar{\psi} \alpha_i \psi, \quad A_4 = \bar{\psi} \alpha_4 \psi \quad (4^*)$$

are in variance. This fact is expressed in equations as follows. If S is used to designate Transformations (2):

$$\psi' = S\psi, \quad \bar{\psi}' = \bar{\psi}S^+, \quad (5)$$

where their S^+ stands for the adjoint (transposed-conjugated) matrix to S , then the following equations hold:

$$S^+ \alpha_i S = \sum_{k=0}^3 a_{ik} \alpha_k; \quad S^+ \alpha_4 S = \alpha_4; \quad S^+ \alpha_5 S = \alpha_5, \quad (6)$$

where a_{ik} are the coefficients of a general Lorentz transformation. Because

$$\bar{\psi}' \alpha_i \psi' = \bar{\psi} S^+ \alpha_i S \psi$$

the quantities (4) and (4*) are therefore transformed according to the equations

$$A'_i = \sum_{k=0}^3 a_{ik} A_k; \quad A'_4 = A_4; \quad A'_5 = A_5, \quad (7)$$

i.e. like a four-vector of like invariance. Since the A_1 ($i = 0, 1, 2, 3$) in the ψ quadratic quantities form a four-vector, we want to designate the ψ quantities with transformation properties (2) as "half vectors."¹

The explicit expressions for the quantities A_1 ($i = 0, 1, 2, 3, 4, 5$) are as follows: /263

$$\begin{aligned} A_0 &= \bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2 + \bar{\psi}_3 \psi_3 + \bar{\psi}_4 \psi_4, \\ A_1 &= \bar{\psi}_1 \psi_2 + \bar{\psi}_2 \psi_1 + \bar{\psi}_3 \psi_4 + \bar{\psi}_4 \psi_3, \\ A_2 &= -i\bar{\psi}_1 \psi_3 + i\bar{\psi}_3 \psi_1 + i\bar{\psi}_2 \psi_4 - i\bar{\psi}_4 \psi_2, \\ A_3 &= \bar{\psi}_1 \psi_4 - \bar{\psi}_4 \psi_1 + \bar{\psi}_2 \psi_3 - \bar{\psi}_3 \psi_2, \\ A_4 &= -\bar{\psi}_1 \psi_4 + \bar{\psi}_4 \psi_1 + \bar{\psi}_2 \psi_3 - \bar{\psi}_3 \psi_2, \\ A_5 &= -i\bar{\psi}_1 \psi_4 + i\bar{\psi}_4 \psi_1 - i\bar{\psi}_2 \psi_3 + i\bar{\psi}_3 \psi_2. \end{aligned}$$

Using these expressions, we confirm the following identical

¹ This term was introduced by L. Landau.

relationship between the quantities A_i :

$$A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 = A_0^2. \quad (8)$$

2. We have considered the transformation properties of ψ functions for a Lorentz transformation in the context of the special relativity theory. If we now assume the standpoint of the general relativity theory, then in order to be able to introduce the half vector concept, we must have an orthogonal (more precisely pseudo-orthogonal) reference system in each space-time point. For this purpose we introduce a network of four orthogonal curve congruences and, after Einstein, designate the directions of these congruences as "legs." The observations made in the above paragraphs then also remain valid for the general relativity theory case if by A_i we mean the components of a vector along the legs.

We number the legs with Roman indices and the coordinates with Greek indices which everywhere run through the values 0, 1, 2, 3. In the summation according to the Roman indices, the sum symbol is given explicitly, whereas in the summation using the Greek indices, it is suppressed. We designate the parameters of the curve congruences with small h_k^α and the moments with $h_{k,\alpha}$. Since we are dealing here with an indefinite metric, we introduce along with Eisenhart¹ the quantities $e_1 = e_2 = e_3 = -1$; $e_0 = +1$. The components of a vector along the coordinate directions (A_α) and along the legs (A'_k)² are then expressed in terms of each

¹ See [1] and also the excellent collection of the most important equations and facts in the study of T. Levi-Civita [8], p. 3.

² In what follows, the leg and coordinate components are more frequently designated with one and the same letter; in order to avoid confusions, the leg components are provided with a prime accent.

other as follows:

$$A'_k = A_\sigma h_k^\sigma; \quad A_\sigma = \sum_k e_k A'_k h_{k,\sigma} \quad (9)$$

If we designate the leg components of an infinitesimal shift with ds_k , then for the change of the components of a vector in a parallel shift, from the following equation /264

$$\delta A_\sigma = \Gamma_{\sigma\sigma}^f A_\rho dx^\rho; \quad \Gamma_{\sigma\sigma}^f = \left\{ \begin{smallmatrix} \sigma \\ \rho \end{smallmatrix} \right\} \quad (10)$$

results the following expression for the change of its leg components:

$$\delta A'_i = \sum_{k,l} e_k e_l \gamma_{ikl} A'_k ds_l \quad (11)$$

where γ_{ikl} are the rotation coefficients introduced by Ricci:

$$\gamma_{ikl} = (\nabla_\sigma h_i^\rho) h_{k,\rho} h_l^\sigma = (\nabla_\sigma h_{i,\rho}) h_k^\rho h_l^\sigma. \quad (12)$$

Here ∇_σ stands for the covariant derivation with respect to x^σ .

3. We now want to consider the change of the components of a half vector ψ for an infinitesimal parallel shift. For this change, we set up the equation

$$\delta \psi = \sum_i C_i ds_i \psi. \quad (13)$$

The C_1 's are matrices with the elements $(C_1)_{mn}$, and by $C_1 \psi$ we mean four functions whose m th degree is given by the equation

$$(C_1 \psi)_m = \sum_{n=1}^4 (C_1)_{mn} \psi_n$$

The conjugate complex equation for (13) is as follows:

$$\delta \bar{\psi} = \bar{\psi} \sum_i c_i C_i^\dagger ds_i \quad (13^*)$$

where $C_1^\dagger$ stands for the adjoint matrix. Now by means of Eq. (13) for the parallel shift of a half vector, the equation of a vector is already determined; specifically we must have:

$$\begin{aligned} \delta A'_i &= \delta (\bar{\psi} \alpha_i \psi) = \delta \bar{\psi} \alpha_i \psi + \bar{\psi} \alpha_i \delta \psi \\ &= \bar{\psi} \sum_i c_i (C_i^\dagger \alpha_i + \alpha_i C_i) ds_i \psi. \end{aligned} \quad (14)$$

Should this change agree with that given by (11), then the C_1 's must satisfy the conditions

$$C_i^\dagger \alpha_i + \alpha_i C_i = \sum_k c_k \alpha_k \gamma_{ik} \quad (15)$$

Furthermore, since $A'_4 = \bar{\psi} \alpha_4 \psi$ and $A'_5 = \bar{\psi} \alpha_5 \psi$ are invariants, then

$$\delta A'_i = \bar{\psi} \sum_i c_i (C_i^\dagger \alpha_i + \alpha_i C_i) ds_i \psi \quad (16)$$

and likewise $\delta A'_5$ must disappear, from which follow the further conditions

$$C_i^\dagger \alpha_i + \alpha_i C_i = 0; \quad C_i^\dagger \alpha_5 + \alpha_5 C_i = 0 \quad (17)$$

We immediately convince ourselves that the general solution /265 to Eqs. (15) and (17) is given by the equation

$$C_i = \frac{1}{2} \sum_{m,k} \alpha_m \alpha_k c_k \gamma_{mki} + i \Phi_i \quad (18)$$

in which ϕ'_1 's are Hermitian matrices, which must be interchangeable with all α_1 's as well as with α_4 and α_5 . If we remain in the region of the four-row matrices, then proportionality with the unit matrix follows from the interchangeability with all α matrices. By contrast, if we consider matrices with more than four rows,¹ then the case is not excluded that the ϕ'_1 are not proportional to the unit matrix. We want to stay with the four-row matrices and consider the ϕ'_1 's as real numbers.

It must be borne in mind that the C_1 's do not contain the α_4 and α_5 matrices so that the first two ψ functions are transformed between themselves and the last two between themselves. Because of Eq. (2) this was also to be expected a priori.

4. After we have proposed the concept of the parallel shift of a half vector, we can define the covariant derivation $D'_1\psi$ of a half vector ψ along leg direction 1 by the following equation

$$D'_1\psi = \frac{\partial\psi}{\partial s_1} - C_1\psi \quad (19)$$

where $\frac{\partial\psi}{\partial s_1} = h_1^\sigma \cdot \frac{\partial\psi}{\partial x^\sigma}$ stands for the derivation in the direction of the 1th leg. We designate the covariant derivation of a half vector along coordinate x^σ as follows

$$D_\sigma\psi = \frac{\partial\psi}{\partial x^\sigma} - \Gamma_\sigma\psi, \quad (19^*)$$

where for purposes of abbreviation

$$\Gamma_\sigma = \sum_k e_k h_k \cdot C_k \quad (20)$$

¹ Such matrices could perhaps occur in certain generalizations of the Dirac equation, for example as applied to the two body problem.

If for a moment we consider the space as pseudo-euclidian and set γ_{ikl} equal to zero, then Eq. (20) for D'_1 equals

$$D'_i \psi = \frac{\partial \psi}{\partial s_i} - i \phi'_i \psi.$$

But this is the same expression which appears in the Dirac equation if ϕ'_1 is taken to mean the quantity

$$\phi'_i = \frac{2\pi e}{hc} \varphi_i \quad (21)$$

where ϕ'_1 stands for the leg components of the vector potential. /266
In what follows we intend to adhere to this physical interpretation of the geometric quantities ϕ'_1 . Thus we have obtained a geometric interpretation for the occurrence of the vector potential in the Dirac equation, and indeed this interpretation is such that the potential can also be distinguished from zero when the gravitation terms containing the quantities γ_{ikl} disappear.

If we now turn to Eq. (13) for $\delta\psi$, we see that precisely here the Weylian linear differential form appears:

$$\sum_i e_i \phi'_i ds_i = \varphi \cdot dx$$

This is in agreement with the assumption expressed by Weyl.¹ The occurrence of the Weylian differential form in the equation for the parallel shift of a half vector is closely related to the fact noted by the author [4] and also by Weyl (loc. cit.) that the addition of a gradient to the four-potential corresponds to the multiplication of the ψ -function by a factor with an absolute value of 1. This fact was designated by Weyl as the "principle of gauge invariance."

¹ [13] Section 19, page 88.

If for a moment we consider the space as pseudo-euclidian and set γ_{ikl} equal to zero, then Eq. (20) for D'_1 equals

$$D'_1\psi = \frac{\partial\psi}{\partial s_1} - i\Phi'_1\psi.$$

But this is the same expression which appears in the Dirac equation if Φ'_1 is taken to mean the quantity

$$\Phi'_i = \frac{2\pi e}{hc} \varphi'_i \quad (21)$$

where φ'_1 stands for the leg components of the vector potential. /266 In what follows we intend to adhere to this physical interpretation of the geometric quantities Φ'_1 . Thus we have obtained a geometric interpretation for the occurrence of the vector potential in the Dirac equation, and indeed this interpretation is such that the potential can also be distinguished from zero when the gravitation terms containing the quantities γ_{ikl} disappear.

If we now turn to Eq. (13) for $\delta\psi$, we see that precisely here the Weylian linear differential form appears:

$$\sum_i a_i \varphi'_i ds_i = \varphi_0 dx^0.$$

This is in agreement with the assumption expressed by Weyl.¹ The occurrence of the Weylian differential form in the equation for the parallel shift of a half vector is closely related to the fact noted by the author [4] and also by Weyl (loc. cit.) that the addition of a gradient to the four-potential corresponds to the multiplication of the ψ -function by a factor with an absolute value of 1. This fact was designated by Weyl as the "principle of gauge invariance."

¹ [13] Section 19, page 88.

5. Using the concept of covariant derivation of a half vector, it is possible to set up the Dirac wave equation for the electron in the general relativity theory. For this purpose we consider the operator

$$F\psi = \frac{\hbar}{2\pi i} \sum_k e_k \alpha_k \left(\frac{\partial \psi}{\partial x_k} - C_k \psi \right) - mc \alpha_4 \psi. \quad (22)$$

We want to show that it is a self-adjoint operator.¹ In order to see this, we pass from the legs to the coordinates and introduce the matrices

$$\gamma^s = \sum_k e_k \alpha_k h_k^s \quad (23)$$

and the Γ_σ matrices defined by (20). Analogous relationships follow from Eqs. (15) for the just-introduced matrices

$$\Gamma_\sigma^* \gamma^s + \gamma^s \Gamma_\sigma = -\nabla_\sigma \gamma^s. \quad (24)$$

This equation can easily be proved by going back to Definition (12) of γ_{ikl} .

Expressed in terms of the coordinates, the operator F appears as follows:

/267

$$F\psi = \frac{\hbar}{2\pi i} \gamma^s \left(\frac{\partial \psi}{\partial x^s} - \Gamma_s \psi \right) - mc \alpha_4 \psi. \quad (25)$$

Taking into consideration (24) one can easily prove the identity

$$\bar{\psi} F\psi - (\bar{F}\psi)\psi = \frac{\hbar}{2\pi i} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^s} (\bar{\psi} \sqrt{g} \gamma^s \psi), \quad (26)$$

¹ The word "self-adjoint" is meant here in a somewhat extended sense. Specifically, we mean that the expression $\bar{\psi} F\psi - (\bar{F}\psi)\psi$ can be written in the form of a "in general four-dimensional" divergence.

where g stands for the absolute value of the determinant $||g_{\rho\sigma}||$. This identity expresses the fact that the operator F is a self-adjoint operator. This fact permits us to set up the equation

$$F\psi = 0 \quad (27)$$

for the Dirac equation in the general relativity theory. If ψ satisfies this equation, then it follows from identity (26) that the divergence of the current vector

$$S^{\rho} = \bar{\psi} \gamma^{\rho} \psi, \quad (28)$$

which is obviously real because of the Hermitian character of the γ^{ρ} matrices, disappears:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\rho}} (\sqrt{g} S^{\rho}) = 0. \quad (29)$$

It is easy to prove¹ that Eqs. (25) and (27) are invariant (more precisely, covariant) for the Dirac equation not only with respect to the choice of coordinates but also with respect to the choice of orthogonal curve congruences.

By way of proof, let us first observe that the values of Γ_{σ} can be uniquely defined in agreement with the above Definitions (18), (20), and (21) by the following equations:

$$\left. \begin{aligned} \Gamma_{\sigma} \gamma^{\sigma} + \gamma^{\sigma} \Gamma_{\sigma} &= -\nabla_{\sigma} \gamma^{\sigma} \\ \frac{1}{4} \text{trace} \Gamma_{\sigma} &= \frac{2\pi i e}{hc} \varphi_{\sigma} \end{aligned} \right\} \quad (30)$$

If we now introduce any new network whatsoever of curve congruences

¹ This section (to the end of section 5.) was added during the corrections.

and designate the quantities on this network with a star, then the new Γ_{σ}^* solutions to the analogous equations are as follows:

$$\left. \begin{aligned} \Gamma_{\sigma}^{*+} \gamma^{\sigma} + \gamma^{*\sigma} \Gamma_{\sigma}^* &= -\nabla_{\sigma} \gamma^{*\sigma} \\ \frac{1}{4} \text{trace} \Gamma_{\sigma}^* &= \frac{2\pi i e}{hc} \varphi_{\sigma} \end{aligned} \right\} \quad (30^*)$$

However the transition to the new leg directions in each /268
space-time point looks like a local Lorentz transformation. Therefore the new components of the half vector ψ^* and the new matrices $\gamma^{*\sigma}$ are related to the old ψ and γ^{σ} by equations of the form

$$\psi^* = S\psi; \quad \gamma^{\sigma} = S^* \gamma^{*\sigma} S \quad (31)$$

[cf. Eqs. (5) and (6)], where S stands for a matrix of the form

$$S = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \bar{\alpha} & \bar{\beta} \\ 0 & 0 & \bar{\gamma} & \bar{\delta} \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1$$

with variable elements.

But the transformation equation for the Γ_{σ} coefficients of the parallel displacement is as follows:

$$\Gamma_{\sigma}^* = S \Gamma_{\sigma} S^{-1} + \frac{\partial S}{\partial x^{\sigma}} S^{-1}, \quad (32)$$

for this expression is the unique solution of $(30^*)^1$.

¹ It holds that: $\text{trace} \frac{\partial S}{\partial x^{\sigma}} S^{-1} = 0$.

Furthermore it holds that

$$\frac{\partial \psi^*}{\partial x^\sigma} - \Gamma_\sigma \psi^* = S \left(\frac{\partial \psi}{\partial x^\sigma} - \Gamma_\sigma \psi \right). \quad (33)$$

If we let $F^* \psi^*$ stand for the analogous terms in (25) which we get if in that equation a star is assigned to γ^σ , Γ_σ and ψ , then from (31) and (33) it follows that

$$F\psi = S^* F^* \psi^*. \quad (34)$$

The equation $F\psi = 0$ is thus equivalent to $F^* \psi^* = 0$, which was to be proved.

6. In this section we want to present the operator F in another form in which we calculate the sum $\sum_k c_k \alpha_k C_k$ in Eq. (22).

In order to present the result in a clear form, we proceed as follows. We introduce the quantities ϵ_{ijkl} , which should disappear if among the $ijkl$ indices two appear which are identical, and in case different indices are equal to +1 or -1, according as the $ijkl$ sequence arises from 0 1 2 3 by means of an even or odd permutation. With the help of these quantities we form the "leg vector"

$$f_i = \frac{1}{2} \sum_{jkl} c_j c_k c_l \epsilon_{ijkl} \gamma_{jkl} \quad (35)$$

with the components

$$\left. \begin{aligned} f_0 &= -c_0(\gamma_{123} + \gamma_{231} + \gamma_{312}), \\ f_1 &= -c_1(\gamma_{023} + \gamma_{032} + \gamma_{320}), \\ f_2 &= -c_2(\gamma_{301} + \gamma_{013} + \gamma_{130}), \\ f_3 &= -c_3(\gamma_{102} + \gamma_{021} + \gamma_{210}). \end{aligned} \right\} \quad (35^*)$$

If we consider the identities

$$\left. \begin{aligned} \alpha_1 \alpha_2 \alpha_3 &= i \varrho_3 \alpha_1, \\ \alpha_2 \alpha_3 \alpha_1 &= i \varrho_1 \alpha_2, \\ \alpha_3 \alpha_1 \alpha_2 &= i \varrho_2 \alpha_3, \\ \alpha_1 \alpha_2 \alpha_3 &= i \varrho_3 \alpha_1, \end{aligned} \right\} \quad (1^{**})$$

which follow from Definition (1) of the α_i matrices, then we can write the sum $\sum_k e_k \alpha_k C_k$ in the form

$$\sum_i e_i \alpha_i C_i = \sum_i e_i \alpha_i \left(i \Phi_i - \frac{1}{2} \sum_j e_j \gamma_{ij} - \frac{i}{2} \varrho_i f_i \right) \quad (*)$$

We let

$$k_i = - \sum_j e_j \gamma_{ij} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} h_i^0) \quad (36)$$

and introduce the term (*) into (22). We then obtain

$$\begin{aligned} F\psi = \sum_j e_j \alpha_j \left(\frac{\hbar}{2\pi i} \frac{\partial \psi}{\partial x_j} - \frac{e}{c} \varphi_j \psi + \frac{\hbar}{4\pi i} k_j \psi \right) \\ + \frac{\hbar}{4\pi} \varrho_i \sum_j e_j \alpha_j f_j \psi - m c \alpha_i \psi. \end{aligned} \quad (22^*)$$

We note that in this equation the first and second sums are single self-adjoint operators.

In the event that all these congruences are normal congruences, then the "leg" factor f_1 disappears, since each Ricci symbol γ_{1kl} disappears individually with three different indices. Furthermore, we can then choose the hypersurfaces, whose perpendiculars are given by the curve congruences, as coordinate surfaces. We then have

$$ds^2 = \sum_j e_j H_j^2 dx_j^2; \quad \sqrt{g} = H_1 H_2 H_3; \quad (37)$$

$$h_i^0 = \frac{1}{H_i}; \quad k_i = e_i H_i; \quad f_i = 0, \quad (37^*)$$

while all $h_{1,\sigma}$ and h_1^σ parameters disappear with different indices /270 indices. The equation for the operator F is then as follows

$$F\psi = \sum_j e_j \alpha_j \frac{1}{H_j} \left(\frac{\hbar}{2\pi i} \frac{\partial \psi}{\partial x_j} - \frac{e}{c} \varphi_j \psi + \frac{\hbar}{4\pi i} \frac{\partial}{\partial x_j} \left(\lg \frac{\sqrt{g}}{H_j} \right) \psi \right) - mc \alpha_4 \psi. \quad (38)$$

This equation permits us to immediately write the Dirac equation in any curvilinear orthogonal coordinates. In so doing, the following must be borne in mind. If for example in the case of an ordinary euclidian space, we write Eq. (38) first in cartesian coordinates and then in curvilinear coordinates, then the ψ functions occurring in both cases in (38) are not identical, but they are related to one another by a transformation of the form (2) with variable coefficients $\alpha, \beta, \gamma, \delta$. This fact must be kept in mind in setting up the uniqueness postulates for the ψ functions.

In concluding this section it should be noted that as everyone knows, it is not always possible in a general Riemann space to select all the curve congruences as normal congruences. This is possible, however, in the important special cases of a static gravitation field with central and axial symmetry as has been shown by the solutions to the Einsteinian equations found by Schwarzschild and Levi-Civita.

7. We now want to try to find the energy tensor. To do this we consider the tensor

$$A^\sigma_\alpha = \bar{\psi} \gamma^\sigma \left(\frac{\partial \psi}{\partial x^\alpha} - \Gamma_\alpha \psi \right) = \bar{\psi} \gamma^\sigma D_\alpha \psi \quad (39)$$

and calculate its divergence.¹

¹ The results of this section could also be derived in a more elegant form by considering an infinitesimal transformation (cf. [11]). However, we prefer to proceed in a more elementary manner.

We write the Dirac equation with its conjugate complexes in the following form:

/271

$$\gamma^\mu \left(\frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi \right) - \frac{2\pi i}{h} mc \alpha_\mu \psi = 0, \quad (40)$$

$$\left(\frac{\partial \bar{\psi}}{\partial x^\mu} - \bar{\psi} \Gamma_\mu^+ \right) \gamma^\mu + \frac{2\pi i}{h} mc \bar{\psi} \alpha_\mu = 0. \quad (40^*)$$

We differentiate (40) with respect to x^α and multiply on the left by $\bar{\psi}$. We multiply Eq. (40*) on the right with $\frac{\partial \psi}{\partial x^\alpha}$ and add the results. If we take into account the following equation, resulting from (24)

$$\Gamma_\mu^+ \gamma^\mu + \gamma^\mu \Gamma_\mu = -\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} \gamma^\mu}{\partial x^\mu}, \quad (41)$$

then we can write the sum in the following form:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left(\bar{\psi} \gamma^\mu \frac{\partial \psi}{\partial x^\mu} \right) + \bar{\psi} \frac{\partial \gamma^\mu}{\partial x^\alpha} D_\mu \psi - \bar{\psi} \gamma^\mu \frac{\partial \Gamma_\mu^+}{\partial x^\alpha} \psi = 0 \quad (42)$$

Furthermore, we multiply (40) on the left with $-\bar{\psi} \Gamma_\alpha^+$, (40*) on the right with $\Gamma_\alpha \psi$ and add the results. In the sum we replace the terms $\Gamma_\alpha^+ \gamma^\sigma$ and $\Gamma_\sigma^+ \gamma^\alpha$ by their equivalent terms from Eqs. (24) and (41). In this way we get

$$\begin{aligned} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} (\bar{\psi} \sqrt{g} \gamma^\sigma \Gamma_\sigma \psi) + \bar{\psi} (\nabla_\alpha \gamma^\sigma) D_\sigma \psi \\ - \bar{\psi} \gamma^\sigma \frac{\partial \Gamma_\sigma^+}{\partial x^\alpha} \psi + \bar{\psi} \gamma^\sigma (\Gamma_\sigma \Gamma_\alpha - \Gamma_\alpha \Gamma_\sigma) \psi = 0. \end{aligned} \quad (43)$$

Here if we replace $\nabla_\alpha \gamma^\sigma$ with

$$\nabla_\alpha \gamma^\sigma = \frac{\partial \gamma^\sigma}{\partial x^\alpha} + \Gamma_{\alpha\sigma}^\rho \gamma^\rho,$$

then subtracting (43) from (42) gives

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} (\sqrt{g} A^a_{;a}) - F^a_b A^b_{;a} = \psi \gamma^a D_{aa} \psi, \quad (44)$$

where by way of abbreviation we set

$$D_{aa} = \frac{\partial \Gamma_a}{\partial x^a} - \frac{\partial \Gamma_a}{\partial x^a} + \Gamma_a \Gamma_a - \Gamma_a \Gamma_a \quad (45)$$

We must now calculate the $D_{\sigma\alpha}$ matrix. We have

$$D_{\sigma\alpha} = D_\sigma D_\alpha - D_\alpha D_\sigma = \sum_{kl} e_k e_l h_{k,\sigma} h_{l,\alpha} D'_{kl}, \quad (46)$$

where we set

$$D'_{kl} = D'_k D'_l - D'_l D'_k + \sum_m e_m (\gamma_{mkl} - \gamma_{mlk}) D'_m \quad (47)$$

The operator (47) is equal to

$$D'_{kl} = \frac{1}{4} \sum_{ij} a_i a_j e_j \gamma_{ijkl} + \frac{2\pi i e}{hc} M'_{kl}, \quad (48)$$

where γ_{ijkl} stand for the leg components of the Riemann tensor:

$$\gamma_{ijkl} = \frac{\partial \gamma_{ijk}}{\partial s_l} - \frac{\partial \gamma_{ijl}}{\partial s_k} + \sum_m e_m (\gamma_{ijm} (\gamma_{mkl} - \gamma_{mlk}) + \gamma_{mil} \gamma_{mjk} - \gamma_{mkl} \gamma_{mjl}), \quad (49)$$

and the skewsymmetric tensor M'_{kl} :

1272

$$M'_{kl} = \frac{\partial \varphi'_k}{\partial s_l} - \frac{\partial \varphi'_l}{\partial s_k} + \sum_m e_m (\gamma_{mkl} - \gamma_{mlk}) \varphi'_m, \quad (50)$$

represents the electromagnetic field.

First of all we express the $\gamma^\sigma D_{\sigma\alpha}$ matrix in terms of D'_{kl} :

$$\gamma^\sigma D_{\sigma\alpha} = \sum_{kl} e_k e_l \alpha_k h_{l,\alpha} D'_{kl}. \quad (51)$$

The sum $\sum_k e_k \alpha_k D'_{ki}$ which appears here can be calculated with the aid of (48), whereby the cyclic symmetry of the Riemann tensor must be taken into account. We get

$$\sum_k e_k \alpha_k D'_{ki} = \sum_k e_k \alpha_k \left(-\frac{1}{2} R_{ki} + \frac{2\pi i e}{hc} M'_{ki} \right), \quad (52)$$

where

$$R_{ki} = -\sum_l e_l \gamma_{likl} \quad (53)$$

denote the leg components of the reduced Riemann tensor. If we set (52) in (51) then we get

$$\gamma^e D_{ee} = \gamma^e \left(-\frac{1}{2} R_{ee} + \frac{2\pi i e}{hc} M_{ee} \right). \quad (51^*)$$

Thus for the divergence of the $A^{\delta}_{\cdot\alpha}$ we get the following equation

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\sigma} (\sqrt{g} A^{\sigma}_{\cdot\alpha}) - \Gamma^{\sigma}_{\alpha\sigma} A^{\sigma}_{\cdot\alpha} = S^e \left(-\frac{1}{2} R_{ee} + \frac{2\pi i e}{hc} M_{ee} \right). \quad (54)$$

If we set

$$\frac{c\hbar}{2\pi i} A^{\sigma}_{\cdot\alpha} = W^{\sigma}_{\cdot\alpha} = T^{\sigma}_{\cdot\alpha} + iU^{\sigma}_{\cdot\alpha}, \quad (55)$$

where $T^{\sigma}_{\cdot\alpha}$ and $U^{\sigma}_{\cdot\alpha}$ stand for the real and the imaginary component of the complex tensor $W^{\sigma}_{\cdot\alpha}$, then Eq. (54) can be written in the following form:

$$\nabla_\sigma W^{\sigma}_{\cdot\alpha} = S^e \left(e M_{ee} - \frac{\hbar c}{4\pi i} R_{ee} \right) \quad (56)$$

or if we separate the real and the imaginary components:

$$\left. \begin{aligned} \nabla_\sigma T^{\sigma}_{\cdot\alpha} &= e S^e M_{ee}, \\ \nabla_\sigma U^{\sigma}_{\cdot\alpha} &= \frac{\hbar c}{4\pi} S^e R_{ee}. \end{aligned} \right\} \quad (57)$$

The second of these two equations is an easy-to-prove identity, for the $U^{\sigma}{}_{,\alpha}$ tensor is equal to

$$U^{\sigma}{}_{,\alpha} = -\frac{\hbar c}{4\pi} \nabla_{\alpha} S^{\sigma}, \quad (58)$$

and the divergence of the S^{σ} vector disappears according to (29).

Eq. (57) states that the divergence of tensor $T^{\alpha}{}_{,\alpha}$ is equal /273 to the Lorentz force. We can therefore interpret $T^{\sigma}{}_{,\alpha}$ as an energy tensor. Eq. (57) are then the equations of motion of the general relativity theory. Perhaps it would be more consistent to interpret not just the real component $T^{\sigma}{}_{,\alpha}$ but the entire complex tensor $W^{\sigma}{}_{,\alpha}$ as an energy tensor. We will not discuss here which of these interpretations is preferable.

What is surprising here is the appearance of the electromagnetic tensor $M_{\rho\alpha}$ along with the Riemann tensor $R_{\rho\alpha}$ in the form of a Hermitian matrix.

$$\left\| R_{\rho\sigma} - \frac{4\pi i c}{\hbar c} M_{\rho\sigma} \right\|.$$

8. From the results obtained in order to derive the quantum mechanic equations of motion which correspond to those of a mass point (geodesic), we proceed as follows.

In the range of spatial variables x_1, x_2, x_3 we choose a complete system of functions:

$$\psi_n(x_1, x_2, x_3; \xi) \quad (\xi = 1, 2, 3, 4), \quad (59)$$

each of which satisfies the Dirac equation¹ and is normalized by the postulate:

¹Cf [5] and [2].

$$\iiint \bar{\psi}_n \gamma^0 \psi_n \sqrt{g} dx_1 dx_2 dx_3 = 1 \quad (60)$$

Because of (26) and (27) it follows from the existence of this equation for a special value of x_0 that it is valid for any other value of x_0 . We define the matrix element for an operator L by the equation

$$L_{mn} = \iiint \bar{\psi}_m L \psi_n \sqrt{g} dx_1 dx_2 dx_3. \quad (61)$$

We bear in mind that the operations performed in the preceding section and in particular Eq. (54) remain unchanged if in $A^{\sigma \cdot \alpha}$ and in S^{ρ} we replace ψ by ψ_n and $\bar{\psi}$ by $\bar{\psi}_m$, hence by two different solutions of the Dirac equation. We now write Eq. (54) in the form

$$\begin{aligned} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^0} (\bar{\psi}_m \sqrt{g} \gamma^0 D_0 \psi_n) &= \Gamma_{\alpha\beta}^0 \bar{\psi}_m \gamma^0 D_0 \psi_n \\ &+ \bar{\psi}_m \gamma^0 \left(-\frac{1}{2} R_{\beta\alpha} + \frac{2\pi i e}{\hbar c} M_{\beta\alpha} \right) \psi_n. \end{aligned} \quad (62)$$

If we multiply (62) with $\sqrt{g} dx_1 dx_2 dx_3$ and integrate over the entire space, then only one single term remains left over from the sum on the left side of (62), and we obtain /274

$$\begin{aligned} &\frac{d}{dx^0} \left\{ \iiint \bar{\psi}_m \gamma^0 D_0 \psi_n \sqrt{g} dx_1 dx_2 dx_3 \right\} \\ &= \iiint \bar{\psi}_m \left[\Gamma_{\alpha\beta}^0 \gamma^0 D_0 + \gamma^0 \left(-\frac{1}{2} R_{\beta\alpha} + \frac{2\pi i e}{\hbar c} M_{\beta\alpha} \right) \right] \psi_n \sqrt{g} dx_1 dx_2 dx_3, \end{aligned} \quad (63)$$

which we can also write symbolically in the form

$$\frac{d}{dx^0} (\gamma^0 D_0) = \Gamma_{\alpha\beta}^0 \gamma^0 D_0 + \gamma^0 \left(-\frac{1}{2} R_{\beta\alpha} + \frac{2\pi i e}{\hbar c} M_{\beta\alpha} \right) \quad (64)$$

or also, if we set

$$P_\alpha = \frac{\hbar}{2\pi i} D_\alpha \quad (65)$$

in the form

$$\frac{d}{ds}(\gamma^\sigma P_\sigma) = \Gamma_{\sigma\tau}^\sigma \gamma^\tau P_\sigma + \gamma^\sigma \left(\frac{e}{c} M_{\sigma\tau} - \frac{\hbar}{4\pi i} R_{\sigma\tau} \right). \quad (66)$$

Now we can interpret the γ^σ operators as representing the classical velocity $\frac{dx^\sigma}{dx^0}$ and P_σ as that particular quantity of the covariant motion quantities $mg_{\sigma\rho} \frac{dx^\rho}{ds}$. This interpretation makes it possible to complete the transition to the classical theory. If we do this and consistently ignore the $\hbar$ term on the right side, then we obtain precisely the classical equation of motion for a charged mass point in a gravitation field and in particular -- when no electromagnetic field is present -- the differential equation of the geodesic.

9. The pure covariant tensor

$$W_{\sigma\alpha} = g_{\sigma\tau} W^\tau{}_\alpha = c \bar{\psi} \gamma_\sigma P_\alpha \psi \quad (55*)$$

is not symmetric with respect to its indices. Because of the significance of the $c\gamma_\sigma$ and P_α operators (velocity and momentum) the quantum mechanic parameter $W_{\sigma\alpha}$ corresponds to the classic parameter $\rho_0 u_\sigma u_\alpha$

$$W_{\sigma\alpha} \rightarrow \rho_0 u_\sigma u_\alpha, \quad (67)$$

where u_α stands for the classical covariant component of the four-velocity and ρ_0 stands for the rest density of the matter. However, the parameter $\rho_0 u_\sigma u_\alpha$ is symmetric with respect to the indices.

The Dirac equation (27) can be derived from a variation principle which can be formulated as follows with the aid of

the energy tensor:

$$\delta \iiint (W^0_0 - mc^2 \bar{\psi} \alpha_4 \psi) \sqrt{g} dx_0 dx_1 dx_2 dx_3 = 0. \quad (68)$$

This equation produces a simple physical interpretation of the invariance $m \bar{\psi} \alpha_4 \psi$ as the rest density of the matter. /275

10. We now want to summarize the results of our study.

Our starting point is the concept of the parallel shift of a half vector. By means of this concept, the appearance of the ϕ_α potentials along with the p_α impulses in the Dirac equation can be interpreted purely geometrically. The purely formal transfer of the expression $p_\alpha - \frac{e}{c} \phi_\alpha$ from classical mechanics into quantum mechanics thus became superfluous. Furthermore, this concept allowed us to easily arrange the potentials in the geometrical scheme of the general relativity theory, and this can be useful for setting up a unified theory of electricity and gravitation.

Moreover, the Dirac equations were set up in the general relativity theory. These are invariant with respect to the choice of coordinates and "legs." This produced a secondary result, namely an explicit representation of the Dirac operator in curvilinear orthogonal coordinates. A tensor was constructed whose divergence is equal to the Lorentz force. This tensor was interpreted as an energy tensor and the equation satisfying it was interpreted as a macroscopic equation of motion. In addition, the quantum mechanic equations of motion for the electron were derived which correspond to the classical equations for a charged mass point or -- in the absence of an electromagnetic field -- the equations of a geodesic. Finally, the variation principle from which the Dirac equation can be derived was written.

Our aim was to geometrize the Dirac theory of the electron and integrate it into the general relativity theory. In so doing, the difficulties attached to the Dirac theory -- such as the occurrence of negative energy values and a non-disappearing probability of charge exchange of the electron -- were not at all encountered. But perhaps our observations can contribute indirectly to solving these difficulties by showing what the original, unchanged Dirac theory can accomplish.

Leningrad, University Physics Institute, May/June 1929.

After completing this paper, I discovered the very interesting paper of H. Weyl [12]. Weyl's basic mathematical idea is essentially identical to the concept of the parallel shift of a half vector. However the physical content of Weyl's work is completely different from mine.

The essential features of Weyl's approach can be summarized as follows.

1. Weyl considers the Dirac equation to be a wave equation not for the electron but for the electron-proton system.
2. In the additive gravitation terms, Weyl thinks he has found a substitute for the $m\alpha_4$ term, the latter simply being cancelled.

In my view, both of these theses can hardly be supported, for they run into considerable difficulties which I would here like to draw attention to.

The quantum mechanic equations of motion resulting from the Dirac equation are completely analogous with the classic equations of motion for a charged particle (and not by the way for a two-body system), as has already been shown in my earlier work [2].

The Dirac equation, and indeed with the $m\alpha_4$ terms, is perfectly suited for describing the force-free motion of an electron as a wave in the sense of the original de Broglie view.

The splitting up of the current vector S by Weyl into two summands $S^{(+)}$ and $S^{(-)}$, which are interpreted as positive current and negative electricity, cannot be upheld, for these

summands are null vectors, and only their sum $S = S^{(+)} + S^{(-)}$ is a time-like vector.¹ However, the current is a static-macroscopic quantity and as such must have the same character as in the classical theory, hence must necessarily be time-like. /277

Weyl's equations are supposed to describe the electron-proton system. We may therefore demand that they accurately reflect the energy level of the hydrogen atom. However, since the $m\alpha_4$ term has been left out, this is hardly possible and in any case is not proven.

The gravitation terms ["leg vector" f_1 in our equation (35)] interpreted by Weyl as a substitute for the mass can be made to disappear as soon as a system of normal congruences exists and especially in the case of spherical symmetry as well as in the static case of axial symmetry. However, one can expect a large degree of symmetry from the electron-proton system.

Finally, it remains completely unclear just how the constants m and M -- the mass of the electron and of the proton -- should be produced from the gravitation terms.

Because of these difficulties, I cannot consider Weyl's attempt to tackle the quantum mechanics problem of mass and the two body problem as successful. On the other hand, I gladly concur with Weyl's general idea that both problems are closely

¹ Proof: The time-like character of S follows from identity (8) (where now S_1 is to be read instead of A_1), for it gives

$$S_0^2 - S_1^2 - S_2^2 - S_3^2 = S_4^2 + S_5^2 \quad (*)$$

$S_1^{(+)}$ or $S_1^{(-)}$ is obtained from S_1 if ψ_3 and ψ_4 or ψ_1 and ψ_2 are set equal to zero. In both cases S_4 and S_5 disappear, thus also the left side of (*), which was to be proved.

related to one another and to the problem of gravity.

In conclusion, I would like to make a few general remarks on the physical content of the Dirac equations and on the two body problem in quantum mechanics.

In my opinion, the Dirac equation describes only the electron in terms of quantum mechanics while it describes the rest of the world (perhaps also the mass of the electron) macroscopically. In this case the rest of the world also includes the proton. The solution to the two body problem must consist in finding a quantum mechanical description of the electron, the proton, the electromagnetic field and the mass. The quantum mechanical problem of mass seems to me to defy solution as long as only one body is considered. By contrast, for the macroscopic description of gravity and electricity, the quantum mechanical one body problem seems to render good service.

BIBLIOGRAPHY

1. Eisenhart, L., Riemannian Geometry, Princeton, 1926.
2. Fock, V., "The concept of velocity in the Dirac theory of the electron. Appendix." Zeitschrift für Physik 55, 127 (1929).
3. Fock, V., "Dirac equations in the general relativity theory," C.R. 189, 25 (1929).
4. Fock, V., "The invariant form of wave and motion equations for a charged mass point," Zeitschrift für Physik 39, 226 (1926).
5. Fock, V., Zeitschrift für Physik 49, 323 (1928).
6. Fock, V. and Iwanenko, D., "Linear quantic geometry and parallel displacement," C.R. 188, 1470 (1929). This paper was delivered on May 20, 1929 at the Physics Conference in Charkow.
7. Fock, V. and Iwanenko, D., "A possible geometric interpretation of the relativistic quantum theory," Zeitschrift für Physik 54, 798 (1929).
8. Levi-Civita, T., Vereinfachte Herstellung der Einsteinschen einheitlichen Feldgleichungen [Simplified Production of Einsteinian Unified Field Equations], Berl. Ber. 1929, p. 3.
9. Möglich, F., "Quantum Theory of the Rotating Electron," Zeitschrift für Physik 48, 852 (192-).
10. Neumann, J. von, "Some observations on the Dirac theory of the relativistic rotating electron," Ebenda [this journal: Zeitschrift für Physik] p. 868.
11. Tetrode, H., "General relativity quantum theory of the electron," Zeitschrift für Physik 50, 336 (1928).
12. Weyl, H., "Gravitation and the electron," Proc. Nat. Acad. Amer. 15, 323 (1929).
13. Weyl, H., Gruppentheory und Quantumechanik [Group Theory and Quantum Mechanics], Section 19, pp. 88. Leipzig, 1928.